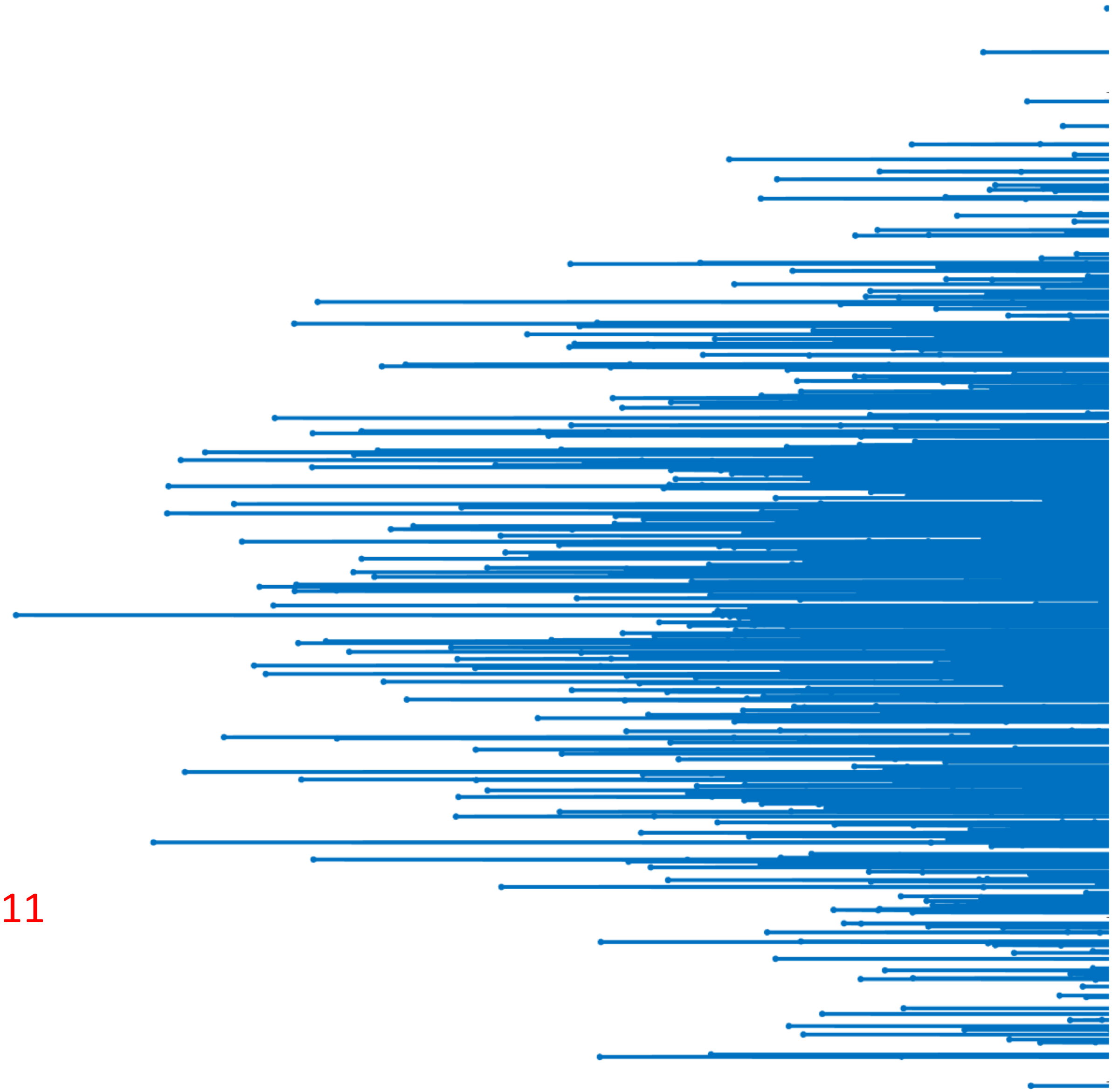
PHYS 2010 (W20)
Classical Mechanics



2020.02.04

Relevant reading:

Knudsen & Hjorth: 15.4-15.6, 15.11

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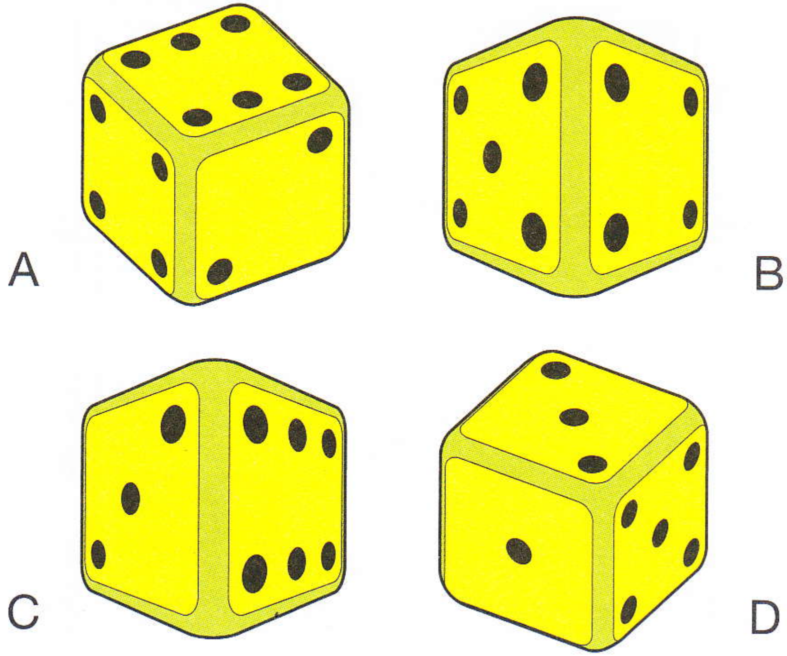
Ref.s:
Knudsen & Hjorth (2000), Fowles & Cassidy (2005), French (1971)

77. Four Dice

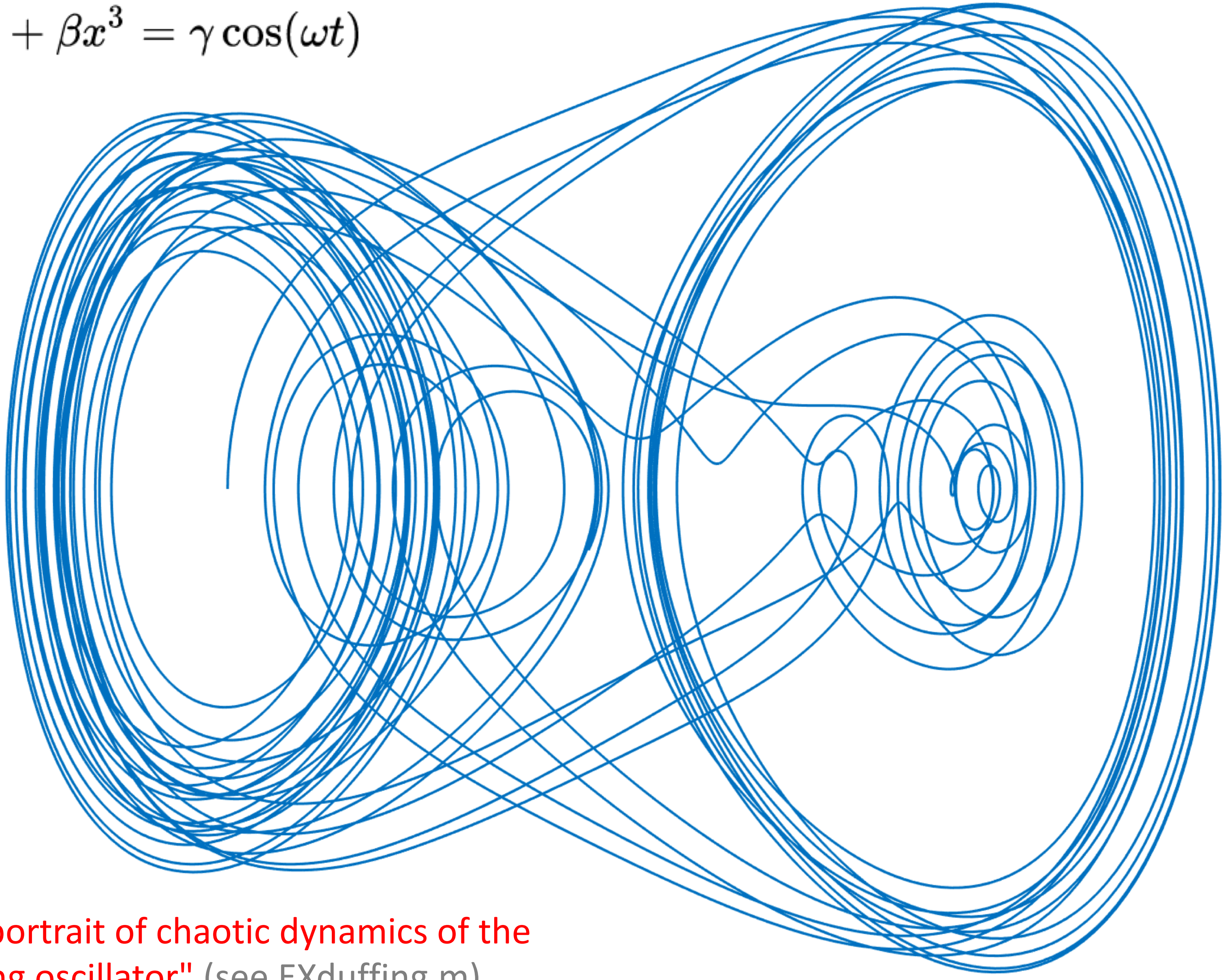


The four dice are identical. Which one does not belong?

A | B | C | D

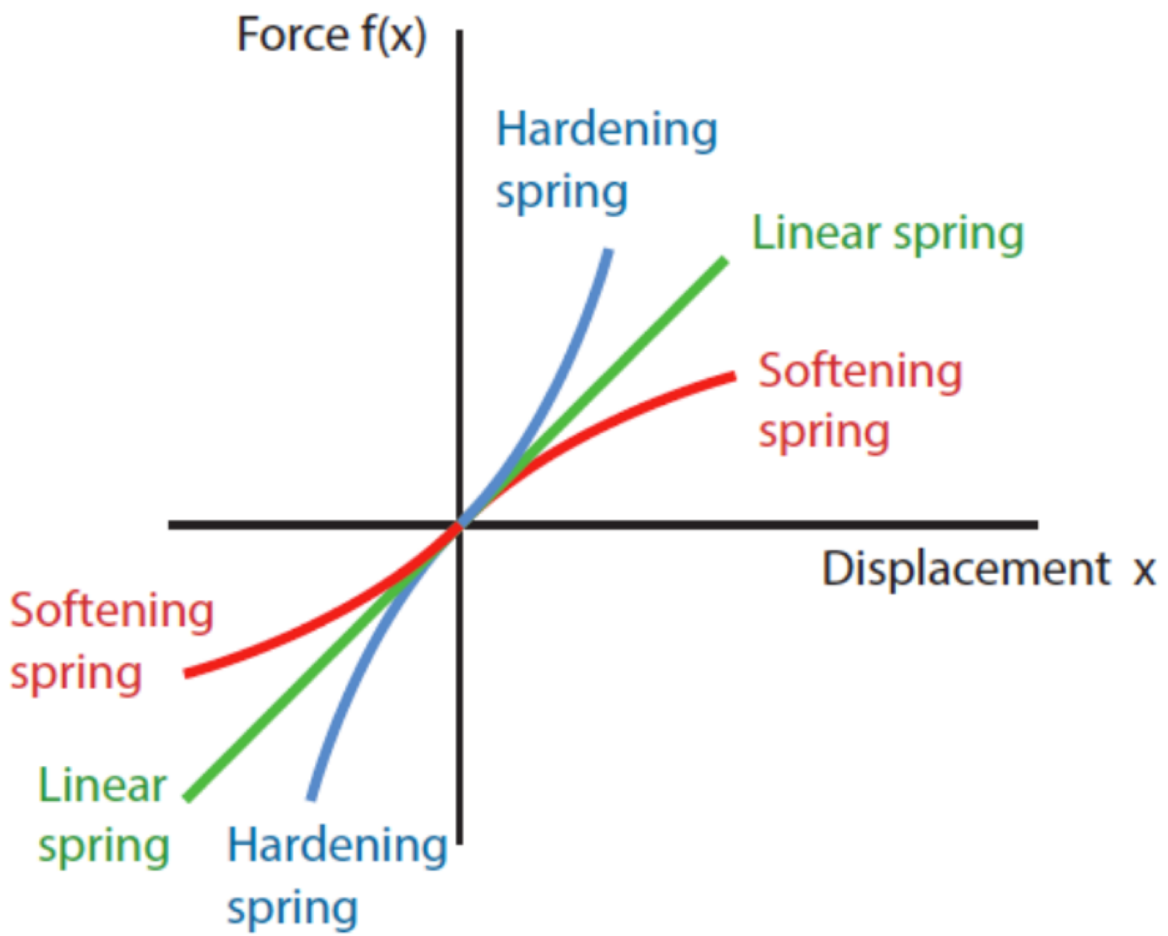
Duffing Equation

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma\cos(\omega t)$$



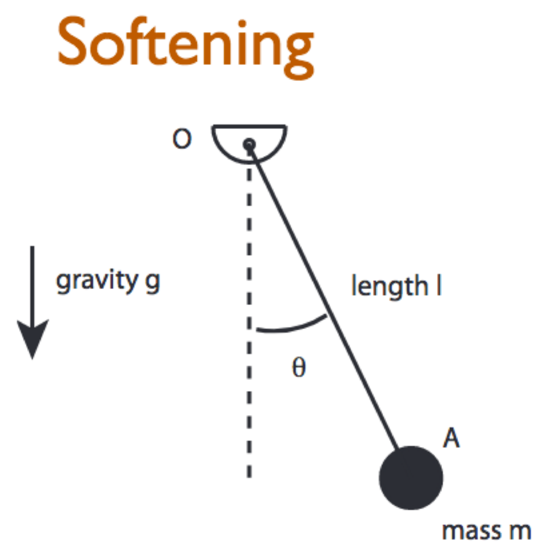
Phase plane portrait of chaotic dynamics of the driven "Duffing oscillator" (see EXduffing.m)

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma\cos(\omega t)$$



Hardening – $\beta > 0$

Softening – $\beta < 0$



https://movement.osu.edu/Courses/ME8230NonlinearDynamics_Sp2016/LectureNotes/L14_ForcedNonlinearSystemsPhenomena.pdf

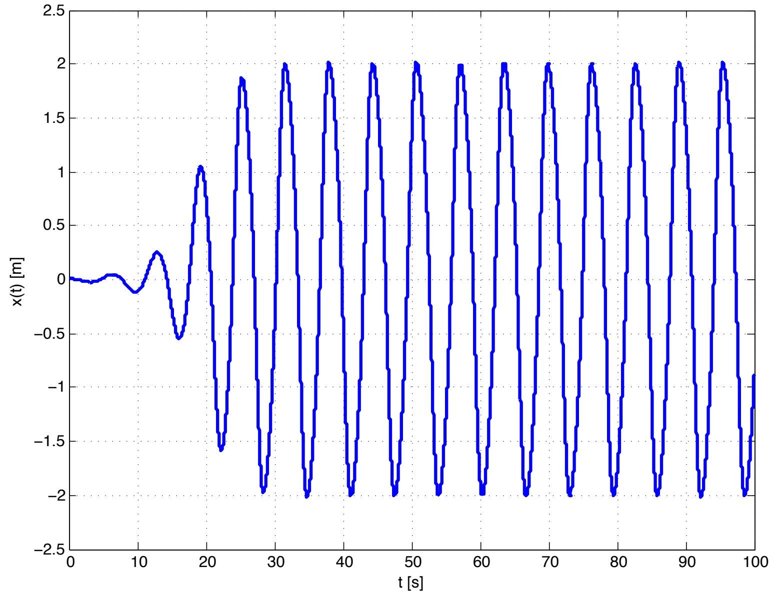
These are the Phenomena of Springs and springy bodies, which as they have not hitherto been by any that I know reduced to Rules, so have all the attempts for the explications of the reason of their power, and of springiness in general, been very insufficient.

ROBERT HOOKE, *De Potentia Restitutiva* (1678)

French (1971)

VDPode45EX.m

Timecourse



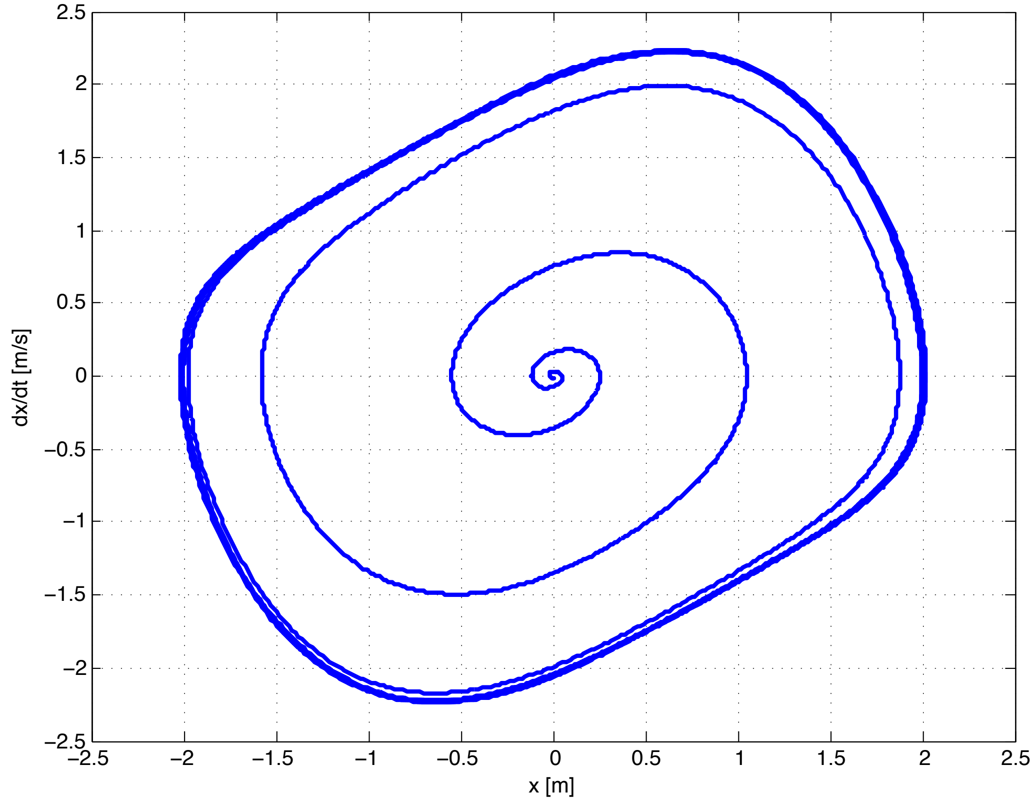
van der Pol Equation

$$\ddot{x} = -x - \varepsilon(x^2 - 1)\dot{x}$$

What about things that oscillate on their own!?!

(e.g., limit cycles)

Phase space



Aside....

What about things that oscillate on their own!?!

(e.g., limit cycles)

Ear actually **EMITS** sound!

otoacoustic emissions – OAEs

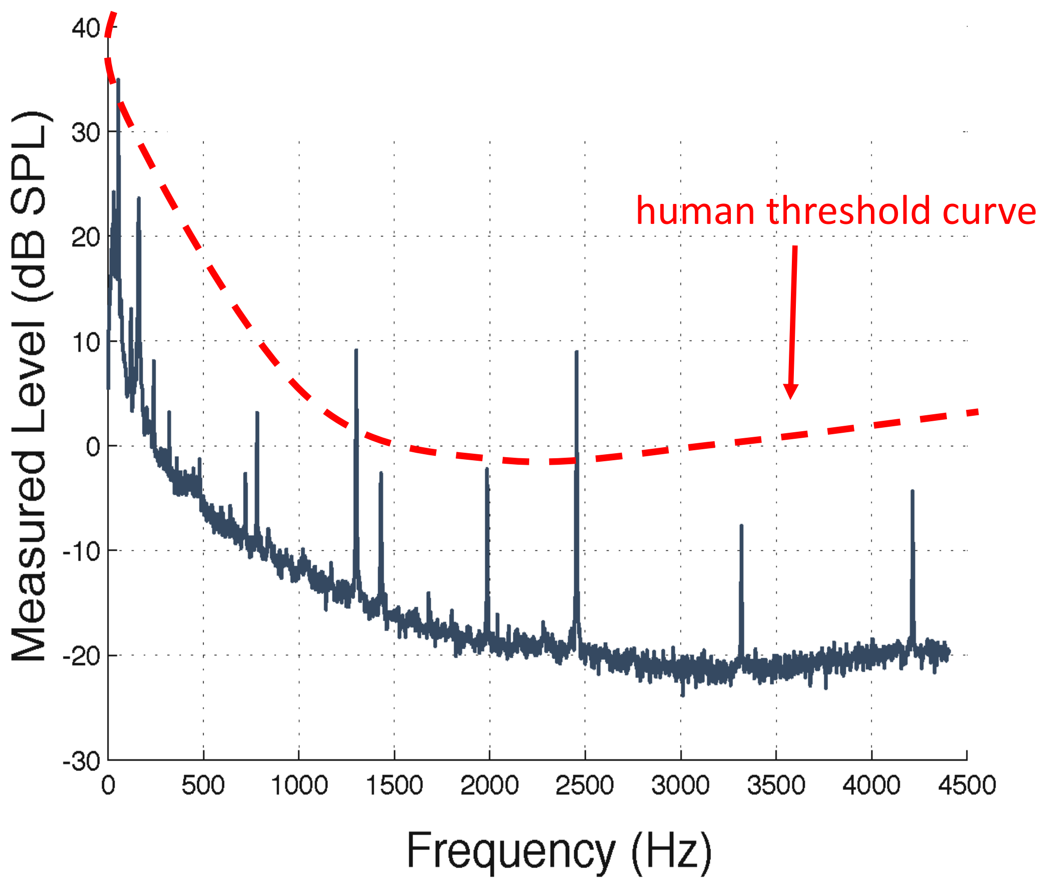
http://www6.miami.edu/UMH/CDA/UMH_Main/

Aside....

- Healthy ears can actually spontaneous *emit* sound! ("SOAEs")

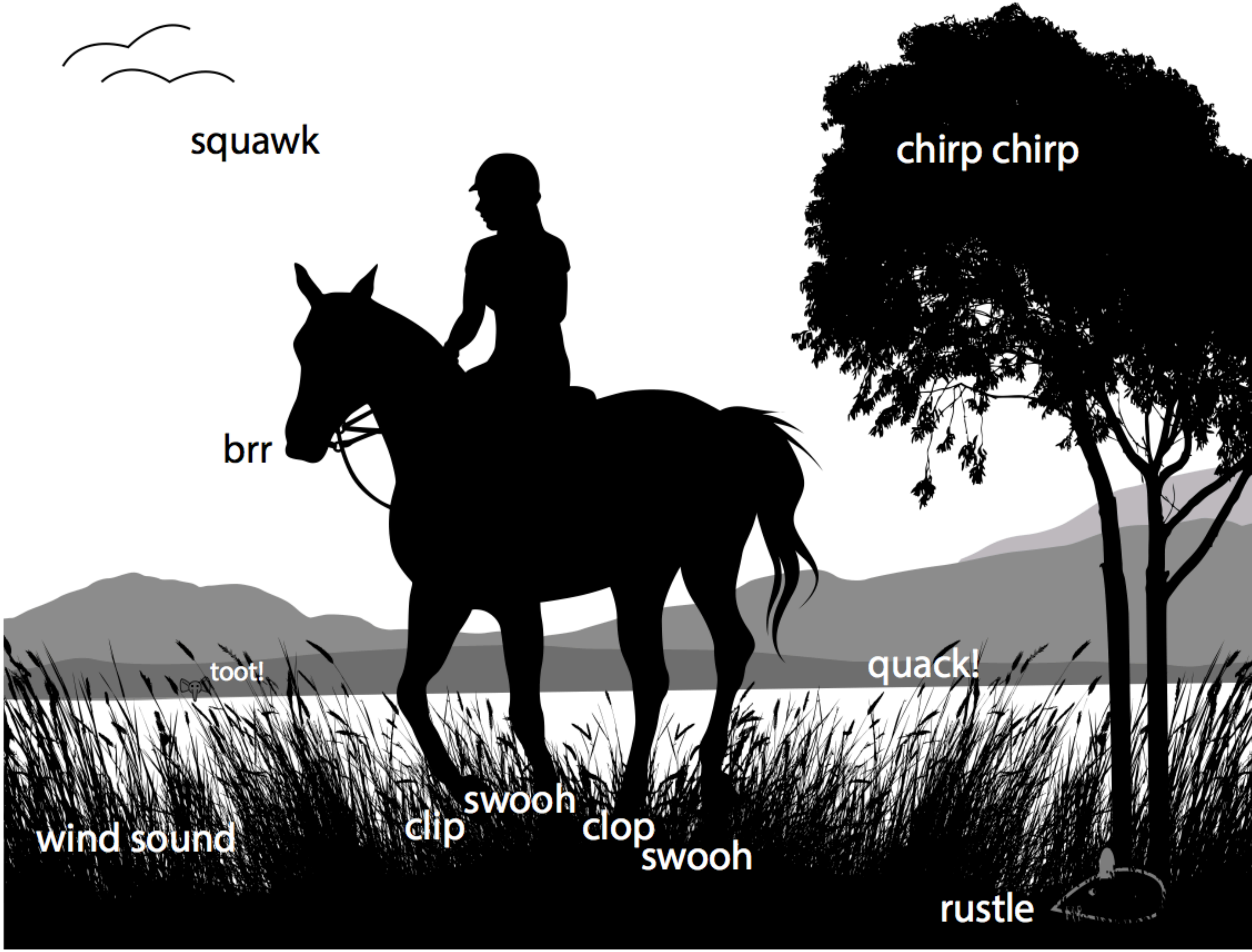
What about things that oscillate on their own!?!

(e.g., limit cycles)



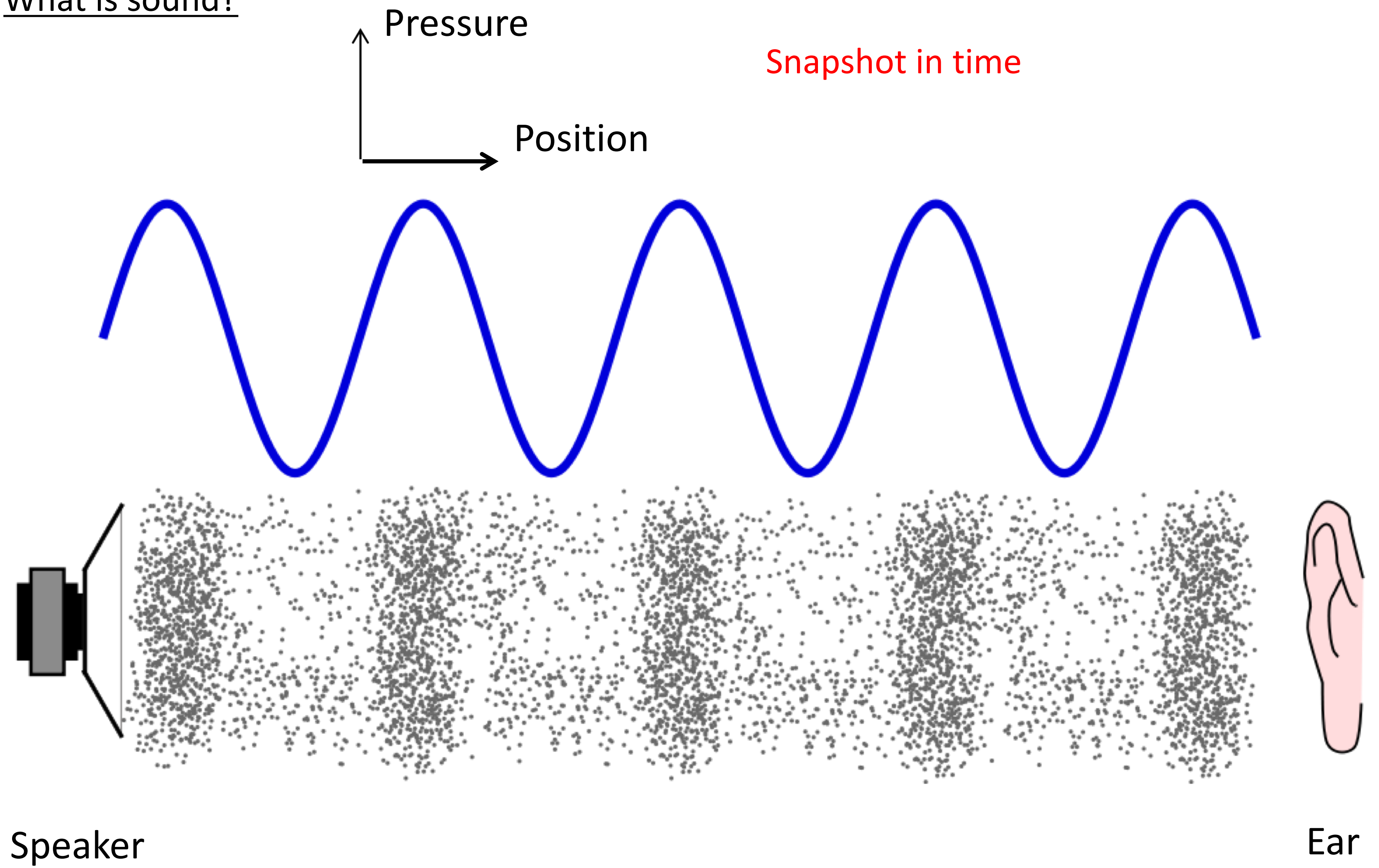
Note the abscissa (i.e., horizontal "x" axis) here: *Frequency*

Moving along....



Pulkki & Karjalainen (2015)

What is sound?



Snapshot in time

→ Note the periodic nature present....

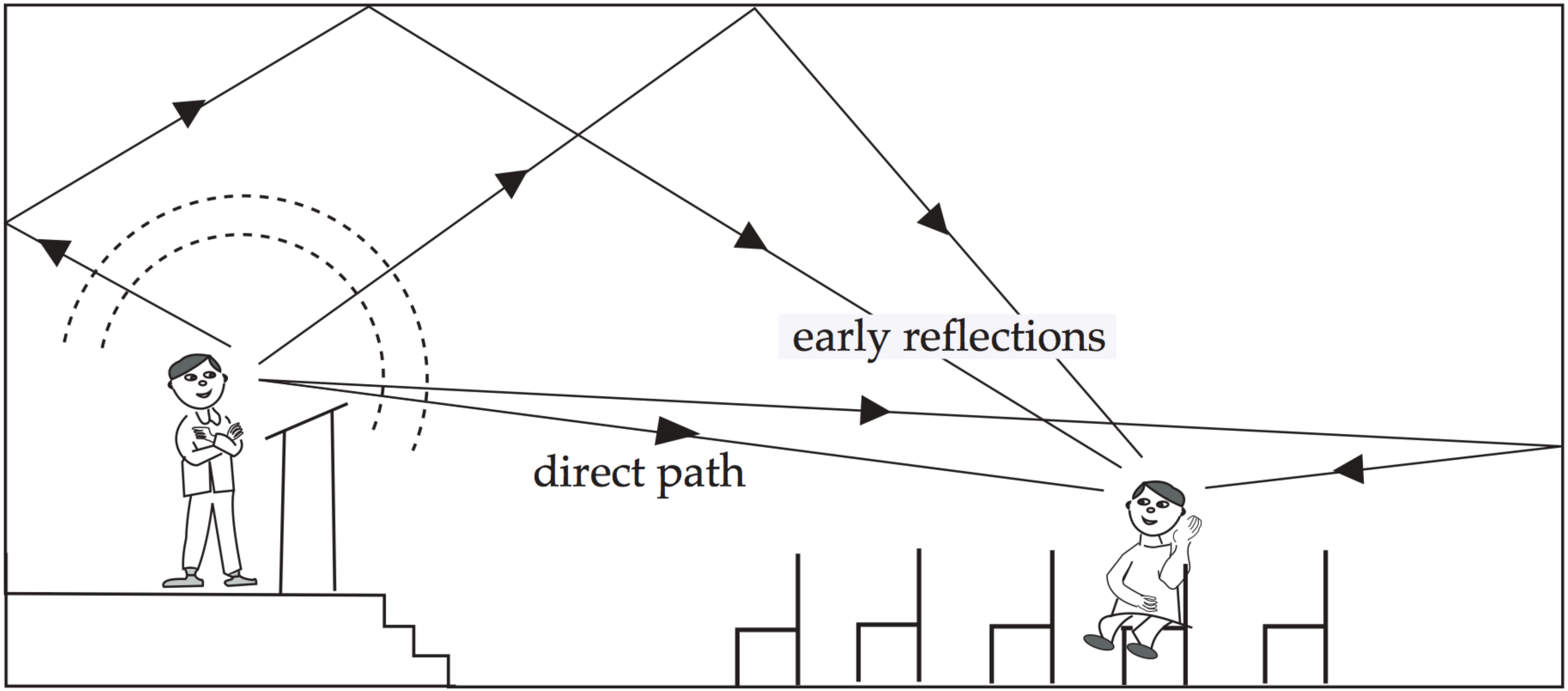
Pulkki & Karjalainen (2015)

Ex.

Why does the sound in a hall filled with people sound deader than in the same hall empty?

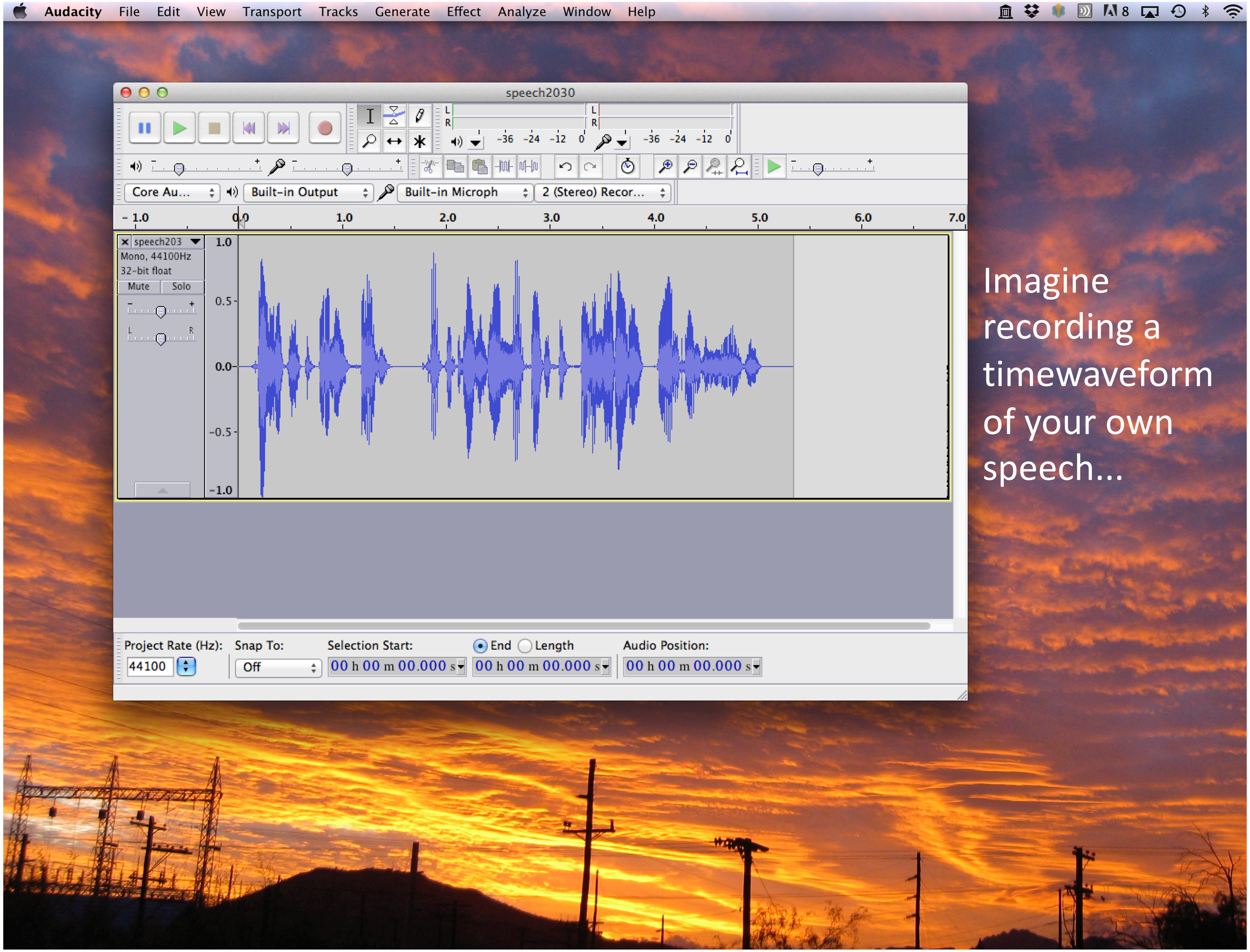
Shaskol'Skaya & El'Tsin (1963)

What is sound? (REVISITED)

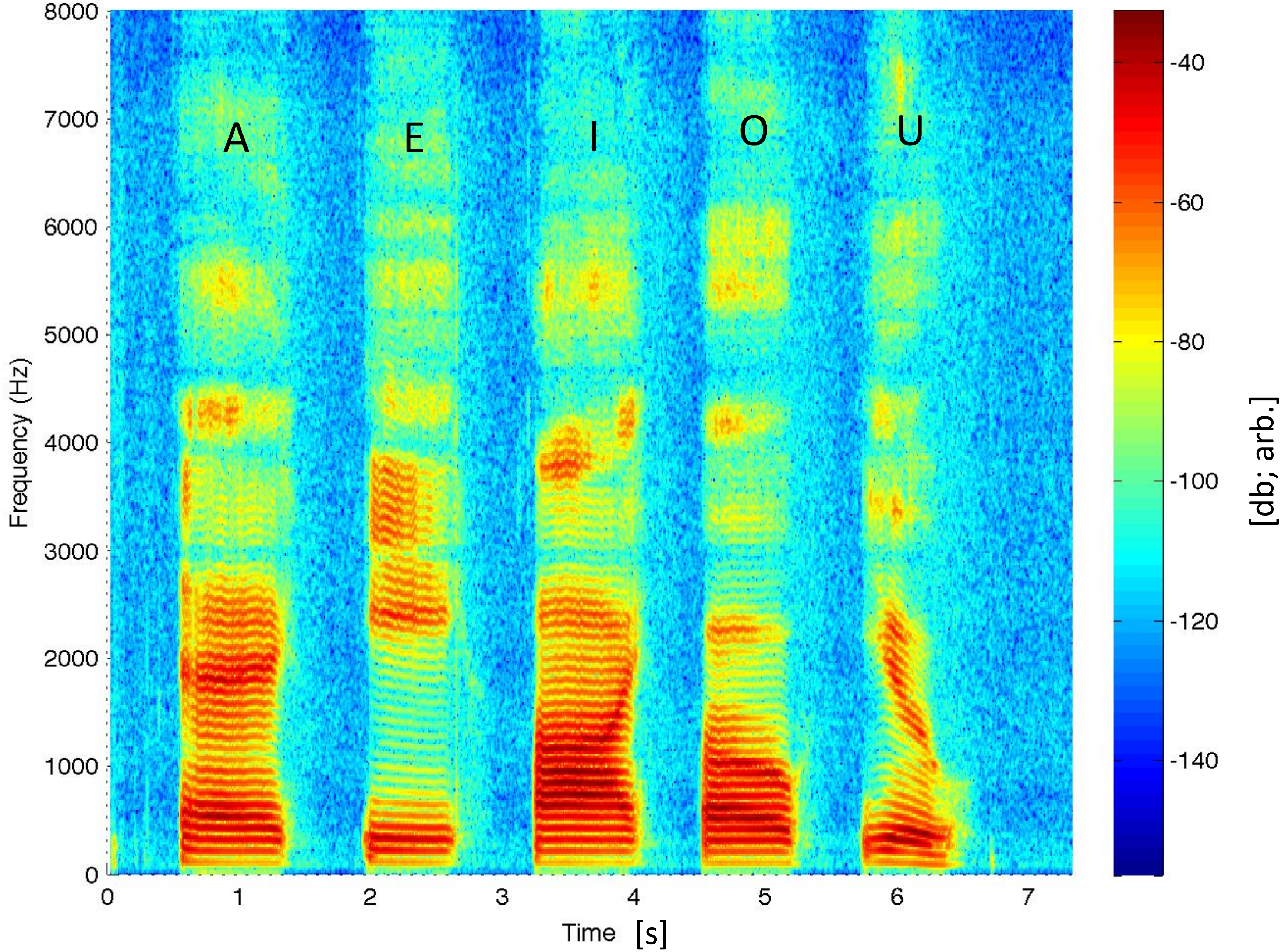


→ The notion of acoustics deals not just with oscillations, but *waves* as well....

Pulkki & Karjalainen (2015)

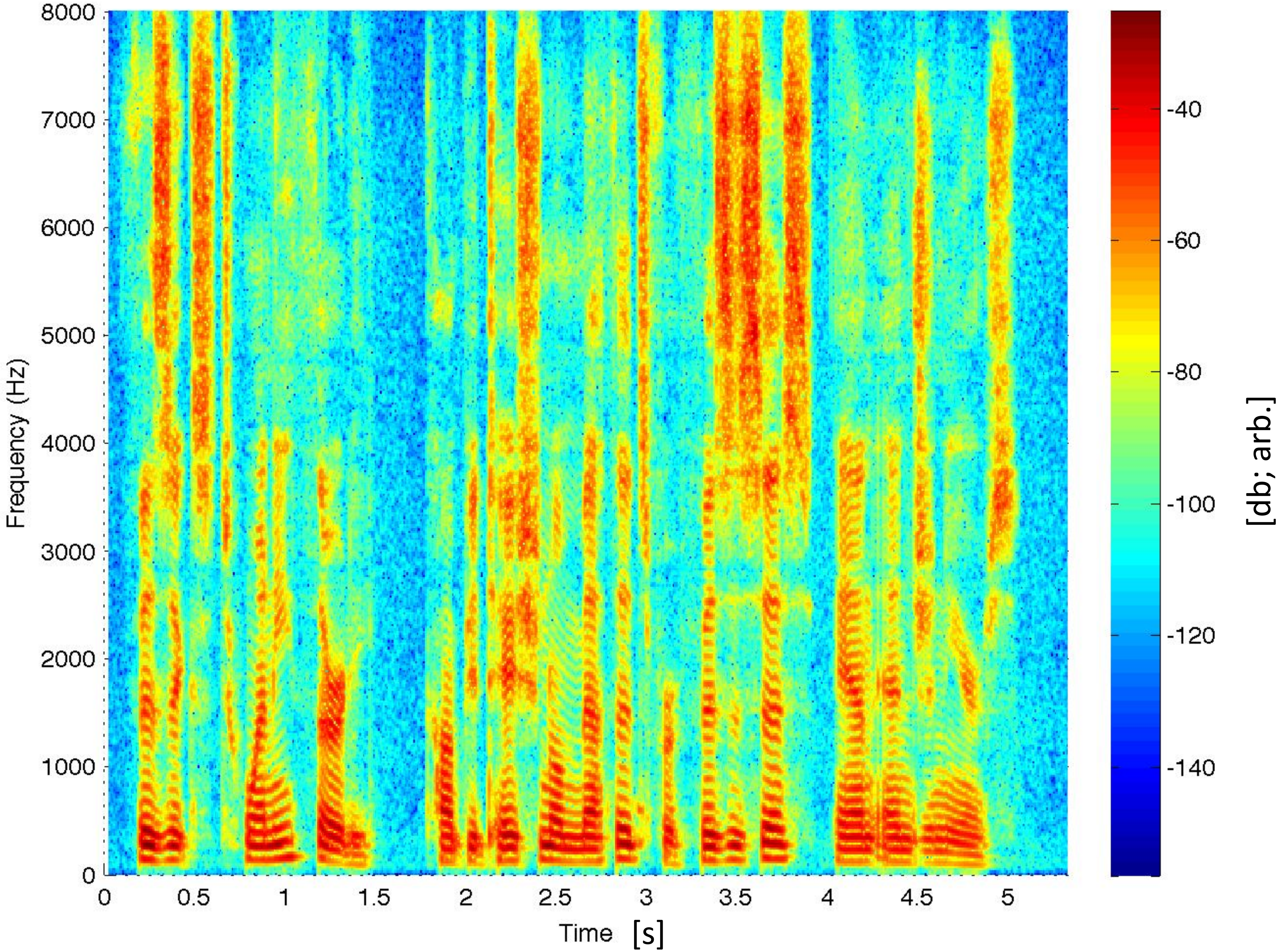


→ We are going to perform a specific type of spectral analysis called the ‘Short Time Fourier Transform’ (STFT) to make what is called a *spectrogram*



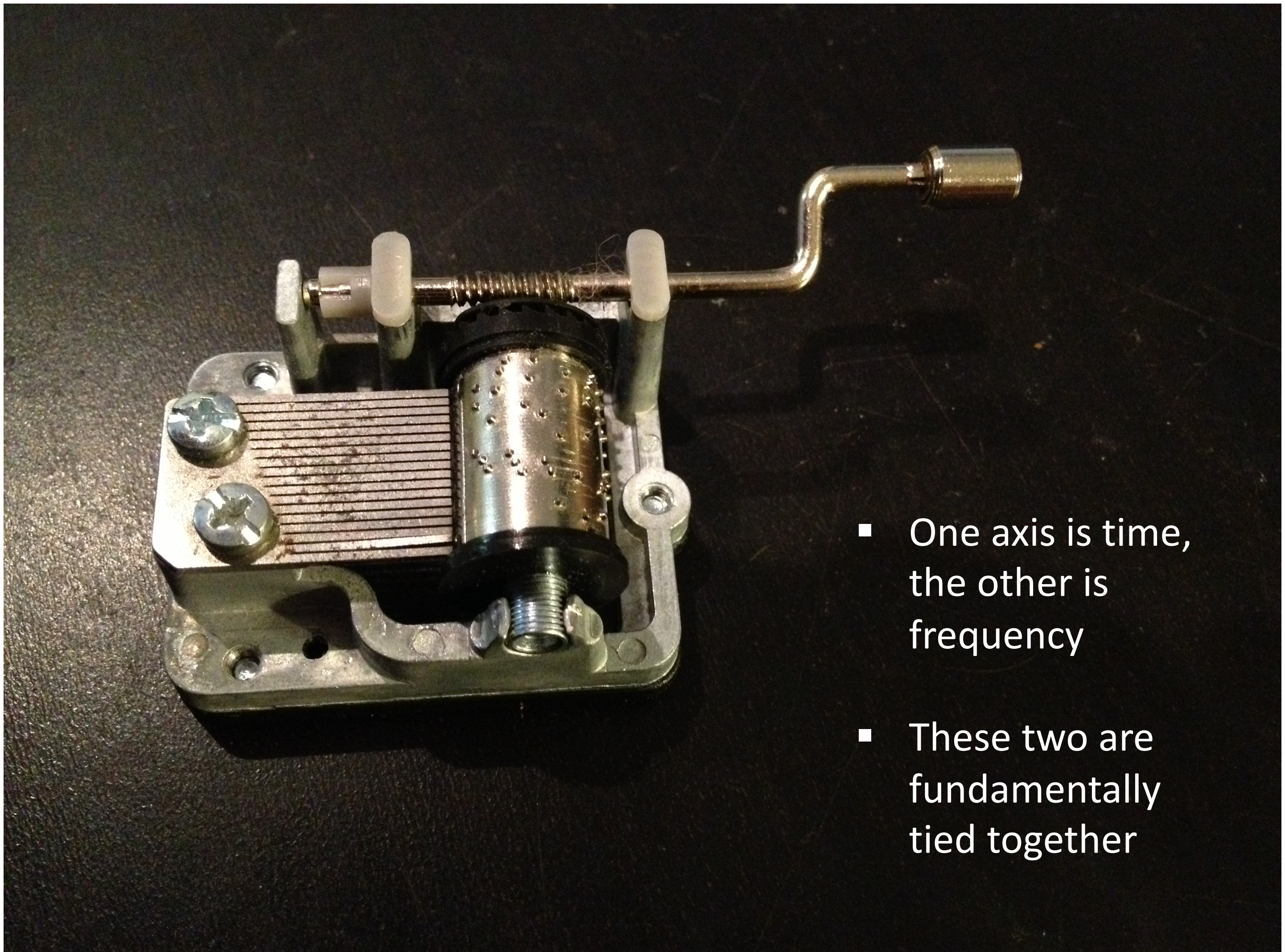
A
E
I
O
U
Frequency (Hz)
8000
7000
6000
5000
4000
3000
2000
1000
0
0
1
2
3
4
5
6
7
Time [s]
-40
-60
-80
-100
-120
-140
[db; arb.]

EXspectrogram.m



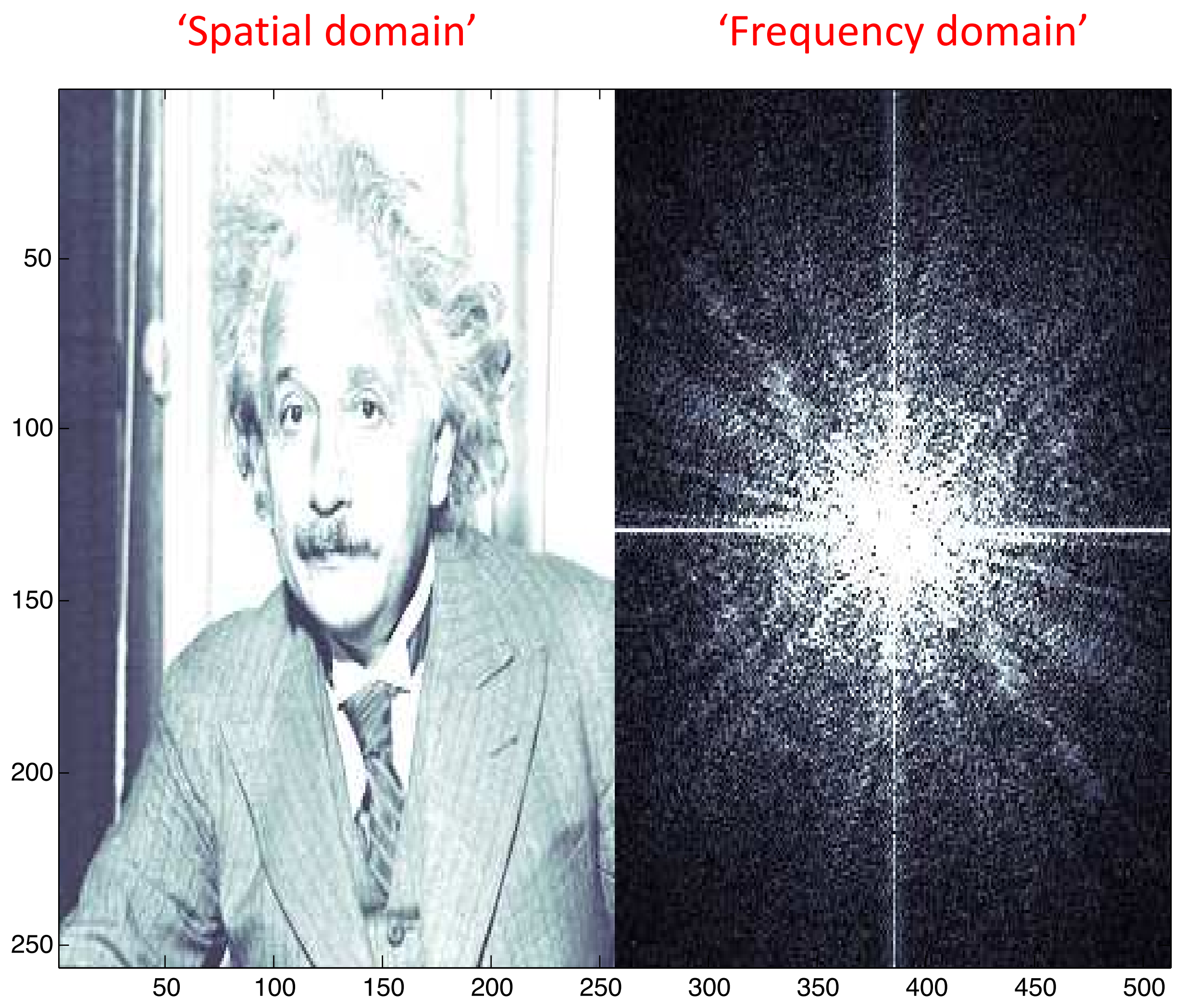
Looking Ahead: Fourier Analysis

- Allows one to go from a time domain description (e.g., our recorded mic signal) to a spectral description (i.e., what frequency components make up that signal)

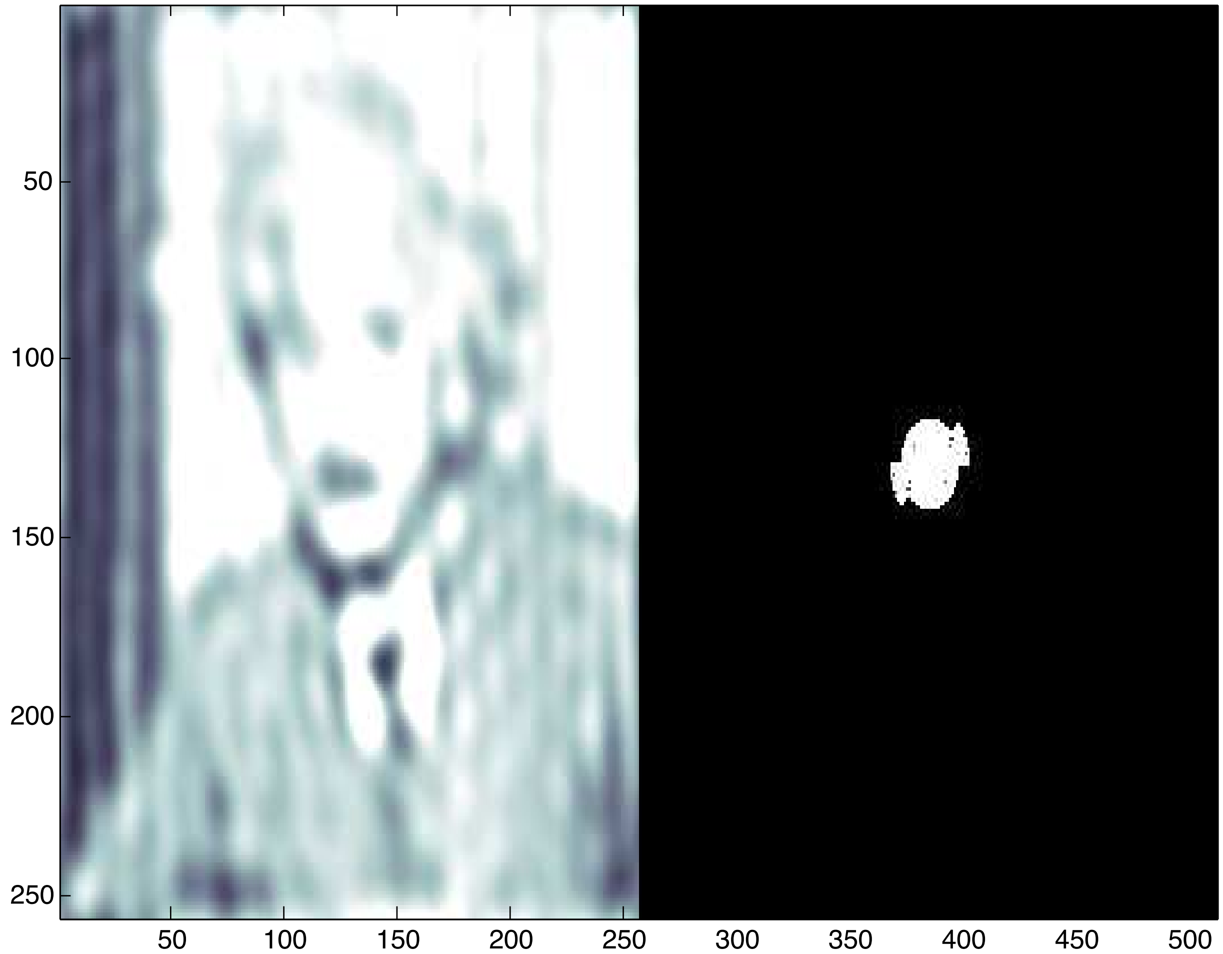


EXfourier2D.m

<u>Aside:</u> This stuff scales up to higher dimensions too...

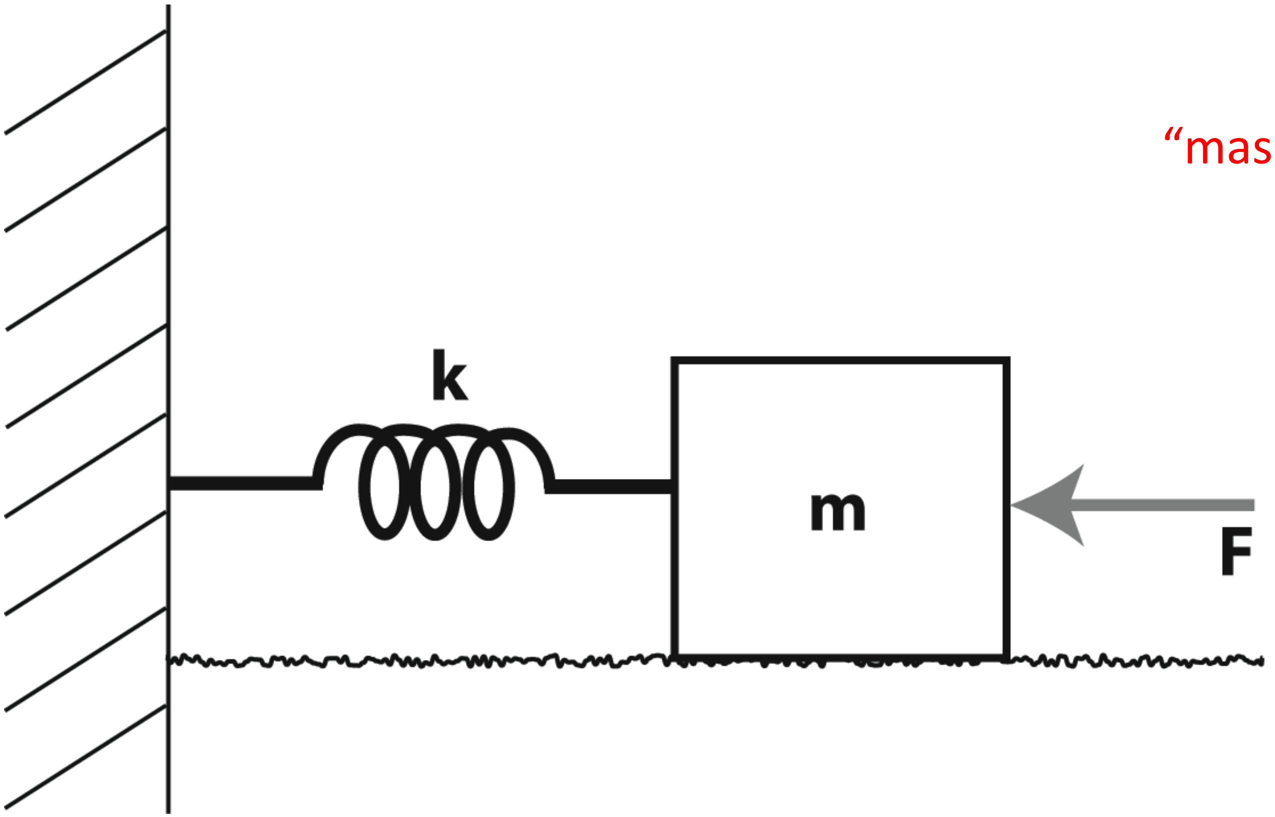


<u>Note</u>: Only ½ of the information is shown on the right (amplitude only; phase not shown)



→ 'Low-pass filtered' version of the image

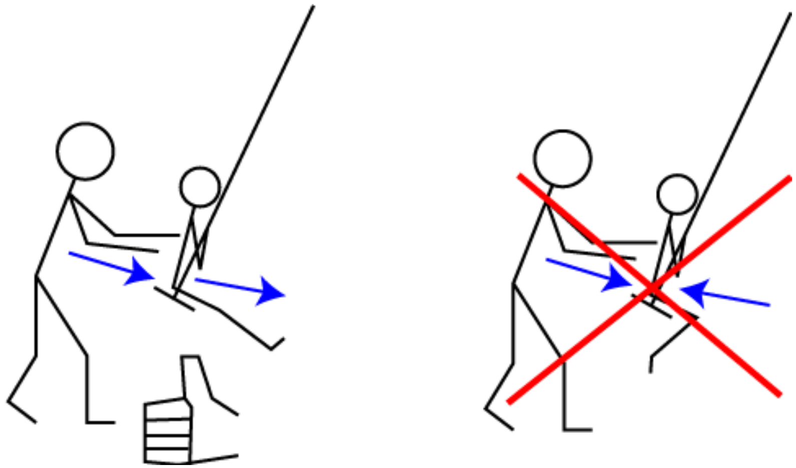
Canonical Anchor Point: Harmonic oscillator



"mass-on-a-spring"

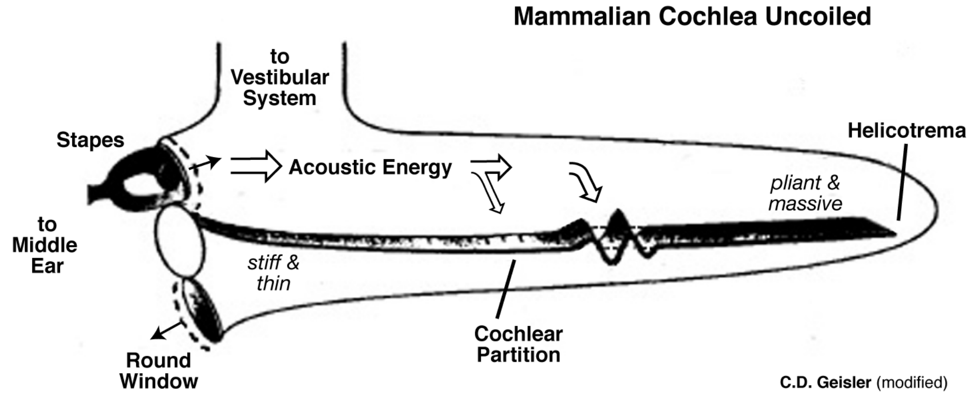
- One of the more fundamental/canonical problems in all areas of physics...

Looking Ahead: (examples of) Resonance



http://physics.stackexchange.com/questions/159728/forced-oscillations-resonance

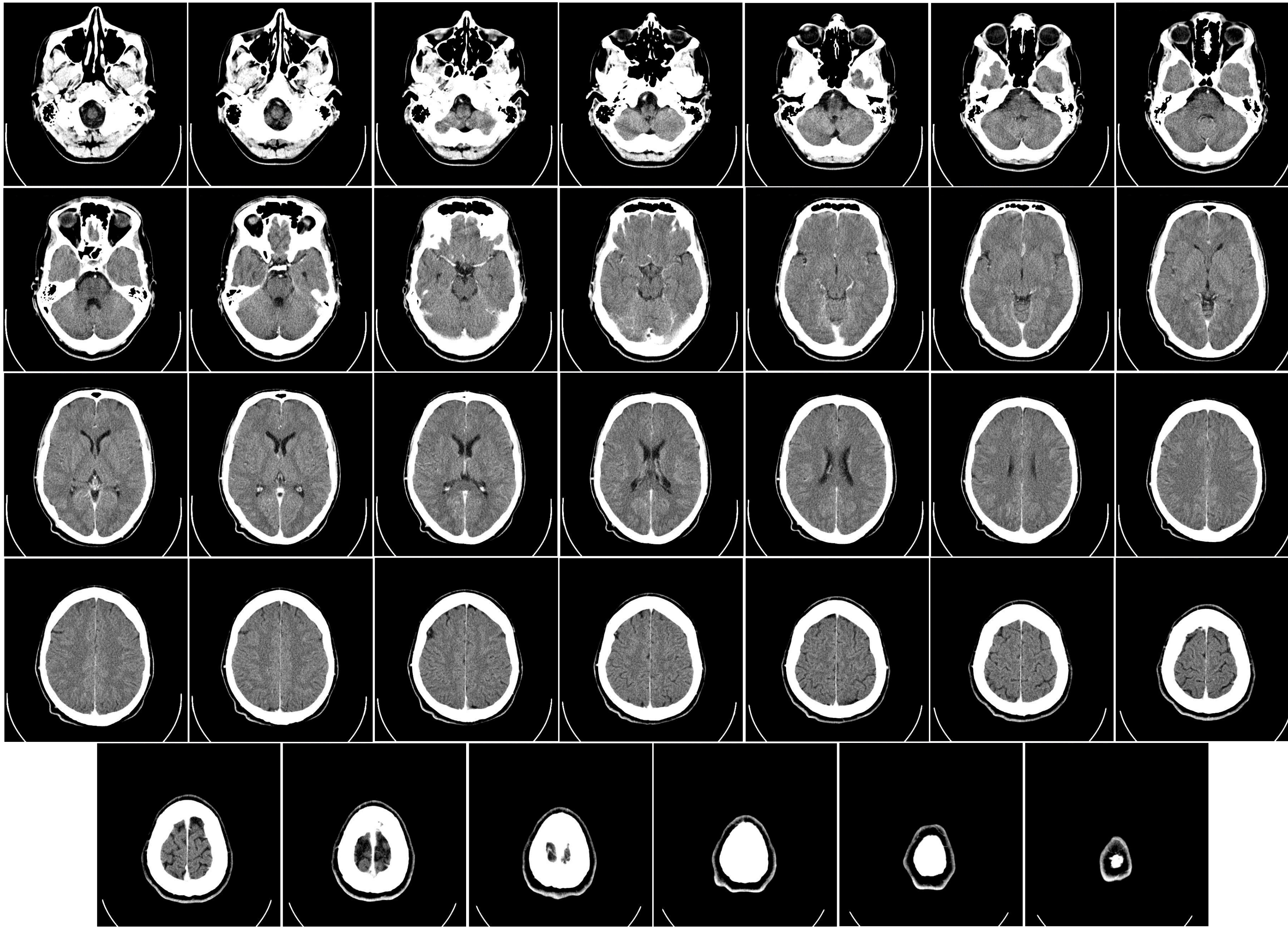
"Tonotopy" of the inner ear



Slightly different type of "resonance"...

MRI

Looking Ahead: (examples of) Resonance



Wikipedia

Looking Ahead: (examples of) Resonance

Wikipedia

Looking Ahead: (examples of) Resonance

MEMS (**M**icro**e**lectro**m**echanical **s**ystems)

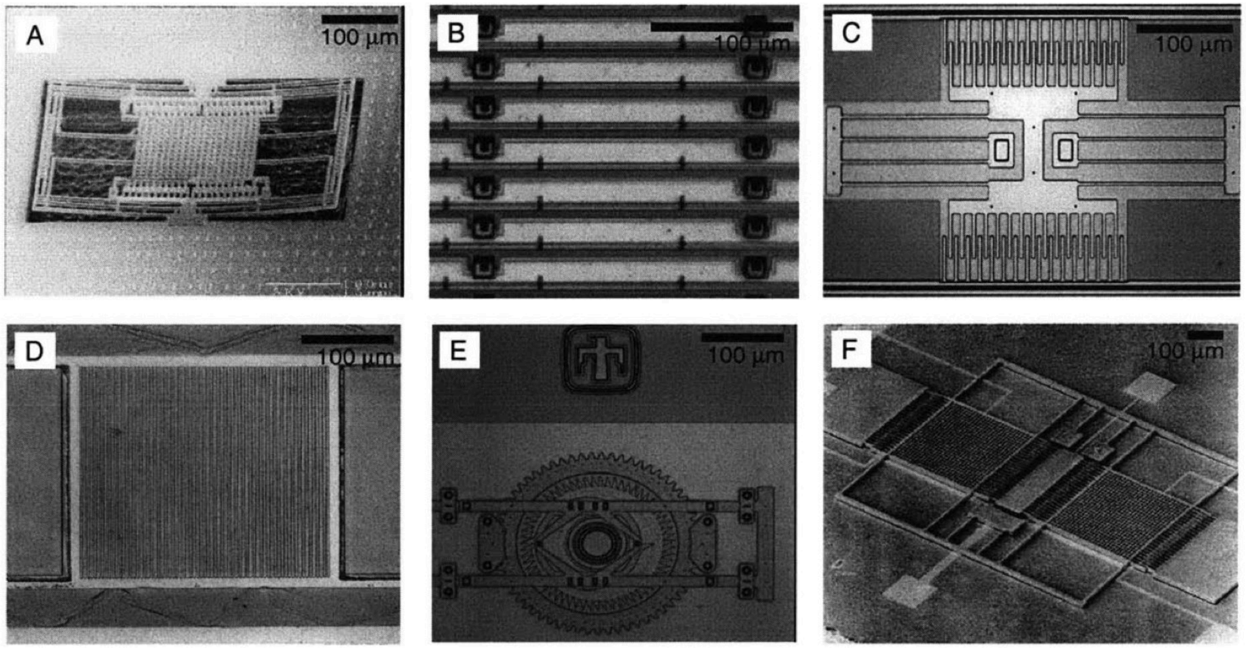


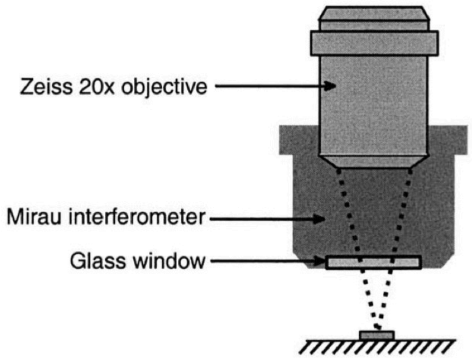
Figure 1-1: Microelecromechanical systems encompassing a wide variety of applications and a broad spectrum of fabrication processes: (A) SEM image of CMOS MEMS (Fedder et al., 1996) multiple degree-of-freedom microresonator fabricated at Carnegie Mellon University, (B) Optical micrograph of surface micromachined polysilicon diffraction gratings fabricated at the State University of New York at Albany, (C) Optical micrograph of surface micromachined lateral resonator fabricated using Cronos MUMPs, (D) Optical micrograph of platinum diffraction gratings fabricated at the Massashusetts Institute of Technology, (E) Optical micrograph of indexing motor fabricated using Sandia SUMMiT4 fabrication process, and (F) SEM image of Draper Laboratory tuning fork gyroscope (SEM picture courtesy Charles Stark Draper Laboratory, Cambridge MA).

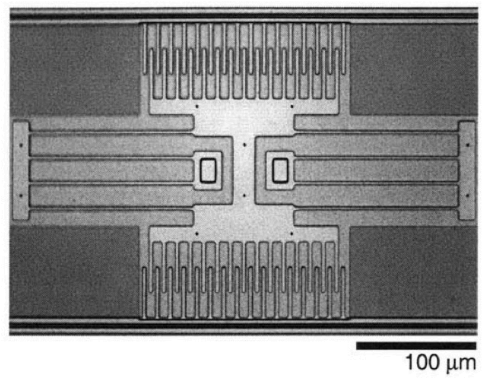
$\Rightarrow$ Resonant behavior

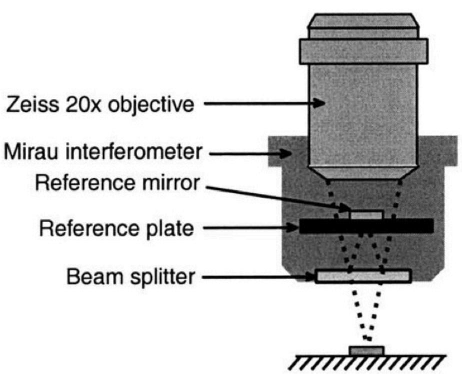
http://dspace.mit.edu/handle/1721.1/87312#files-area

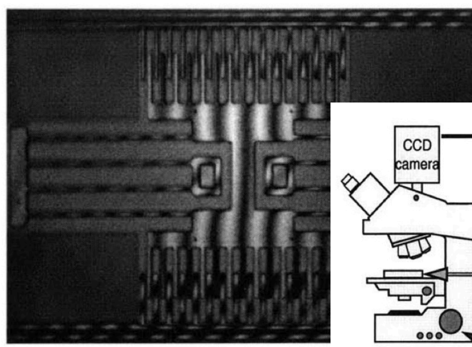
Looking Ahead: (examples of) Resonance

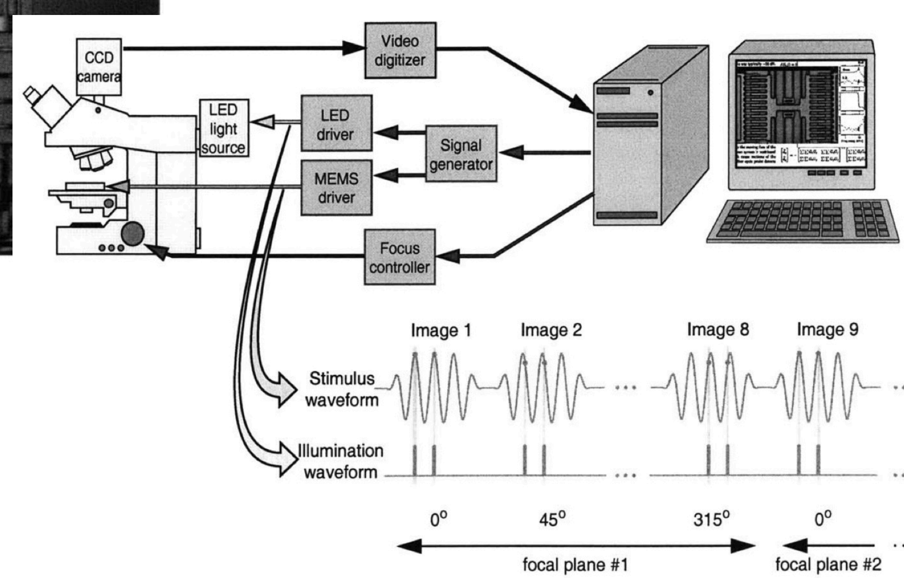
MEMS (**M**icro**e**lectro**m**echanical **s**ystems)











⇒ Stroboscopic imaging allows dynamics to be characterized

http://dspace.mit.edu/handle/1721.1/87312#files-area

Looking Ahead: (examples of) Resonance

MEMS (**M**icro**e**lectro**m**echanical **s**ystems)

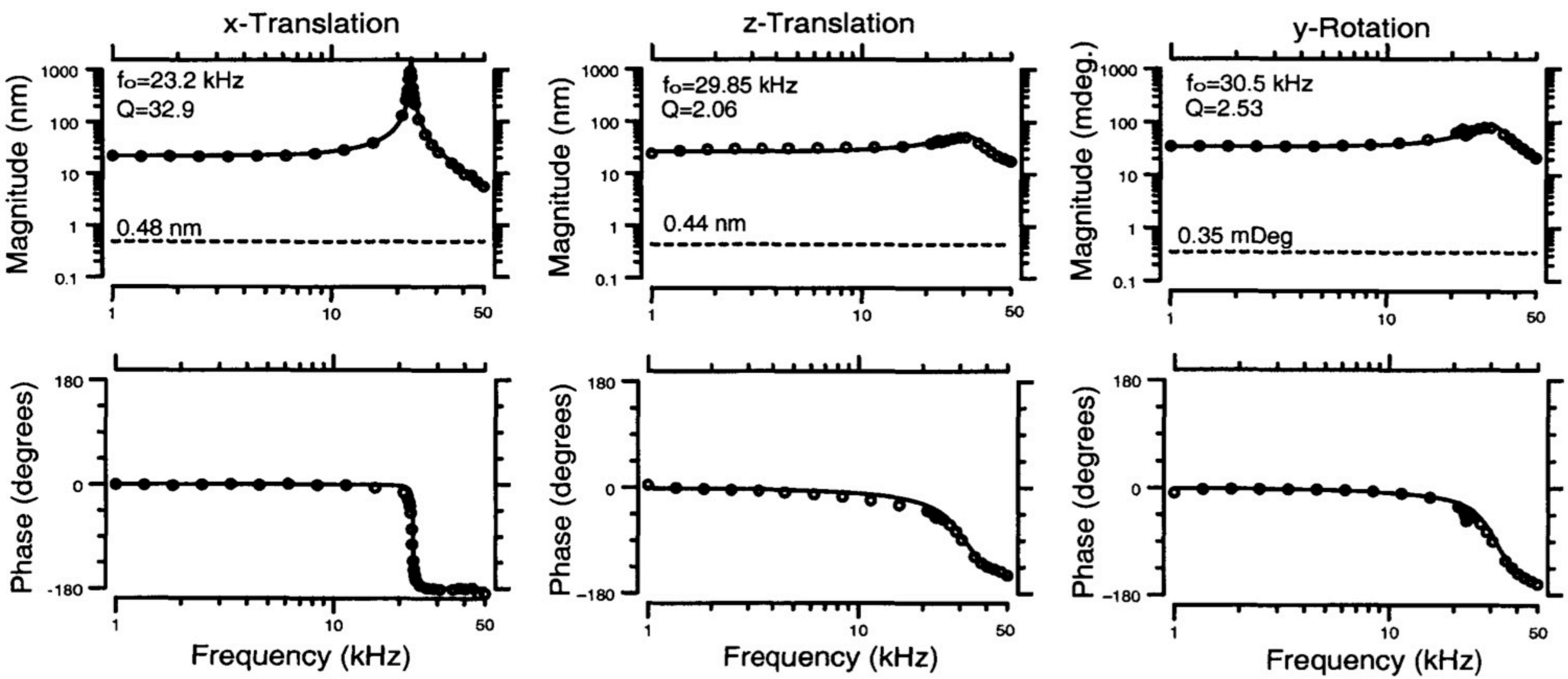
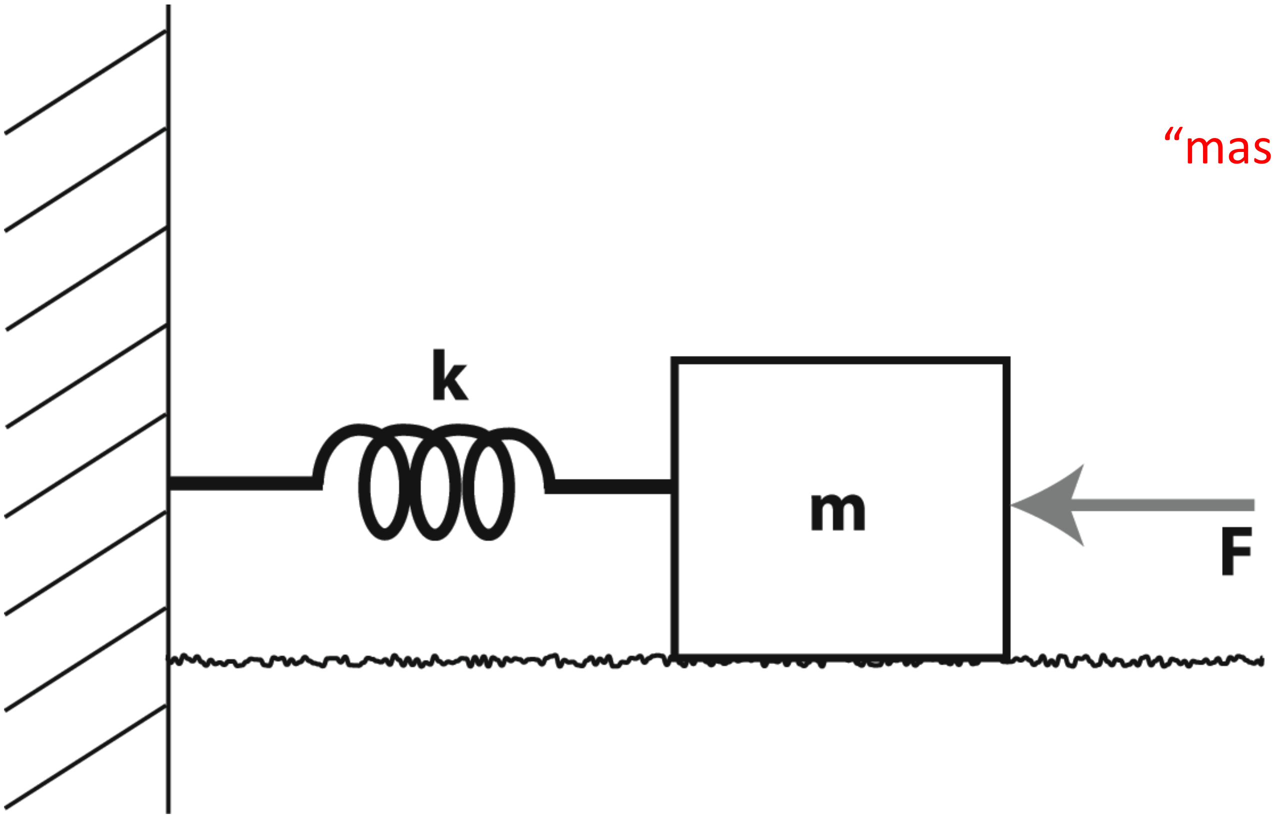


Figure 4-5: Magnitude and phase of frequency response for translation along x (left panel), translation along z (center panel), and rotation about y (right panel)

http://dspace.mit.edu/handle/1721.1/87312#files-area

All this brings us back to here....



"mass-on-a-spring"

- One of the more fundamental/canonical problems in all areas of physics...

(Rough/Tentative) Outline looking ahead.... (BEFORE THE MIDTERM)

- Review SHO → Foreshadows linear superposition + Fourier
- Notion of amplitude vs period (or frequency) vs "phase"
- Geometric representations (e.g., rotating vector) and complex #s
- Superposition of sinusoids and examples (e.g., beats, Lissajous)
- SHO via complex exponentials → Eigenvalues!
- Several canonical examples (e.g., buoyancy "bobbing", Helmholtz resonator)
- Damping and decayed oscillations
- Sinusoidal driving forces → **Resonance**

Note:
Will start weaving Matlab code/results into lectures to help set stage for project
(as well as help strengthen connections to PHYS 2030 and MATH 2271)

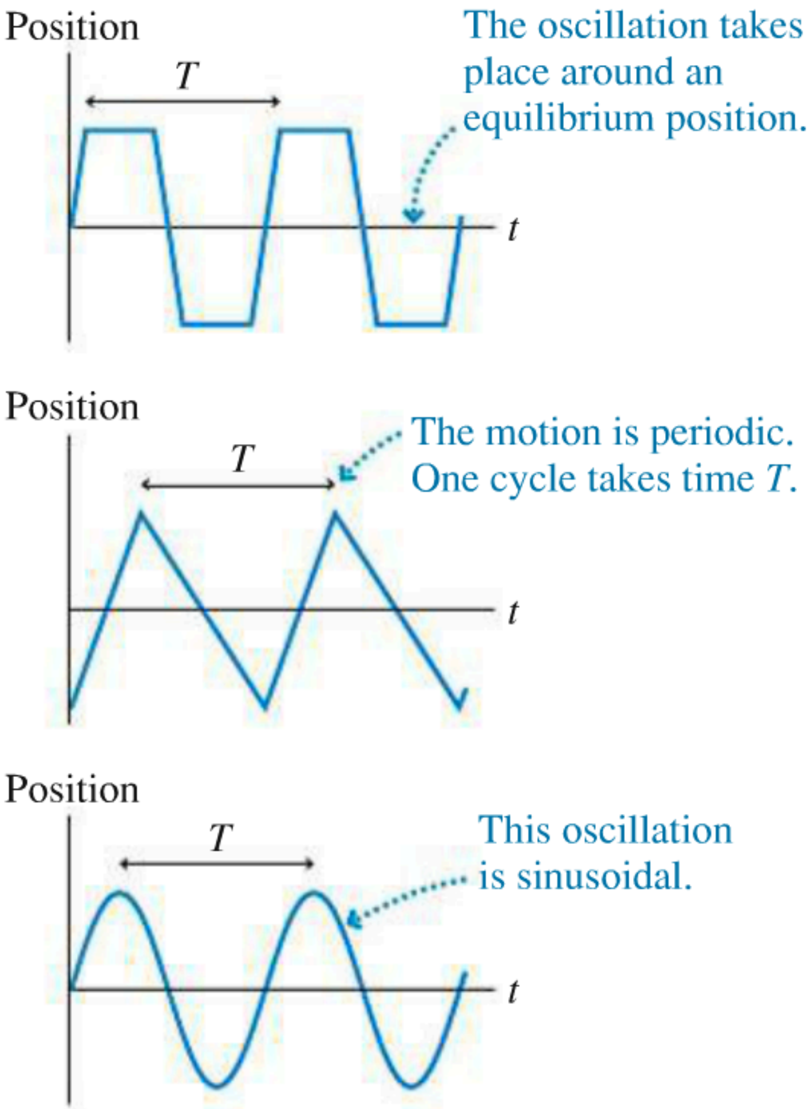
(Rough/Tentative) Outline looking ahead.... (BEFORE/AFTER THE MIDTERM)

- *Tuning* and effects of damping → "Filtering"
- Transient behavior → Impulse Response
- Notion of a *transfer function* → Connection to convolutions....
- Variety of examples: RLC circuits, optical, NMR, ... ← Our likely break point re the midterm
- Fourier analysis and approaches
- Coupled oscillators (+ excursion to *normal modes*)
- Pendulum → Transition to nonlinear systems (+ excursion to nonlinear dynamics)
- Duffing (→ Chaos!) & van der Pol (→ Limit Cycles!)

Note:
Will start weaving Matlab code/results into lectures to help set stage for project
(as well as help strengthen connections to PHYS 2030 and MATH 2271)

Review: Periodicity

Examples of position-versus-time graphs for oscillating systems.



Frequency & period

$$f = \frac{1}{T}$$

$$1 \text{ Hz} \equiv 1 \text{ cycle per second} = 1 \text{ s}^{-1}$$

Units of frequency

Frequency	**Period**
10^3 Hz = 1 kilohertz = 1 kHz	1 ms
10^6 Hz = 1 megahertz = 1 MHz	1 μs
10^9 Hz = 1 gigahertz = 1 GHz	1 ns

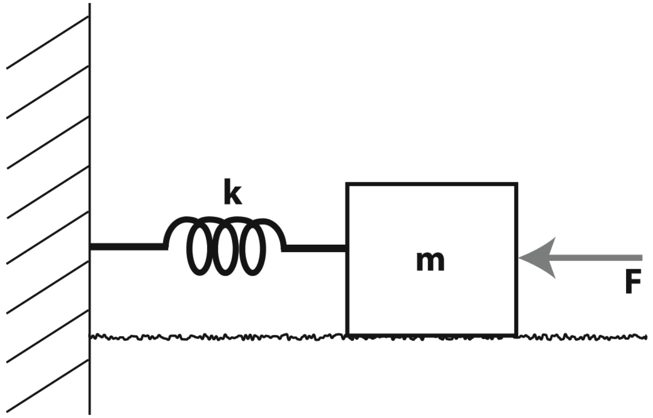
Angular frequency

$$\omega \text{ (in rad/s)} = \frac{2\pi}{T} = 2\pi f \text{ (in Hz)}$$

Knight

Review: SHO → *Natural Frequency*

Undamped, Undriven



$$F = ma = m\ddot{x} = -kx$$

Newton's Second Law
Hooke's Law

$$\ddot{x} + \frac{k}{m}x = 0$$

Second order ordinary differential equation
(no need worrying about how to "solve", yet...)

$$x(t) = A\cos\left(\omega_o t + \phi\right)$$

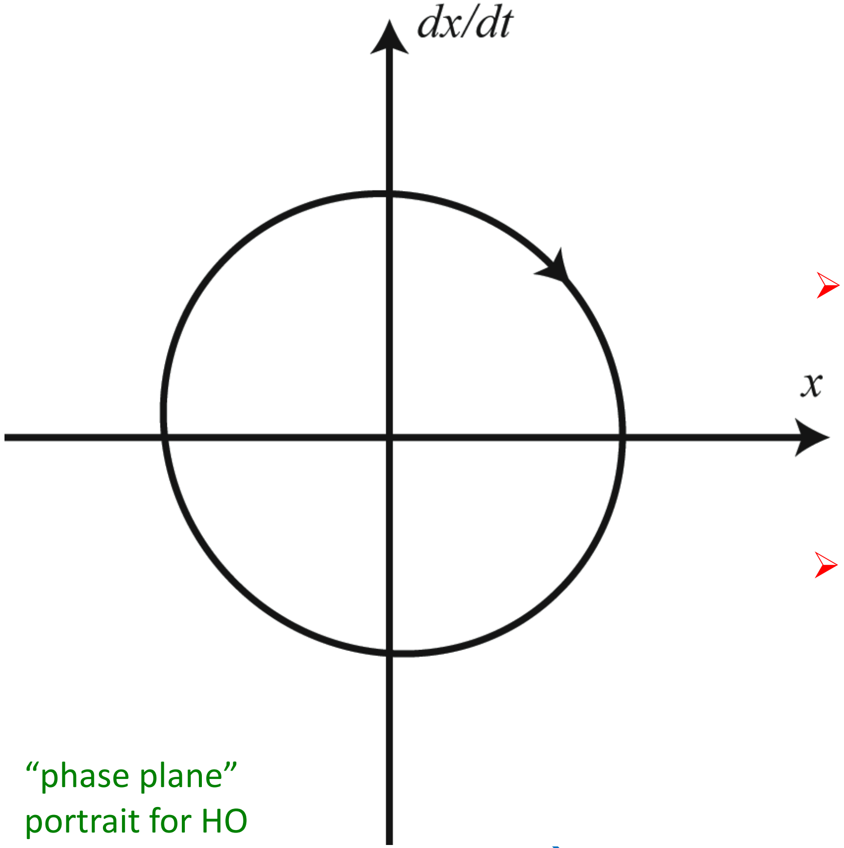
$\Rightarrow$ Solution is oscillatory!

$$\omega_o = \sqrt{k/m}$$

System has a *natural* frequency

Review: Energy & Natural Frequency

Consider the system's energy:



$$E = T + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

- Two means to *store* energy: mass & spring
- Oscillation results as energy transfers back and forth between these two *modes* (i.e., system is considered second-order)

"phase plane" portrait for HO

→ Through this lens, the natural frequency represents an optimal rate at which energy is swapped between these two

$$\omega_o = \sqrt{k/m}$$

Exploring alternative representations....

Equation of motion:

$$m\frac{d^2x}{dt^2} = -k_1 x$$

One possible form of solution:

$$x = A\sin(\omega t + \varphi_0)$$

Initial conditions (i.e., at $t = 0$):

$$x_0 = A\sin\varphi_0$$

$$v_0 = \omega A\cos\varphi_0$$

Magnitude

$$A = \left[x_0^2 + \left(\frac{v_0}{\omega}\right)^2\right]^{1/2}$$

Phase

$$\varphi_0 = \tan^{-1}\left(\frac{\omega x_0}{v_0}\right)$$

Note we could have gone w/ $\cos$ instead:

$$A\sin(\omega t + \varphi_0) = A\cos(\omega t + \alpha)$$

where $\varphi_0 = \alpha + \frac{\pi}{2}$

French (1971)

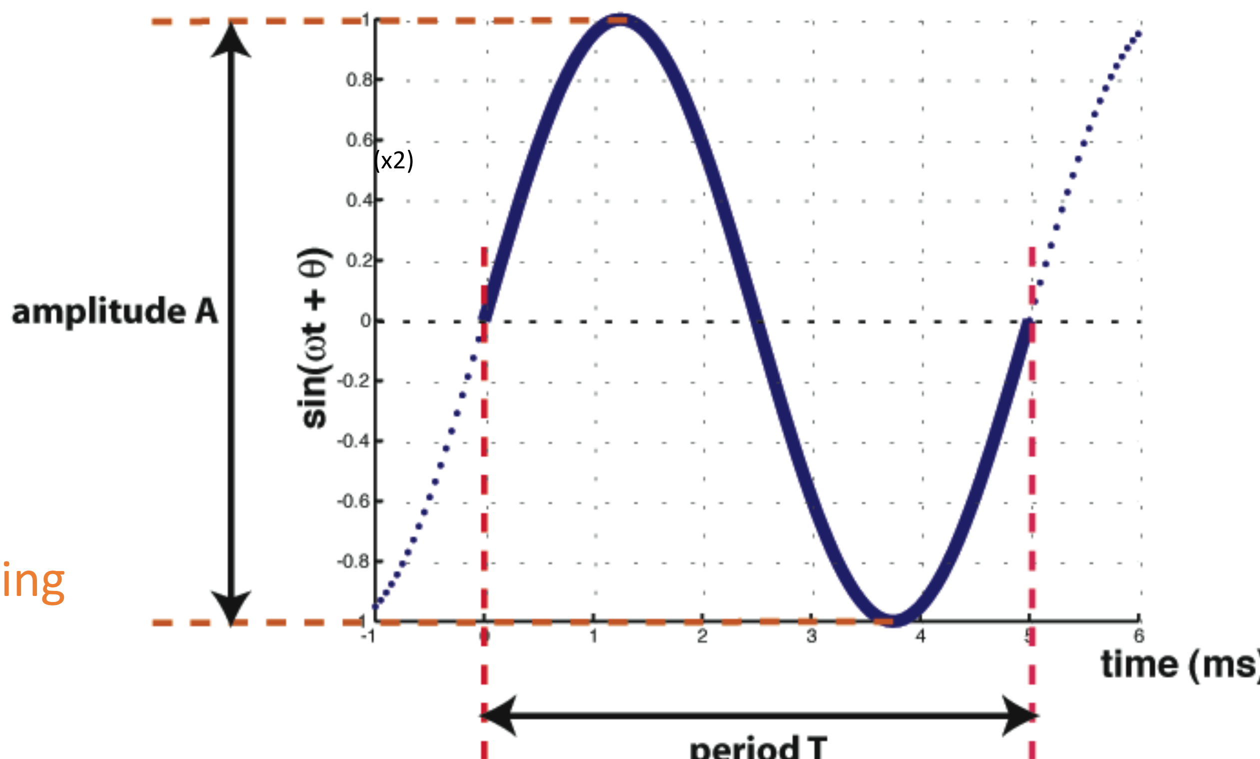
Review: Sinusoids

Sinusoid has 3 basic properties:

i. Amplitude - height

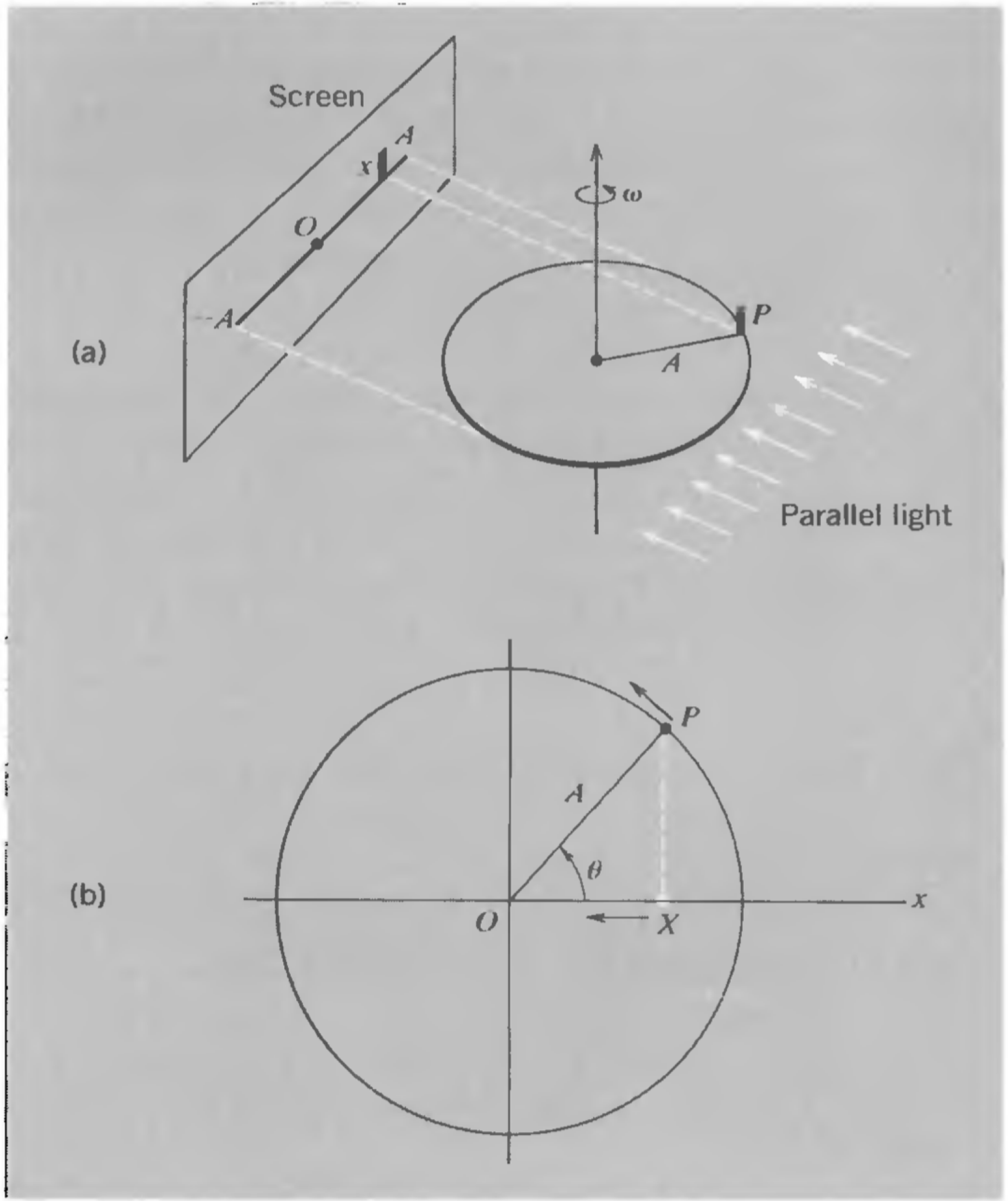
ii. Frequency = 1/T [Hz]

iii. Phase - tells you where the peak is (needs a reference)



⇒ Phase reveals timing information

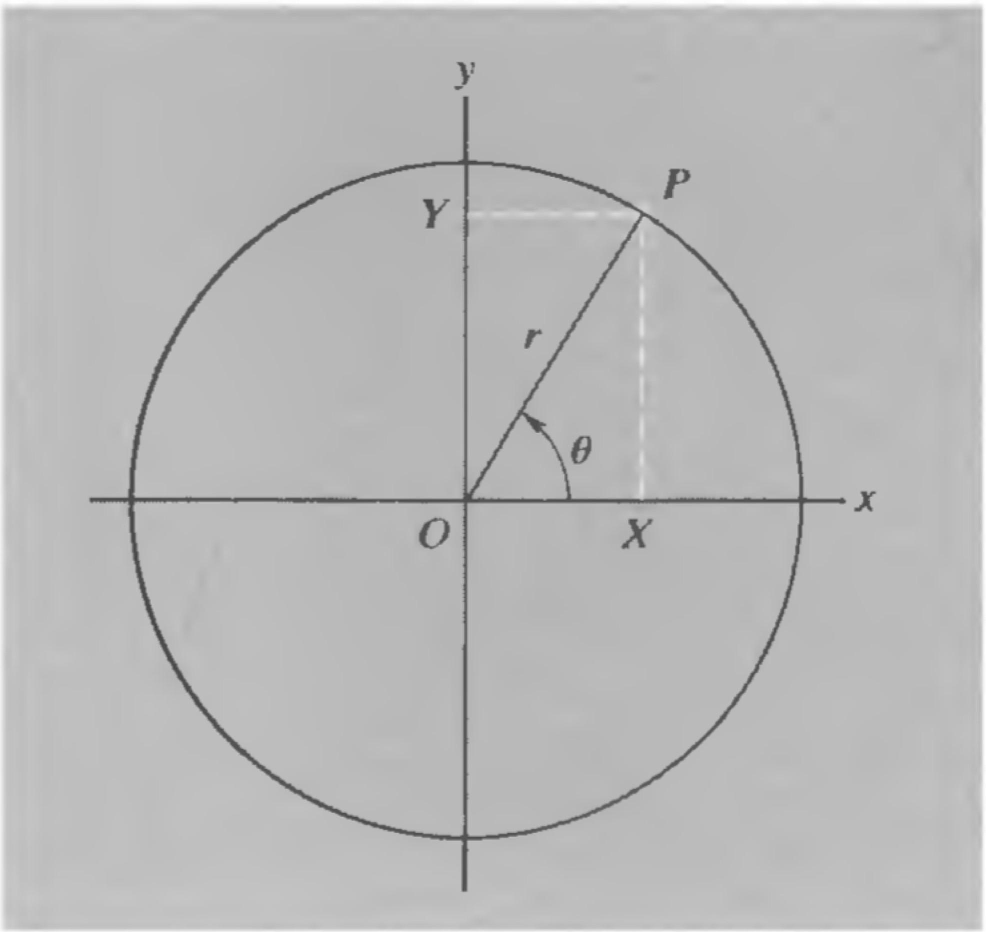
Exploring alternative representations....



French (1971)

Exploring alternative representations....

Consider "motion" as a rotating vector



Coords. given by:

$$x = A\cos(\omega t + \alpha)$$
$$y = A\sin(\omega t + \alpha)$$

In "cartesian form":

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Exercise: What are they then in "polar form"?

In vector form:

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y$$

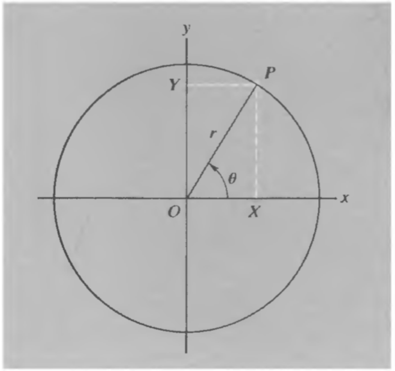
Or better yet....

$$\mathbf{r} = x + jy$$

French (1971)

Complex Representations

$$\mathbf{r} = x + jy$$



Basic stipulations:

1. A displacement, such as x, without any qualifying factors, is to be made in a direction parallel to the x axis.

2. The term jy is to be read as an instruction to make the displacement y in a direction parallel to the y axis. It is, in fact, customary to dispense with the usual vector symbolism altogether, by introducing a quantity z, understood to be the result of adding jy to x—i.e., identical with $\mathbf{r}$ as defined above. Thus we put

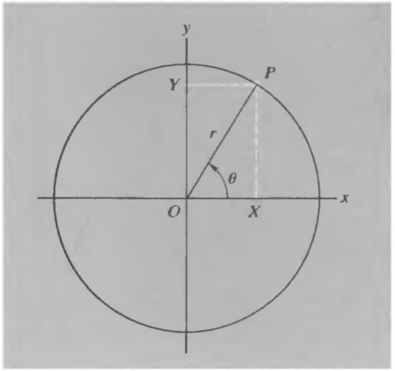
$$z = x + jy$$

We now proceed to broaden the interpretation of the symbol j, by reading it as an instruction to perform a counterclockwise rotation of 90° upon whatever it precedes.

French (1971)

Complex Representations

$$\mathbf{r} = x + jy$$

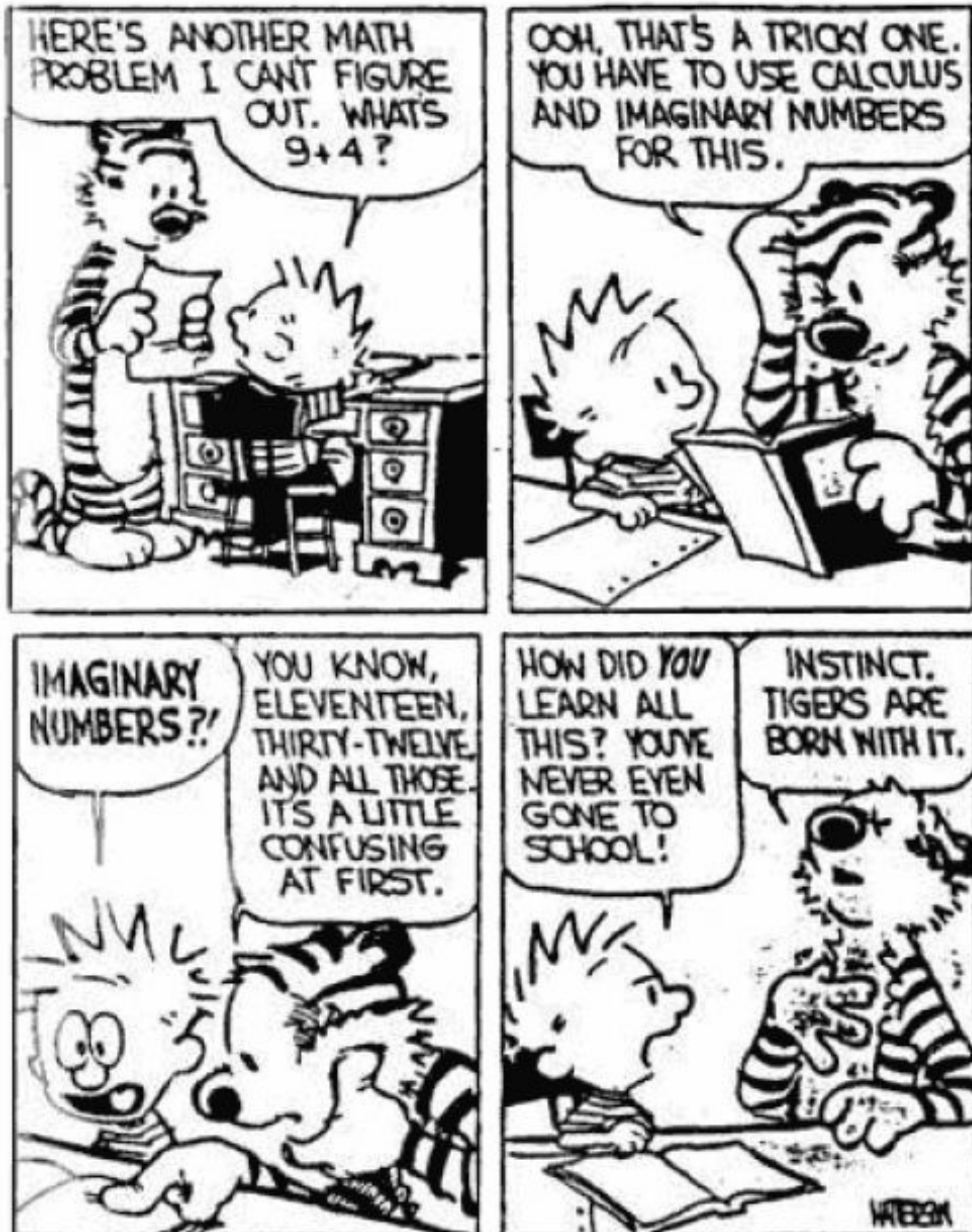


And thus we transition into the realm of complex numbers....

$$j^2 = -1$$

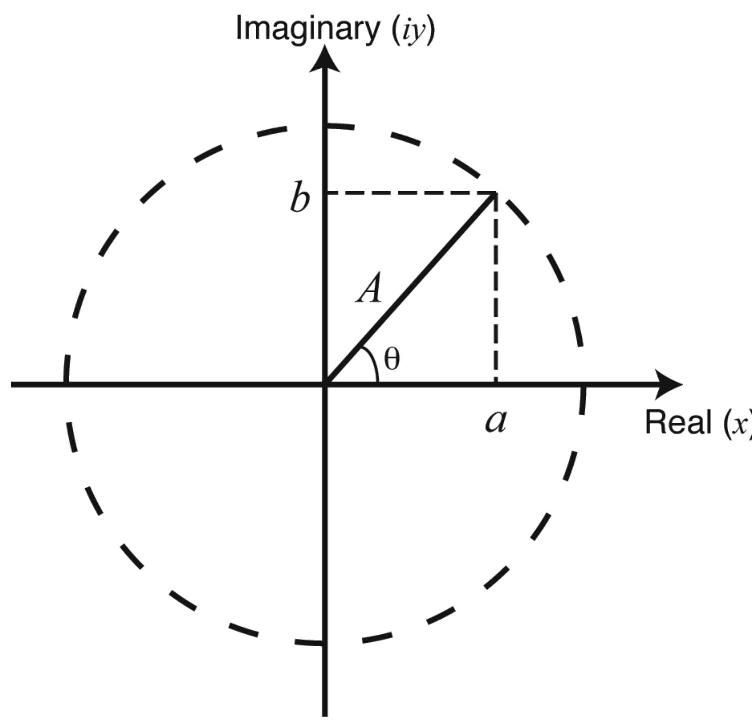
[1]The use of the symbol j for $\sqrt{-1}$ has emerged rather naturally from our quasi-geometrical approach. Very often, however, in mathematics texts, one will find the symbol i used for this purpose. Physicists and engineers tend to prefer the j notation, so as to reserve the symbol i for electric current—a not insignificant consideration because the mathematical techniques we are developing here find some of their most important uses in connection with electrical circuit problems.

French (1971)


HERE'S ANOTHER MATH PROBLEM I CAN'T FIGURE OUT. WHAT'S 9+4?
OOH, THAT'S A TRICKY ONE. YOU HAVE TO USE CALCULUS AND IMAGINARY NUMBERS FOR THIS.
IMAGINARY NUMBERS?!
YOU KNOW, ELEVENTEEN, THIRTY-TWELVE, AND ALL THOSE. IT'S A LITTLE CONFUSING AT FIRST.
HOW DID YOU LEARN ALL THIS? YOU'VE NEVER EVEN GONE TO SCHOOL!
INSTINCT. TIGERS ARE BORN WITH IT.
WATERSON

Whether or not you have been introduced to this kind of analysis previously, you will be able to recognize that we are walking along a dividing line—or, more properly, a bridge—between geometry and algebra. If the quantities a and b are real numbers, as we have assumed in example c, then the combination $z = a + jb$ is what is known as a complex number. But in geometrical terms it can be regarded as a displacement along an axis at some angle θ to the x axis, such that $\tan\theta = b/a$

$$j^2 = -1$$



French (1971)

Complex #s

Definition:

$$i = \sqrt{-1} \;\; (= j)$$

$$i^2 = -1$$

$$i^3 = -i$$

$$i^4 = 1$$

$$i^5 = i$$

$$\ldots$$

Cartesian form:

$$z = a + ib$$

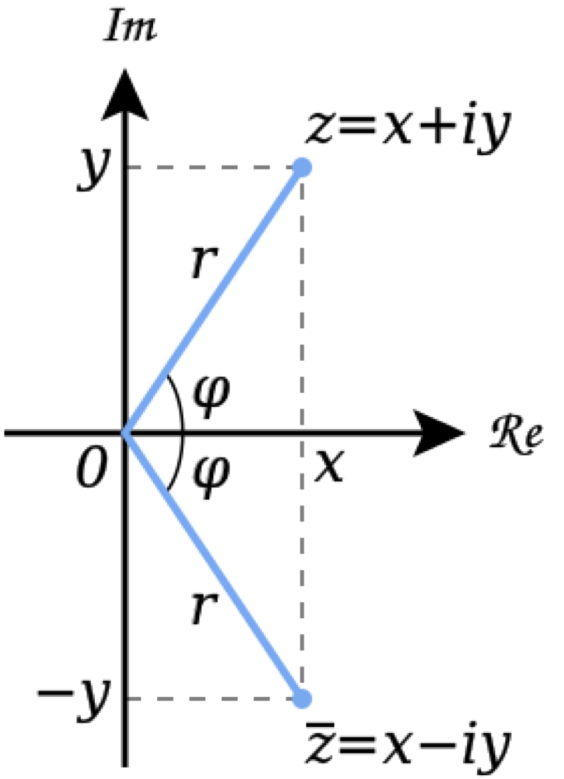
$$\mathrm{Re}(z) = a, \;\; \mathrm{Im}(z) = b$$

Communicative rule:

$$z_1 + z_2 = (a + ib) + (c + id) = (a + b) + i(c + d) = z_3$$

Multiplicative rule:

$$z_1 z_2 = (a + ib)(c + id) = (ac - bd) + i(ad + bc) = z_3$$



Polar form &
complex conjugate

→ Other basic algebraic manipulations? (e.g., division)

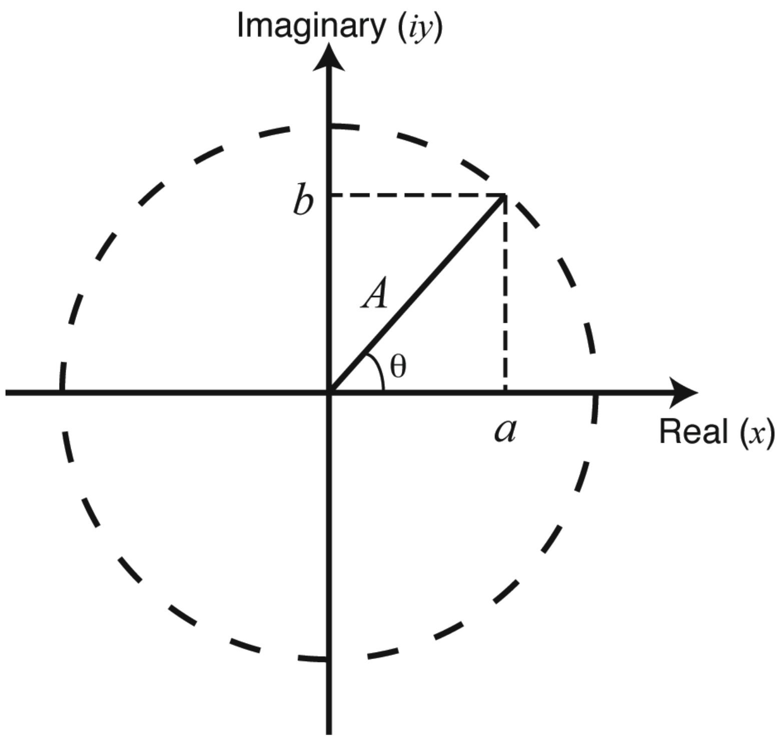
wikipedia (complex number)

Complex Exponentials

Euler's formula → Polar form

$$a + ib = Ae^{i\theta}$$

$$= A(\cos\theta + i\sin\theta)$$



Cartesian Form

$$a = A\cos(\theta)$$

$$b = A\sin(\theta)$$

⇔

Polar Form

$$A = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

→ Very useful to consider complex numbers geometrically via a circle centered about the origin in the complex plane

Magnitude

$$|a + ib| = A$$

Phase

$$\angle(a + ib) = \theta$$

- At the most basic level, simply consider a complex number as a means to compactly express two real numbers (along with the remarkable number i)

Complex Exponentials

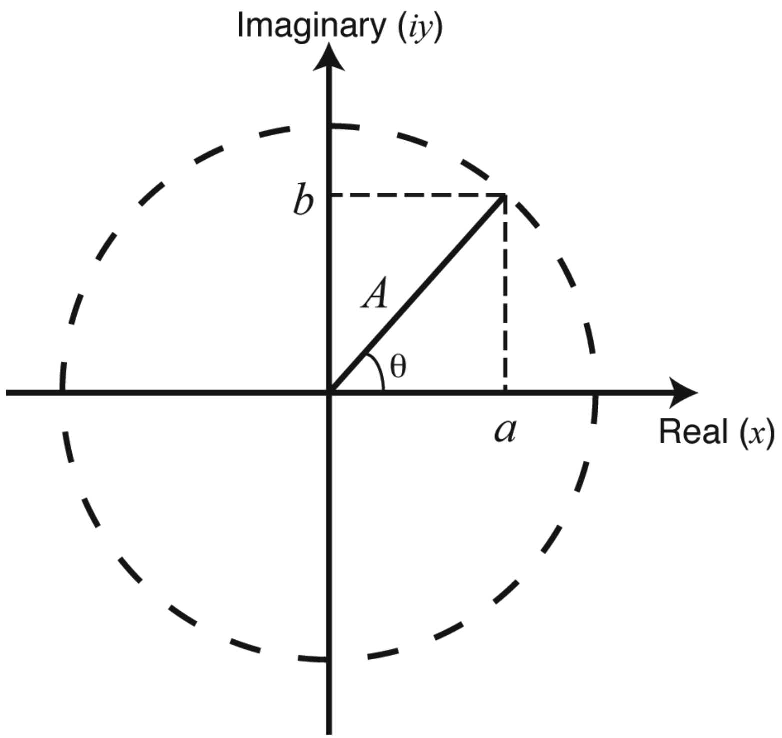
$$a + ib = Ae^{i\theta}$$

$$e^{i0} = 1$$

$$e^{i\pi/2} = i$$

$$e^{i\pi} = -1$$

$$e^{i3\pi/2} = -i$$



Ex.

$$i^i = (e^{i\pi/2})^i = e^{i^2\pi/2} = e^{-\pi/2} \approx 0.2079$$

→ Very useful to consider complex numbers geometrically via a circle centered about the origin in the complex plane

Ex.

A load of mass m lies on a perfectly smooth plane, being pulled in opposite directions by springs 1 and 2, whose coefficients of elasticity are k_1 and k_2 respectively (Fig. 60). If the load be forced out of its state of equilibrium (by being drawn aside), it will begin to oscillate with period T. Will the period of oscillation be altered if the same springs be fastened not at points A_1 and A_2, but at B_1 and B_2? Assume that the springs are subject to Hooke's law for all strains.

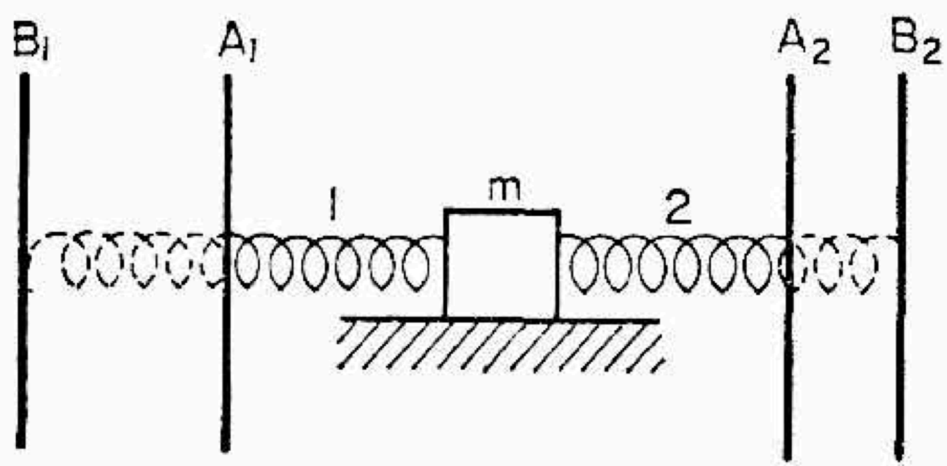


FIG. 60

Shaskol'Skaya & El'Tsin (1963)

Ex.

Why is a tuning-fork made with two prongs (Fig. 69)? Would a tuning-fork be of any use for its normal purpose if one of the prongs were sawn off?

Shaskol'Skaya & El'Tsin (1963)

Ex. (SOL)

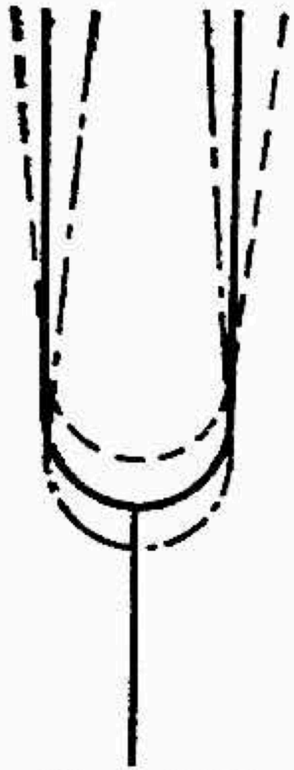


FIG. 213

The prongs of a tuning-fork working normally move in opposite phase, i.e. they are always moving in opposite directions (Fig. 213). Therefore the centre of gravity of the tuning-fork remains stationary and consequently no external force is required to cause these oscillations. The tuning-fork can make its oscillations without being rigidly fixed.

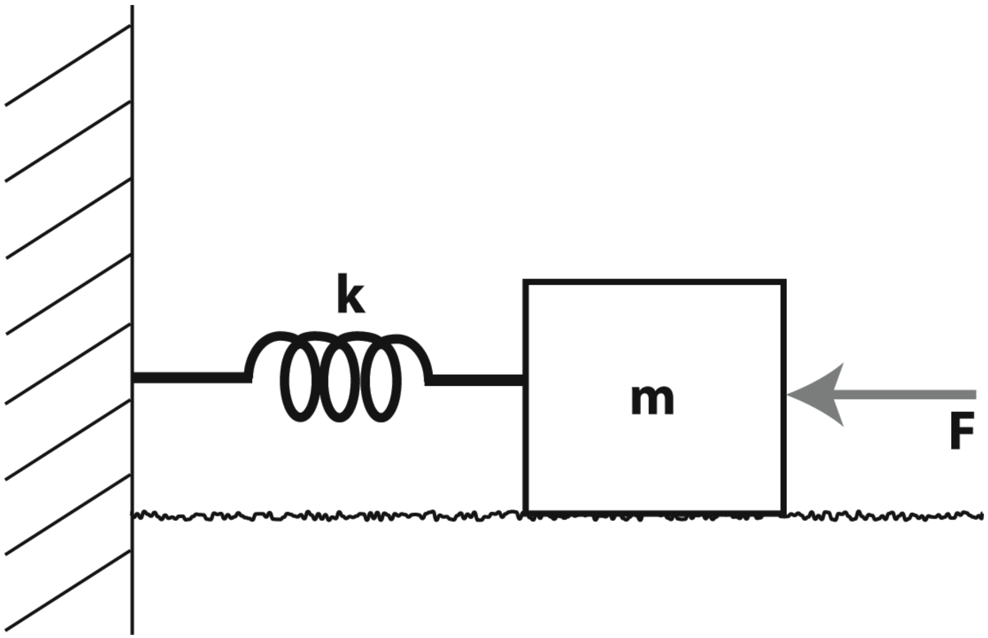
If one of the prongs be cut off, and the remaining prong makes oscillations of the same sort as before, then the centre of gravity will no longer remain stationary. Consequently an external force must act in order that these oscillations should occur, i.e. the tuning-fork must be rigidly fixed (e.g. the handle should be clamped in a vice); there is then an outside force acting, on the part of the clamp, which brings the centre of gravity into motion. But if the handle be simply held in the hand, the fork will not be sufficiently rigidly fixed and oscillations of the previous type cannot occur.

Thus the existence of two prongs makes it unnecessary to clamp the tuning-fork rigidly, i.e. it allows the instrument to be used when the handle is held in the hand.

→ This is actually a center-of-mass question!

Shaskol'Skaya & El'Tsin (1963)

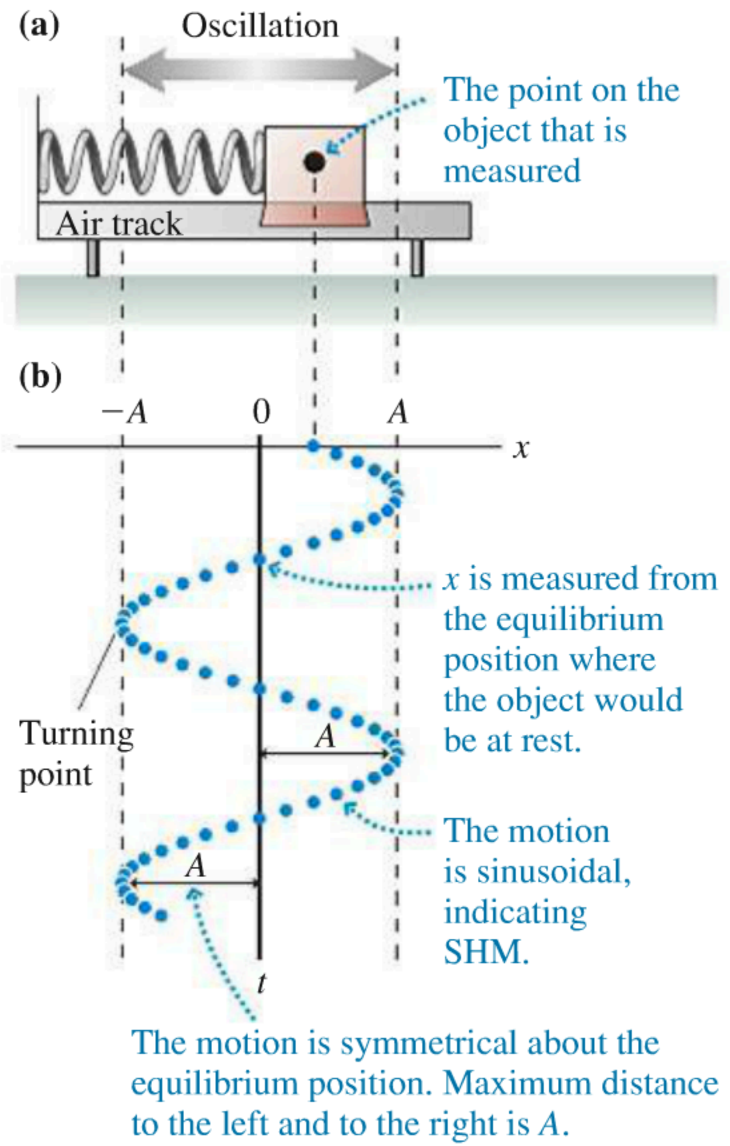
Review: Harmonic oscillator 101



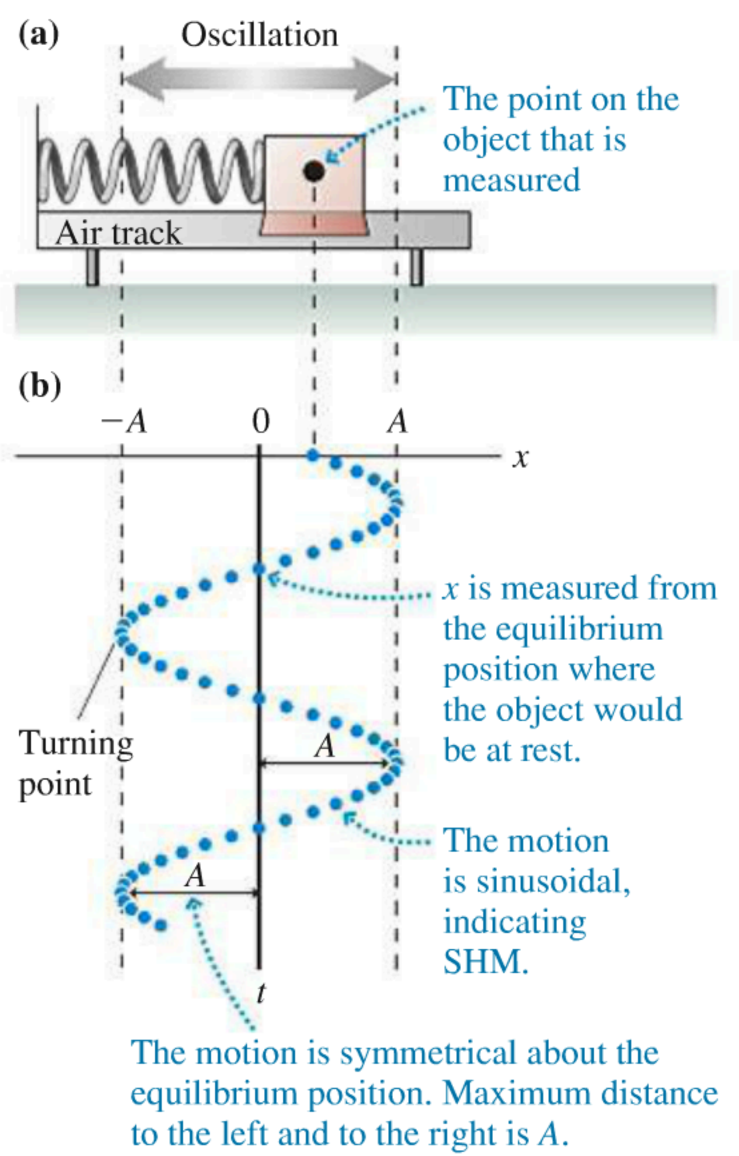
$$x(t) = A\cos(\omega_o t + \phi)$$

$$\omega_o = \sqrt{k/m} \qquad f = \frac{1}{T}$$

A prototype simple-harmonic-motion experiment.



A prototype simple-harmonic-motion experiment.



Position and velocity graphs of the experimental data.