York University
PHYS 2010 HW2 (Winter 2020) SOLUTIONS

1 Problems

1. (6 pts) An important consideration in launching a rocket is *escaping* the gravitational potential energy of earth.

   (a) Briefly explain what gravitational potential energy is, and what the *escape speed* (v_{esc}) represents.
   SOL: Gravitational potential energy ($\equiv U_G$) stems from the attractive force due to gravity between two bodies. For a given pair of masses (m_1 and m_2) separated by distance r, the typical definition (stemming from Newton's Law of Gravitation and the general definition of potential energy as the integral of work over distance) is

   $$U_G = -\frac{Gm_1 m_2}{r} \tag{1}$$

   with the convention that it is negative (in that it is "stored" energy): Consider that as an object approaches a planet, gravity does positive work on that object. $U_G = 0$ represents the case when $r = \infty$ (i.e., when the masses have been infinitely separated). v_{esc} represents the condition when kinetic energy of one of the objects (say, m_1) is equal to U_G. That is, it has enough energy to escape the bound state due to the gravitational attraction. Carrying that calculation through, one finds $v_{esc} = \sqrt{2Gm_2/r}$ (note that this does not explicitly depend upon m_1).

   (b) Consider a rocket that is launched vertically upward at $\sqrt{2}v_{esc}$ from planet Earth, whose mass is M_E and approximate radius is R_E. Derive an expression for its speed as a function of the distance from the planet [i.e. $v(r)$]. [*Hint: Remember conservation of energy!*.]
   SOL: Neglecting air resistance, energy will be conserved, meaning that $K_0 + U_0 = K(v) + U(r)$. Here we know K_0 [=$\frac{1}{2}m(\sqrt{2}v_{esc})^2$, since $v_0 = \sqrt{2}v_{esc}$], U_0 (=$-\frac{GM_E m}{R_E}$, where m is the mass of the rocket), and $U(r)$ [=$-\frac{GM_E m}{r}$], so we can readily solve for the velocity v [since $K(v) = \frac{1}{2}mv^2$]:

   $$v(r) = \pm\sqrt{v_0^2 + 2GM_E\left(\frac{1}{r} - \frac{1}{R_E}\right)} = \pm\sqrt{2GM_E\left(\frac{1}{r} + \frac{1}{R_E}\right)} \tag{2}$$

   Note that if $r = R_E$, this reduces to $v(R_E) = \sqrt{2}v_{esc}$ as expected.

   (c) Find numerical values for M_E and R_E, and write a Matlab script to plot $v(r)$. Make sure to clearly comment your code, and that your plot is clearly labeled.
   SOL: Sample code is included at end of the document. Plot included below.

   (d) Like the above problem, a popular first-year physics text physics text (Wolfson) states "*A rocket is launched vertically upward at 3.1 km/s*". Briefly explain conceptually what is wrong with the basis of these problems.
   SOL: A rocket starts with zero velocity and is *accelerated* to some velocity. So one can have a rocket moving at that stated speed, but technically it is "launched" from rest (launch is defined as to "start or set in motion").

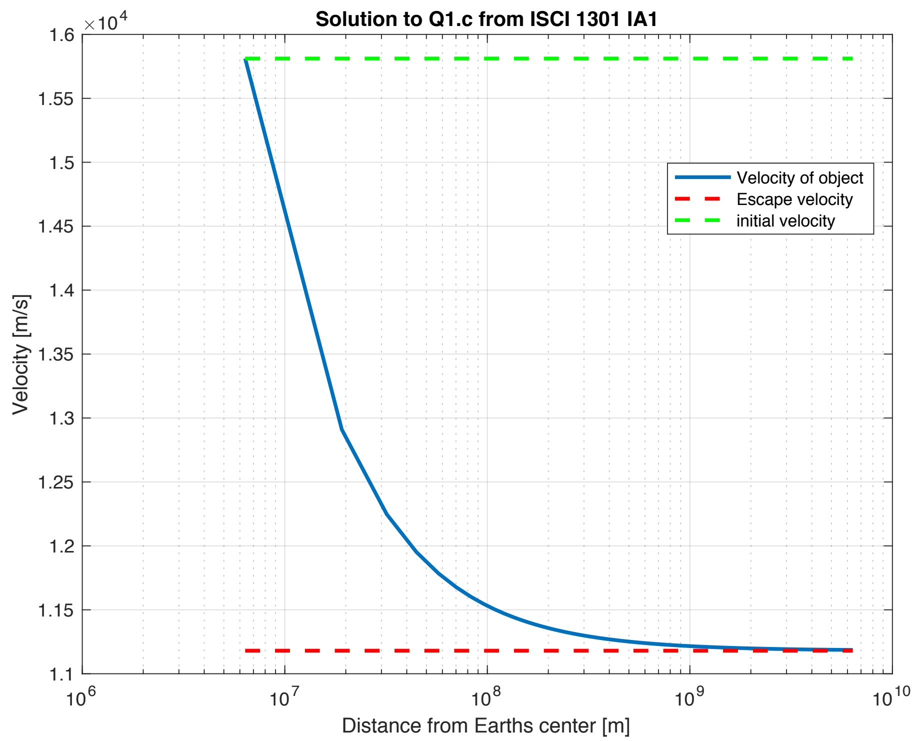

Figure 1: Answer to Q.1c. See code at end of document.

2. (8 pts) Estimate the energy required to send a rocket of mass M_o to Mars. Clearly state assumptions made and outline all the relevant calculations in detail.

   [*Hint: This is a "Fermi-type" problem. There are many paths one could take and possible considerations to make, such as gravitational potential energy tied to Earth/Mars/Moon/Sun, engine efficiency, etc.*]

   SOL: Many different ways to approach this problem. Let's focus on the change in gravitational potential energy, where what we seek we will call ΔU. A simple approach would be as follows, though we first introduce several variables: R_E is the radius of the Earth, M_E is the mass of the Earth, R_{EM} is the distance between the surfaces of Earth and Mars, R_M is the radius of the Mars, and M_M is the mass of the Mars.

   - Let U_E represent the gravitational potential energy binding the rocket to Earth. This is
   $$U_E = -\frac{GM_E M_o}{R_E} \tag{3}$$
   - Now ultimately this needs to be changed by
   $$\Delta U_E \equiv U_{EM} - U_E \tag{4}$$
   where
   $$U_{EM} = -\frac{GM_E M_o}{R_E + R_{EM}} \tag{5}$$
   That is, ΔU_{EM} represents the amount of energy we need to give the rocket to move it distance R_{EM} due to Earth's gravitational potential. Note that $\Delta U_E > 0$.
   - This energy is in part offset somewhat* by the gain in potential due to Mars (ΔU_M). That is
   $$\Delta U_M \equiv U_M - U_{ME} = -GM_M M_o \left(\frac{1}{R_M} - \frac{1}{R_M + R_{EM}} \right) \tag{6}$$
   Note that $\Delta U_{ME} < 0$. Also, $|\Delta U_{ME}| < |\Delta U_{EM}|$, since $M_E > M_M$ and $R_E > R_M$.
   - It is also worth noting that $R_{EM} >> R_M, R_E$ such that $1/R_M - 1/(R_M + R_{EM}) \approx 1/R_M$ (ditto for Earth). This yields
   $$\Delta U = \Delta U_E + \Delta U_M \approx GM_o \left(\frac{M_E}{R_E} - \frac{M_M}{R_M} \right) \tag{7}$$
   So to first order, we have a rough ballpark estimate.
   - Extending further, this is (obviously) not the whole story. When leaving Earth and entering Mars, there are atmospheric conditions that will give rise to drag. That is, we will need to impart a kinetic energy greater than predicted from ΔU alone. How much so? Put another way, how might one estimate the relative contribution of drag forces? Additionally, engines are not 100% efficient. That is, they burn more energy than they output. So such would also need to be factored in to revised estimates.

*Practically speaking, there is no obvious means to "harvest" the gain in gravitational potential energy due to moving closer to Mars. But it is fun to include here nonetheless!

3. (4 pts) Rockets move about via *thrust*. That is, say in outer space, there is nothing to "push against" so to induce an external force and thereby accelerate something else is required. So the relevant forces have to come from "inside" the system (and hence thrust). A fundamental principle underlying thrust is the *conservation of momentum* (COM).

   (a) State the law of COM. Identify several different interdisciplinary scenarios where COM plays an important role (e.g., kinesiology, car crashes, etc.).
   SOL: COM states that in a closed system not subject to external forces, the total momentum is a constant. COM arises in an enormous variety of situations. A salient interdisciplinary example is that of *optical tweezers*, where diffraction of incident light cause a spring-like (i.e., restorative) force that allows one to "trap" microscopic objects. Such allows one to measure biologically-relevant forces such as those tied to molecular motors such as kinesin (e.g., `http://www.nature.com/nature/journal/v400/n6740/abs/400184a0.html`) and the forces that hold DNA together
   (e.g., `https://www.ncbi.nlm.nih.gov/pmc/articles/PMC1184516/`).

   (b) Explain conceptually why COM is relevant for rockets. For example, the momentum of what? Does a rocket's mass change with time? How is such related back to COM? Feel free to use equations as needed.
   SOL: A rocket, to first order, can be considered a *closed system*. By expelling propellant, preferably at some high (exhaust) velocity in an oppositely oriented direction, COM dictates that the rocket must be propelled forward as a result. Same basic idea with a balloon that deflates and flies about the room. Correspondingly, a rocket's mass does change with time. We can firm this up analytically a bit more. Consider a rocket whose mass (including fuel) is m. At some instant, moves with velocity v_0. At a later time, it has expelled some fraction of fuel (of mass Δm) in the opposite direction at velocity v_p and as a result now moves forward with velocity v_R. From COM, we have:
   $$(m+\Delta m)v_0 = \Delta m v_p + (m-\Delta m)v_R \tag{8}$$
   Note that while the ejected fuel moves opposite *relative to* the rocket, it can still move in the same direction as viewed by a stationary observer. So its possible v_p and v_R have the same sign (though the sign of the difference between them is key!).

   (c) Examine Figs.1a and 1b. Outline what parallels there are between the two different scenarios shown there. Additionally, what differences are there?
   SOL: The two scenarios are directly analogous. A main difference is that the bullets come out in discrete "chunks", as opposed to a more continuous exhaust in the case of the rocket (and hence the factor of n that arises in the figure).

4. (12 pts) Consider a rocket in deep space (like that in Fig.1b), where there are negligible external forces such as gravity or drag. The mass of the entire rocket (i.e., including unused fuel) is M and it burns fuel (i.e., expels the fuel in an opposite direction) at a constant rate μ until all the fuel is used up (call that the *burnout time*, or t_B). Assume that the velocity v_{rel} of the ejected gas particles relative to the rocket is constant.

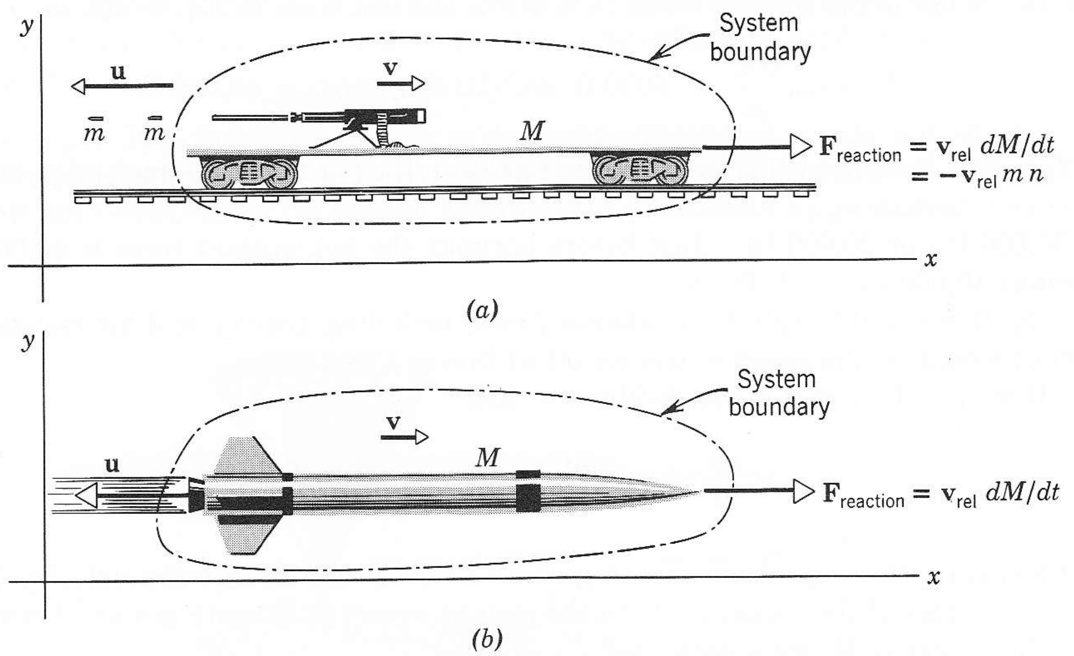

Fig. 9–11 (*a*) Example 8. A machine gun is fixed to a car that rolls with negligible friction. The gun fires bullets of mass m at a rate (number per unit time) n, the velocity of the bullets with respect to the gun being $\mathbf{u} - \mathbf{v}$. At the instant shown some bullets have already left the system. The velocities indicated for the car and the bullets are those that would be measured by an observer in a reference frame fixed to the rails as shown. The reaction force on the system is $\mathbf{F} = -mn\mathbf{v}_{\text{rel}} = (dM/dt)\mathbf{v}_{\text{rel}}$. (*b*) A rocket moves through space with negligible external forces. Gas particles are ejected from the exhaust, the particles having a velocity $\mathbf{u} - \mathbf{v}$ with respect to the rocket. The rate at which mass is expelled at the exhaust is $-dM/dt$. The reaction force on the rocket is $\mathbf{F} = (dM/dt)\mathbf{v}_{\text{rel}}$. The velocities indicated for the rocket and exhaust gases are relative to the ground.

Figure 2: From Resnick and Halliday (1966).

(a) Draw a free-body diagram and set up the equation of motion via Newton's 2nd law). Specifically, you should find that

$$M\frac{d\mathbf{v}}{dt} = \mathbf{F}_T \tag{9}$$

where $\mathbf{F}_T$ is the thrust force. How does $\mathbf{F}_T$ depend upon the prescribed variables?

SOL: Free-body diagram is similar to that shown in the previous figure (i.e., rocket along w/ machine gun atop train car). The thrust force is directly tied to the ejected gas and the changing mass of the rocket. Putting the pieces together, we have:

$$F_T = -\frac{dM}{dt}v_{\text{rel}} \tag{10}$$

where dM/dt describes the change in the rocket's mass and v_{rel} the relative velocity by which that mass is ejected. Note that the units of dM/dt are mass per unit time and v_{rel} distance per unit time. Also note that given the assumptions made (chiefly that μ is a constant), dM/dt is just a linear function of time.

(b) Determine the change of the velocity [i.e., $v_{tB}-v_0$] of the rocket during the time interval $t \in [0, t_B]$ as a function of the change of the rocket's mass during that interval. Note that your answer does not have an explicit time dependence and velocity depends upon the *mass ratio* (a useful measure of time into the burn of a rocket). A common approach to solving this is to use integrals, but you should try to solve it without integrals. Use only your knowledge of derivatives and solutions to the (differential) equation of the form $x' = a + bx$ (which you saw earlier in the semester). [*Hint: Your answer should be the so-called* **rocket equation**.]†

SOL: For simplicity, we will assume the rocket starts from rest (i.e., $v_0 = 0$) and that we'll take the "integral" approach here. Let M be the mass of the rocket (and fuel) at some time t. From the last part, we have

$$M\frac{dv}{dt} = -\frac{dM}{dt}v_{\text{rel}} \tag{11}$$

Rearranging and "cancelling" the dt, we have:

$$dv = -\frac{dM}{M}v_{\text{rel}} \tag{12}$$

Now for simplicity, to integrate we assume v_{rel} is a constant and that at $t = 0$, the rocket has zero velocity and mass M_0. Furthermore, we do not explicitly consider time (e.g., $v = v(t)$):

$$\int_0^v dv = -v_{\text{rel}}\int_{M_0}^{M}\frac{dM}{M} \tag{13}$$

Solving through, we obtain:

$$v = v_{\text{rel}}\ln\left(\frac{M_0}{M}\right) \tag{14}$$

While time (e.g., t_B) is not explicitly included here, implicitly it is (see next part). As long as the fuel is being burned and is done so at a constant rate, Eqn.14 is valid. The velocity at t_B would

†Kudos if you didn't Google this! But since you likely did, who was Konstantin Tsiolkovsky?

simply just be $v_{\rm rel}\ln\left(M_0/M_{final}\right)$ Note that Eqn.14 indicates that the speed of the rocket can increase to any value provided that the rocket expels enough propellant so that the final remaining mass is sufficiently small. However, relativistic mechanics tells us that there ultimately is an upper limit: c (the speed of light).

(c) Comment briefly on how time factors into your answer to the last part. How does the thrust change as a function of time?
SOL: In our last equation, we implicitly have $v = v(t)$ and $M = M(t)$. Furthermore, the thrust is assumed to be constant here.

(d) Rearrange your answer to the last part to express the rocket's mass as a function of its velocity.
SOL:

$$M = M_0 e^{-v/v_{\rm rel}} \tag{15}$$

(e) A rocket, weighing 30000 lbs. before liftoff, is fired vertically upward. At burnout, it weighs 10000 lbs. Gases are exhausted at a rate of 10 slugs/s, at a velocity of 5000 ft/s (relative to the rocket). Assume both those two quantities are constant. What is the thrust force? Additionally, what is the rocket's speed and kinetic energy at t_B?
SOL: Here we will neglect all external forces (e.g., gravity, air resistance). Relating back to the rocket equation we derived, $M_0 = 30000$ lbs., $M = 10000$ lbs, $v_{\rm rel} = 5000$ ft/s, and $dM/dt = 10$ slugs/s. The thrust force is just $v_{\rm rel}dM/dt = 50000$ lbs. (since 1 lb of force equals 1 slug ft/s^2). The resulting velocity at burnout is 5493 ft/s $\approx$ 3800 miles/hour. The kinetic energy is $1/2Mv^2 = (0.5)(10000)(5493)^2 = 1.51 \times 10^{11}$ foot pounds ($\approx 2.11 \times 10^{11}$ J). Presumably if air resistance and/or gravity were factored in, these values would be smaller.

(f) At what mass ratio is the kinetic energy (T) of the rocket, including fuel, maximal? Calculate the velocity, mass, kinetic energy of the rocket, and the time at which that occurs [assuming $v(0) = 0$].
SOL: Let's make a change of variables. From earlier, we had

$$F_T = -\frac{dM}{dt}v_{\rm rel} \equiv \mu\gamma \tag{16}$$

We just made the change $\mu \equiv -\dot{M}$ (the diacritical dot indicates a time derivative) and $\gamma \equiv v_{\rm rel}$. Now assuming no external forces (like gravity), we simply have from Newton's 2nd Law:

$$M\dot{v} = \mu\gamma \tag{17}$$

which leads to the familiar rocket equation we derived:

$$v = \gamma\ln\left(\frac{M_o}{M}\right) \tag{18}$$

Now the kinetic energy (T) is

$$T = \frac{1}{2}Mv^2 \tag{19}$$

where both M and v implicitly depend upon t. Let us determine the *time* that T is maximal:

$$\frac{dT}{dt} = \frac{1}{2}(\dot{M}v^2 + 2Mv\dot{v}) = 0 \tag{20}$$

Recall that $\dot{M} = -\mu$ and $M\dot{v} = \mu\gamma$, allowing us to rewrite the last equation as

$$\frac{\mu v}{2}(-v + 2\gamma) = 0 \tag{21}$$

This indicates the velocity is maximal when $v = 2\gamma$. Back to the rocket equation, the kinetic energy is maximal at:

$$2\gamma = \gamma \ln\left(\frac{M_o}{M}\right) \rightarrow \frac{M_o}{M} = e^2 \tag{22}$$

Note that we can also explicitly solve for the time at which this occurs. Since the exhaust rate (μ) is constant, $M(t) = M_0 - \mu t$ (i.e., just a linear function, w/ the caveat that M cannot become negative). Solving then for t_T (i.e., the time at which T is maximal), we find

$$2 = \ln\left(\frac{M_0}{M_0 - \mu t_T}\right) \rightarrow t_T = \frac{M_0}{\mu}(1 - e^{-2}) \tag{23}$$

Correspondingly:

$$T(t_T) = 2M_0\left(\frac{\gamma}{e}\right)^2 \tag{24}$$

To visualize this, consider the output of the code appended to the solutions. To summarize, note that a more generally, given the assumptions made, the rocket equation can be written with time-dependence as

$$v(t) = \gamma \ln\left(\frac{M_0}{M_0 - \mu t}\right) \tag{25}$$

(g) In terms of mass ratios, is a rocket like a car? Why or why not?
SOL: Not really. The mass of the car is far greater than that of the gasoline it carries. Such is unlike a rocket, where a significant fraction of the total mass (at launch) is fuel.

(h) Now consider the rocket in the *constant* gravity field near Earth, where the goal is to get off the surface. Draw the relevant free-body diagram. Set up the equation of motion (i.e., what modifications to Eqn.9 need to be made?). Which condition has to be satisfied to allow the rocket to take off?
SOL: Thrust is no longer the only relevant force. Consider inclusion of gravity for example (your free-body diagram should indicate that it acts opposite to thrust and hence):

$$M\frac{d\mathbf{v}}{dt} = \mathbf{F}_T - Mg \tag{26}$$

where g is the acceleration due to Earth's gravitational field. In order for take-off to occur, there must a positive acceleration (i.e., $F_T = \mu\gamma > M_0 g$; see previous parts for variable definitions).

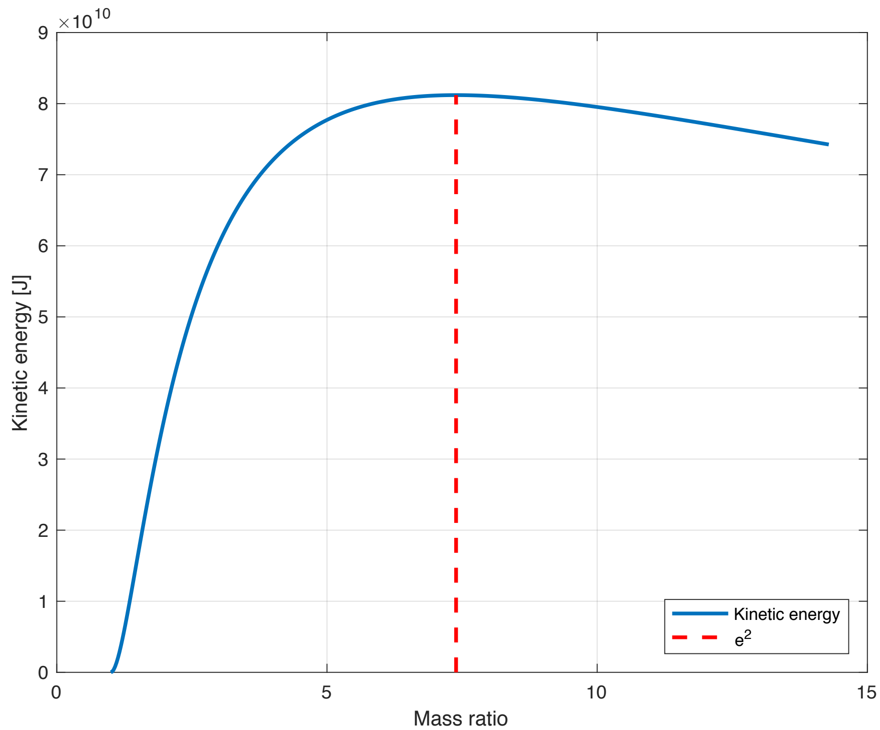

Figure 3: Kinetic energy as a function of mass ratio. Assumes that $g = 0$ m/s^2. Note the maximum occurs when the mass ratio is e^2.

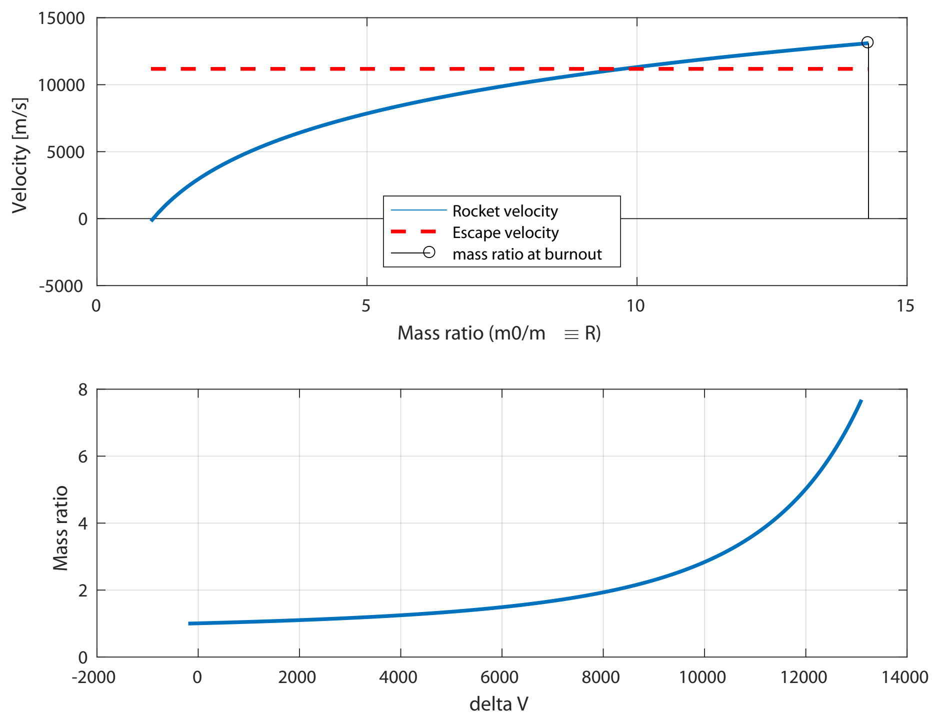

Figure 4: Figure computed via EXrocketM1.m (see code appended to end). Here it was assumed that $v_{\rm rel} = 5000$ m/s and that $g = 9.8$ m/s^2 (constant).

[*Hint: Ch.7.8 of Hawkes (Physics for Scientists and Engineers: An Interactive Approach) might be helpful! Try heading over to Steacie!*]

5. (6 pts) This question builds off your derivation of the rocket equation in the last part and asks you to computationally explore several aspects via Matlab. Along with your answers, you should turn in any relevant code. Make sure it is concise and commented (so to make clear what is what).

   (a) Write a code that plots the velocity of the rocket as a function of the mass ratio for several different exhaust velocities ($v_{\rm rel}$).
   SOL: Sample code is included at end of the document. Figure for one choice of exhaust velocity ($v_{\rm rel} = 5000$ m/s) is shown here. Lower velocities can be observed to not allow the rocket to achieve escape velocity.

   (b) Pick a set of parameters (e.g., $v_{\rm rel}$) and determine at what point the rocket's velocity exceeds that of the exhaust velocity. Comment (e.g., How does such depend upon $v_{\rm rel}$? Do the relevant values seem *special* in some way?). Additionally, how would such appear to an observer on the ground?
   SOL: From the upper panel in the figure shown for the last part, one can observe that the mass ratio is approximately e when the rocket velocity surpasses the exhaust velocity (it is precisely e

when $g = 0$ m/s^2). This should make sense as the rocket equation (neglecting gravity) states that the rocket velocity (V) is

$$V = v_{\rm rel} \ln \frac{M_0}{M} \tag{27}$$

and the natural log of e is just 1. At that point, a stationary observer on Earth would see both exhaust and rocket moving in the same direction, but the rocket moving relatively faster.

[*Hint: A book entitled* **Rocket and Spacecraft Propulsion** *by Martin J.L. Turner might be helpful. You should be able to find a soft copy via York's subscription to SpringerLink (if you have trouble, ask a librarian in Steacie!).*]

Sample Matlab code for Q.1c that plots the velocity as a function of distance from Earth:

```
% code to produce plot to ISCI F16 IA1 Q1c (12.20.16)

clear
% -----
RE= 6.371* 10^(6);     % approx. radius of Earth [m]
ME= 5.97* 10^(24);     % approx. mass of Earth [kg]
G= 6.67* 10^(-11);       % Gravitational const. [m^3 kg^-1 s^-2]
% -----
r=linspace(RE,1000*RE,500);
v= sqrt(2*G*ME*(1./r + 1/RE));
figure(1); clf;
h1= semilogx(r,v,'LineWidth',2); grid on; hold on;
h2= plot([r(1) r(end)],sqrt(2*G*ME/RE)*[1 1],'r--','LineWidth',2);
h3= plot([r(1) r(end)],sqrt(2)*sqrt(2*G*ME/RE)*[1 1],'g--','LineWidth',2);
legend([h1 h2 h3],'Velocity of object','Escape velocity','initial velocity')
xlabel('Distance from Earths center [m]'); ylabel('Velocity [m/s]');
title('Solution to Q1.c from ISCI 1301 IA1');
```

Sample Matlab code for Q.5a that plots the velocity of the rocket as a function of the mass ratio for a specified exhaust velocity:

```
% ### EXrocketM1.m ###       [08.10.16 CB]

% NOTE: Not sure if this code is correct for g \neq 0 (i.e., the
% time-dependent aspect needs to be handled more carefully)

% Simple code to demonstrate 1-D "dynamics" tied back to a rocket launch.
% Governing eqn. (assumed 1-D here) is as follows:
% m*dv/dt = F + c*dm/dt
% v is the rocket's velocity and m the total mass at any given instant
% o left-hand term is just Newton's 2nd law
% o the term c*dm/dt (=T) is the "thrust" due to gas expulsion, c is const.
%   and assumed to be parallel and opposite v; dm/dt <0 (i.e., mass is lost
%   due to rocket burning fuel)
% o F represents any additional external forces such gravity and/or drag

% NOTE: In some texts, the thrust force is written as c*dm/dt = v_e* mR
% where v_e (=c) is the "effective exhaust velocity" and mR (=dm/dt) is the "mass flow
% rate", with the notion that mR= dM/dt (M is the total instantaneous mass
% of the rocket, i.e., that of rocket + fuel). Presumbaly done because both of these are
% assumed to be constant?; see note below re "mass ratio"

% Here we assume:
% - c is constant
% - v0=0 (i.e., rocket starts from rest) and t0=0
% - the time at which the rocket burns all its fuel is called "time of burn" (t= tB)

% Several different cases:
% 1. No external forces
%    dv/dt = -c* (1/m)* dm/dt
%  SOL: m= m0* exp(-deltaV/c)
%    + conversely: v= v0+ c* ln(m0/m) (commonly known as the "rocket equation")
%    + m0/m is called the "mass ratio" (>= 1) and is sometime called R
%    + obtained via integrating between initial time t0 and final time t
%    + here deltaV = v-v0 the change in velocity over the interval t-t0
%    + m=m(t) and m0= m(t0)
%    + rearrange to bring in time-depend.: v(t)= v0- c*ln[1+ t*(dm/dt)/m0]
%    + propellant mass (massF) equals m0-m= m0(1- exp(-deltaV/c))
%    + from an engineering POV, one can control c; dm/dt<0
% 2. Gravity present
%    m*dv/dt = -c* (1/m)* dm/dt - g
%  SOL: v= v0+ c*ln(m0/m)- g*tB = v0+ c*ln(m0/m)- g*[(m-m0)/(dm/dt)]
%    + conversely: m= m0* exp[-(deltaV+g*tB)/c]
%    + *** assumes c*(dm/dt) > m0*g at t=0 (otherwise rocket won't take off;
% it will sit on launch pad burning fuel until this req. is met)
% 3. Gravity and drag present (NOT CONSIDERED numerically by this code)
%    m*dv/dt = -c* (1/m)* dm/dt - g - D/m
%  SOL: {see note below; hard since D is a function of t}
```

```
%   + here D is the drag force
%   + D= (1/2)* rho* v^2* A* Cd
%   + rho is the density of air, A the effective area, Cd drag coefficient
%   + rho is a function of the rocket's height (call it z) and can be
%   approximated by rho= rho0* exp(-z/H) where rho0 is the density at
%   sea-level and H the "scale height" (~8000 m)
%   + hard to determine an explicit analytic solution and the ODE needs to
%   be integrated numerically
%   + realisitcally, the relative difference between the drag force and
%   gravity is quite small (~2%)

% =========================================================
% From Turner (pg.16)
% "The rocket equation shows that the final speed depends upon only two numbers:
% the final mass ratio, and the exhaust velocity. It does not depend on the thrust,
% rather surprisingly, or the size of the rocket engine, or the time the rocket burns,
% or any other parameter. Clearly, a higher exhaust velocity produces a higher rocket
% velocity, and much of the effort in rocket design goes into increasing the exhaust
% velocity. [...] To achieve a high rocket velocity, the mass ratio has to be large."

% =========================================================
% QUESTIONS (for students)

% o When using the escape velocity as a benchmark, does it matter that the
% rocket's mass is changing as it rises? Or that the distance is changing
% as it accelerates?
% o How might we change our approach here if the exhaust velocity (c or
% v_e) is not a constant?
% o How would you modify the code below to bring in explicit time
% dependence?
% o The exhaust has some mass (otherwise the rocket wouldn't move!). Can
% you plot that as some sort of mass flow rate?
% o At what point does the rocket speed exceeds the exhaust speed? (Hint:
% DOes the mass ratio equal a special number? Why?)



% Ref. (see eqns.12ff)
% o [Peraire & Widnall] http://ocw.mit.edu/courses/aeronautics-and-astronautics/
% 16-07-dynamics-fall-2009/lecture-notes/MIT16_07F09_Lec14.pdf
% o book by Turner (Rocket and Spacecraft Propulsion)

clear
% --------------------
% User params.

m0= 300000;      % initial total rocket mass [kg] {300000}
massfrac= 0.07; % fraction of rocket's mass not fuel {0.07}
c= 4000; % exhaust velocity [m/s] {4000}

g= 9.8;  % acceleration due to gravity [m/s^2] {0}
tB= 20;    % time of burn [s]
```

```
G= 6.67*10^(-11);    % Gravitational const. [m^3/(kg*s^2)]
rE= 6.37*10^(6);    % Earth's radius [m]
massE= 5.97*10^(24);  % Earth's mass [kg]

% --------------------


% [derived quantities]
Isp= c/g;   % "specific impulse" [s]
massRO= m0*massfrac;    % rocket's total mass (const) [kg]
massF= m0- massRO;    % mass of rocket's fuel at launch [kg]
massR= linspace(1,1/massfrac,100);  % mass ratio (as a function between liftoff and burnout)
mass= m0./massR;    % rocket's total mass for a given mass ratio [kg]
vE= sqrt(2*G*massE/rE); % escape velocity from Earth [m/s]
mrBO= m0/massRO;    % mass ratio at time of burnout [same as massR(end)]

mdot= massF/tB;         % mass flow rate [kg/s]
time= -(mass-m0)/mdot;   % proxy for time (see P&W eqn.16)
vt= c*log(1+ time*mdot/m0);  % velocity as function of time (assumes g=0!)

% determine relevant quantities (see notes above)
v= c*log(massR)- g*tB;      % velocity versus mass ratio
m= m0- mdot*time;               % mass as a function of time
T= 0.5* m.*v.^2;    % kinetic energy of rocket
%T= 0.5* (m0./massR).*v.^2;    % equivalent to the last line


exFlow= (massF-massRO)/tB;

% plot
figure(1); clf;
subplot(211)
h1= plot(massR,v,'LineWidth',2); grid on; hold on;
h2= plot([massR(1) massR(end)],[vE vE],'r--','LineWidth',2);
h3= stem(mrBO,v(end),'k');
xlabel('Mass ratio (m0/m \equiv R)'); ylabel('Velocity [m/s]');
legend([h1 h2 h3],'Rocket velocity','Escape velocity',...
    'mass ratio at burnout','Location','West');
subplot(212)
plot(v,m0./m,'LineWidth',2); grid on; hold on;
xlabel('delta V'); ylabel('Mass ratio');

figure(2); clf;
k1= plot(massR,T,'LineWidth',2); grid on; hold on;
xlabel('Mass ratio'); ylabel('Kinetic energy [J]');
k2= plot([exp(2) exp(2)],[0 max(T)],'r--','LineWidth',2);
legend([k1 k2],'Kinetic energy','e^2','Location','SouthEast');
```