Standard Model Solution 5:

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1 The Eightfold Way

a) Key facts about Young Tableaux:

- Vertical representations are completely antisymmetric
- Horizontal representations are completely symmetric
- To tensor product with a fundamental representation, add a box to the right or below in each possible place. A valid Young Tableau should look like upside down stairs.
- For $SU(N)$ put an N in the upper left. The number in each box should be one more than than the box to its left, and one less than the box above it.
- To find the dimension of the representation multiply all of the boxes together and divide by the hooks.

Now let's have some fun with Young Tableaux:

b) Fun with Young Tableaux:

Please use your knowledge to prove the following important fun facts

$$\begin{aligned} 8 &= 4 + 2 + 2 \\ 27 &= 10 + 8 + 8 + 1 \end{aligned} \tag{1}$$

Show that they are more important for particle physicists than most of the other splittings, e.g.,

$$\begin{aligned} 8 &= 2 + 3 + 3 \\ 27 &= 20 + 2 + 2 + 3 \end{aligned} \tag{2}$$

Every Baryon is made of 3 quarks and it is a fermion, so according to Pauli the wavefunction should be completely antisymmetric under the exchange of two quarks. The wavefunction of a baryon is always factorized as the following

$$\psi_{\text{space}}\psi_{\text{spin}}\psi_{\text{flavor}}\psi_{\text{color}} \tag{3}$$

c) Let us consider $l_1 = l_2 = 0$ case (why two angular momentum?), so that ψ_{space} is always completely symmetric. In 1963, there were 9 baryons known which all had similar masses and spin $\frac{3}{2}$. According to the fun facts, which multiplet of the $SU(3)$ flavor (aka isospin) group they must live in? Which representation of the $SU(2)$ spin group?

d) Using the Young Tableau you drew, what is the symmetry of this multiplet under spin?

e) Using the Young Tableau you drew, what is the symmetry of this multiplet under flavor?

f) Now argue that there must be another symmetry, color, under which the wavefunction is completely antisymmetric.

g) If color were not there in some different universe, how many spin $\frac{3}{2}$ baryon could there be in a multiplet?

In 1964 Ω^-, the tenth baryon was discovered, and the representations of the $SU(3)$ approximate flavor symmetry or the 8-fold way were completed. We will have to do a bit more work to figure out which of the other representations are relevant. Since the wavefunction is symmetric in space, and completely antisymmetric in color, we are looking for representations that are overall symmetric in spin and flavor.

You have already argued that the singlet of $SU(3)$ color is antisymmetric. The same logic shows that the singlet of $SU(3)$ flavor is antisymmetric. The two 8's of $SU(3)$ flavor and two 2's of $SU(2)$ spin have mixed symmetry. We can choose one of the 8's to be antisymmetric under the exchange of the first two quarks and the other to be symmetric under the exchange of the first two quarks, and similarly for the the 2's.

h) Why are there no $(10, 2)$, $(8, 4)$, $(1, 4)$, or $(1, 2)$ multiplets of the flavor and spin groups $(SU(3),\ SU(2))$?

Solution:

a) By addition of the numbers, the fun facts speak truth. See the other file for the Young Tableau construction and calculation. The fun facts in particle physics represents the three spin $\frac{1}{2}$ quark can make one spin $\frac{3}{2}$ multiplet with 4 baryons and two spin $\frac{1}{2}$ multiplet with 2 baryons. And three quarks that come in three flavors can make one multiplet with 10 baryons and two multiplets with 8 baryons and a singlet.

b) There are two angular momenta, because there will be a rotation around the center of the mass of quark 1 and quark 2. And another rotation around the center of (the center of quark 1 and 2) and quark 3. Since 9 baryons could not be in the octet, they must be in the multiplet with 10 baryons.

c) under spin it is totally symmetric

d) under flavor it is also totally symmetric

e) so under space$\times$ spin $\times$ flavor, it is totally symmetric, conflict with Pauli exclusion principle. The resolution is the color singlet.

f) In different universe, if we have spin totally symmetric, we have to have to flavor totally anti-symmetric without color, thus we have only one baryon.

g) a) (10,2) is completely symmetric under flavor, but only partially symmetric under spin, then together can never be completely symmetric.

   b) (8,4) is completely symmetric under spin, but only partially symmetric under flavor, then together can never be completely symmetric.

   c) (1,4) is completely anti-symmetric under flavor, completely symmetric under spin, then together can never be completely symmetric.

   d) (1,2) is completely anti-symmetric under flavor, but only partially symmetric under spin, then together can never be completely symmetric.

2 Magnetic Moment of the Proton and Neutron

In this problem you will compute the ratio of the magnetic moment of the proton and neutron in the nonrelativistic limit. To do this, we need to construct the wavefunction of the proton and neutron. We know from the previous problem that the spin and flavor part of the wavefunction should be completely symmetric. For example, the spin and flavor parts of the wavefunction of an Ω^- with $J_3 = \frac{3}{2}$ is

$$s \uparrow s \uparrow s \uparrow \tag{4}$$

a) The symmetric combination of u, u, and d quarks is:

$$\Delta = \sqrt{\frac{1}{3}} \left[uud + udu + duu\right] \tag{5}$$

There are also two combinations of mixed symmetry. We can take one to be the combination antisymmetric under exchange of the first two quarks:

$$p_A = \sqrt{\frac{1}{2}} \left[ud - du\right] u \tag{6}$$

Show that the following state (which is symmetric under the exchange of the first two quarks) is orthogonal to both p_A and Δ:

$$p_S = \sqrt{\frac{1}{6}} \left[(ud + du)u - 2uud\right] \tag{7}$$

b) By making the replacements $u \rightarrow\uparrow$ and $d \rightarrow\downarrow$ we immediately get the symmetric spin-up combination of three spins:

$$\chi_S = \sqrt{\frac{1}{3}} \left[\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow\right] \tag{8}$$

as well as the spin-up combinations of mixed symmetry that are symmetric under exchange of the first two spins:

$$\chi_{MS} = \sqrt{\frac{1}{6}}\,[(\uparrow\downarrow + \downarrow\uparrow)\uparrow -2\uparrow\uparrow\downarrow] \tag{9}$$

and the one antisymmetric under exchange of the first two spins:

$$\chi_{MA} = \sqrt{\frac{1}{2}}\,[\uparrow\downarrow - \downarrow\uparrow]\uparrow \tag{10}$$

Show that the combination

$$\sqrt{\frac{1}{2}}\,(p_S\chi_{MS} + p_A\chi_{MA}) \tag{11}$$

is completely symmetric. You have found the wavefunction for a spin-up proton. Ta da.

c) In the non-relativistic approximation, we can write the magnetic moment of the proton as

$$\mu_p = \sum_{i=1}^{3} \langle p\uparrow|\mu_i(\sigma_3)_i|p\uparrow\rangle \tag{12}$$

where the sum is over the constituent quarks. The (classical) magnetic moment of a point charge is

$$\mu_i = Q_i\frac{e}{2m_i} \tag{13}$$

Compute the magnetic moment of the proton in terms of the magnetic moments of the up and down quarks.

d) We can find the neutron magnetic moment by exchanging u and d. By assuming $m_u \approx m_d$, determine the ratio of the proton to neutron magnetic moments. Experimentally the ratio is measured to be $\frac{\mu_p}{\mu_n} = -1.45989806(34)$. Does your calculation agree?

e) Note that suppose again we do not have color symmetry, and we can indeed construct a totally antisymmetric proton wavefunction, such as

$$|p\uparrow\rangle = \sqrt{\frac{1}{2}}[p_A\chi_{MS} - p_S\chi_{MA}]$$

Calculate the magnetic moment ratio again, and show that it is excluded by experiment. Long Live Color.

Solution:

a) Let us compute the dot product of p_S and Δ,

$$\begin{aligned}\langle p_S|\Delta\rangle &= \langle\sqrt{\frac{1}{6}}\left[(ud+du)u-2uud\right]|\sqrt{\frac{1}{3}}\left[uud+udu+duu\right]\rangle \\ &= \frac{1}{3}\sqrt{\frac{1}{2}}\langle udu+duu-2uud|uud+udu+duu\rangle \\ &= \frac{1}{3}\sqrt{\frac{1}{2}}(0+1+0+0+0+1-2-0-0) \\ &= 0\end{aligned} \tag{14}$$

Now compute the dot product of p_S and p_A,

$$\begin{aligned}\langle p_S|p_A\rangle &= \langle\sqrt{\frac{1}{6}}\left[(ud+du)u-2uud\right]|\sqrt{\frac{1}{2}}\left[ud-du\right]u\rangle \\ &= \frac{1}{2}\sqrt{\frac{1}{3}}\langle udu+duu-2uud|udu-duu\rangle \\ &= \frac{1}{2}\sqrt{\frac{1}{3}}(1+0+0-1+0+0) \\ &= 0\end{aligned} \tag{15}$$

b) It is obviously symmetric under the exchange of the first two quarks by construction. We only need to show that it is also symmetric under the exchange of the first and third quarks by construction. Then by transportation, it is also symmetric under the exchange of second and third quark. Let us exchange the first and the third quark then,

$$\begin{aligned}\sqrt{\frac{1}{2}}\left(p_S\chi_{MS}+p_A\chi_{MA}\right) &= \sqrt{\frac{1}{2}}\left(\sqrt{\frac{1}{6}}\left[(udu+duu)-2uud\right]\right)\left(\sqrt{\frac{1}{6}}\left[(\uparrow\downarrow\uparrow+\downarrow\uparrow\uparrow)-2\uparrow\uparrow\downarrow\right]\right) \\ &\quad+\sqrt{\frac{1}{2}}\left(\sqrt{\frac{1}{2}}\left[udu-duu\right]\right)\left(\sqrt{\frac{1}{2}}\left[\uparrow\downarrow\uparrow-\downarrow\uparrow\uparrow\right]\right) \\ &\xrightarrow{1\leftrightarrow 3}\sqrt{\frac{1}{2}}\left(\sqrt{\frac{1}{6}}\left[(udu+uud)-2duu\right]\right)\left(\sqrt{\frac{1}{6}}\left[(\uparrow\downarrow\uparrow+\uparrow\uparrow\downarrow)-2\downarrow\uparrow\uparrow\right]\right) \\ &\quad+\sqrt{\frac{1}{2}}\left(\sqrt{\frac{1}{2}}\left[udu-uud\right]\right)\left(\sqrt{\frac{1}{2}}\left[\uparrow\downarrow\uparrow-\uparrow\uparrow\downarrow\right]\right)\end{aligned} \tag{16}$$

Let us then compare coefficient of each term, the term that is individually symmetric under the exchange is of course symmetric, one of the first terms that does not obey this is (we ignore the overall factor of $\sqrt{\frac{1}{2}}$),which showed up twice in the original wave function

$$\sqrt{\frac{1}{6}}\sqrt{\frac{1}{6}}duu\downarrow\uparrow\uparrow+\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}}duu\downarrow\uparrow\uparrow$$

and only once in the exchanged one

$$\sqrt{\frac{1}{6}}2\sqrt{\frac{1}{6}}2duu \downarrow\uparrow\uparrow$$

By fun fact of

$$\frac{1}{6} + \frac{1}{2} = \frac{1}{3}$$

this term is ok. Similarly we can check all the other terms that are not manifestly invariant under the exchange of the quarks. We can also just expand and write out the wavefunction of a proton with an up spin,

$$|p \uparrow\rangle = \sqrt{\frac{1}{18}} \left(uud(\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow - 2 \uparrow\uparrow\downarrow) + udu(\uparrow\uparrow\downarrow + \downarrow\uparrow\uparrow - 2 \uparrow\downarrow\uparrow) + duu(\uparrow\downarrow\uparrow + \uparrow\uparrow\downarrow - 2 \downarrow\uparrow\uparrow)\right) \tag{17}$$

This is manifestly symmetric under change of first and third quark.

c) Using the last expression, we can explicitly express proton's magnetic moment as,

$$\begin{aligned} \mu_p &= \frac{1}{18}\left((\mu_u - \mu_u + \mu_d) + (-\mu_u + \mu_u + \mu_d) + 4(2\mu_u - \mu_d)\right) \times 3 \\ &= \frac{1}{3}(4\mu_u - \mu_d) \end{aligned} \tag{18}$$

Note from the definition, the magnetic moment is proportional to the charge, and consider the limit that the mass of the up and down quarks is the same, then we have

$$\begin{aligned} \mu_u &= \frac{2}{3}\mu \\ \mu_d &= -\frac{1}{3}\mu \\ \mu_u &= -2\mu_d \end{aligned} \tag{19}$$

and Then we have

$$\mu_p = -3\mu_d$$

d) So neutron must have magnetic moment of

$$\mu_n = \frac{1}{3}(4\mu_d - \mu_u) = 2\mu_d$$

So the ratio is

$$\frac{\mu_p}{\mu_n} = -\frac{3}{2}$$

This is very close.

e) The new wave function is given by

$$
\begin{aligned}
|p\uparrow\rangle &= \sqrt{\frac{1}{2}}[p_A\chi_{MS} - p_S\chi_{MA}] \qquad (20)\\
&= \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}}\,[ud-du]\,u\sqrt{\frac{1}{6}}\,[(\uparrow\downarrow+\downarrow\uparrow)\uparrow - 2\uparrow\uparrow\downarrow]\\
&- \sqrt{\frac{1}{2}}\sqrt{\frac{1}{6}}\,[(ud+du)u-2uud]\,\sqrt{\frac{1}{2}}\,[\uparrow\downarrow-\downarrow\uparrow]\uparrow\\
&= \frac{1}{2}\sqrt{\frac{1}{6}}\,[(udu-duu)(\uparrow\downarrow\uparrow+\downarrow\uparrow\uparrow-2\uparrow\uparrow\downarrow)-(udu+duu-2uud)(\uparrow\downarrow\uparrow-\downarrow\uparrow\uparrow)]\\
&= \frac{1}{2}\sqrt{\frac{1}{6}}\,[2udu(\downarrow\uparrow\uparrow-\uparrow\uparrow\downarrow)-2duu(\uparrow\downarrow\uparrow-\uparrow\uparrow\downarrow)+2uud(\uparrow\downarrow\uparrow-\downarrow\uparrow\uparrow)]\\
&= \sqrt{\frac{1}{6}}\,[udu(\downarrow\uparrow\uparrow-\uparrow\uparrow\downarrow)-duu(\uparrow\downarrow\uparrow-\uparrow\uparrow\downarrow)+uud(\uparrow\downarrow\uparrow-\downarrow\uparrow\uparrow)]
\end{aligned}
$$

This is indeed completely anti-symmetric. Now let us calculate the magnetic moment, which is given by

$$
\begin{aligned}
\mu_p &= \frac{1}{6}((-\mu_u+\mu_d+\mu_u)+(\mu_u+\mu_d-\mu_u)) \qquad (21)\\
&+\frac{1}{6}((\mu_d-\mu_u+\mu_u)+(\mu_d+\mu_u-\mu_u)+(\mu_u-\mu_u+\mu_d)+(-\mu_u+\mu_u+\mu_d))\\
&= \mu_d
\end{aligned}
$$

So for neutron, we have

$$\mu_n = \mu_u$$

Thus the ratio is

$$\frac{\mu_p}{\mu_n} = -\frac{1}{2}$$

which is not what we observe. Or we can simply note that proton has positive magnetic moment not negative one. Long live color.

3 How the Higgs Low Energy Theorem Saves Trees: 50% accurate? We will take it.

In this problem we explore how the Higgs Low Energy Theorem makes calculations easier.

In class we saw how to choose a gauge such that the would-be Goldstone bosons are "eaten up" and we are left with one real higgs field and three of the vector bosons become massive. This choice of gauge is known as the unitary gauge and in this gauge the massive vector boson propagator is given by:

$$D_{\mu\nu\ (\mathrm{W,Z})}(q) = \frac{-i\left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{M^2_{\mathrm{W,Z}}}\right)}{q^2 - M^2_{\mathrm{W,Z}}}.$$

For large values of q^2 the propagator goes as q^0 instead of the usual q^-2. This scaling means that all of the divergences are worse, and loop calculations in unitary gauge are usually significantly more difficult than in other gauges. There is another class of gauge choice, known as the R_ξ gauge, in which the physical degrees of freedom are not manifest but loop calculations are generally easier.

We start by setting:

$$H = \begin{pmatrix} w^+ \\ \frac{1}{\sqrt{2}}(v + h + iz) \end{pmatrix}.$$

where v is the Higgs VEV, h is the physical higgs, and z and $w^\pm$ are the would-be Goldstone bosons that were "eaten" in unitary gauge. Our goal is to just to draw the Feynman diagrams for $h \to \gamma\gamma$ (not compute them), so we will first determine (some of) the new vertices that are present in R_ξ gauge. We will not be interested in the Feynman rules, only which vertices are present.

a) Show that the quartic higgs self-interaction term gives rise to a w^+w^-h interaction.

b) Show that the higgs kinetic term gives rise to w^+w^-A, $w^\pm hW^\mp$, $w^\pm W^\mp A$, $hw^\pm W^\mp A$, and w^+w^-AA interactions.

c) Draw the diagrams that contribute to the higgs to two photon decay $h \to \gamma\gamma$ at one loop in R_ξ gauge.

d) Draw the diagrams that you would need to compute to use the Higgs Low Energy Theorem to find the higgs to two photon partial decay rate.[1] (*Hint*: Don't be afraid....) Do you think the diagrams in part (c) or part (d) are easier to compute?

Solution:

a) The higgs self-interaction $(H^\dagger H)^2$ contains two factors of $H^\dagger H$. We can write

$$H^\dagger H = w^+w^- + \frac{1}{2}|v + h + iz|^2 \tag{22}$$

so the cross term between w^+w^- and hv is the term we are looking for.

b) The Higgs kinetic term is $(D_\mu H)^\dagger(D^\mu H)$. The covariant derivative contains the ordinary derivative plus the gauge fields. If we choose the upper charged components of the higgs doublet and the $U(1)_Y$ gauge boson parts of the covariant derivative we get the w^+w^-AA interaction (since the $U(1)_Y$ gauge boson is a linear combination of the photon and Z boson).

If we choose the partial derivative instead of one of the $U(1)_Y$ gauge bosons we get the w^+w^-A interaction.

Two of the $SU(2)_L$ gauge bosons $W^{1,2}$ mix the upper and lower components of the higgs doublet. These are linear combinations of $W^\pm$.

We get the $w^\pm hW^\mp$ interaction from choosing $W^\pm$ from one covariant derivative, the partial derivative from the other covariant derivative, and h from the lower component of the higgs doublet.

[1] If you followed Sean's advice on how to become a beta person, you may have already computed some of these.

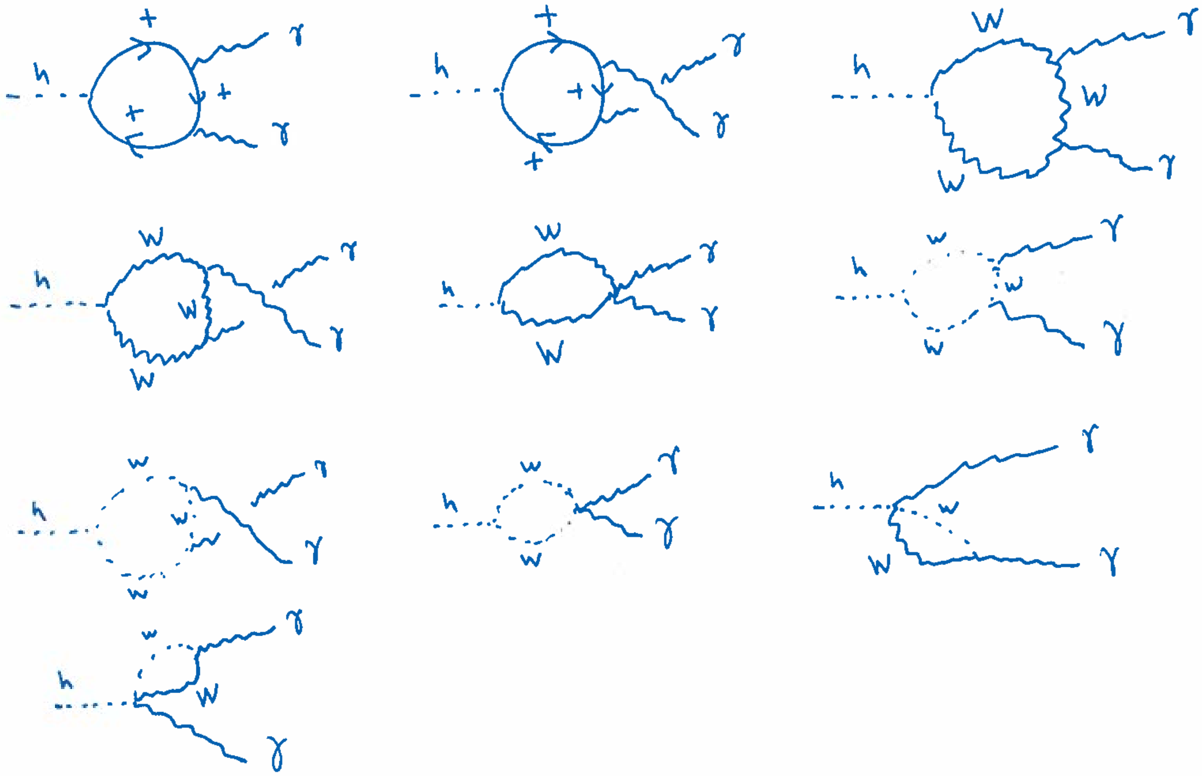
We get the $w^{\pm}W^{\mp}A$ term from choosing $W^{\pm}$ from one covariant derivative, W^3 or B from the other, and the vev v from the lower component of the higgs doublet.

We get the $hw^{\pm}W^{\mp}A$ term from choosing $W^{\pm}$ from one covariant derivative, W^3 or B from the other, and the physical higgs h from the lower component of the higgs doublet.

c) The digarams are drawn on the next page.

d) The diagrams are drawn on the final page. Note that apart from weak mixing angles, the three diagrams on the right are the same diagrams you would compute to find the beta function of a pure $SU(2)$ gauge theory.

$h \to \gamma\gamma$



Higgs Low Energy Theorem