HW5 Prob.1

Assume the eqn. of motion $m \ddot{x} + b \dot{x} + kx = F_0 \cos(\omega t)$.
We are told that $\frac{b}{m} (\equiv \xi) = \frac{\omega_0}{5}$.

As derived in class, the steady-state amplitude is given by

$$A(\omega) = \frac{F_0/m}{\left[ (\omega_0^2 - \omega^2)^2 + (\omega_0 \xi \omega)^2 \right]^\frac{1}{2}}$$

When driven at $\omega_0$ (i.e., $\omega = \omega_0$), we have

$$A(\omega_0) = \frac{F_0/m}{8(\omega_0)} = \frac{F_0}{b \omega_0}$$

Now we want an expression for the ratio $\frac{A(\omega)}{A(\omega_0)}$ (as we need that for what the problem asks)

$$A(\omega) = \frac{b \omega_0}{F_0} \cdot \frac{F_0/m}{\left[ (\omega_0^2 - \omega^2)^2 + (\omega_0 \xi \omega)^2 \right]^\frac{1}{2}} = \frac{\omega_0 b/m}{\left[ (\omega_0^2 - \omega^2)^2 + (\frac{b}{m} \omega_0 \xi \omega)^2 \right]^\frac{1}{2}}$$

$$A(\omega) = \frac{m}{m b \frac{b \omega_0}{m}} \cdot \frac{1}{\left[ (\frac{m \omega_0}{b})^2 (\omega_0^2 - \omega^2)^2 + (\frac{b}{m} \omega_0 \xi \omega)^2 \right]^\frac{1}{2}}$$

$$\text{Note:} \quad \frac{m \omega_0}{b} = \frac{5 \omega_0}{\omega_0} = 5$$

$$\text{Note:} \quad \frac{m \omega_0}{b} \frac{b \omega_0}{m} = 5$$

Now we can compute the necessary ratios!

$$\frac{A(1.1 \omega_0)}{A(\omega_0)} = \frac{[1.1 \omega_0]^2 \left[ 1 - \left( \frac{1.1 \omega_0}{\omega_0} \right)^2 \right]^2 + (1.1)^2}{[\omega_0^2 - (1.1 \omega_0)^2]^2 + (1.1 \xi \omega_0)^2} \approx 0.658$$

$$\frac{A(0.9 \omega_0)}{A(\omega_0)} = \frac{[0.9 \omega_0]^2 \left[ 1 - (0.9)^2 \right]^2 + (0.9)^2}{[\omega_0^2 - (0.9 \omega_0)^2]^2 + (0.9 \xi \omega_0)^2} \approx 0.764$$

$$\Rightarrow \quad \frac{A(1.1 \omega_0)}{A(\omega_0)} = \frac{65.8\%}{\text{and}} \frac{A(0.9 \omega_0)}{A(\omega_0)} = \frac{76.4\%}{:}$$
HW5 Prob.2

Assume collision occurs at $t=0$

- $V_1(t=0) = 0$, $m_1 = 1.2 \text{ kg}$
- $k = 23 \text{ N/m}$
- $V_2(t=0) = -1.7 \text{ m/s}$, $m_2 = 0.80 \text{ kg}$

At $t=0$, $m_2$ collides w/ $m_1$ at point $x = A_1$ (where $A_1 = 10 \text{ cm}$) when $m_1$ is (instantaneously) at rest under its SHO motion.

We can immediately derive a few relevant quantities of interest:

- $\omega_1 = \sqrt{\frac{k}{m_1}} = \sqrt{\frac{23}{1.2}} = 4.38 \text{ rad/s}$ or $f_1 = 0.70 \text{ Hz}$

  \[ \text{Freq. of block 1 before collision} \]

- $\omega_{oc} = \sqrt{\frac{k}{m_1 + m_2}} = \sqrt{\frac{23}{1.2 + 0.8}} = 3.39 \text{ rad/s}$ or $f_{oc} = 0.54 \text{ Hz}$

  \[ \text{Freq. of oscillation AFTER collision (since } k \text{ presumably doesn't change)} \]

- During an inelastic collision (which occurs at $t=0$), energy is not conserved, but momentum is, thus:

  $m_2 V_2(0) = (m_1 + m_2) V_c(0)$

  \[ \Rightarrow V_c(0) = \frac{m_2 V_2(0)}{m_1 + m_2} = \frac{0.80}{1.2 + 0.8} (-1.7) = -0.68 \text{ m/s} \]

Now after the collision, the combined masses will still exhibit SHO and thus we can express such as

$X_c(t) = A_c \cos(\omega_{oc} t + \phi_c)$ and $V_c(t) = -\omega_{oc} A_c \sin(\omega_{oc} t + \phi_c)$
HWS Prob.2 (cont)

We know \( \omega_c \), but still need to determine \( A_c \) and \( \phi_c \).

- We do know \( x(t=0) \) and \( v(t=0) \), leading to

\[
x(t=0) = 10 = A_c \cos(\omega_c \cdot 0 + \phi_c) \quad \Rightarrow \quad 10 = \cos \phi_c
\]
\[
v(t=0) = -68 \left[ \frac{cm}{s} \right] = - \omega_c A_c \sin(\omega_c \cdot 0 + \phi_c) \quad \Rightarrow \quad 20.1 = A_c \sin \phi_c
\]

\( \Rightarrow \) two equations, two unknowns

- \( \frac{20.1}{10} = \frac{A_c \sin \phi_c}{A_c \cos \phi_c} = \tan \phi_c \quad \Rightarrow \quad \phi_c = \tan^{-1}(2.01) = 1.11 \text{ radians} \)

And plugging back into the first equation:

\[
10 = A_c \cos(1.11) \quad \Rightarrow \quad A_c = 22.5 \text{ cm}
\]

Thus after the collision, the combined masses move according to

\[
x_c(t) = 22.5 \cos[3.39t + 1.11]
\]
HW5 Prob. 3

Here we have an object that'll undergo periodic motion (since there is no friction). This is due to gravity and the normal force of the track creating a "restoring force" ($F_R$) that always points to $x = 0$ (except at $x = 0$, where $F_R = 0$).

Two approaches to this problem.

1) Let $G(x) = ax^2$ act as the potential function governing the motion. Then

$$F_R = -\frac{dG}{dx} = -2ax$$

the constant of proportionality being $mg$. $F_R$ is a 2-D vector, but considering just the $x$-component ($= F_{Rx}$), we have

$$F_{Rx} = -2mgax = m \frac{d^2x}{dt^2} = -kx$$

Thus the effective spring constant is $k = 2mg$. This leads to $w_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{2mg}{m}} = \sqrt{2ga} = 2\pi f_0 = \frac{2\pi}{T}$

$$T = \frac{2\pi}{\sqrt{2ga}}$$

2) The potential energy of the mass relative to the bottom of the track is

$$U(x) = mgy = mgax^2$$. For SHO, $U(x) = \frac{1}{2}kx^2$ $\Rightarrow k = 2mg$

(which leads to the same result).
HWS Prob. 4

- Best place to start is with the presumed equation of motion. Take this to be:

\[ m \ddot{x} + c \dot{x} + kx = F_{ext} = F_0 \sin \omega t \]

- What we care about here is the steady-state solution, which has the form \( x(t) = A(w) \sin (\omega t - \phi(w)) \). Our job is to derive \( A(w) \) and \( \phi(w) \).

- Let us use complex exponentials as follows:

\[ z(t) = Ae^{i(\omega t - \phi)} \quad \text{where} \quad x(t) = \text{Re}(z) \]

- Now we have: \( \dot{z} = i\omega z \) and \( \ddot{z} = -\omega^2 z \) leading to

\[ m \ddot{z} + c \dot{z} + kz = F_0 e^{i\omega t} \rightarrow z \left[ -m \omega^2 + i\omega c + k \right] = F_0 e^{i\omega t} \]

but \( z = Ae^{i(\omega t - \phi)} = Ae^{-i\phi} e^{i\omega t} \rightarrow A e^{-i\phi} \left[ k - m \omega^2 + i\omega c \right] = F_0 \)

- While there are two unknowns \( (A, \phi) \), because the equation is complex, there are effectively two eqns. (e.g. real and imaginary parts). Breaking such up, we have

\[ A \left[ k - m \omega^2 + i\omega c \right] = F_0 e^{i\phi} = F_0 (\cos \phi + i \sin \phi) \]

now just equate real and imaginary parts

\[ \rightarrow A \left( k - m \omega^2 \right) = F_0 \cos \phi \]
\[ A \omega c = F_0 \sin \phi \]

- Right off the bat, we can divide both equations to obtain

\[ \tan \phi = \frac{\omega c}{k - m \omega^2} = \frac{\omega c}{m \left( \frac{k}{m} - \omega^2 \right)} = \frac{\omega \cdot \frac{c}{m}}{\omega_0^2 - \omega^2} = \frac{\omega \cdot \frac{2c}{\omega_0^2}}{\omega_0^2 - \omega^2} \]

\[ \tan \phi = \frac{\omega c}{k - m \omega^2} \]
where $X = \frac{q}{2m}$ and $\omega_0 = \sqrt{\frac{k}{m}}$. Thus $Q = \tan^{-1}\left(\frac{2Ym}{\omega_0^2 - \omega^2}\right)$

To determine $A$, we use the identity $\sin^2 \theta + \cos^2 \theta = 1$

$$(F_0 \sin \phi)^2 + (F_0 \cos \phi)^2 = F_0^2 (\sin^2 \phi + \cos^2 \phi) = F_0^2$$

$$= (A \omega c)^2 + (A \pm \omega m)^2 = A^2 \left[ (\omega c)^2 + (k - \omega m)^2 \right]$$

$$\Rightarrow A = \frac{F_0}{\left[\omega^2 c^2 + k^2 + \omega^2 m^2 - 2k\omega m\right]^{\frac{1}{2}}} = \frac{F_0}{\left[(\omega^2 m^2 - 2k\omega m + k^2) + \omega^2 c^2\right]^{\frac{1}{2}}}$$

$$= \frac{F_0}{\left[(\omega^2 m - k)^2 + \omega^2 c^2\right]^{\frac{1}{2}}} = \left[\frac{\epsilon m (\omega^2 - k^2)}{\epsilon^2 m^2 + m^2 + \omega^2 \omega^2}\right]^{\frac{1}{2}}$$

$$= \frac{F_0}{\left[(\omega^2 - \omega_0^2)^2 + 4\omega^2 \omega_0^2\right]^{\frac{1}{2}}}$$

$\Rightarrow A(\omega) = \frac{F_0/m}{\left[(\omega^2 - \omega_0^2)^2 + 4\omega^2 \omega_0^2\right]^{\frac{1}{2}}}$

**Note:** $(\omega^2 - \omega_0^2)^2 = (\omega_0^2 - \omega^2)^2$
Assume we can treat \( f(t) \) as an infinite series of sinusoids using complex exps:

\[
  f(t) = \sum_{n} c_n e^{i\omega n t}
\]

where \( c_n \in \mathbb{C} \) and

\[
  c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-i\omega n t} \, dt
\]

\( n = 0, \pm 1, \pm 2, \pm 3, \ldots \)

\[
  T = \frac{2\pi}{\omega} \quad \left( \text{thus } \frac{T}{n} = \frac{\omega}{2\pi} \right)
\]

\[
  \Rightarrow c_n = \frac{\omega}{2\pi} \int_{-\pi/\omega}^{\pi/\omega} f(t) e^{-i\omega n t} \, dt
\]

\[
  = \frac{\omega}{2\pi} \left[ \int_{-\pi/\omega}^{0} (-1) e^{-i\omega n t} \, dt + \int_{0}^{\pi/\omega} (1) e^{-i\omega n t} \, dt \right]
\]

\[
  = \frac{\omega}{2\pi} \left[ \frac{1}{i\omega} e^{-i\omega n t} \bigg|_{-\pi/\omega}^{0} - \frac{1}{i\omega} e^{-i\omega n t} \bigg|_{0}^{\pi/\omega} \right]
\]

\[
  = \frac{\omega}{2\pi} \left[ \frac{1}{i\omega} (1 - e^{-i\pi n}) - \frac{1}{i\omega} (e^{-i\pi n} - 1) \right]
\]

\[
  = \frac{\omega}{2\pi i\omega} \frac{1}{1 - e^{-i\pi}} - \frac{\omega}{2\pi i\omega} \frac{1}{1 - e^{i\pi}}
\]

\[
  = \frac{4}{2\pi i \omega} \text{ for } n = \pm 1, \pm 3, \pm 5, \ldots
\]

(i.e. \( c_{2n+1} = 0 \) for even \( n \))

\[
  \Rightarrow f(t) = \sum_{n} \frac{4}{2\pi i \omega} e^{i\omega n t}
\]

(for odd \( n \) as noted above)

\[
  = \sum_{n} \frac{4}{\pi} \frac{1}{n} \frac{1}{2i} \left( e^{i\omega n t} - e^{-i\omega n t} \right) \quad \text{for } n = 1, 3, 5, \ldots
\]

\[
  = \frac{4}{\pi} \sum_{n} \frac{1}{n} \sin(n\omega t)
\]

(since \( \sin \omega = \frac{e^{i\omega} - e^{-i\omega}}{2i} \))

\[
  = \frac{2}{\pi} \left[ \sin \omega t + \frac{1}{3} \sin 3\omega t + \frac{1}{5} \sin 5\omega t + \cdots \right]
\]
HWS Prob. 6 \[ \ddot{x} + 2\beta \dot{x} + \omega_0^2 x = f_0 \cos(\omega t) \]

(a) Assuming a solution of the form \( x(t) = A \cos(\omega t - \delta) \), we can connect back to our solution to prob. 4. Here, we have (comparing back to the eqn. of motion in prob. 4)

\[ c = m \cdot 2\beta \rightarrow \beta = \frac{c}{m} \text{ (since } c = \frac{\text{force}}{\text{mass}} \text{)} \text{ and } f_0 = \frac{F_0}{m} \]

\[ A(\omega) = \frac{f_0/m}{[(\omega_0^2 - \omega^2) + 4\beta^2 \omega^2]^2} = \frac{f_0}{(\omega_0^2 - \omega^2)^2 + 4\beta^2 \omega^2} \]

Now we simply have \[ A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2 \omega^2} \]

(b) Now for \( A^2 \) to have a max, the denominator must be a minimum. To show this, let \[ f(\omega) = (\omega_0^2 - \omega^2)^2 + 4\beta^2 \omega^2 \]

we want \( \omega_m \) such that \[ \frac{df}{d\omega} \bigg|_{\omega=\omega_m} = 0 \]

\[ \frac{df}{d\omega} = 2(\omega_0^2 - \omega^2)(-2\omega) + 8\beta^2 \omega = 8\beta^2 \omega - 4\omega(\omega_0^2 - \omega^2) \]

Now \( 8\beta^2 \omega_m - 4\omega_m(\omega_0^2 - \omega_m^2) = 0 \rightarrow 2\beta^2 = \omega_0^2 - \omega_m^2 \)

or \[ \omega_m = \sqrt{\omega_0^2 - 2\beta^2} \]

Note: \[ \frac{d^2f}{d\omega^2} = 8\beta^2 - 4(\omega_0^2 - \omega^2)(-2\omega) \]

\[ = 8\beta^2 + 8\omega(\omega_0^2 - \omega^2) > 0 \]

(because \( \omega_m < \omega_0 \)) guaranteeing we found a max. (remember that \( f \) is the denominator!)
c) Here, we interpret the "∞" as \( \omega_0 = \omega \). Thus

\[
A_{\text{max}} \propto \left[ \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta \omega_0^2} \right]^{1/2} = \frac{f_0}{2\beta \omega_0}
\]

d) see EX4hw5Prob6.m

e) Ignoring part c, we can determine the true \( \text{max} \) for \( A^2 (\approx A_{\text{max}}^2) \) by plugging in \( \omega = \sqrt{\omega_0^2 - 2\beta^2} \)

\[
A^2_{\text{max}} = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2(\omega_0^2 - \beta^2)} = \frac{f_0^2}{4\beta^2(\omega_0^2 - \beta^2)}
\]

Now we want \( \omega_{\text{FWHM}} \) such that \( A^2(\omega_{\text{FWHM}}) = \frac{1}{2} A^2_{\text{max}} \)

\[
\frac{f_0^2}{(\omega_0^2 - \omega_{\text{FWHM}}^2)^2 + 4\beta^2 \omega_{\text{FWHM}}^2} = \frac{f_0^2}{8\beta^2(\omega_0^2 - \beta^2)}
\]

\[
8\beta^2(\omega_0^2 - \beta^2) = (\omega_0^2 - \omega_{\text{FWHM}}^2)^2 + 4\beta^2 \omega_{\text{FWHM}}^2 = \omega_0^4 + \omega_{\text{FWHM}}^4 - 2\omega_0^2 \omega_{\text{FWHM}}^2 + 4\beta^2 \omega_{\text{FWHM}}^2
\]

\[
\Rightarrow \omega_{\text{FWHM}}^4 + \omega_{\text{FWHM}}^2 (4\beta^2 - 2\omega_0^2) + \left[ \omega_0^4 - 8\beta^2(\omega_0^2 - \beta^2) \right] = 0
\]

→ Getting a tad messy (but still doable). But let's try another approach: Let's plug in the answer and verify the equality.

\[
\text{FWHM} = \Theta \beta \quad \Rightarrow \quad \omega = \omega_0 \pm \beta
\]
\[ H W 5 \text{ Prob. 6 (cont.\#)} \]

\[ \Delta_w^2 = \Delta^2 (\omega = \omega_0 \pm \beta) \]

\[ \frac{f_0^2}{\left[ \omega_0^2 - (\omega_0 \pm \beta)^2 \right]^2 + 4\beta^2 (\omega_0 \pm \beta)^2} \]

\[ = \frac{f_0^2}{\left[ \omega_0^2 - \omega_0^2 - \beta^2 \mp 2\omega_0 \beta \right]^2 + 4\beta^2 (\omega_0^2 + \beta^2 \pm 2\omega_0 \beta)} \]

\[ = \frac{f_0^2}{\beta^4 + 4\omega_0^2 \beta^2 \pm 4\omega_0 \beta^3 + 4\beta^2 \omega_0^2 \pm 4\beta^4 \pm 8\beta^2 \omega_0} \]

\[ \text{Note:} \quad (\omega_0 \pm \beta)^2 = \omega_0^2 + \beta^2 \pm 2\omega_0 \beta \]

\[ \Rightarrow \Delta_{max}^2 = \frac{1}{\Delta^2} \]

\[ \text{I seem to be mentally stalling on this one. Too much coronavirus...} \]
HWS Prob. 7

a) See attached

b) Note that if $w$ is close to $w_0$, then $w + w_0 \approx 2w_0$

$\Rightarrow w^2 - w^2 = (w_0 - w)(w_0 + w) \approx 2w_0 (w_0 - w)$

$\Rightarrow p_* = \frac{1}{2} \frac{F_0^2}{8m} \frac{8^2 w^2}{(w_0^2 - w^2)^2 + 8^2 w^2} \approx \frac{1}{2} \frac{F_0^2}{8m} \frac{8^2 w^2}{4w^2 (w_0 - w)^2 + 8^2 w^2}$

$= \frac{1}{2} \frac{F_0^2}{8m} \frac{1}{4w_0^2} \frac{8^2 w^2}{(w_0 - w)^2 + 8^2 w^2}$

$= \frac{1}{2} \frac{F_0^2}{8m} \frac{8^2 w^2}{(w_0 - w)^2 + 8^2 w^2}$

(since we assume $\frac{w_0}{w} \approx 1$)

**Note:** The general form for a Cauchy/Lorentz probability distribution function is

$$\frac{1}{\pi Y \left[ 1 + \frac{(x-x_0)^2}{Y^2} \right]}$$

where $Y$ affects the width and $x_0$ the peak location. If we manipulated our expression for $p_*$ further, we would obtain

$$p_* \approx \frac{F_0^2}{2m} \frac{1}{8} \frac{1}{(w_0 - w)^2 + 1}$$

which has the same basic form.
c) To determine the FWHM, we note that \( P_L \) takes on its max. value when \( \omega = \omega_0 \) (since \( \omega_0 - \omega = 0 \) and thus the denominator is the smallest). Hence

\[
P_L(\omega_0) = \frac{F_0^2}{2\lambda m} \]

so we want \( \omega_n \) such that \( P(\omega_n) = \frac{1}{2} \cdot \frac{F_0^2}{8m} \)

\[
\frac{1}{2} \cdot \frac{F_0^2}{8m} = \frac{\frac{F_0^2}{8m} \cdot \frac{\sqrt{2}}{4}}{(\omega_0 - \omega_n)^2 + \frac{\sqrt{2}}{4}} \implies \frac{1}{2} = \frac{\frac{\sqrt{2}}{4}}{(\omega_0 - \omega_n)^2 + \frac{\sqrt{2}}{4}}
\]

So \((\omega_0 - \omega_n)^2 + \frac{\sqrt{2}}{4} = \frac{\sqrt{2}}{2} \implies (\omega_0 - \omega_n)^2 = \frac{\sqrt{2}}{4} = \omega_0^2 + \omega_n^2 - 2\omega_0 \omega_n\)

\[
\omega_n^2 - \omega_n 2\omega_0 + \omega_0^2 - \frac{\sqrt{2}}{4} = 0 \implies \text{quadratic for } \omega_n
\]

\[
\omega_n = \frac{2\omega_0 \pm \sqrt{4\omega_0^2 - 4(\omega_0^2 - \frac{\sqrt{2}}{4})}}{2} = \omega_0 \pm \frac{\sqrt{2}}{2}
\]

Thus \( \Delta \omega = \omega_{n+} - \omega_{n-} = \omega_0 + \frac{\sqrt{2}}{2} - (\omega_0 - \frac{\sqrt{2}}{2}) = \sqrt{2} \)

\[\implies \text{FWHM}(P_L) = \sqrt{2} \]