PHYS 1420 (F19)
Physics with Applications to Life Sciences

2019.09.16
Relevant reading:
Kesten & Tauck ch.3.4

Ref. (re images):
Wolfson (2007), Knight (2017)
The four dice are identical. Which one does not belong?
Announcements & Key Concepts (re Today)

→ (online) HW #3 posted → Due next Monday, 9/23

→ (written) HW #1 (posted later today) → Due next Wednesday, 9/25

→ Labs: Start this week! (Sept.16-20)

Some relevant underlying concepts of the day...

- 2-D kinematics: vector “components” of $x$, $v$, and $a$

- projectile motion
Connecting Vectors to Mechanics

3.2 Velocity and Acceleration Vectors

We defined velocity in one dimension as the rate of change of position. In two or three dimensions it's the same thing, except now the change in position—displacement—is a vector. So we write

\[ \vec{v} = \frac{\Delta \vec{r}}{\Delta t} \]  \hspace{1cm} \text{(average velocity vector)} \tag{3.3}

for the average velocity, in analogy with Equation 2.1. Here division by \( \Delta t \) simply means multiplying by \( 1/\Delta t \). As before, instantaneous velocity is given by a limiting process:

\[ \vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} \]  \hspace{1cm} \text{(instantaneous velocity vector)} \tag{3.4}

Again, that derivative \( d\vec{r}/dt \) is shorthand for the result of the limiting process, taking ever smaller time intervals \( \Delta t \) and the corresponding displacements \( \Delta \vec{r} \). Another way to look at Equation 3.4 is in terms of components. If \( \vec{r} = x\hat{i} + y\hat{j} \), then we can write

\[ \vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} = v_x\hat{i} + v_y\hat{j} \]

where the velocity components \( v_x \) and \( v_y \) are the derivatives of the position components.

Acceleration is the rate of change of velocity, so we write

\[ \vec{a} = \frac{\Delta \vec{v}}{\Delta t} \]  \hspace{1cm} \text{(average acceleration vector)} \tag{3.5}

for the average acceleration and

\[ \vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} \]  \hspace{1cm} \text{(instantaneous acceleration vector)} \tag{3.6}

for the instantaneous acceleration. We can also express instantaneous acceleration in components, as we did for velocity:

\[ \vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} = a_x\hat{i} + a_y\hat{j} \]

\[ \vec{v} = \vec{v}_0 + \vec{a}t \]  \hspace{1cm} \text{(for constant acceleration only)} \tag{3.8}

\[ \vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2} \vec{a}t^2 \]  \hspace{1cm} \text{(for constant acceleration only)} \tag{3.9}

\[ \rightarrow \text{Our previous (1-D) expressions can be generalized to 2-D (or higher) via vector notation} \]
You should be comfortable bouncing back & forth between the both versions of the eqns.

\[ \vec{v} = \vec{v}_0 + \vec{a}t \quad \text{(for constant acceleration only)} \quad (3.8) \]

\[ \vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2} \vec{a}t^2 \quad \text{(for constant acceleration only)} \quad (3.9) \]

\[
\begin{aligned}
\vec{v}_x &= \vec{v}_{x0} \\
\vec{v}_y &= \vec{v}_{y0} - gt \\
x &= x_0 + \vec{v}_{x0}t \\
y &= y_0 + \vec{v}_{y0}t - \frac{1}{2}gt^2 \\
\end{aligned}
\]

(for constant gravitational acceleration)
A classic example: “Maria” riding a Ferris wheel

**FIGURE 4.1 Using Tactics Box 4.1 to find Maria’s acceleration on the Ferris wheel.**

(a) The lengths of all the velocity vectors are the same, indicating constant speed.

The direction of each vector is different. This is a changing velocity.

Maria moves at constant speed but not at constant velocity. Thus she is accelerating.

(b) No matter which dot is selected, finding $\Delta \vec{v}$ like this will show that it points to the center of the circle.

Velocity vectors

Acceleration vectors

All acceleration vectors point to the center of the circle.

Maria’s acceleration is an acceleration of changing direction, not of changing speed.
This acceleration will cause the particle to

a. Speed up and curve upward.  
b. Speed up and curve downward.

c. Slow down and curve upward.  
d. Slow down and curve downward.

e. Move to the right and down.  
f. Reverse direction.
This acceleration will cause the particle to

a. Speed up and curve upward.

b. Speed up and curve downward.

c. Slow down and curve upward.

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e. Move to the right and down.

f. Reverse direction.

d
Projectile Motion

- Ok, now we have the pieces in place to turn a hard problem into an easier one...

Akira Kurosawa’s Throne of Blood

Katniss Everdeen

Daryl Dixon
Remarkably, the same principle applies for a bullet fired from a gun
(i.e., it takes the same amount of time for a bullet to fall to the ground independent of whether you shoot it or drop it!)
Be careful re intuition....

A hungry bow-and-arrow hunter in the jungle wants to shoot down a coconut that is hanging from the branch of a tree. He points his arrow directly at the coconut, but as luck would have it, the coconut falls from the branch at the exact instant the hunter releases the string. Does the arrow hit the coconut?

Yes, it does(!!!). Why/how?

**Figure 4.18** A projectile follows a parabolic trajectory because it “falls” a distance $\frac{1}{2}gt^2$ below a straight-line trajectory.
Our ball on the track is a bit different. Why? (we’ll come back to this soon)

Much easier to study a jet of water than a falling ball (they behave the same!)

**Figure 3.11** Two marbles, one dropped and the other projected horizontally.

**Figure 3.14** Water droplets—each an individual projectile—combine to form graceful parabolic arcs in this fountain.

Vertical spacing is the same, showing that vertical and horizontal motions are independent.
Projectile Motion

Vector representation of the problem

**FIGURE 4.14** A projectile launched with initial velocity \( \vec{v}_0 \).

- \( \theta \): Launch angle
- \( v_0 \sin \theta \): Initial speed \( v_0 \)
- \( v_0 \cos \theta \): Horizontal component \( v_{0x} \)

**Parabolic trajectory**

**FIGURE 4.15** The velocity and acceleration vectors of a projectile moving along a parabolic trajectory.

- The vertical component of velocity decreases by 9.8 m/s every second.
- The horizontal component of velocity is constant throughout the motion.

Velocity vectors are shown every 1 s. Values are in m/s.

When the particle returns to its initial height, \( v_y \) is opposite its initial value.
Projectile Motion

\[ x = \frac{v_0^2}{g} \sin 2\theta_0 \quad \text{(horizontal range)} \]

\[ y = x \tan \theta_0 - \frac{g}{2v_0^2 \cos^2 \theta_0} x^2 \quad \text{(projectile trajectory)} \]

\[ \Rightarrow \text{You should feel comfortable deriving these formulae...} \]

Note: \[ \sin 2A = 2 \sin A \cos A \]

\[ \text{FIGURE 3.17 Parabolic trajectory of a projectile. Wolfson} \]

... and Tartaglia’s discovery should also be readily apparent
Projectile Motion

\[ x = \frac{v_0^2}{g} \sin 2\theta_0 \]  
(horizonal range)

Now you have the basic pieces & w/ some practice, we can do harder problems:
• What happens if there is “drag” (i.e., air resistance)?
• Circular motion...
• Force, energy, and all that....
• How does one launch a rocket to Mars?
A bird flies 100 m due east from a tree, then 200 m northwest (that is, 45° north of west). What is the bird’s net displacement?
A bird flies 100 m due east from a tree, then 200 m northwest (that is, 45° north of west). What is the bird’s net displacement?

**Solve**

The two displacements are \( \vec{A} = \text{100 m, east} \) and \( \vec{B} = \text{200 m, northwest} \). The net displacement \( \vec{C} = \vec{A} + \vec{B} \) is found by drawing a vector from the initial to the final position. But describing \( \vec{C} \) is a bit trickier than the example of the hiker because \( \vec{A} \) and \( \vec{B} \) are not at right angles. First, we can find the magnitude of \( \vec{C} \) by using the law of cosines from trigonometry:

\[
C^2 = A^2 + B^2 - 2AB \cos 45°
\]

\[
= (100 \text{ m})^2 + (200 \text{ m})^2 - 2(100 \text{ m})(200 \text{ m}) \cos 45°
\]

\[
= 21,720 \text{ m}^2
\]

Thus \( C = \sqrt{21,720 \text{ m}^2} = 147 \text{ m} \). Then a second use of the law of cosines can determine angle \( \phi \) (the Greek letter phi):

\[
B^2 = A^2 + C^2 - 2AC \cos \phi
\]

\[
\phi = \cos^{-1} \left[ \frac{A^2 + C^2 - B^2}{2AC} \right] = 106°
\]

It is easier to describe \( \vec{C} \) with the angle \( \theta = 180° - \phi = 74° \). The bird’s net displacement is

\[
\vec{C} = (147 \text{ m, 74° north of west})
\]
Carolyn drives her car north at 30 km/h for 1 hour, east at 60 km/h for 2 hours, then north at 50 km/h for 1 hour. What is Carolyn’s net displacement?
Carolyn drives her car north at 30 km/h for 1 hour, east at 60 km/h for 2 hours, then north at 50 km/h for 1 hour. What is Carolyn's net displacement?

SOLVE Chapter 1 defined velocity as
\[ \vec{v} = \frac{\Delta \vec{r}}{\Delta t} \]
so the displacement \( \Delta \vec{r} \) during the time interval \( \Delta t \) is \( \Delta \vec{r} = (\Delta t)\vec{v} \). This is multiplication of the vector \( \vec{v} \) by the scalar \( \Delta t \). Carolyn's velocity during the first hour is \( \vec{v}_1 = (30 \text{ km/h, north}) \), so her displacement during this interval is
\[ \Delta \vec{r}_1 = (1 \text{ hour})(30 \text{ km/h, north}) = (30 \text{ km, north}) \]
Similarly,
\[ \Delta \vec{r}_2 = (2 \text{ hours})(60 \text{ km/h, east}) = (120 \text{ km, east}) \]
\[ \Delta \vec{r}_3 = (1 \text{ hour})(50 \text{ km/h, north}) = (50 \text{ km, north}) \]

In this case, multiplication by a scalar changes not only the length of the vector but also its units, from km/h to km. The direction, however, is unchanged. Carolyn's net displacement is
\[ \Delta \vec{r}_{\text{net}} = \Delta \vec{r}_1 + \Delta \vec{r}_2 + \Delta \vec{r}_3 \]
This addition of the three vectors is shown in FIGURE 3.8, using the tip-to-tail method. \( \Delta \vec{r}_{\text{net}} \) stretches from Carolyn's initial position to her final position. The magnitude of her net displacement is found using the Pythagorean theorem:
\[ r_{\text{net}} = \sqrt{(120 \text{ km})^2 + (80 \text{ km})^2} = 144 \text{ km} \]
The direction of \( \Delta \vec{r}_{\text{net}} \) is described by angle \( \theta \), which is
\[ \theta = \tan^{-1}\left(\frac{80 \text{ km}}{120 \text{ km}}\right) = 34^\circ \]
Thus Carolyn's net displacement is \( \Delta \vec{r}_{\text{net}} = (144 \text{ km, } 34^\circ \text{ north of east}) \).
**Figure 3.23** shows three forces acting at one point. What is the net force \( \mathbf{F}_{\text{net}} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 \)?

**Note:** Be a bit careful here. Notice that the coord. system is tilted....
One strategy is to “rotate” the given coordinate system...

... and then re-express the vectors

Note: Be careful w/ angle in degrees versus radians

\[
\text{radians} = \text{degrees} \cdot \frac{\pi}{180^\circ}
\]
Be careful! How you define the angle and/or keep track of “bookkeeping” can have huge consequences!
**VISUALIZE** Figure 3.23 show the forces and establishes a tilted coordinate system.

**SOLVE** The vector equation $\vec{F}_{\text{net}} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$ is really two simultaneous equations:

$$(F_{\text{net}})_x = F_{1x} + F_{2x} + F_{3x}$$

$$(F_{\text{net}})_y = F_{1y} + F_{2y} + F_{3y}$$

The components of the forces are determined with respect to the axes. Thus

$$F_{1x} = F_1 \cos 45^\circ = (50 \, \text{N}) \cos 45^\circ = 35 \, \text{N}$$

$$F_{1y} = F_1 \sin 45^\circ = (50 \, \text{N}) \sin 45^\circ = 35 \, \text{N}$$

$\vec{F}_2$ is easier. It is pointing along the $y$-axis, so $F_{2x} = 0 \, \text{N}$ and $F_{2y} = 20 \, \text{N}$. To find the components of $\vec{F}_3$, we need to recognize—because $\vec{F}_3$ points straight down—that the angle between $\vec{F}_3$ and the $x$-axis is $75^\circ$. Thus

$$F_{3x} = F_3 \cos 75^\circ = (57 \, \text{N}) \cos 75^\circ = 15 \, \text{N}$$

$$F_{3y} = -F_3 \sin 75^\circ = -(57 \, \text{N}) \sin 75^\circ = -55 \, \text{N}$$

The minus sign in $F_{3y}$ is critical, and it appears not from some formula but because we recognized—from the figure—that the $y$-component of $\vec{F}_3$ points in the $-y$-direction. Combining the pieces, we have

$$(F_{\text{net}})_x = 35 \, \text{N} + 0 \, \text{N} + 15 \, \text{N} = 50 \, \text{N}$$

$$(F_{\text{net}})_y = 35 \, \text{N} + 20 \, \text{N} + (-55 \, \text{N}) = 0 \, \text{N}$$

Thus the net force is $\vec{F}_{\text{net}} = 50 \hat{i} \, \text{N}$. It points along the $x$-axis of the tilted coordinate system.

**ASSESS** Notice that all work was done with reference to the axes of the coordinate system, not with respect to vertical or horizontal.
Frogs, with their long, strong legs, are excellent jumpers. And thanks to the good folks of Calaveras County, California, who have a jumping frog contest every year in honor of a Mark Twain story, we have very good data on how far a determined frog can jump.

High-speed cameras show that a good jumper goes into a crouch, then rapidly extends his legs by typically 15 cm during a 42 ms push off, leaving the ground at a 30° angle. How far does this frog leap?
FIGURE 4.19 Pictorial representations of the jumping frog.

**Pushing off**

Known:
- \( x_0 = 0 \text{ m} \)
- \( v_{0x} = 0 \text{ m/s} \)
- \( t_0 = 0 \text{ s} \)
- \( x_1 = 0.15 \text{ m} \)
- \( t_1 = 0.042 \text{ s} \)

Find:
- \( v_{ix} \)

**In flight**

Known:
- \( x_0 = 0 \text{ m} \)
- \( y_0 = 0 \text{ m} \)
- \( t_0 = 0 \text{ s} \)
- \( v_0 \theta = 30^\circ \)
- \( y_1 = 0 \text{ m} \)
- \( a_y = -g \)

Find:
- \( x_1 \)
Represent the frog as a particle. Model the push off as linear motion with constant acceleration. A bullfrog is fairly heavy and dense, so ignore air resistance and consider the leap to be projectile motion.

This is a two-part problem: linear acceleration followed by projectile motion. A key observation is that the final velocity for pushing off the ground becomes the initial velocity of the projectile motion. Figure 4.19 shows a separate pictorial representation for each part. Notice that we’ve used different coordinate systems for the two parts; coordinate systems are our choice, and for each part of the motion we’ve chosen the coordinate system that makes the problem easiest to solve.

While pushing off, the frog travels $15 \text{ cm} = 0.15 \text{ m}$ in $42 \text{ ms} = 0.042 \text{ s}$. We could find his speed at the end of pushing off if we knew the acceleration. Because the initial velocity is zero, we can find the acceleration from the position-acceleration-time kinematic equation:

$$x_1 = x_0 + v_{0x} \Delta t + \frac{1}{2} a_x (\Delta t)^2 = \frac{1}{2} a_x (\Delta t)^2$$

$$a_x = \frac{2x_1}{(\Delta t)^2} = \frac{2(0.15 \text{ m})}{(0.042 \text{ s})^2} = 170 \text{ m/s}^2$$

This is a substantial acceleration, but it doesn’t last long. At the end of the 42 ms push off, the frog’s velocity is

$$v_{1x} = v_{0x} + a_x \Delta t = (170 \text{ m/s}^2)(0.042 \text{ s}) = 7.14 \text{ m/s}$$

We’ll keep an extra significant figure here to avoid round-off error in the second half of the problem.
The end of the push off is the beginning of the projectile motion, so the second part of the problem is to find the distance of a projectile launched with velocity \( \vec{v}_0 = (7.14 \text{ m/s}, 30^\circ) \). The initial \( x \)- and \( y \)-components of the launch velocity are

\[
\nu_{0x} = v_0 \cos \theta \quad \nu_{0y} = v_0 \sin \theta
\]

The kinematic equations of projectile motion, with \( a_x = 0 \) and \( a_y = -g \), are

\[
\begin{align*}
x_1 &= x_0 + \nu_{0x} \Delta t \\
&= (v_0 \cos \theta) \Delta t
\end{align*}
\]

\[
\begin{align*}
y_1 &= y_0 + \nu_{0y} \Delta t - \frac{1}{2} g (\Delta t)^2 \\
&= (v_0 \sin \theta) \Delta t - \frac{1}{2} g (\Delta t)^2
\end{align*}
\]

We can find the time of flight from the vertical equation by setting \( y_1 = 0 \):

\[
0 = (v_0 \sin \theta) \Delta t - \frac{1}{2} g (\Delta t)^2 = (v_0 \sin \theta - \frac{1}{2} g \Delta t) \Delta t
\]

and thus

\[
\Delta t = 0 \quad \text{or} \quad \Delta t = \frac{2v_0 \sin \theta}{g}
\]

Both are legitimate solutions. The first corresponds to the instant when \( y = 0 \) at the launch, the second to when \( y = 0 \) as the frog hits the ground. Clearly, we want the second solution. Substituting this expression for \( \Delta t \) into the equation for \( x_1 \) gives

\[
x_1 = (v_0 \cos \theta) \frac{2v_0 \sin \theta}{g} = \frac{2v_0^2 \sin \theta \cos \theta}{g}
\]

We can simplify this result with the trigonometric identity

\[
2 \sin \theta \cos \theta = \sin(2\theta)
\]

Thus the distance traveled by the frog is

\[
x_1 = \frac{v_0^2 \sin(2\theta)}{g}
\]

Using \( v_0 = 7.14 \text{ m/s} \) and \( \theta = 30^\circ \), we find that the frog leaps a distance of 4.5 m.

**ASSESS** 4.5 m is about 15 feet. This is much farther than a human can jump from a standing start, but it seems believable. In fact, the current record holder, Rosie the Ribeater, made a leap of 6.5 m!
A bomb is dropped from an aeroplane flying horizontally at a constant speed. Where will the aeroplane be when the bomb hits the ground?
Be careful! Does “air resistance” matter? (we’ll return to this soon)

The plane flies horizontally with constant speed $v$. The bomb follows the path of a parabola, since its motion is compounded of horizontal motion with initial velocity $v$ and uniformly accelerated vertical fall. If there were no air-resistance, the bomb’s horizontal velocity would be no different from that of the plane, and the plane would be directly above the bomb the whole time—in particular, when the bomb hits the ground. But in fact, as a result of air-resistance, the bomb’s horizontal velocity is decreasing all the time, and so it falls behind the plane (Fig. 163). Therefore the fall to earth and explosion of the bomb take place not underneath the plane, but considerably behind it.
We want $v_0$ so that the hammer will just clear the point $x = 3.1\,\text{m}$, $y = 1.6\,\text{m}$.

FIGURE 3.18 Our sketch for Example 3.5.

→ Practice these “projectile motion” problems, keeping careful track of what assumptions are stated (or need to be presumed!)