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A MATHEMATICAL MODEL FOR THE TRANSITION RULE IN PIAGET'S DEVELOPMENTAL STAGES

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ABSTRACT

Piaget's cognitive-developmental variable is conceptualized as a quantitative construct, the central processor *M*. The set measure of *M*, i.e., the maximum number of discrete "chunks" of information or schemes that *M* can control or integrate in a single act, is assumed to grow in an all-or-none manner as a function of age in normal subjects. The *M* measure is taken as the quantitative characteristic of each developmental stage. *M* values for the Piagetian stages were inferred from Piagetian data and postulated as experimental hypotheses.

A new compound-stimuli visual information (CSVI) type of task was designed for testing quantitatively the *M* construct. The stochastic model used for these predictions in the CSVI task is the Bose-Einstein occupancy model of combinatorial analysis.

Successful results from two different series of studies with 5-, 7-, 9- and 11-year-olds are reported. In addition, the manifestation at the performance level of the "hidden parameter" *M* is shown to be mediated by a number of moderator variables such as Witkin's cognitive style, attentional learning-sets, etc.

'Transformation of information and its coordination' is Inhelder's characterization of the function of 'intelligence' as it is studied by the Geneva school (INHELDER et al., 1966, p. 162).

Under this interpretation the construct 'intelligence' in piagetian [1] theory stands for the functioning and the growth of what in other current literature is called 'mediation', 'serial processor of information', 'finite central computing space', 'integrator mechanism', 'comprehension operator', etc.

The distinctive features of the Geneva position stem from the following two methodological innovations. (a) Its original developmental method

[1] Henceforth a distinction shall be made between the adjective "Piagetian" written with a capital, which strictly refers to the work of Piaget's own team, and the adjective "piagetian," without capital, referring to work also done in other laboratories, on the type of problems which Piaget's team has inaugurated.

which led the school to postulate (cf. INHELDER, 1964) that intelligence, although it may very well grow in a continuous manner, passes through a series of discontinuous functional stages. These stages are supposed to present the property that, once achieved, their functional structures become relatively independent of their generating histories. (b) The school's choice of symbolic logic as a tool for organizing the data and for describing the intellectual capabilities of the construct 'epistemic subject' of Piaget (BETH and PIAGET, 1961). This 'epistemic subject' has been abstracted by describing in logical terms the invariant behavioral properties exhibited by the majority (50 to 75%) of normal subjects of the same age across a large variety of *different* situations. Clearly this approach facilitates the unitary *general-stages* interpretation of cognitive development. This seems to have been the original position of PIAGET (cf. 1947, 1956) and the position of many other researchers in their earlier discussions of the Piagetian system (e.g., WOHLWILL, 1963, 1966; HUNT, 1961). Such a structural and content-free interpretation of intellectual development bears some resemblance to Spearman's general factor theory of intelligence as HUNT (1961) among others (cf. VERNON, 1955) has noted. Indeed it has already been shown (VERNON, 1965) that Piagetian tasks constitute good measures of G.

It is well known that this heuristically powerful notion of general stages offers the possibility of constructing a natural ordinal scale of intellectual development. The dimension on which this scale is located appears to the present writer to be the informational complexity of the task considered from the *subject's point of view*. A similar interpretation was quantitatively tested by NASSEFAT (1963) at Geneva in an important thesis. Indeed the informational interpretation is suggested in Piaget's discussion of inferential processes (PIAGET, 1958, p. 18; PIAGET and MORF, 1958, pp. 81–91). An important corollary of this view is that any general stage of cognitive development could in principle have one numerical characteristic: the number of separate schemes (i.e. separate chunks of information) on which the subject can operate simultaneously using his mental structures.

Piaget refers explicitly to an attentional construct or central computing space in which this integration of information is supposed to take place. In his early work (PIAGET, 1928) he refers to it as an 'attention span' or 'field of centration'. Later (PIAGET, 1956b) he uses the term 'field of equilibrium'. In both instances PIAGET (1928, 1956b) states that as the child grows older the extension of this 'field of centration' increases.

This view has been frequently advanced by child psychologists using different terminologies. The present writer's proposal of a numerical characteristic for each piagetian general stage could thus be interpreted as the set measure of Piaget's field of equilibrium. If the existence of this numerical characteristic was proven, and if a recursive function was found which generates the numerical characteristics corresponding to the piagetian stages, this model could perhaps be used as a rule explaining (or at least formulating more clearly) the transition from one stage to the next (cf. SIMON, 1962; KESSEN, 1962; PIAGET, 1966; WOHLWILL, 1966). The purpose of the present paper is to present and to test a quantitative model satisfying the conditions laid out above. However, before this model is presented it is convenient to discuss the current state of opinion with regard to the notion of general stages.

1. THE MULTIPLE, LOCAL-STAGE INTERPRETATION

Despite the simplicity of the general-stage structural interpretation Piaget and Inhelder, compelled by their findings, have accepted the existence of more specific 'group factors' (to use a factor-analytical metaphor) which are somehow related to the situational structure. One of these is the 'horizontal decalages' invoked to explain the fact that conservation of weight is always acquired after conservation of substance in spite of the similar logical structure of both tasks (PIAGET, 1956a). It is assumed that this decalage is due to the need to reconstruct the logical structures when they are applied in a different concrete context. Also related to context but of a different nature is a 'figural factor' occasionally used by PIAGET and INHELDER (1966; INHELDER and PIAGET, 1964) as an explanation for the performance on some tasks which would not be predicted by the logical analysis. Another 'factor' considered by them is the degree of familiarity with the content area which influences the subject's performance level to some extent.

Piaget and Inhelder seem to interpret all these more specific factors as mere modulators of the manifestation in performance of the main factor 'intelligence' (i.e., the structural-stage level attained by the subject).

This is not the position taken by all investigators, however. Certainly many researchers have recorded similar observations. For instance the following generalization summarizes an impressive body of data: the degree of sophistication of children's responses to a great many piagetian situations is, to some extent, an inverse function of the importance of misleading perceptual cues within the situation (WOHLWILL, 1962;

GRECO, 1962; SMEDSLUND, 1964; AEBLI, 1963; FEIGENBAUM, 1963; BRUNER, 1964; BEILIN, 1964; UZGIRIS, 1964; ZIMILES, 1965; HALPERN, 196[illegible], BRAINE and SHANKS, 1965, etc.). These findings evidently refer to the figural factor of Piaget and Inhelder.

There is another frequent finding which leads many investigators (e.g. HUNT, 1965) to doubt the structural-stage notion. This is the failure to find the expected (within-subjects) high intercorrelations among piagetian tasks belonging to the same developmental level (LOVELL and OGILVIE, 1961; SMEDSLUND, 1964; LUNZER, 1960; DODWELL, 1960, 1962, 1963).

BEILIN (1965) has expressed the puzzlement of American investigators as follows: 'Inferring both structural invariance and response variability from the same body of data, however, places the Geneva group in the position of offering either a paradox or a contradiction'. Although Piagetian general-stage theory appears to be more flexible than Spearman's two-factor theory, American investigators experience again great difficulty in accepting the notion of a general intellectual factor.

Thus it seems that any attempt to save the general-stages construct must account *separately* for the general structural invariants and for the response variability.

The present proposal attempts to explain cognitive growth by means of a 'hidden parameter': the size of a central computing space M which increases in a lawful manner during normal development. The *general* structural characteristics of the piagetian stages would then be interpretable as qualitative manifestations of this internal computing system or M operator. The frequently found response variability can be explained in the following three ways:

(a) Assuming that M operates upon the units or behavioral segments available *in the subject's repertoire*, as the repertoire changes with learning so will the level of performance even if the subject's M value remains constant.

(b) A distinction can and should be made between the subject's maximum capacity or 'structural M' (M_s) and his 'functional M' (M_f) or amount of M_s space actually used by him at any particular moment of his cognitive activity (cf. INHELDER and PIAGET, 1958, p. 260; WITKIN et al., 1962, p. 54; etc.).

It seems reasonable to assume that the value taken by M_f oscillates between zero and M_s. This functional M_f would constitute a discrete random variable which can be influenced by a multiplicity of factors, from the degree of motivational arousal and the degree of fatigue to some individual-differences variables such as Witkin's field-dependence-independence (cf. WITKIN et al., 1962; PASCUAL-LEONE, 1966).

(c) It has been frequently shown that the probability of a cue s (i.e., a releasing response for a scheme, see below) depends on its 'salience', which in turn depends both on learning and on the 'innate' perceptual organizational laws. It has also been frequently shown that there are 'innate' SR compatibility factors assigning different probability weights to the different schemes in the subject's repertoire (cf. FITTS and POSNER, 1967). Although space limitations do not allow pursuit of this issue, it is reasonable to assume that: the lower the 'innate' salience and/or SR compatibility of a scheme, the higher the level of the M operation required to bring about its activation. This 'innate' modulator or construct-variable corresponds probably to the figural factor (and the 'field effects') of Piaget and Inhelder.

The M-operator theory was inductively derived from cognitive-developmental (mainly Piagetian) data by means of a semantic-pragmatic analysis (PASCUAL-LEONE and SMITH, 1969) of tasks using symbolic logic. However, space limitations make it impossible to illustrate here how such an analysis can be conducted. A hypothetical-deductive style of presentation will thus be adopted below. After a theoretical framework (which includes the inferred M values), a new experimental situation and a mathematical model connecting theory and data, shall be offered. In this manner the M values previously inferred and postulated here can be quantitatively tested.

2. THEORETICAL FRAMEWORK

The child is viewed as a relatively autonomous psychological system; he is assumed to be an independent source of data (i.e. of behavior) which is relatively self-consistent across situations.

At any given time this psychological system can (in principle) be described by means of the following three components: (1) a repertoire H of behavioral units or schemes; (2) a central computing space M (i.e. M operator) in which the information processed by the set H^* of activated schemes is transformed or integrated into novel behavior; (3) a number of organizational laws, such as learning laws, field organization laws, etc.

Only the first two components will be briefly discussed. The third component shall not be used in what follows.

2.1. *The repertoire H*

A scheme is an organized set of reactions (i.e., a behavioral, perceptual or mental blueprint) susceptible to being transferred from one situation to another by assimilation of the second to the first (PIAGET and MORF, 1958, p. 86). This functional definition of Piaget's can be translated into structural language by defining a scheme either as an *ordered pair* of implicit responses (i.e., response classes) such as: s, r (see VON UEXKÜLL, 1934) or, in some cases, as an ordered triplet: (s, r, s'), following the tradition of TOLMAN (1959).

The first response s is a releasing response which elicits the activation of the 'effecting response' r. The releasing response s corresponds to the simple or patterned cues discussed in modern discrimination learning theories (e.g., NEIMARCK and ESTES, 1967; TRABASSO and BOWER, 1968). Before the input is categorized or 'assimilated' by the subject, a probabilistic 'choice' takes place among the different releasing responses activated. This 'choice' is based on the weights of the different cues. The outcome of this choice is the sampling of a set of compatible releasing responses which in turn will activate their corresponding effecting responses r. The (perceptual or behavioral) overt response R is a function of the weighted integration of all the compatible r's activated at the given time. Thus this output is preceded by another probabilistic 'decision process'. The terminal response or 'stop rule' s' of the ordered triplets (s, r, s') can be considered to be a special (inhibitory) type of releasing response.

The schemes intervening in human processes are of many content-defined sorts: perceptual, cognitive, behavioral, motivational, etc. Independently of content, it is important to emphasize the recursive character of the scheme's structure: it is possible to have schemes constituted by schemes constituted by schemes constituted . . . The schemes capable of functioning as releasing responses of other superordinate schemes are called *figurative schemes* or *schemas* (PIAGET and INHELDER, 1966); the schemes capable of functioning as effecting responses of superordinate schemes are called *operative schemes* (PIAGET and INHELDER, 1966). A superordinate scheme or *superscheme* is analogous to a computer program which uses subroutines (i.e., subordinate schemes or *subschemes*) which are stored elsewhere in the subject's memory or repertoire H (e.g., REITMAN, 1965). This concept of superscheme is recursive so that it is possible to have superschemes constituted by superschemes constituted by superschemes constituted . . . Well-known *compound cognitive superschemes* of this type are the operational structures responsible for the logical operational thinking which have been studied by Piaget and Inhelder.

Note that for any ordinarily learned superscheme (i.e., for superschemes which have not been overlearned, cf. MANDLER, 1962) the actual use of its subroutines requires the simultaneous and/or serial activation of the corresponding (subordinate) schemes stored elsewhere in the repertoire H.

This simultaneous and/or serial activation of elsewhere-stored subschemes requires the executive service of a central processor or finite computing space M.

2.2. *The central processor M*

Piaget's cognitive-developmental variable is conceptualized as a quantitative construct, the central processor or computing space M. This computing space M, together with the operative superschemes discussed above, is responsible for the *transformation* and *coordination* of the information initially available to the psychological system at any time. (This initial information corresponds to the schemes which the input and/or other schemes have activated in H.)

Whenever the task requires the subject to process or transform information (i.e., process one or more activated schemas z_i) conforming to a plan (or operative superscheme) ϕ_{zi} in order to obtain new information (to activate a new scheme or schemes z'_i), the processing is carried out as follows: placing each one of the different relevant subschemas (i.e., $z_i, \ldots z_n$) in one of the channels or 'centration places' of the central processor M, together with the schemas representing the task instructions (ψ_I) and the general task situation (ψ_S). In this manner ψ_I and ψ_S together with subschemes placed in the channels of the M operator activate the corresponding superschemes ϕ_{zi} which in turn transform any centrated z_i into its corresponding z'_i.

The set measure of M, i.e., the maximum number of schemes or discrete 'chunks' of information that M can attend to or integrate in a single act, is assumed to grow in an all-or-none manner as a function of age in normal subjects. This M measure is considered as a quantitative characteristic of each developmental stage. With respect to a type of task including verbal or other kind of symbolic instructions the values of M are predicted to be the following:

(1) For the last substage of Piaget's pre-operational period (PIAGET,

1956a) the value is $M = a + 2$, where a is an *unknown but constant-across-ages* quantity which corresponds to the processing space taken by ψ_I and ψ_S. This stage represents on the average the 5th and 6th years of chronological age. Because of its relevance to the 'verbal mediation' hypothesis note that this stage was summarized by CRONBACH (1965, p. 120) as: 'The point near age 5 where the child allegedly begins to use words to direct his own actions, so that verbal mediation for the first time makes his learning different from animal learning'.

(2) Piaget's substage of low concrete operations (PIAGET, 1956a) exhibits an M operator of value $M = a + 3$. The corresponding chronological ages are, on the average, 7 and 8 yr.

(3) Piaget's substage of high concrete operations shows an $M = a + 4$. (Approximate ages 9 and 10 yr.).

(4) At Piaget's substage introductory to formal operations, which corresponds with the acquisition of volume, the M value is $M = a + 5$. The corresponding normative chronological ages are 11 and 12 yr.

A logical generalization of the model would be to suggest that the well-known average information-processing capacity of adults which is described by MILLER's (1956) 'magical number seven', represents the upper limit of our computing space M. With this view the piagetian data could be integrated with the very important work on channel capacity conducted in other experimental laboratories (e.g., GARNER, 1962; ERIKSEN and LAPPIN, 1967; FITTS and POSNER, 1967, etc.). There are reasons to believe that both approaches could benefit from this integration (cf. GREEN and COURTIS, 1966; PISHKIN et al., 1967).

The importance of this M-operator system is that it postulates that the difference between the M values of any two contiguous substages is a constant. If this feature was confirmed it would mean that the M operator construct offers the possibility of measuring intelligence against a *normative* system of measurement with the eventual power of an *interval scale* (this is because the set of possible M values for the different stages when considered together, would constitute a finite equal difference system in the sense of SUPPES and ZINNES (1963). This last feature, together with the basic testability of the present model, shows its advantage over the many previous attempts to use a 'mental span' construct for explaining cognitive development (e.g., MCLAUGHLIN, 1963).

Four remarks should be made before discussing the experimental tasks.

(1) The M operator has been said to be constituted by two parts:

a and k ($k = 2, 3, 4, 5, \ldots$). This distinction is being made only in the observer's language and does not presuppose any assumption with regard to whether the M system itself is divided in subsystems. This is an empirical problem which will not be studied here.

(2) In the experiments which follow the subjects will be assumed to operate consistently at their maximum (i.e., structural) computing capacity M (e.g., 7-year-olds with $M = a + 3$, the 11-year-olds with $M = a + 5$ etc.). This is by no means a necessity. As discussed above a subject can very well operate with only a fraction of his structural computing space. One of the variables believed by this writer to act as 'moderators' in the 'choice' of functional M value is the cognitive style field-dependence-independence (FDI) of Witkin (cf. PASCUAL-LEONE, 1966). More direct experimental evidence in support of this assumption has been recently produced by ECCLES (1968) and will be summarized later. According to these findings field dependent subjects frequently function with an M size inferior to their structural capacity. The methodological corollary is that only field-independent subjects can be included in the experimental samples, if, as is now the case, the structural computing space of each age group is being studied.

(3) The computing space M is assumed to operate upon the schemes existing in the subject's repertoire. The performance level will thus depend on the content of this repertoire independently of the power of the subject's M operator. Thus a good experimental strategy will be to have the subject acquire an artificial repertoire of simple schemes and then to test the M operator's capacity by the subject's ability to integrate these schemes.

(4) The construct M is assumed to be the only central computing space available to the subject. It follows that any sense modality and any effector channel could in principle be used for constructing the input-output functions. Under this circumstance a compound-stimuli visual information task (CSVI) has the advantage of experimental simplicity.

3. DESCRIPTION OF THE CSVI TASK

A two-step procedure was followed. In the first step all subjects learned a small repertoire of S–R units. The number of units to be learned varied across age groups as a function of the predicted M operator size for the group. As mentioned above, the M operator is assumed (in the observer's language) to be constituted by two parts:

one of size value a, an unknown quantity constant across ages, and the other of size value k which varies across developmental stages and is considered to be a numerical characteristic of these stages. The number of S–R units learned by each age group was equal to $3 + k$. That is, 5-year-olds learned $3 + 2 = 5$ units; 7-year-olds learned $3 + 3 = 6$ units; 9-year-olds learned 7 units and 11-year-olds learned 8 units. According to the M-operator hypotheses, the difference δ between the number of units-to-be-learned and the corresponding M values remains invariant across ages. Thus it can be said that, should the hypotheses be upheld, the 'difficulty' of the total task would remain invariant across ages. This point is important and shall be discussed further when the data have been presented.

The stimuli forming the repertoire were easily discriminable simple visual cues such as: 'square', 'red', 'dot-inside-the-figure', etc. The corresponding responses were overlearned motor behaviors such as: 'raise-the-hand', 'hit-the-basket', 'clap-hands', etc. The universe of simple S–R units used in the whole series of studies is presented in table 1.

TABLE 1
Universe of simple S–R units.

Simple S_n–R_n	Dimension	Positive instance	Negative instance	Response to positive instance
S_1–R_1	Shape	Square	Circle, triangle, cross	Raise hand
S_2–R_2	Color	Red	Blue, yellow green	Clap hands
S_3–R_3	Size	Large	Small	Open mouth
S_4–R_4	Closure	Open spaces in contour	Closed	Close eyes
S_5–R_5	Circle in the center of the figure	Present	Absent	Kick basket
S_6–R_6	Outline	Present	Absent	Stand up
S_7–R_7	X in the center of the figure	Present	Absent	Nod head
S_8–R_8	Purple background	Present	Absent	Hit table

The choice of these input components was guided by the following three constraints: (a) the intention of minimizing learning difficulties; (b) the desire to *exclude* material which would require the use of verbal mediation. This is because, according to the theory outlined above, mediation is not necessarily verbal (cf. INHELDER and PIAGET, 1964);

(c) the need to control for the receptor-orienting (or 'scanning') component of attention during the presentation of the second step. This was done by carefully selecting simple stimuli which can be 'nested' into a single compound unit. In this way failure to respond to only some of the cues could not be explained in terms of lack of peripheral reception. There is also the fact reported by SHEPARD et al. (1961), by GARNER (1962), and by LAPPIN (1967), that cues added to a single object (or compound stimulus) are more easily processed than if added to separate objects.

When the subject had learned his repertoire the second step was introduced. A new set of cards (set # 3) was presented, one after the other, exhibiting compound stimuli of the form:

$$S^n = (S_1, \ldots, S_n)$$

where all the n simple cues belong to the repertoire of N simple stimuli previously learned by the subject. The subject's task was to respond to any recognized stimulus. All possible values of n greater than one up to N simple stimuli were used in the randomly ordered set of cards. Cards were presented manually and the exposure time was fixed at 5 sec but the subject had free time for responding.

The experimental setting and instructions were as follows:

(1) *First step:* Two different sets of cards constituted the stimulus material. Set # 1 consisted of N (i.e., the number of simple stimuli) pairs of $6\frac{1}{2} \times 8\frac{1}{2}$ inch cards each one containing a fourfold division with one figure in each cell. Every pair of cards illustrated a simple S–R unit. The first card of the pair contained three positive and one negative instance of the stimulus to be learned. The second card offered three negative and one positive instance. As the series proceeded other already learned positive instances were included in the cards, besides the primary ones. Set # 2 consisted of 4×6 inch cards each one showing, among other negative instances (see table 1), one and only one positive instance belonging to the repertoire to be learned. The total number of cards in set # 2 changed slightly with the age group of the sample: 5 yr (= 21), 7 yr (= 24), 9 yr (= 24), 11 yr (= 27). Experimenter (E) and subject (S) set facing each other across a table at a distance of about three feet. The task was introduced as a 'spy' game. E would teach S a code and when he knew it well, E would send him some secret messages. The set # 1 or 'introduction series' was then started. E presented each card by saying 'Here is one clue; whenever

I show a ________ *1*, let me know you have received the message by ________ *2*' (here and below, blank *1* stands for the description of a positive instance and blank 2 stands for the description of a motoric response. Shaping of the subject's motor response by means of verbal reinforcement was used at this point if necessary). *E* pointed to each one of the four figures on the card and asked: 'What will you do here?' (Eventually the question was replaced by an interrogating gaze at *S*.) According to *S*'s behavior *E* said: (a) 'that's right' or (b) 'no, it is not a ________ *1*,' or (c) 'no, you have to ________ *2*, because it is a ________ *1*.' When a secondary positive instance was present and *S* did not respond to it *E* would eventually add as a casual remark: 'That's good. You could also have ________ *2* for the ________ *1* on that one.' When set # 1 was completed *E* said: 'Now we will have some more practice before I send you the secret messages'. The set # 2 or 'learning series' was then started. Every time *S* made an error he was told: 'No, for ________ *1* you ________ *2*.' The whole series was repeated until it was passed without error.

(2) *Second step:* set 3 was formed by 4 × 6 inch cards offering a compound stimulus figure. The actual set used with each group of subjects appears in table 2.

TABLE 2
Compound stimuli (S^n) distribution for each age group.

Age group	Simple stimuli*	S^x distribution							Total
		S^2	S^3	S^4	S^5	S^6	S^7	S^8	
5	S_1–R_1 to S_5–R_5	28	24	9	1				62
7	S_1–R_1 to S_6–R_6	15	24	24	10	1			74
9	S_1–R_1 to S_7–R_7	10	10	25	26	10	1		82
11	S_1–R_1 to S_8–R_8	10	10	10	26	24	12	1	93

* See table 1.

The second step or 'testing series,' was started without interruption after successful completion of set # 2. *E* said 'Now I will send you some secret messages. This time there can be several messages for the same card and you have to try to respond to all of them. I will show you each card for a few seconds but you may send the signals back for as long as you wish. I don't know how many messages are on the cards, so I will only know that you are finished when you stop sending signals. Then I will show you the next card.' Each card was manually presented for about 5 sec and all the responses elicited were manually recorded. Older subjects showed some tendency to say the rule defining the response to-be-produced aloud. When this occurred they were asked to keep quiet so that 'the enemy spies could not hear'.

Equivalent standard instructions were used for all age groups. Only minor rewordings which conserve the informational content and the serial organization of the instructions were permitted across age groups.

4. An M-operator model for the CSVI task

By virtue of the initial paired-associates learning each unit pair (cue S, response R) of the CSVI task constitutes a separate scheme ϕ_{zi} available in the subject's repertoire H.

Each one of these schemes ϕ_{zi} is actually a superordinate scheme specifying a contingency relation between two subschemes: the perceptual schema z_i corresponding to the cue S_i and the motor operative scheme z'_i corresponding to R_i. Clearly any superordinate scheme ϕ_{zi} constitutes the blueprint for a transformation of the form: $z_i \rightarrow z'_i$.

By virtue of the initial (first-step) training the subject has learned the different schemes $z_1, z_2, \ldots z_n, z'_1, z'_2, \ldots, z'_n, \phi_{z1}, \phi_{z2}, \ldots \phi_{zn}$ and their corresponding orienting and releasing responses. As in addition the different cues $S_1, S_2, S_3, \ldots S_n$ are nested in the stimulus compound S^n, it seems safe to conclude that: (a) The sensorial input includes information corresponding to *each one and all* the available cues $S_1, \ldots S_i, \ldots S_n$; and (b) each one of the cues S_i represented in the input activates in the subject's repertoire the corresponding perceptual subschema z_i. *Call H_S* this set of activated subschemas and observe that for the reasons stated above H_S* is a perfect representation of the stimulus compound* S^n.

The subject remembers the task instructions (call ψ_I his representation of these instructions) because he keeps the scheme ψ_I activated in the central processor M; at the same time and for the same reason he has a representation ψ_S of the testing situation. Thus as soon as the input activates the set H_S* of schemes the subject attempts to transform the perceptual information conveyed by H_S* i.e., $z_1, z_2, \ldots z_n$ into the corresponding motoric information $z'_1, z'_2, \ldots, z'_n$, as stipulated by the activated superschemes $\phi_{z1}, \phi_{z2}, \ldots, \phi_{zn}$. It was stated above that transformations of information of this sort require the use of the M

operator unless (and this *is not* the case here) the superschemes ϕ_{zi} have been overlearned (cf. MANDLER, 1962).

As the size of the M operator (call this size m) is limited, the number of perceptual subschemas which M can sample at a time (i.e., centrate or attend to) from the set $H_S{}^*$ is quite small. The schemes corresponding to the task instructions (i.e., ψ_I) and to the general structure of the situation (i.e., ψ_S) must be continuously centrated by M ('kept in mind'), therefore the M space left for the sampling is equal to m-a, where a is the M space taken by ψ_I and ψ_S. This amount of space m-a was called above k and its actual value was said to be a numerical characteristic of the Piagetian substages.

An intuitive model which clearly illustrates this is provided by the following analogy.

4.1. *Intuitive representation of the model and first 'attending act'*

Consider that the repertoire H of schemes is a panel containing a matrix of nonordered light bulbs; each bulb will stand for a scheme existent in H. The present light intensity of these bulbs will represent the degree of activation of the corresponding schemes. The set of activated subschemas $H_S{}^*$ will therefore correspond to a set of dimly lighted bulbs existent in the matrix H. The operator M can be conceptualized as a given number m of light intensity units (i.e., m discrete and equal energy quantities) which the subject has at his disposal and can use to turn on or to increase the luminosity of one or more bulbs. This operation of turning on or increasing the light intensity of bulbs is subject to the following restriction: the resulting light patterns have to expend no more than m energy units of the M operator per unit of time (this time-unit corresponds to the 'psychological moment' frequently described in the literature, e.g., VON UEXKÜLL, 1934; STROUD, 1955; FRAISSE, 1967; PARKINSON, 1969). As long as this restriction is observed, any conceivable pattern of light can be generated by the subject's application of M to the matrix of bulbs. Clearly, permissible patterns of light can range from one single bulb lighted with the intensity of m units, to m different bulbs each one lighted by one single energy unit.

In the present task (and generally in all directed-thinking activities) the bulbs ψ_I and ψ_S are lighted throughout the cognitive process. The number of energy units spent on ψ_I and ψ_S is equal to a as noted above; the remaining k units can be used to increase the light of the bulbs which belong to H_S*. Following the programme stated by ψ_I and ψ_S the operator M proceeds to apply its remaining k energy units (channels or centration places) on the bulbs of the set H_S*. Note that each of the k energy units is *randomly* applied upon one of the bulbs in H_S* simultaneously with and *independently from* all other k-1 energy units available in the M system.

This type of application of the *m equal and indistinguishable* energy units (or centration places) to ψ_I, ψ_S, and to the bulb matrix H_S* is called an 'attending act' A_j or an M-sampling. As mentioned above any one of these 'attending acts' A_j will lead to the activation of the motoric subschemes $z'_1 \ldots z'_i$ previously activated by M. The model's point of contact with the empirical data takes place now, when, as a consequence of the activation of $z_1', \ldots, z_i'$ the subject will produce the overt motor responses $R_{z'_1}, \ldots, R_{z'_i}$. In this regard however it should be noted that *the subject produces the overt responses* $R_{z'_1}, \ldots, R_{z'_i}$ *only for those motoric responses to which he has not yet responded in the present item.*

It is easy to verify that for any attending act A_j the outcome, a set u of bulbs ($u = 1, 2, \ldots k$) lighted at different degrees of intensity, can be exhaustively characterized by means of the following three pieces of information:

(1) The actual list c_j of bulbs which have been turned on or energized by A_j (i.e. the *content* of the centration A_j).

(2) The number x_j of bulbs other than ψ_I or ψ_S energized by A_j (i.e., the number of items in the list c_j other than ψ_I or ψ_S).

(3) The numerical function f_j* which assigns to every item in the list c_j a positive integer (1, 2, ...) indicating the number of energy units from M which have energized the corresponding light bulb.

Thus the event category A (the general class of attending acts) can be represented by the triplet (C, X, F *), where C, X and F * are the variables corresponding respectively to the three pieces of information enumerated above.

These three variables have an obvious psychological interpretation in terms of classic attentional constructs. The variable C corresponds to the concept of 'content of attention', X corresponds to 'span of attention'; finally F * corresponds to the concept of 'intensity of attention'. Although at least the first two variables can easily be measured by means of the CSVI task, the data presented below have exclusively used X as the dependent variable. Clearly the empirical correlate of X

is the number x of different motoric responses of the form R_z, which the subject produces after each attending act.

Note that the random variable X is dependent upon the following two parameters: (1) the value of k, i.e., the number of energy units used in energizing bulbs other than ψ_I or ψ_8. (2) the value of n, i.e., the number of simple stimuli included in the compound S^n presently being shown to the subject and represented by him in $H_8{}^*$. The basic empirical data to be presented below will be constituted by probability estimates of the form: $p\ (x;\ n,\ k)$, i.e., the probability of the subject producing x relevant motoric responses $R_z{}'_1, \ldots, R_z{}'_x$ to the stimulus compound, given the fact that the number of simple stimuli presented is n and the power of the M operator is $a + k$ (remember that a is supposed to be a constant invariant across ages).

Before attempting to show how the theoretical probabilities for $p\ (x;\ n,\ k)$ can be computed, consider that the subject is not likely to stop his information-processing activity after the first attending act. He will rather 'evaluate' the state of his CSVI processing (evaluation E_1) and eventually will start attending again.

4.2. *Subsequent attending acts and the stop rule*

After having produced the first attending act A_1 and the corresponding number x_1 of motoric responses the content of the repertoire (i.e., the memory) H of the subject will have obviously changed. Now it will include a new (although perhaps ephemeral) scheme $\phi_{Ax.\,1}$ recoding the representation of the previous attending-and-responding activities. At this point the subject is assumed to 'evaluate' his CSVI processing by energizing (attending to) the schemes ('bulbs') ψ_I, ψ_8, and $\phi_{Ax.\,1}$ i.e., the schemes which represent the previous activities. It follows from the description of the M operator given above that the centration places (or energy units) required for attending to (i.e., centering) the three schemes do not exhaust the power of M: some centration places or energy units of the M operator remain empty (exactly k–1 of these units) unless they are redundantly applied upon $\phi_{Ax.\,1}$, ψ_8 and ψ_I. It can thus be said that the M operator is left 'unsaturated'. Because of this 'unsaturatedness' of the M operator and the high number of energy units (i.e., intensity of attention) thus applying upon ψ_I and ψ_8, the subject is assumed to start a new attending act A_2. (Note in this connection that the number n of cues presented in the stimulus compound S^n varies randomly from trial to trial and it is unknown to the subject

– i.e., the subject ignores the size of the H_S* from which he is sampling).

The act A_2 follows the same rules stated above in connection with A_1. First the subject scans the stimulus compound S^n generating anew the 'field of activation' H_S* which mirrors the composition of S^n; second, the m indistinguishable energy units (i.e., centration places) of the M system randomly energize subschemas ('bulbs') from H_S* following the procedure described above; third, a number x_2 of motoric responses are produced corresponding to the *different* perceptual subschemas which had been energized in H_S*. Finally, closing this cycle, the subject will produce a new 'evaluation act' E_2 similar to the first evaluation described above: the M operator is applied upon the schemes which represent the previous activities: ψ_S, $\phi_{Ax.\,1}$, $\phi_{Ax.\,2}$. Whether the subject stops at this point or proceeds to generate a third, a fourth, etc. cycles of 'attending', 'operating' (i.e., production of motoric responses) and 'evaluation' acts depends entirely on the degree of 'saturation' (see above) of M at the moment of evaluation. For as long as M will remain 'unsaturated', the 'decision' or output of the evaluation has to be starting a new cycle. The reasons for this 'decision' were indicated above. It thus follows that the subject will 'decide' to stop his information-processing activity precisely at the time of the evaluation E_k when a total of k *successive* attending acts will have been produced. Indeed as the schemes ψ_I and ψ_S occupy the space a of the M operator, the remaining space (i.e., $m-a = k$) will be completely occupied by the k successive attending acts.

Summing up:

The *empirical* dependent variable is the *total* number x ($x = x_1 + x_2 + \ldots + x_k$) of *different* and relevant motoric responses produced by the subject when the attending-and-responding process has come to an end. As only *different* motoric responses are considered, the range of x is from 1 to n; the range of its partial components is from 0 to n (where the zero value arises when the subject has exhausted the set of relevant responses ϕ_z', which can be produced vis-a-vis S^n). In addition it will be observed that the random process by which the responses $x_1, x_2, \ldots x_k$ are generated (i.e., $A_1, A_2 \ldots, A_k$) is invariant. If *the random process is invariant* for the successive attending acts, if the stimulus compound S^n and therefore the *sampling space* H_S* *is invariant* for successive attending acts which belong to the same trial (item), and if *the result* of the successive attending acts belonging to the same trial *is cumulative*, then the outcome of a series of attending acts $A_1 + A_2 + \ldots + A_k$

should be the same as the outcome of a single act in which the number of energy units or centration places utilized is equal to the sum of energy units used in each one of the partial attending acts in the series. Since the number of energy units was k for each partial attending act, the following conclusion can be made: the probability of any total value x when the subject's information-processing activity comes to an end, is equivalent to the probability of the value x obtained from a single attending act in which the sampling space was the same H_s^* and the number of energy units was k^2.

The M-operator electromechanical interpretation in terms of bulbs and energy units conveys well the psychological significance of the model. However there is another interpretation of the CSVI task which is classic in probability theory and leads to easy computations.

5. The Bose-Einstein occupancy model

Any occupancy model of combinatorial analysis concerns itself with the outcomes generated by the process of throwing randomly a number k of balls into a number n of cells. Of the many interesting questions which can be asked in this context the only one relevant to the present discussion is: how many cells will be filled (with at least one ball) after having thrown k balls into n cells?

It may be noticed that if the cells are assumed to be equiprobable and if the balls are assumed to be indistinguishable the logical structure of this task becomes identical to our M-operator interpretation of the CSVI task. Now the bulbs (or subschemes) have been transformed into cells and the energy units transformed into balls; the act of energizing a bulb corresponds to a ball entering into a cell; the number x of different relevant responses produced by the subject will thus correspond to the number of filled cells in the question written above. This is a very fortunate situation because statistics are available which answer the problem beforehand. In order to keep the correspondence with the sampling space of the CSVI task we shall only consider the case where all the balls thrown are indistinguishable (so were the energy units) and only distinguishable arrangements of balls into cells (i.e., different patterns of light) are considered to be different outcomes.

The sampling scheme thus defined is known as the Bose-Einstein statistics (Feller, 1957). In this sampling scheme the probability of any distinguishable arrangement (i.e., pattern of light or centration) is equal to:

$$\binom{n+k-1}{k}^{-1} \tag{1}$$

where n is the number of cells available (i.e. number of cues in the stimulus compound S^n) and k is the number of balls thrown (i.e., number of energy units energizing schemes other than ψ_I and ψ_S).

The probability that exactly x cells are filled is:

$$\Pr(x) = \binom{n}{n-x}\binom{k-1}{x-1} \div \binom{n+k-1}{k} \tag{2}$$

With this formula available it becomes very easy to obtain the probability distribution of the random variable X (i.e. number of different relevant responses produced by the subject) described in the M-operator CSVI model.

It was said above that for any given stimulus compound S^n the values x of the variable X would range from n to 1 where n is the total number of cues available in S^n. Furthermore it was indicated that the attending process would come to an end either when the number of attending acts in the series is equal to k. Since in any one of the attending acts the number of energy units is equal to k, the *total* number of different energy units used by the subject at the end of the attending process will be equal to the square of k.

Thus according to the M-operator CSVI model the probability that *at the end of his attending process* a subject will have responded to x different cues from the compound S^n is:

$$\Pr(x) = \binom{n}{n-x}\binom{k^2-1}{x-1} \div \binom{n+k^2-1}{k^2} \tag{3}$$

where the value n corresponds to the total number of cues available in the compound stimulus being used. The value k, according to the developmental hypotheses stated above, is inferred from the chronological age of the sample of *normal and field independent* subjects. Finally the value x is systematically varied along the possible positions in the range of values X can take (i.e., from 1 to n).

A concrete example will illustrate this computational procedure. The example can be found in table 3 below. This table shows a matrix of theoretical and empirical probabilities corresponding to the 5-year-olds' performance on the CSVI task; in addition it exhibits the expected values and the variances for the different distributions. The rows of the probability matrix describe the different values taken by n (i.e. 2, 3, 4, 5) in the set of compound stimuli S^n presented to the subjects. For instance, the first row corresponds to the class of compound stimuli S^2 i.e., the class of those stimuli presenting two cues. The columns R^1, R^2, R^3, etc. represent the different acceptable [2] values taken by X, i.e., from 1 to n. As an example, consider the theoretical probability corresponding to a number of four relevant responses (i.e. R^4) produced vis-a-vis a stimulus compound which presents five different cues (i.e. S^5). Table 3 shows that the theoretical probability in question is equal to 0.071. This probability value has been obtained with the help of equation 3 by way of assigning the following values to the variable: $n = 5$; $x = 4$; $k = 2$. The value of k is given by the hypothesis # 1 of the M-operator model in which the normal and field independent five-year-olds were assumed to have a value of k equal to 2. Any other theoretical probability in the tables has been computed in a similar manner. As pointed out above the predicted values of k were 2 for 5- to 6-year-olds; 3 for 7–8; 4 for 9–10; and 5 for 11– to 12-year-olds.

The theoretical probabilities for the total task were obtained from those of each class of stimuli S^n after having transformed the probabilities of each column into frequencies by using the empirical 'total number of responses' recorded on the first column of the tables. The theoretical frequencies thus obtained generate, when added by columns and divided by the grand total, the desired theoretical probabilities for the total task.

Mathematical expectations and variances as well as the empirical statistics were computed in the usual manner.

6. Samples

5-, 7-, 9- and 11-year-olds were used in the main study. They shall be described successively.

6.1. *Group 5*

Thirteen children from the Child Study Centre of the University of British Columbia, eight of whom were girls. The Goodenough draw-a-man test was used as a measure of intelligence and/or field dependence-independence (cf. Witkin et al., 1962). The test was scored following

[2] Despite previous training it happened on rare occasions that no response was produced vis-a-vis a compound stimulus. For obvious reasons these exceptional R^0 responses were excluded from the computations.

HARRIS' point scale manual (1963) by a nonexperienced graduate assistant who was instructed to underestimate rather than overestimate in his rating. Under these conditions the mean standard score (IQ) for the group was 101 with a range from 82 to 144 (no sex difference). The mean age of the group was 5.28 (range: 5.0–6.0).

6.2. *Group* 7

Fourteen children from the superior second grade class in a public school in Vancouver. The school (GG) is located in a middle-class neighborhood. Under the scoring conditions described above the mean standard score for the Goodenough draw-a-man test was 106 with a range from 91 to 135. There was an equal number of boys and girls fairly matched in Harris-Goodenough IQ. The mean age of the group was 7.12 with range from 6.75 to 7.75.

6.3. *Group* 9

Fifteen grade 4 or grade 5 children (eight girls) coming from the same GG school. Henmon Nelson or Otis IQ scores were available from the school testing service. According to these data the mean IQ was 125 (range: 118–136). The 15 subjects were selected from a larger sample pretested individually with an FDI measure: the children's embedded figures test (CEFT) of KARP and KONSTADT (1963). The CEFT mean score [3] for the selected children was 21 with a range from 18 to 23. The mean age for the sample was 9.95 (9.17–10.17). No sex differences in the selected sample.

6.4. *Group* 11

Fourteen grade 6 children (nine girls) coming from another school (BV) located in the same neighborhood. The mean of their Otis (or Henmon-Nelson) IQ scores was 118.7 (106–131). The mean of their CEFT scores was 19.5 (16–24).[4] Their age mean was 11.82 (11.–12.41). Both sexes were similar in all parameters.

7. MAJOR RESULTS AND DISCUSSION

The three major simplifying assumptions being made in the M-operator CSVI model were the following:

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[3] In KARP and KONSTADT (1963) standardization of New York the mean score for this age is 16.43.

[4] In KARP and KONSTADT's standardization the mean score for 11-year-olds is 18.

(1) The subject will consistently function using his whole structural (i.e. maximum size) M-operator.
(2) The sampling of subschemas in H_s* is random, i.e., that the subject does not use any systematic exploratory routine or learning set.
(3) All the subschemas (simple stimuli) are equiprobable, i.e. they are equally salient and equally SR compatible with their respective responses.

An attempt was made to satisfy the first assumption by controlling in the sample for one individual-differences variable, Witkin's field-dependence-independence (FDI). This variable was expected to insure that only high M-operating subjects (i.e. FI) were accepted in the experimental groups. The selection procedure however, was unsatisfactory in this respect with the groups 5 and 7 (practical reasons made it impossible to use the CEFT measure with these subjects). The second assumption has been empirically handled by constructing a novel task and by avoiding any training of the subjects to the compound-stimuli section of the CSVI task. An attempt was made to meet the third assumption statistically by counterbalancing as much as possible the distribution of S–R units in the different classes of compounds.

Some independent experimental evidence in support of the solution given to the assumptions (1) and (2) will be offered in a later section after the main results have been presented. It remains now to decide empirically to what extent the simple S–R units violate the equiprobability assumption. The relevant data for testing this third assumption appear in fig. 1 which shows the proportion of correct simple responses (over the total number of possible responses) generated by each age group during the total task.

The differences in difficulty among S–R units are surprisingly consistent across ages, although the absolute differences decrease with age. It can be seen that two of the units, S_1–R_1 and S_3–R_3 are much more difficult (i.e. less probable) than any other unit. A Kolmogorov-Smirnov two-sample test shows that S_1–R_1 and S_3–R_3 do not differ statistically from each other but exhibit a highly significant difference from any other unit. The units S_2–R_2 and S_4–R_4 present an intermediate difficulty and they differ statistically between themselves as well as with regard to the other units.

Given the low relative probability of units # 1 and # 3 the empirical low values of X should become more frequent than expected, thus creating a displacement of the empirical probability distribution of R^x (i.e., of the total number x of relevant motoric overt responses produced

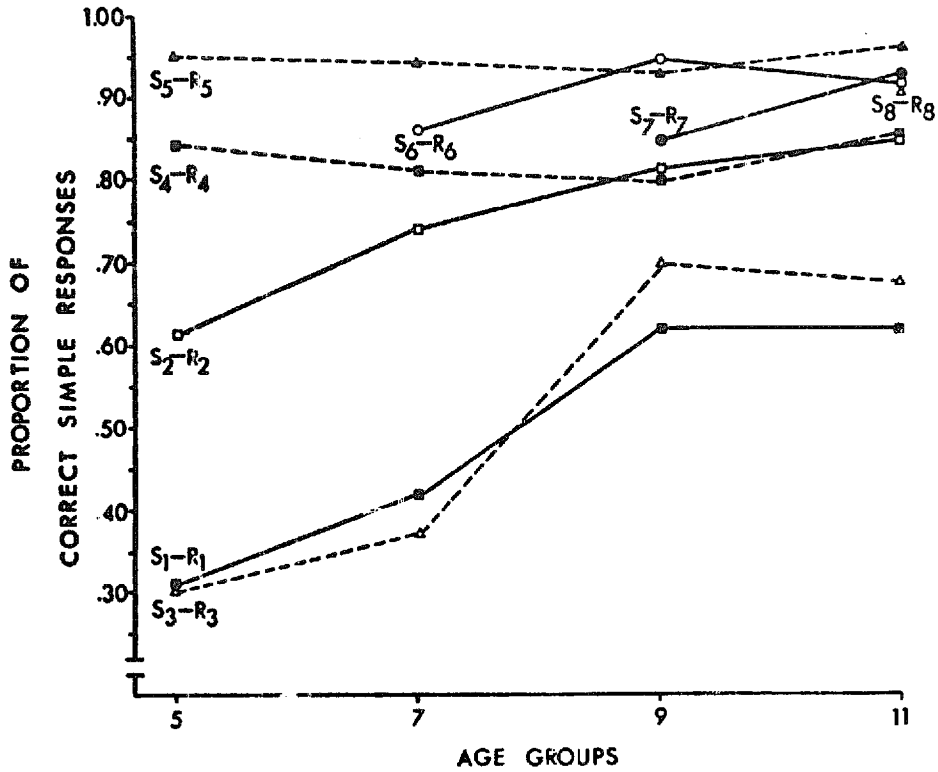

Fig. 1. The proportion of correct simple responses over the total number of possible for each S–R unit.

by the subject) towards the left (i.e., toward the lower values of X). With this warning in mind we turn now to discuss the major results.

Tables 3, 4, 5 and 6 present the theoretical and empirical probabilities corresponding to any pair of values (R^x, S^n), including those of the

TABLE 3

Theoretical (Pr) and empirical ($\hat{Pr}$) probabilities of compound responses. Predicted and empirical mathematical expectations and variances: sample of 5-year-olds. ($N = 13$)

Compound stimulus	Total no. of responses	R^1 Pr	R^1 $\hat{Pr}$	R^2 Pr	R^2 $\hat{Pr}$	R^3 Pr	R^3 $\hat{Pr}$	R^4 Pr	R^4 $\hat{Pr}$	R^5 Pr	R^5 $\hat{Pr}$	E (R^x) Pred.	E (R^x) Emp.	Var (R^x) Pred.	Var (R^x) Emp.
S^2	351	0.400	0.527	0.600	0.473							1.600	1.473	0.240	0.249
S^3	309	0.200	0.297	0.600	0.586	0.200	0.117					2.000	1.820	0.400	0.382
S^4	117	0.114	0.162	0.514	0.573	0.343	0.248	0.029	0.017	0.000	0.000	2.287	2.120	0.491	0.464
S^5	13	0.071	0.077	0.429	0.308	0.429	0.538	0.071	0.077	0.000	0.000	2.500	2.615	0.534	0.545
Total task	806	0.274	0.376	0.584	0.529	0.136	0.091	0.005	0.004	0.000	0.000	1.868	1.723	0.409	0.406

total task. Empirical and theoretical expected values and variances for the random variable X at any compound class S^n have also been included. In order to give an overall picture of these results figs. 2, 3, 4 and 5 plot the predicted and empirical probability distribution of the total task for each one of the age groups.

TABLE 4. Theoretical (Pr) and empirical ($\hat{Pr}$) probabilities of compound responses. Predicted and empirical mathematical expectations and variances: sample of 7-year-olds. ($N = 14$)

Compound stimuli	Total no. responses	R^1 Pr	R^1 $\hat{Pr}$	R^2 Pr	R^2 $\hat{Pr}$	R^3 Pr	R^3 $\hat{Pr}$	R^4 Pr	R^4 $\hat{Pr}$	R^5 Pr	R^5 $\hat{Pr}$	R^6 Pr	R^6 $\hat{Pr}$	E (R^x) Pred.	E (R^x) Emp.	Var (R^x) Pred.	Var (R^x) Emp.
S^2	208	0.200	0.341	0.800	0.659									1.800	1.659	0.160	0.225
S^3	333	0.055	0.066	0.436	0.568	0.509	0.366							2.454	2.300	0.358	0.342
S^4	336	0.018	0.021	0.218	[illegible]	0.509	0.586	0.255	0.063					3.001	2.691	0.545	0.382
S^5	139	0.007	0.014	0.112	0.094	0.392	0.525	0.392	0.331	0.097	0.036			3.460	3.281	0.708	0.546
S^6	14	0.003	0.000	0.060	0.214	0.280	0.357	0.420	0.429	0.210	0.000	0.027	0.000	3.855	3.215	0.844	0.597
Total task	1036	0.055	0.099	0.390	0.440	0.387	0.385	0.142	0.071	0.016	0.005	0.000	0.000	2.652	2.443	0.737	0.617

TABLE 5. Theoretical (Pr) and empirical ($\hat{Pr}$) probabilities of compound responses. Predicted and empirical mathematical expectations and variances: sample of 9-year-olds. ($N = 15$)

Compound stimuli	Total no. of responses	R^1 Pr	R^1 $\hat{Pr}$	R^2 Pr	R^2 $\hat{Pr}$	R^3 Pr	R^3 $\hat{Pr}$	R^4 Pr	R^4 $\hat{Pr}$	R^5 Pr	R^5 $\hat{Pr}$	R^6 Pr	R^6 $\hat{Pr}$	R^7 Pr	R^7 $\hat{Pr}$	E (R^x) Pred.	E (R^x) Emp.	Var (R^x) Pred.	Var (R^x) Emp.
S^2	150	0.118	0.187	0.882	0.813											1.882	1.813	0.104	0.152
S^3	150	0.020	0.033	0.294	0.333	0.686	0.633									2.666	2.598	0.262	0.312
S^4	375	0.004	0.011	0.093	0.112	0.433	0.440	0.470	0.437							3.369	3.303	0.443	0.501
S^5	390	0.001	0.000	0.031	0.051	0.217	0.254	0.470	0.467	0.281	0.228					3.999	3.872	0.631	0.669
S^6	150	0.000	0.000	0.011	0.027	0.103	0.133	0.335	0.333	0.402	0.413	0.148	0.093			4.569	4.408	0.833	0.875
S^7	15	0.000	0.000	0.004	0.000	0.049	0.000	0.213	0.200	0.384	0.733	0.282	0.067	0.067	0.000	5.088	4.867	1.012	0.249
Total task	1230	0.018	0.030	0.183	0.193	0.298	0.308	0.336	0.325	0.143	0.132	0.021	0.012	0.001	0.000	3.470	3.372	1.147	1.136

TABLE 6. Theoretical (Pr) and empirical ($\hat{Pr}$) probabilities of compound responses. Predicted and empirical mathematical expectations and variances: sample of 11-year-olds. ($N = 14$)

Compound stimuli	Total no. of responses	R^1 Pr	R^1 $\hat{Pr}$	R^2 Pr	R^2 $\hat{Pr}$	R^3 Pr	R^3 $\hat{Pr}$	R^4 Pr	R^4 $\hat{Pr}$	R^5 Pr	R^5 $\hat{Pr}$	R^6 Pr	R^6 $\hat{Pr}$	R^7 Pr	R^7 $\hat{Pr}$	R^8 Pr	R^8 $\hat{Pr}$	E (R^x) Pred.	E (R^x) Emp.	Var (R^x) Pred.	Var (R^x) Emp.
S^2	140	0.077	0.079	0.923	0.921													1.923	1.921	0.071	0.073
S^3	140	0.009	0.000	0.205	0.114	0.786	0.886											2.777	2.886	0.191	0.101
S^4	130	0.001	0.000	0.044	0.029	0.337	0.279	0.618	0.669									3.572	3.571	0.339	0.579
S^5	374	0.000	0.005	0.010	0.029	0.116	0.128	0.426	0.414	0.448	0.422							4.312	4.213	0.507	0.698
S^6	334	0.000	0.000	0.003	0.003	0.039	0.045	0.213	0.228	0.447	0.431	0.298	0.293					4.998	4.966	0.694	0.727
S^7	168	0.000	0.000	0.000	0.000	0.013	0.036	0.096	0.065	0.303	0.369	0.404	0.363	0.183	0.167			5.645	5.560	0.875	0.926
S^8	11	0.000	0.000	0.000	0.000	0.005	0.000	0.042	0.000	0.177	0.364	0.354	0.364	0.320	0.091	0.102	0.182	6.248	6.097	1.057	1.138
Total task	1297	0.009	0.010	0.130	0.124	0.164	0.179	0.252	0.254	0.285	0.284	0.132	0.126	0.026	0.022	0.001	0.001	4.169	4.149	1.871	1.745

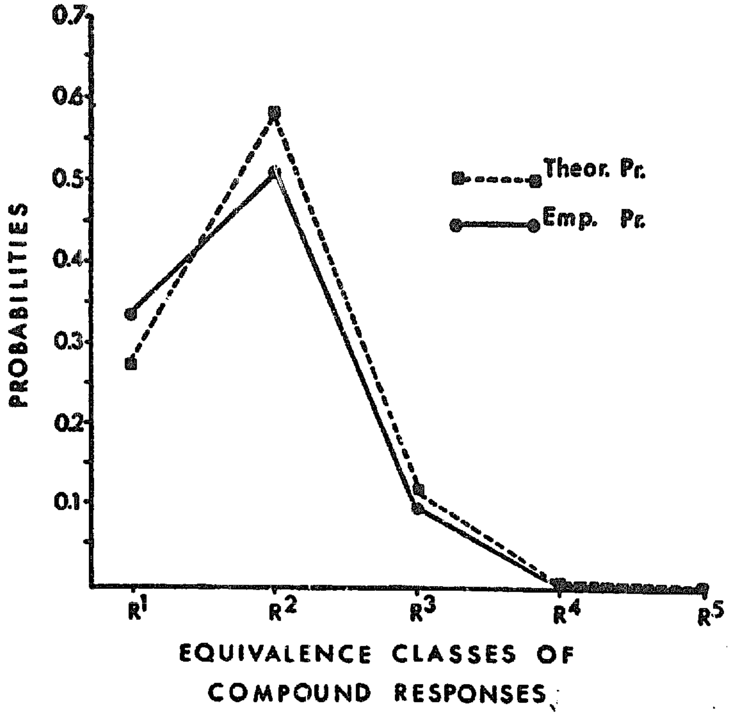

Fig. 2. Theoretical and empirical probabilities of R^x for the total task. The random variable X is the number of different correct responses produced vis-a-vis a stimulus compound. Sample of 5-year-olds ($N=13$; observations$=806$).

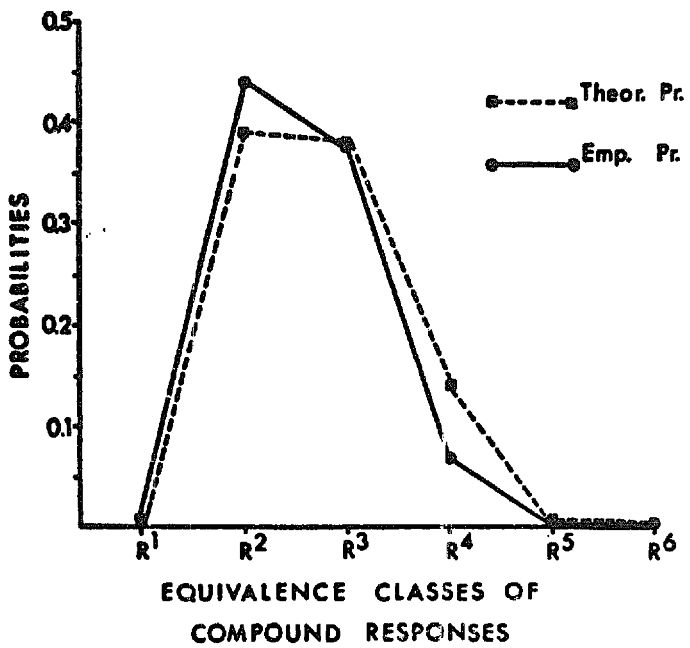

Fig. 3. Theoretical and empirical probabilities of R^x for the total task. The random variable X is the number of different correct responses produced vis-a-vis a stimulus compound. Sample of 7-year-olds ($N=14$; observations$=1036$).

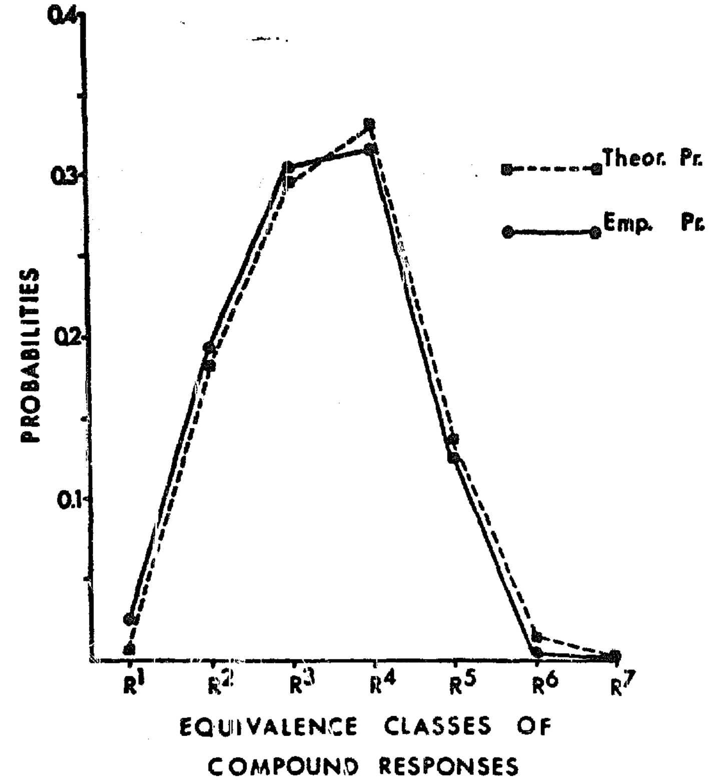

Fig. 4. Theoretical and empirical probabilities of R^x for the total task. The random variable X is the number of different correct responses produced vis-a-vis a stimulus compound. Sample of 9-year-olds ($N=15$; observations$=1230$).

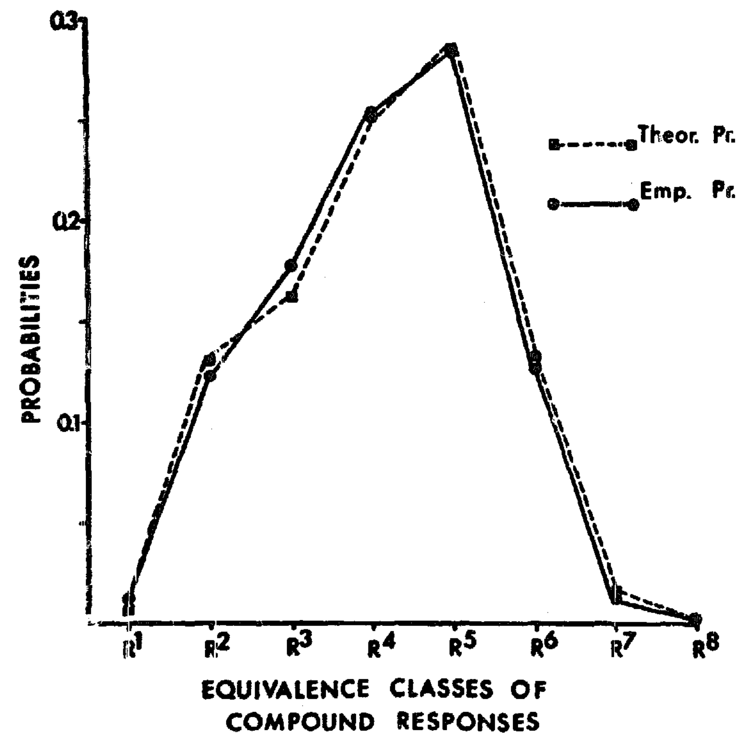

Fig. 5. Theoretical and empirical probabilities of R^x for the total task. The random variable X is the number of different correct responses produced vis-a-vis a stimulus compound. Sample of 11-year-olds ($N=14$; observations$=1297$).

As expected from the results discussed above, the empirical distributions are all displaced towards the left with respect to the theoretical distributions. However, despite the noise introduced by the violation of assumptions the theoretical predictions appear to be good for all ages and in particular for 9- and 11-year-olds. Because the theoretical model being used is the same for all ages and because the M construct integrates within one system the different age-group values, it seems reasonable to say that the results from any age group lends empirical support to the predictions about any other age group. Under the present conditions this across-ages validation is more important than the actual accuracy of the predictions. After all no parameter has been estimated in the samples, and the values used for M were obtained from different populations and from quite different types of tasks. These circumstances make a goodness-of-fit test irrelevant, particularly when the present results cannot be predicted or explained by any other model available in the literature (cf. BOWER, 1966, p. 375; BUSH, 1963, p. 432).

Expected values of X are good for all four ages and for any S^n distribution. With regard to the variances the only possible objection could be the tendency of theoretical variances to be higher than the empirical ones, a trend which is more apparent for 5- and 7-year-olds. This finding would usually suggest (cf. BUSH, 1963) that something is wrong with the assumptions of the model. In the present case a possible explanation is offered by the violations of assumptions (1), (2) and (3) mentioned above.

As the M-operator model is a developmental construct concerned with the processing capacity of individuals it seems important to ask how well the model predicts individual performance. Figs. 6 and 7 present the scatter of expected values and variances for individual subjects with regard to the total task.

The individual performance scores correspond fairly well to the group performance and the model predicts on the average their results. However there is some dispersion of the individuals and it appears that the theoretical variance is usually larger than the individual variances.

7.1. *A test of the finite equal difference assumption*

The developmental postulates of the M-operator model assume that the value of M grows in discrete steps with the cognitive Piagetian stages. It is also assumed that the difference between the (structural) values of any two contiguous stages is a constant. Thus the theoretical

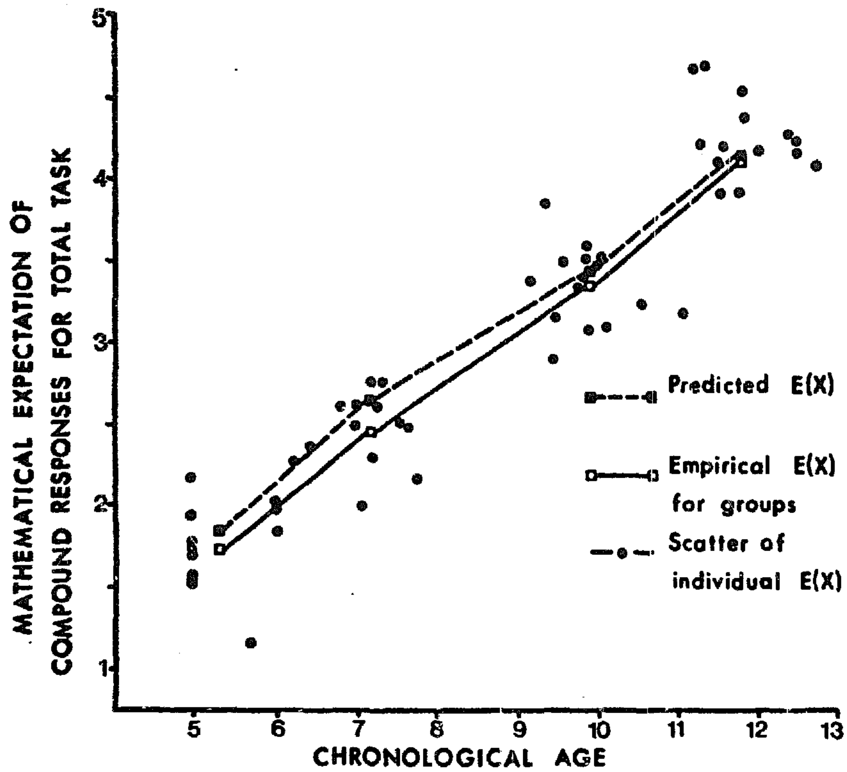

Fig. 6. Theoretical and empirical expected values of X (i.e., R^x) for the total task as a function of chronological age. Scatter of the expected values for individual subjects.

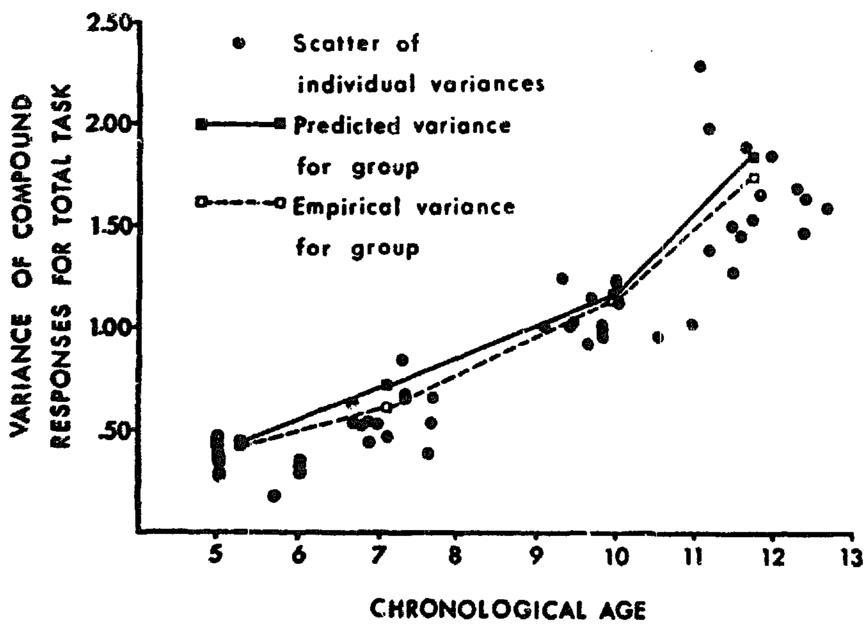

Fig. 7. Theoretical and empirical variances of X (i.e., R^x) for the total task as a function of chronological age. Scatter of the variances corresponding to individual subjects.

system constituted by the different values m taken by the structural M across development form a finite equal difference system (fd system) similar to the one described by SUPPES and ZINNES (1963). The novelty of the present fd system consists in its being defined on the values taken by *a theoretical construct* and not on the values used in making stimuli for a 'standard fd set', as is the usual procedure. In addition, the total set of compound stimuli used in the present CSVI task constitute a standard fd set in the following sense: the compound stimuli being presented have been systematically varied with regard to the number n of simple stimuli included in the compound. As the reader will remember, the total set of compound stimuli S^n corresponding to the total task is constituted by a number of subsets, each one of them defined by one of the possible values taken by n ($n = 2, 3, 4, \ldots N$). The value taken by N, i.e. the total number of different simple stimuli being presented, was increased proportionally to the predicted increment of the component k in the M operator. For each age group the value of N was by construction equal to $3 + k$. At this point the reader may find it useful to look at tables 3, 4, 5 and 6 in order to observe how the number of different distributions or subsets of data S^2, S^3, S^4, ... increase with age as stated above.

In sum, the total set of compound stimuli constitutes by construction a fd system which is correlated with the hypothetical fd system generated by the values taken by k for the different age groups; i.e., *the differences between N and k would be invariant across ages if – and only if – the finite equal difference assumption for the M operator was upheld.* Psychologically the invariance of δ could perhaps be interpreted as reflecting task difficulty invariance. Mathematically, the invariance-of-δ assumption leads to the derivation that the value of k in formula 3 is equal to N minus three: this derivation is a particular case of a general pattern of derivation which applies to any across-ages fd series of data.

In order to make this line of reasoning explicit it is convenient to define the concept of an across-ages fd series. If n (or $n + i$) stands for the number of single stimuli available in S^n, X for the random variable corresponding to the number of motoric responses produced vis-a-vis S^n, g (or $g + i$) for the stage-bound chronological age of the subject (i.e., 5 yr, 7 yr, 9 yr, etc.) and $\hat{E}(x)$ for the expected value of X, then an across-ages fd series of expected values is:

$$\hat{E}(X; n, g), \hat{E}(X; n+1, g+2), \hat{E}(X; n+2, g+4), \ldots, \hat{E}(X; n+i, g+2i), \ldots \quad (4)$$

that is: the expected value of X given the fact that the number of stimuli in S^n is $n + i$ and the subject's age is $g + 2i$. Observe that according to the developmental postulates of the M-operator the series of values: g, $g + 2$, $g + 4$ ..., $g + 2i$, ... are the 'apparent' manifestation of the theoretically 'real' series of values: k, $k + 1$, $k + 2$. ..., $k + i$, ... In addition, for any expected value $\hat{E}(X; \ldots)$ in this across-ages fd series it is true by construction that

$$n + i = k + i + \delta, \text{ and thus: } \delta = n - k$$

for any values of n, i and k in the across-ages fd series. Therefore δ is a constant invariant across ages for any fd series. It follows that as long as we remain within the same fd series k is equal to $n - \delta$, a linear function of n. Replacing k by n–δ in formula 3, the value $\Pr(x)$ becomes an exclusive function of n. Since by construction the growth of n across-ages in the fd series is linear, the following conjecture can be made: 'Any *short* across-ages fd series of expected values of X grows according to a function of n which approximates a linear function'.

As mentioned above the across-ages series of total tasks is organized into an fd system as tables 3, 4, 5 and 6 respectively, clearly show. The other fd across-ages series in our data are: (S^2, S^3, S^4, S^5), (S^3, S^4, S^5, S^6), (S^4, S^5, S^6, S^7,), (S^5, S^6, S^7, S^8). Each one of the four positions in any one of these series corresponds to one distribution from one of the different samples: group 5, group 7, group 9 and group 11, in that order. The conjecture derived above predicts that if the corresponding mathematical expectations are plotted against their respective age groups the curve thus formed will be close to a straight line.

Fig. 6 illustrates these results with regard to the total tasks. It can be observed that both the theoretical and the empirical curves approximate a straight line. Anticipating the new data to be summarized below it is convenient to indicate now that in *two* replication series of experiments conducted by ECCLES (1968), one with field-independent and another with field-dependent subjects, the curves obtained were straight lines. The relevant data are reported in table 8 and the reader may verify this assertion by plotting the expected values obtained by Eccles.

By using the data in tables 3, 4, 5 and 6 the reader may similarly verify that the other four fd across-ages series enumerated above, satisfy the conjecture by generating a straight line. This is true for both theoretical and (except for one sampling error) empirical expected values. Similar empirical data (not reported here) from the two series of Eccles replicate the same findings twice.

A SUMMARY DISCUSSION OF OTHER RELEVANT STUDIES

The theory described earlier in the paper attempts to explain the response variability frequently found among subjects belonging to the same developmental stage by means of a number of 'moderator variables' which change the behavioral manifestations of the 'hidden parameter' M. The degree of familiarity with the task (i.e., learning) and individual-differences variables such as field-dependence-independence (FDI) were among the moderator variables suggested above. Another variable which in principle could influence the performance level together with the ones mentioned, is exposure time, i.e., the length of time the compound stimuli are shown. (Notice that for the long exposure times being used here the M-operator model predicts a negligible effect, because the decision to stop attending is supposed to be elicited by the temporal saturation of M in the 'evaluation acts'.)

Finally the role of systematic exploratory routines or attentional learning-sets in the CSVI task should be studied because it was invoked in both the explanation of results and the theoretical model. (The model predicts that subjects possessing an attentional learning-set will perform better than subjects without it. This is because in the first case the temporal integration of attending acts is done by the attentional learning-set and consequently the decision to stop will be delayed until the exploratory routine has been completed.)

Four different series of studies using the CSVI task were designed to test predictions drawn from the model with regard to the four variables mentioned above. The general hypothesis was that these variables would be 'modulators' of performance in the CSVI task; however, it was expected that the M operator (i.e., the chronological age of the samples) would account for most of the variance. The new twelve samples are entirely comparable to the ones described above and therefore will not be described here. Experimental procedures were identical to procedures previously described for the CSVI task except for modifications to be indicated.

8.1. *The role of learning and of exposure time*

The first study to be reported concerns the variables 'learning during the acquisition of simple S–R units' (L) and 'exposure time during the compound stimulus presentation' (T). These two variables were combined in a two by two design. Three samples of 11-year-olds were studied. These samples were called L^+T^+, L^-T^+, and L^-T^- according to the values of the independent variables involved. The base line for these conditions was the standard 11-year-old group reported before, here representing the condition L^+T^-. L^- and T^- constitute the lower values while L^+ and T^+ are the higher values of L and T. L^+ is the standard procedure as previously described. L^- is the standard procedure modified by eliminating the set # 1 of cards. (The subjects were started directly with set # 2 and were simply told what response to produce vis-a-vis each simple stimulus as it appeared.) T^- is the standard exposure time, that is, 5 sec. T^+ is an exposure time of 10 sec. For obvious reasons it was predicted that L^+T^+ would generate the highest performance level while L^-T^- was expected to generate the lowest performance. Table 7 summarizes these results by presenting the expected values and variances of the total task together with the number of subjects in the samples and the number of stimuli used.

Neither of the two effects seems to be very large and T appears to have an even smaller effect than L. When analyses were conducted on the total pool of responses obtained for each group, Kolmogorov-Smirnov two-samples tests comparing the total-task frequency distributions (i.e., R^1, R^2, R^3 . . . R^N) of the samples with the standard group, showed the following[5]: the comparison L^-T^- vs. L^+T^- was significant at the $p < 0.001$ level (two-tailed). L^-T^+ vs. L^+T^- was significant at $p < 0.05$, L^+T^+ vs. L^+T^- was significant at $p < 0.01$. However, a different result appears when the analysis is conducted on the expected values obtained by the subjects to the different conditions. A two-way analysis of variance computed on these data (seven randomly selected subjects per cell) shows that neither L nor T exhibits statistically significant effects (F values 0.675 and 0.168 respectively).

8.2. *The role of attentional learning sets*

A second study conducted on 9-year-olds tested the effect of acquiring an exploratory learning-set (L^s) for the compound-stimuli section of the CSVI task. The procedure used with the L^s group was the same one described in the main study with the exception that during

[5] In order to equalize the total frequencies of the samples being compared, an appropriate number of subjects was selected randomly from the larger sample when necessary.

presentation of card set # 1 the experimenter made a systematic effort in training the subject to respond to all the cues available in the compound. A summary of these results appears in table 7. A Kolgomorov-Smirnov two-sample test comparing the total task frequency distribution of these data with the one for the standard 9-year-olds shows that the difference is significant at the $p < 0.01$ level (two tails).

8.3. *A partial replication of the main data and the role of cognitive style and of learning sets*

The last series of studies to be discussed was conducted by ECCLES (1968) and by Pascual-Leone and Parkinson. They were designed as a partial replication of the original developmental data and of the L^S data presented above. At the same time, these studies were to test the assumption that field-independent subjects (FI) would perform better in the CSVI task than field-dependent (FD).

In Eccles' experiments the same set of stimuli used originally and a similar testing procedure were employed, but the experimental setting and the set of responses were different. Eccles used slides for projecting the stimuli onto a screen which was placed in front of a panel containing a set of push buttons. The responses of the subject were button pushes; each button had been previously associated with a different visual cue. For more details see ECCLES (1968).

The children of 5, 7, 9 and 11 years of age were studied in two different series. In all groups the series FI (or FD) was constituted exclusively by field-independent (or field-dependent) subjects selected by means of the children's embedded figures test (CEFT) of KARP and KONSTADT (1963). All the FI (or FD) subjects obtained a CEFT score above (or below) the mean reported by Karp and Konstadt in their standardization. The expected values and variances of X for the total task corresponding to the eight groups of Eccles are shown in table 8 together with the number of subjects and number of different stimuli used.[6]

It may be observed that FI subjects performed considerably better than FD subjects for all ages. Eccles tested the statistical significance of these results by means of a two-way analysis of variance computed on the individual subjects' expected values. In order to equalize the

[6] It is important to stress that in this experiment and in the one by Parkinson which follows, the experimenter did not know whether her subjects were field-dependent or field-independent.

number of subjects she selected ten subjects randomly from each one of the eight groups; the analysis was based on their performance up to class S^5 of compound stimuli only, i.e. those classes common to all groups. The FDI dimension has an F value of 3.503, which closely approaches significance at the 0.05 level (significance is attained at $F = 3.98$). Considering the drastic cut in number of subjects and in task difficulty, imposed by the test, this result is satisfactory. The second effect, age, was highly significant ($F = 28.752$, $p < 0.005$) and the interaction effect was negligible.

ECCLES (1968) presents experimental evidence suggesting that the simple S–R units of her task had not been learned well enough by the subjects, which explains the low performance level of these groups. Support for this conclusion is offered by the study of Pascual-Leone and Parkinson that follows.

One month after the testing of Eccles had been completed but before her data had been analyzed, Parkinson tested the same subjects.[7] The testing procedure followed in every detail the original technique described previously in the paper. It had been predicted that Parkinson's data would show a learning-set effect similar to the one found before for the group L^s. This is because the children had already been exposed to the same set of compound stimuli and consequently should have developed attentional routines for exploring them. It is important to observe that the set of responses for Parkinson's CSVI task was different from Eccles'. This provides a control against a possible response-learning interpretation of the L^s data and in favour of our attention-learning interpretation.

The summary descriptive statistics of these new data appear in table 8. The sample comparable to the L^s data of table 7 is Parkinson's FI 9-year-olds. As predicted, both the expected values and the variances of these two samples are very similar. With regard to the performance level of the other samples tested by Parkinson, in all cases the expected values are above the theoretical ones and above those of the standard empirical data (see table 7). Thus the learning-set effect appears at all ages. Nevertheless, as the reader may easily observe by plotting the expected values of Parkinson's data, the curves obtained come very close to the expected theoretical values rewritten in table 7 and plotted

[7] The disagreement in N between the samples of Eccles and Parkinson is due to failure in Eccles' recording device which obliged her to eliminate some subjects after the testing had been completed.

TABLE 7
Mathematical expectations and variances of X: total task. Theoretical data, standard condition data and data from four other special conditions.

Condition	Age	No. of subjects	Max. no. of cues	E (X)	Var (X)
Theoretical					
	5	—	5	1.868	0.409
	7	—	6	2.652	0.737
	9	—	7	3.470	1.147
	11	—	8	4.169	1.871
Standard empirical	5	13	5	1.723	0.406
	7	14	6	2.443	0.617
	9	15	7	3.372	1.136
$L^+ T^-$	11	14	8	4.149	1.745
Special groups					
$L^+ T^+$	11	20	8	4.311	1.961
$L^- T^-$	11	7	8	3.440	1.526
$L^- T^+$	11	20	8	3.973	1.755
L^S	9	15	7	3.954	1.257

TABLE 8
Mathematical expectations and variances of X: total task. Eccles' data and Parkinson's data.

Condition	Age	No. of subjects	Max. no. of cues	E(X)	Var(X)
ECCLES					
Field dependent (FD)	5	10	5	1.645	0.621
	7	14	6	2.308	0.987
	9	10	7	2.909	1.613
	11	11	8	3.510	1.972
Field independent (FI)	5	12	5	1.933	0.757
	7	14	6	2.515	1.106
	9	16	7	3.072	1.729
	11	16	8	3.671	2.216
PARKINSON					
Field dependent (FD)	5	12	5	1.876	0.541
	7	15	6	2.726	0.772
	9	9	7	3.688	1.206
	11	11	8	4.080	1.743
Field independent (FI)	5	16	5	2.153	0.633
	7	18	6	2.990	0.801
	9	18	7	3.725	1.220
	11	16	8	4.089	1.779

in fig. 6. This is particularly true for the FD data which as could have been expected shows a weaker learning-set effect.[8] The Parkinson data can therefore be considered not only to support the hypothesis of an attentional learning-set effect but also to (partially) replicate the original empirical results.

GENERAL DISCUSSION AND CONCLUSIONS

Due to the complexity of the problem under study, it seems appropriate before discussing the data to restate the connections between the *M*-operator model and the theory developed by Piaget and his associates. Piaget's theory constitutes to a large extent a "competence model' in the sense of Chomsky; it defines a corpus of *normative* (i.e., ideal, possible) behaviors for each age group. In contrast the theory outlined and tested above is the sketch of an 'automaton' or 'performance model': it attempts to develop a machine-like (psychological) model capable of generating the type of competence described by Piaget. FLAVELL and WOHLWILL (1969) to whom the writer owes the idea of this distinction, suggest that a psychological theory of complex behavior must include both competence and automaton models. In this sense and only in this sense the *M*-operator model can profitably be considered a development of Piagetian theory. *M* is conceived of as a multi-channel information-processing device able to behave as a Piagetian child.

As mentioned in the introduction, this model was inferred from a careful study of Piagetian tasks conducted with the help of symbolic logic and with the heuristics of an automaton builder's point of view (i.e., the author analyzed the correct responses and the subjects' error patterns and asked questions such as: 'What schemes or units of information must have been activated by the input in order to generate this output?). The functional structure of the different Piagetian tasks studied turned out to be describable by sequences of (one or more) logical formulas of the form:

$$M(\psi_I, \psi_S, \phi_{z_1}, \ldots, \phi_{z_k}) \rightarrow pR \qquad (5)$$

where ψ_I is the subject's representation of the task instructions and ψ_s is his representation of the testing situation. The other constituents ϕ_z are k independent cognitions or actively extracted pieces of information

[8] FD subjects are less analytical in their way of experiencing than FI subjects and their ability for incidental learning is lower (cf. WITKIN et al., 1962). This suggests that their transfer of learning from the previous task will be smaller.

which appear to be logically necessary for generating pR; pR stands for the subject's purpose of producing the correct logical response to he Piagetian problem. This type of analyses of Piagetian tasks is illustrated in PASCUAL-LEONE and SMITH (1969). In addition several papers in preparation by the writer will offer an analysis of Piagetian tasks from the present viewpoint. It should nevertheless be clear that formula 5 simply describes an abstract functional structure which is satisfied by the concrete formulas describing the tasks of Piaget. The qualitative content or meaning assigned to each one of the symbolic constituents is of course different for each one of the concrete formulas.

When the different concrete formulas were compared it was found that *all the tasks solved at about the same age by normal children exhibited formulas of equal maximum complexity.* That is: the maximum number of elements included within the parentheses in any one of the formulas was invariant for the set of tasks solved at about the same age. Call this invariant number *max m*. As formula 5 shows this *max m* value was constituted by the sum $a + k$, where a stands for the processing space taken by ψ_I and ψ_S, and k is the number of independent cognitions ϕ_z required by the task. The value k varied with each Piagetian stage: i.e., $a + 2$, $a + 3$, $a + 4$, $a + 5$, ... These k values constitute the developmental hypotheses tested above.

The next step, after this semantic-pragmatic analysis of tasks had been concluded, was to assume that the inductively found logical operator *max m* was expression of a cybernetic construct: The multichannel central computing space M. This is the origin of the energy-units M model put forward in the present paper. Thus an M operation of the form described in formula 5 *is not* equivalent to one of Piaget's mental operations. Rather Piaget's operations are the possible output pR of these M operations whenever M is applied to solving the Piagetian cognitive tasks.[9] It follows that a test of the M assumption does not require the use of the Piagetian cognitive tasks but it does require the experimental control of the schemes used by S in his information-processing.

As in the CSVI task the subject had learned the different S–R units transforming them into schemes of the form $\phi_{zz'}$ (where z and z' stand respectively for the S and the R), each 'attending act' described in the M-operator model could be interpreted as an M operation satisfying

[9] As the Piagetian operations become automatized and transformed into conceptual schemes they could in their turn be used as constituents ϕ_z of formulas like 5.

formula 5. Thus when the M model was reinterpreted in terms of the Bose-Einstein model the parameter k became a test of the numerical values inductively found in Piagetian tasks.

And how successful has this test been?

For each one of the four different age groups the CSVI task had generated a wealth of quantitative predictions of the form $P(x; n, k)$ where P is a probability function, n is the number of simple stimuli available in the compound and x is the number of responses (i.e. schemes $\phi_{zz'}$) actually produced by the subject vis-a-vis the corresponding S^n class of compound stimuli.

It is important to observe that predictions have been made for each possible triplet $(x; n, k)$ available in the pool of data. The accuracy of these predictions can be verified by examining the tables 3, 4, 5 and 6 and the summary statistics diagrammed in figs 2–7. The internal consistency of the data with respect to the theory was assessed in several independent ways: (a) by showing that the predictions hold across different classes S^n of compound stimuli; (b) by showing that for any *short* across-ages series of S^n classes which is organized into an fd (finite equal difference) system, the corresponding series of expected values grows according to a function of n, which approximates a linear function; (c) by showing that not only the group data but also the expected values and variances of individual subjects can be predicted fairly well by means of the model; (d) by showing that the very same theoretical model is able to generate satisfactory predictions for each one of the different age groups.

Of prime importance from an information-processing viewpoint as well as from the point of view of Piagetian research is the fact, extremely rare in the current literature on mathematical psychology, that estimates for the values of k had been obtained from a completely different corpus of data and were postulated here: *The parameter k appears to have passed the very stringent test of invariance across situations and across samples.*

Independently of their bearing on the Piagetian developmental variable these results may turn out to be a factual and perhaps a methodological contribution to the problem of parallel versus sequential perceptual processing. Consider for instance the work on parallel processing carried out by Eriksen and his associates (e.g., ERIKSEN and SPENCER, 1969). Eriksen believes that tachistoscopic visual detection (and possibly recognition) of overlearned items is achieved by means of a multichannel central encoder. This multichannel encoder seems to be a construct equivalent to the computing space M being proposed here. Indeed a major assumption tested in the M-operator CSVI model was that the sampling of perceptual subschemas (i.e., the 'energy units') is independent and with replacement. ERIKSEN and LAPPIN (1967) and ERIKSEN and SPENCER (1969) have come to the conclusion that there are at least six parallel channels in visual perception (1967) or perhaps as many as nine parallel channels (1969). This is very congruent with the present model where an extrapolation of the theoretical developmental-growth function of M to the older age groups suggests that at the age of 15 (i.e., when the intellectual growth is supposed to level off) the size value of M should be $a + 7$ (i.e., 5 yr $= a + 2$; 7 yr $= a + 3$; 9 yr $= a + 4$; ...; 15 yr $= a + 7$; 17 yr $= a + 7$). Obviously this extrapolated value of m corresponds to the famous 'magical number seven' of George MILLER (1956); a number which represents the average size of the multichannel central processor of human adult subjects. In point of fact, the validity of this extrapolation has been successfully tested by PARKINSON (1969) on 15- and 17-year-olds using the task and the quantitative models described in this paper.

From an information-processing viewpoint, the novelty of the present work consists in the following characteristics: (a) it incorporates as a major feature the cognitive-developmental variable (and other individual-differences variables) which could not be handled by the available information-processing models; (b) it offers new quantitative evidence in support of the existence of both parallel and sequential processes; (c) it offers explicit estimates about the parallel processing power corresponding to the different intellectual levels; (d) it presents a new experimental paradigm and methodology in which parallel information processing can be studied under free-exposure conditions which offer an easy control for short-term memory.

Four remarks are in order with regard to the experimental procedure used in the present studies. The first one refers to the extent to which the differences in salience among simple stimuli may have interfered with the experimental aims. Fig. 1 shows these differences; they are indeed statistically significant and strikingly stable across ages. The explanation of these differences constitutes a challenging unexpected problem which deserves to be pursued. However in the present context those differences are only relevant as a possible source of error and only from this point of view need to be discussed. The following argument justifies this writer's confidence about the methodological harmlessness of the differences in salience. As the model assumes stimulus equiprobability the very low relative salience of, e.g., S_1–R_1 and S_3–R_3 can *only* hinder the accuracy of the predictions. Hence the surprisingly good predictions that were obtained can be retained safely.

A second methodological remark concerns the exposure time of 5 sec used for the compound stimuli. It could be claimed that older subjects may not have had enough time for exploring the cards, as the number of stimuli was increased with age. Against this interpretation the experiment discussed above in which exposure time was a variable showed a very small difference in performance between 5 sec and 10 sec.

The third remark about the experimental procedure refers to the type of experimental subjects selected for the main study. They were all field-independent subjects preselected according to their performance on one test measuring Witkin's cognitive-style variable. Had field-dependent subjects been included in the sample the prediction of results would have been less accurate. This effect was demonstrated in the replication studies summarized above. Eccles' results show that field-dependent subjects tend to be low information-processors or, more shortly, low mediators. This 'stylistic' difference or response-bias in processing information could be methodologically important as a control.

The last, very important, methodological remark concerns the role of learning in the present paradigm. At first view it could be thought that these are learning experiments (e.g. the development of learning abilities) and that the mathematical model is a learning model. However, a closer view will convince the reader that this is not the case. Learning enters here not as an independent or an intervening variable but as a control. In order to ensure that all subjects have similar relevant repertoires at the moment when *the real experiment begins*, i.e., when the compound stimuli are presented, all subjects go through a modified paired-associates learning of the simple S–R units. However no subject received any training or practice in the compound-stimuli information-processing task, and any differential reinforcement was carefully avoided after this task had begun. *Thus what is being measured in the main experiment is not learning ability but a new sort of attention span.* The paired-associates learning control is nevertheless of paramount importance in order to guarantee that the subject's 'grain' or 'unit' of information (cf. GREEN and COURTIS, 1966) is the same as the investigator's. In other words: The previous learning of S_i–R_i pairs ensures that the subject's partition of the input will be in conformity to the stipulated S_i units; furthermore the required unfamiliar transformation from S_i into R_i makes it difficult for the subject to chunk the S_i units into more complex patterns which would be inconsistent with the experimenter's partition. This is of special importance in this paradigm because the visual compound-stimuli presentation, the free time-exposure and the free responding time have in addition ensured a negligible short-term memory load. The need for this procedure has been experimentally demonstrated above in the special studies involving prior learning and attentional learning-sets.

In conclusion, the main experiments have measured an empirical relational system which ordinarily is a hidden parameter: the developmental change of the subject's 'real' attention span; that is, his multichannel capacity or, to use the Piagetian expression, 'field of equilibrium'. This paradigm may be a step toward developing a procedure of fundamental measurement for what is quantifiable in the developmental concept of intelligence: the information-processing capacity.

No less important is the fact that the present mathematical model points towards a bridge connecting cognitive development with the modern work in stochastic learning theories.

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References

AEBLI, H., 1963. Über die geistige Entwicklunge des Kindes. Stuttgart: Klett.

BEILIN, H., 1964. Perceptual-cognitive conflict in the development of an invariant area concept. J. exp. Child Psychol. **1**, 208–226.

———, 1965. Learning and operational convergence in logical thought development. J. exp. Child Psychol. **2**, 317–339.

BETH, E. W. and J. PIAGET, 1961. Épistémologie mathématique et psychologie. Paris: P.U.F.

BOWER, T. G. R., 1966. Mathematical learning theory. In: E. R. Hilgard and G. H. Bower (eds.), Theories of learning. New York: Appleton-Century-Crofts, 334–380.

BRAINE, M. D. and B. L. SHANKS, 1965. The conservation of a shape property and a proposal about the design of the conservations. Canad. J. Psychol. **19**, 197–207.

BRUNER, J. S., 1968. The course of cognitive growth. In: S. Endler, L. R. Boulter and H. Osser (eds.), Contemporary issues in developmental psychology. New York: Holt, Rinehart & Winston, 476–494.

BUSH, R. R., 1963. Estimation and evaluation. In: R. D. Luce, R. R, Bush and E. Galanter (eds.), Handbook of mathematical psychology. Vol. 1. New York: Wiley, 429–469.

CRONBACH, L. J., 1965. Issues current in educational psychology. In: L. N. Morrisett and J. Vinsonhaler (eds.), Mathematical learning. Monogr. Soc. Res. Child Developm. **30** (1), 109–126.

DODWELL, P. C., 1960. Children's understanding of number and related concepts. Canad. J. Psychol. **14**, 191–205.

———, 1962. Relations between the understanding of the logic of classes and of cardinal 162 numbers in children. Canad. J. Psychol. **16**, 152–160.

———, 1963. Children's understanding of spatial concepts. Canad. J. Psychol. **17**, 141–161.

ECCLES, E., 1968. Field dependence and a neopiagetian model of information processing capacity. M. A. Thesis, Department of Psychology, University of British Columbia, mimeo.

ERIKSEN, C. W. and J. S. LAPPIN, 1967. Independence in the perception of simultaneously presented forms at brief durations. J. exp. Psychol. **73**, 468–472.

FEIGENBAUM, K. D., 1963. Task complexity and I.Q. as variables in Piaget's problem on conservation. Child Developm. **34**, 423–432.

FELLER, W., 1957. An introduction to probability theory and its applications. Vol. 1. New York: Wiley.

FITTS, P. M. and M. POSNER, 1967. Human performance. Belmont, California: Brooks/Cole.

FRAISSE, P., 1967. L'integration temporelle des perceptions. In: Hommage á André Rey, Bruxelles: Chas. Dessart, 140–150.

GARNER, W. R., 1962. Uncertainty and structure as psychological concepts. New York: Wiley.

GRÉCO, P., 1962. Quantité et quotité. In: P. Gréco and A. Morf (eds.), Structures numériques élémentaires. Paris: P.U.F., 1–70.

GREEN, R. T. and M. C. COURTIS, 1966. Information theory and figure perception: The metaphor that failed. Acta Psychol. **25**, 12–36.

HALPERN, E., 1965. The effects of incompatibility and logic in Piaget's stage of concrete operations. Child Developm. **36**, 491–497.

HARRIS, D. B., 1963. Children's drawings as measures of intellectual maturity. New York: Harcourt, Brace & World.

HUNT, J. McV., 1961. Intelligence and experience. New York: Ronald Press.

———, 1965. Intrinsic motivation and its role in psychological development. In: D. Levine (ed.), Nebraska Symposium on Motivation: 1965. Lincoln: University of Nebraska, 189–282.

INHELDER, B., 1964. Some aspects of Piaget's genetic approach to cognition. In: J. Cohen (ed.), Readings in psychology. London: Allen & Unwin, 83–103.

———, M. BOVET, H. E. SINCLAIR and C. D. SMOCK, 1966. On cognitive development. Amer. Psychologist **21**, 160–164.

——— and J. PIAGET, 1958. The growth of logical thinking from childhood to adolescence. New York: Basic Books.

——— and ———, 1964. The early growth of logic in the child. New York: Harper & Row.

KARP, S. A. and N. L. KONSTADT, 1963. Manual for the children's embedded figures test. Cognitive tests, P.O. Box 4, Vanderveer Station Brooklyn 10 N.Y.

KESSEN, W., 1962. 'Stage' and 'structure' in the study of children. In: W. Kessen and C. Kuhlman (eds.), Thought in the young child. Monogr. Soc. Res. Child Developm. **27** (2), 150–161.

LAPPIN, J. S., 1967. Attention in the identification of stimuli in complex visual displays. J. exp. Psychol. **75**, 321–328.

LOVELL, K. E. and E. OGILVIE, 1961. A study of the conservation of weight in the junior school children. Brit. J. Psychol. **31**, 138–144.

LUNZER, E. A., 1960. Some points on Piagetian theory in the light of experimental criticism. J. child Psychol. Psychiat. **1**, 191–202.

McLAUGHLIN, G. H., 1963. Psycho-logic: A possible alternative to Piaget's formulations. Brit. J. ed. Psychol. **33**, 61–67.

MANDLER, G., 1962. From association to structure. Psychol. Rev. **67**, 415–427.

MILLER, G. A., 1956. The magical number seven plus or minus two: Some limits on our capacity for processing information. Psychol. Rev. **63**, 81–97.

NASSEFAT, M., 1963. Étude quantitative sur l'évolution des opérations intellectuelles. Neuchatel: Delachoux & Niestlé.

NEIMARK, E. D. and W. K. ESTES, 1967. Stimulus sampling theory. San Francisco: Holden-Day.

PARKINSON, G. M., 1969. The recognition of messages from visual compound stimuli: A test of a quantitative developmental model. M. A. Thesis, Department of Psychology, York University.

PASCUAL-LEONE, J., 1966. Piaget's period of concrete operations and Witkin's field dependence: A study on college students and children. Paper read at C. P. A., Montreal. University of British Columbia, Mimeo.

——— and J. SMITH, 1969. The encoding and decoding of symbols by children: A new experimental paradigm and a neo-Piagetian model. J. exp. Child Psychol. **8**, 328–355.

PIAGET, J., 1928. Judgment and reasoning in the child. Paterson, New Jersey: Littlefield & Adams, 1964.

———, 1947. La psychologie de l'intelligence. Paris: Colin. (The psychology of intelligence. Paterson, New Jersey: Littlefield & Adams, 1960.)

———, 1956a. Les stades du développement mentale chez l'enfant et l'adolescent. In: P. Osterriech, J. Piaget, R. de Saussure, J. M. Tanner, H. Wallon and R. Zazzo (eds.), Le problème des stades en psychologie de l'enfants. Paris: P.U.F., 33–49.

———, 1956b. Logique et équilibre dans les comportements du sujet. In: L. Apostel, B. Mandelbrot and J. Piaget (eds.), Logique et équilibre. Paris: P.U.F., 27–117.

———, 1958. Assimilation et connaissance. In: A. Jonckheere, B. Mandelbrot and J. Piaget (eds.), La lecture de l'expérience. Paris: P.U.F., 49–108.

———, 1966. Problèmes du temps et de la fonction. In: J. B. Grize, K. Henry, M. Meylan-Backs, F. Orsini, J. Piaget and N. van den Bogaert-Rombouts (eds.), Épistémologie du temps. Paris: P.U.F., 1–66.

——— and B. INHELDER, 1966. L'image mentale chez l'enfant. Paris: P.U.F.

——— and A. MORF, 1958. Les isomorphismes partiels entre les structures logiques et les structures perceptives. In: J. S. Bruner, F. Bresson, A. Morf and J. Piaget (eds.), Logique et perception. Paris: P.U.F., 49–116.

PISHKIN, V., A. WOLFGANG and E. RASMUSSEN, 1967. Age, sex, amount and type of memory information in concept learning. J. exp. Psychol. **73**, 121–124.

REITMAN, W. R., 1965. Cognition and thought: an information processing approach. New York: Wiley.

SHEPARD, R. N., C. I. HOVLAND and H. M. JENKINS, 1961. Learning and memorization of classifications. Psychol. Monogr. **75** (whole no. 13).

SIMON, H. A., 1962. An information-processing theory of intellectual development. In: W. Kessen and C. Kuhlman (eds.), Thought in the young child. Monogr. Soc. Res. Child Developm. **27** (2), 150-161.

SMEDSLUND, J., 1964. Concrete reasoning: a study of intellectual development. Monogr. Soc. Res. Child Developm. **29** (2).

STROUD, J. M., 1955. The fine structure of psychological time. In: H. Quastler (ed.), Information theory in psychology. Glencoe, Ill.: Free Press, 174–207.

SUPPES, P. and J. L. ZINNES, 1963. Basic measurement theory. In: R. D. Luce, R. R. Bush and E. Galanter (eds.), Handbook of mathematical psychology. Vol. 1. New York: Wiley, 1–76.

TOLMAN, E. C., 1959. Principles of purposive behavior. In: S. Koch (ed.), Psychology: A study of a science, Vol. 1. New York: McGraw Hill.

TRABASSO, T. and G. M. BOWER, 1968. Attention in learning: Theory and research. New York: Wiley.

UZGIRIS, I. D., 1961. Situational generality of conservation. Child Developm. **35**, 831–841.

VERNON, P. E., 1955. The psychology of intelligence and G. Bull. Brit. psychol. Soc. **26** (1). (Reprinted in: J. Cohen, Readings in psychology. London: Allen & Unwin, 1962, 284–301.)

———, 1965. Ability factors and environmental influences. Amer. Psychologist **20**, 723–733.

VON UEXKÜLL, J., 1934. A stroll through the worlds of animals and men. In: C. H. Schiller (ed.), Instinctive behavior. New York: International Universities Press, 1957, 5–80.

WITKIN, H. A., R. B. DYK, H. A. FATERSON, D. R. GOODENOUGH and S. A. KARP, 1962. Psychological differentiation. New York: Wiley.

WOHLWILL, J. F., 1962. From perception to interference: A dimension of cognitive development. In: W. Kessen and C. Kuhlman (eds.), Thought in the young child. Monogr. Soc. Res. Child Developm. **27** (2), 87–112.

———, 1963. Piaget's system as a source of empirical research. Merrill-Palmer Quart. Behav. Developm. **9**, 253–262.

———, 1966. Piaget's theory of the development of intelligence in the concrete operations period. In: M. Garrison, Jr. (ed.), Cognitive models and development in mental retardation. Monogr. Suppl. Amer. J. ment. Defic. **70**, 57–78.

ZIMILES, H., 1965. The development of differentiation and the conservation of number. New York: Bank Street College of Education, research project, no. 2270, mimeo.