PHYS 2010 (W20)
Classical Mechanics

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Relevant reading:
Knudsen & Hjorth: 15.ff, 16.2

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Refs:
233. Two Sprinters

Which sprinter will run a longer distance to make the full circle and get to the start position (or will they run equal distances)?

A  B  Equal
Looking Ahead....

Damped HO (DHO)

- The steady-state response of the sinusoidally-driven harmonic oscillator acts like a **band-pass filter**
- Connection between steady-state response & **impulse response**
SHO Revisted

Two essential features:

1. An inertial component, capable of carrying kinetic energy.
2. An elastic component, capable of storing elastic potential energy.

Two fundamental laws:

1. By Newton’s law \((F = ma)\),
   \[-kx = ma\]

2. By conservation of total mechanical energy \((E)\),
   \[\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = E\]

Two associated differential equations:

\[m \frac{d^2x}{dt^2} + kx = 0\]

\[\frac{1}{2}m \left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2 = E\]
SHO: Complex exponentials

Rewrite in terms of natural frequency

\[ \frac{d^2 x}{dt^2} + \omega^2 x = 0 \]

Now assume a solution in the form of a (possibly) complex exponential:

\[ x = C e^{pt} \]

Plug in assumed form of solution:

\[ p^2 C e^{pt} + \omega^2 C e^{pt} = 0 \]

Solving the ODE becomes an algebraic problem due to the assumptions we made!

\[ p^2 + \omega^2 = 0 \]

Associated eigenvalues

\[ p^2 = -\omega^2 \]

\[ p = \pm j\omega \]
SHO: Complex exponentials

\[ x = C_1 e^{i\omega t} + C_2 e^{-i\omega t} \]

Plugging it back in...

\[ x = Ce^{i(\omega t + \alpha)} + Ce^{-i(\omega t + \alpha)} \]
\[ = 2C \cos(\omega t + \alpha) \]
\[ \equiv A \cos(\omega t + \alpha) \]

Now there are a couple ways things could play out, but keep in mind the same basic issue it at play: there are two free parameters \((C & \alpha, \text{ or } C_1 & C_2, \text{ or } A \text{ and } \alpha)\)

\[ z = A \cos(\omega t + \alpha) + jA \sin(\omega t + \alpha) \]

\[ x = \text{real part of } z \quad \text{where} \quad z = Ae^{i(\omega t + \alpha)} \]

Note: The imaginary part of \(z\) is not any less "physical". It still contains the two key pieces of information (i.e., \(A\) and \(\alpha\) here)! Choosing the real part here is just a convention.
SHO: Complex exponentials

\[ x = Ce^{i(\omega t + \alpha)} + Ce^{-i(\omega t + \alpha)} = 2C \cos(\omega t + \alpha) \equiv A \cos(\omega t + \alpha) \]

Now there are a couple ways things could play out, but keep in mind the same basic issue it at play: there are two free parameters \((C \text{ and } \alpha)\), or \((C_1 \text{ and } C_2)\), or \((A \text{ and } \alpha)\).

**Note:** For the SHO, those two free parameters (plus our general form of the solution) tell us everything about how the system will behave for all time!!

→ So what determines those two free parameters?

\[ \frac{d^2x}{dt^2} + \omega^2 x = 0 \]

2\textsuperscript{nd} order ODE requires two unique initial conditions (or two unique boundary conditions) to find a specific solution \([\text{e.g., } x(t=0) = x_0 \text{ and } v(t=0) = v_0]\)
Damped HO

Eqn. of motion

\[ m \frac{d^2 x}{dt^2} = -kx - bv \]

Making a change of variables:

\[ \frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = 0 \]

where

\[ \gamma = \frac{b}{m} \quad \omega_0^2 = \frac{k}{m} \]

Now we must deal w/ a necessary reality: Despite solutions (possibly) being oscillatory, they will not/cannot be sinusoidal

French (1971)
Damped HO

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Making a change of variables:

\[ \frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = 0 \]

where (remember these!)

\[ \gamma = \frac{b}{m} \quad \omega_0^2 = \frac{k}{m} \]

Now we must deal w/ a necessary reality: Despite solutions (possibly) being oscillatory, they will not/cannot be sinusoidal

So we shift to a complex form....

\[ \frac{d^2 z}{dt^2} + \gamma \frac{dz}{dt} + \omega_0^2 z = 0 \]

With an assumed solution of the form

\[ z = Ae^{j(\omega t + \alpha)} \]

French (1971)
Complex Exponentials....

\[ \frac{d^2 z}{dt^2} + \gamma \frac{dz}{dt} + \omega_0^2 z = 0 \]

\[ z = Ae^{j(\omega t + \phi)} \]

→ Not only does this assumed form of solution capture the oscillation, it also describes the exponential decay/growth (all of which is encapsulated in the eigenvalues)

French (1971)
Damped HO (via complex exponentials)

Note:
There are a lot of starting points w/ regard to aspects such as the assumed form of the solution (see right & below as different possible examples). They may lead in slightly different directions analysis-wise, but ultimately they lead to the same place. It is worthwhile to spend a bit of to convince yourself of such, especially as you learn new mathematical methods....

\[ x(t) = A e^{i(\omega t + \delta)} \]

\[ x(t) = A e^{-i(\omega t + \delta)} \]

\[ x = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} c_1 e^{\lambda_1 t} + \begin{bmatrix} k_3 \\ k_4 \end{bmatrix} c_2 e^{\lambda_2 t} \]

French (1971)
Damped HO (via complex exponentials)

Combining these two:

\[ z = Ae^{j(p t + \alpha)} \]
\[ \frac{d^2 z}{dt^2} + \gamma \frac{dz}{dt} + \omega_0^2 z = 0 \]

We obtain:

\[ (-p^2 + j p \gamma + \omega_0^2) Ae^{j(p t + \alpha)} = 0 \]

Or more succinctly:

\[ -p^2 + j p \gamma + \omega_0^2 = 0 \]

This is sometimes referred to as the characteristic equation

\[ p = n + js \]

A handful of ways to deal w/ this, such as rewriting in terms of real and imaginary parts and solving each separately:

Real parts: \[ -n^2 + s^2 - s \gamma + \omega_0^2 = 0 \]

Imaginary parts: \[ -2 ns + n \gamma = 0 \]

**Note:** Another approach is to solve the char. eqn. via the quadratic formula
(see additional slides at end)
Damped HO (via complex exponentials)

When the smoke clears:

\[ s = \frac{\gamma}{2} \quad n^2 = \omega_0^2 - \frac{\gamma^2}{4} \]

\[ p = n + js \]

Plugging back in:

\[ z = Ae^{i(nt+st+\alpha)} = Ae^{-st}e^{i(nt+\alpha)} \]

And subsequently, from our convention:

\[ x = Ae^{-st} \cos(nt + \alpha) \]

Using variables from the ODE:

\[ x = Ae^{-\gamma t/2} \cos(\omega t + \alpha) \]

where

\[ \omega^2 = \omega_0^2 - \frac{\gamma^2}{4} = \frac{k}{m} - \frac{b^2}{4m^2} \]

⇒ The system doesn't even oscillate at the natural frequency!

French (1971)
Damped HO: Loss of Energy

For the moment, let's assume (i.e., relatively small damping) $\gamma \ll \omega$,

Recall that for SHO, the total energy is:

Thus for the damped case, we have:

Or more succinctly:

$E(t) = E_0 e^{-\gamma t}$

$E(t) = \frac{1}{2} k A_0^2 e^{-\gamma t}$

$\omega^2 = \omega_0^2 - \frac{\gamma^2}{4} = \frac{k}{m} - \frac{b^2}{4m^2}$

$\frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = 0$

$\gamma = \frac{b}{m} \quad \omega_0^2 = \frac{k}{m}$

$\Rightarrow$ Thus energy leaks out via an exponential decay due to the damping
Damped HO: Loss of Energy

\[ x = Ae^{-\gamma t/2} \cos(\omega t + \alpha) \]

\[ E(t) = E_0 e^{-\gamma t} \]

→ What about other relative damping cases? (i.e., small vs medium vs large damping)

\[ p = n + js \]

\[ s = \frac{\gamma}{2}, \quad n^2 = \omega_0^2 - \frac{\gamma^2}{4} \]

Thus it is more typical to find that the real part of the eigenvalue describes energy loss/gain (rather than the imaginary part, as is the case here)

\[ z = Ae^{j(p t + \alpha)} \quad x = Ce^{pt} \]

Note - Recall that we assumed ...

... but a more common convention is
Let’s consider a simple 2\textsuperscript{nd} order system (all these ideas scale up for higher dimension systems)

Re-express in matrix/vector form:

\[
\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]

\[
\frac{dx}{dt} = Ax
\]

Let’s make an assumption: solutions will have the form of (possibly complex) exponentials

\[
x = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} c_1 e^{\lambda_1 t} + \begin{pmatrix} k_3 \\ k_4 \end{pmatrix} c_2 e^{\lambda_2 t}
\]

This expression explicitly deals with the \textit{eigenvalues} and \textit{eigenvectors} of the system
Reference: Eigen Decomposition

\[ \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \frac{dx}{dt} = Ax \]

Characteristic equation:
\[ \det(A - \lambda I) = 0 \] → determinant (det) is scalar value associated with a square matrix

ODE as combination of eigenvalues and eigenvectors
\[ Ax = \lambda x \] ‘secular equation’

General solution:
\[ x = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} c_1 e^{\lambda_1 t} + \begin{bmatrix} k_3 \\ k_4 \end{bmatrix} c_2 e^{\lambda_2 t} \]

→ Remember, we implicitly assume the solution has this exponential form!
Characteristic equation:

\[
\det(A - \lambda I) = 0
\]

Quadratic equation w/ two roots (for a 2\textsuperscript{nd} order system)

\[
\lambda^2 - \lambda(a + d) + (ad - bc) = 0
\]

Note that complex roots are possible

\[
\lambda = \frac{(a + d) \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2}
\]

\[
\mathbf{x} = \begin{bmatrix}
  k_1 \\
  k_2
\end{bmatrix} c_1 e^{\lambda_1 t} + \begin{bmatrix}
  k_3 \\
  k_4
\end{bmatrix} c_2 e^{\lambda_2 t}
\]

→ Eigenvalues explicitly tell you how the solutions behave!
Reference: Classification of equilibrium points (linear autonomous 2nd order systems)

![Diagram of orbits and stability regions]

- **Nodes**
  - $\Delta > 0$
  - Stable

- **Saddles**
  - $\Delta > 0$
  - Unstable

- **Spirals**
  - $\Delta < 0$
  - Periodic

- **Periodic Orbits**
  - $\Delta = 0$

The system of differential equations is given by:

\[
\begin{align*}
\frac{dx}{dt} &= Ax + By \\
\frac{dy}{dt} &= Cx + Dy
\end{align*}
\]

And the discriminants are:

\[
\begin{align*}
p &= A + D \\
q &= AD - BC \\
\Delta &= p^2 - 4q
\end{align*}
\]
Ex.

\[
\frac{dx}{dt} = 5x - 3y
\]

\[
\frac{dy}{dt} = 2x - 4y
\]

→ Only a single equilibrium point exists (at the origin). Stability?

\[
A = \begin{pmatrix} 5 & -3 \\ 2 & -4 \end{pmatrix}
\]

\[
\det(A - \lambda I) = 0
\]

\[
p = \text{Tr}(A) = 5 + (-4) = 1
\]

\[
q = \det(A) = 5(-4) - (-3)2 = -14
\]

\[
\lambda = \frac{1}{2} \left( 1 \pm \sqrt{1 + 56} \right)
\]

\[
x = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} c_1 e^{\lambda_1 t} + \begin{bmatrix} k_3 \\ k_4 \end{bmatrix} c_2 e^{\lambda_2 t}
\]

\[
\lambda = -3.27, 4.27
\]

→ General solution is a linear combination of a (real-valued) exponentials, one converging and one diverging
Ex. (cont.)

\[ p = \text{Tr}(A) = 5 + (-4) = 1 \]
\[ q = \det(A) = 5(-4) - (-3)^2 = -14 \]
\[ \lambda = -3.27, 4.27 \]

→ Solution curves approach the origin, then diverge away

→ Equilibrium point at origin (where the eigenvectors meet) is said to be a *saddle*
Damped HO (Alternative Approach re complex exponentials)

\[
\ddot{x} + \gamma \dot{x} + \omega_o^2 x = 0
\]

\[
\frac{dx}{dt} = y
\]

\[
\frac{dy}{dt} = -\omega_o^2 x - \gamma y
\]

\[
\lambda = \frac{1}{2} \left( -\gamma \pm \sqrt{\gamma^2 - 4\omega_o^2} \right)
\]

\[
p = -\gamma \quad q = \omega_o^2 (>0)
\]

- What if \( \gamma \) is zero? Negative?
- Depending upon the sign and relative values of \( \gamma \) and \( \omega_o \), \( \lambda \) can be complex

\( \rightarrow \) Eigenvalues characterize behavior of all possible solution types!

\[
x(t) = Ae^{-\gamma t/2} e^{i(\omega t + \alpha)}
\]
% Numerically integrate a general 2nd order linear autonomous system (w/ const. coefficients)
% x' = a*x + b*y
% y' = c*x + d*y

clear
% User input (Note: All parameters are stored in a structure)
P.y0(1) = 1.0;   % initial value for x
P.y0(2) = 1;   % initial value for y
P.A= [-3.9 3; -2 1];  % matrix A to contain coefficients A= [a b c d]
P.t0 = 0.0;   % Start value
P.tf = 10.0;   % Finish value
P.dt = 0.01;  % time step

% determine some basic derived quantities
p= P.A(1,1)+ P.A(2,2);  % Tr(A)
q= P.A(1,1)* P.A(2,2)-P.A(1,2)* P.A(2,1); % det(A)
disp(['Tr(A)= ' num2str(p),' and det(A)= ',num2str(q)]);
eigV1= [0.5*(p+sqrt(p^2-4*q)) 0.5*(p-sqrt(p^2-4*q))];     % calc. eigenvalues directly
eigV2= eig(P.A);    % calculate via Matlab's built-in routine
disp(['eigenvalues= ' num2str(eigV1(1)),' and ',num2str(eigV1(2))]);

% use built-in ode45 to solve
[t y] = ode45('LINfunction', [P.t0:P.dt:P.tf],P.y0,[],P);

function [out1] = LINfunction(t,y,flag,P)
% -------------------------------------------
%   y(1) ... x
%   y(2) ... y
out1(1)= P.A(1,1)*y(1) + P.A(1,2)*y(2);
out1(2)= P.A(2,1)*y(1) + P.A(2,2)*y(2);
out1= out1';
Damped HO (Phase Plane Analysis)

→ Computationally, use our ode45 code or pplane to explore behavior of solution curves
Damped HO (Phase Plane Analysis)

- $\gamma = 0.5$
- $\omega_0^2 = 2$
Damped HO (Phase Plane Analysis)

- $\gamma = 2$
- $\omega_0^2 = 2$
Damped HO (Phase Plane Analysis)

- $\gamma = 2$
- $\omega_0^2 = 20$
Damped HO (Phase Plane Analysis)

- $\gamma = 20$
- $\omega_0^2 = 20$