PHYS 2010 (W20)
Classical Mechanics

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Relevant reading:
Knudsen & Hjorth: 15.6

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Refs:
This is a word game that's trickier than it looks. I would like you to make the longest word you can using only the six letters illustrated below.

Your answer must be a well-known English word, and you are not allowed to use any letter twice.

Think you've got a good answer? I can virtually guarantee that my solution will slay your effort.
1. Time response of ‘system’ when subjected to an impulse
   (e.g., striking a bell w/ a hammer)

2. Fourier transform of resulting response
   (e.g., spectrum of bell ringing)

   ex. Harmonic oscillator

   (Important) Note: The Fourier transform of the impulse response is called the *transfer function*
Case 1: Undamped undriven HO (i.e., SHO)

Newton’s Second Law
Hooke’s Law

\[ F = ma = m\ddot{x} = -kx \]

Second order ordinary differential equation
(no need worrying about how to “solve”, yet...)

\[ \ddot{x} + \frac{k}{m}x = 0 \]

\[ x(t) = A \cos(\omega_o t + \phi) \]

⇒ Solution is oscillatory!

\[ \omega_o = \sqrt{\frac{k}{m}} \]

System has a natural frequency
Case 2: Undamped driven HO

\[ \ddot{x} + \frac{k}{m} x = F_o \cos \omega t \]

**Sinusoidal driving force at frequency \( \omega \)**

**Assumption:** Ignore onset behavior and that system oscillates at frequency \( \omega \)

\[ x(t) = B \cos (\omega t + \alpha) \]

Assumed form of solution

\[-m \omega^2 B \cos \omega t + k B \cos \omega t = F_o \cos \omega t\]

\[ x(t) = \frac{F_o / m}{\omega_o^2 - \omega^2} \cos (\omega t + \alpha) \]
Two Important Concepts Demonstrated Here:

- **Resonance** when system is driven at natural frequency

- **Phase shift** of 1/2 cycle about resonant frequency
Case 3: Damped undriven HO

\[
m\ddot{x} + b\dot{x} + kx = 0
\]

\[
\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0
\]

**Assumption**: Form of solution is a complex exponential

\[x(t) = Ae^{i(\omega t + \delta)}\]

Purely sinusoidal solution no longer works!

Change variables
Trigonometry review ⇒ Sinusoids

Sinusoid has 3 basic properties:

i. **Amplitude** - height

ii. **Frequency** = \( 1/T \) [Hz]

iii. **Phase** - tells you where the peak is (needs a reference)

⇒ Phase reveals timing information
Motivation for complex solution:

$$a + ib = Ae^{i\theta}$$

$$= A(\cos \theta + i \sin \theta)$$

**Cartesian Form**

$$a = A \cos (\theta)$$

$$b = A \sin (\theta)$$

**Polar Form**

$$A = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-1} \left( \frac{b}{a} \right)$$

$\Rightarrow$ Complex solution contains both magnitude and phase information

Case 3: Damped undriven HO
Case 3: Damped undriven HO

\[ \ddot{x} + \gamma \dot{x} + \omega_o^2 x = 0 \]

\[ x(t) = Ae^{i(\omega t + \delta)} \]

\[ x(t) = Ae^{-\gamma t/2} e^{i(\omega t + \alpha)} \]

\[ \omega^2 = \omega_o^2 - \frac{\gamma^2}{4} \]

(A and \( \alpha \) are constants of integration, depending upon initial conditions)

\( \Rightarrow \) Damping causes energy loss from system
Case 4: Damped driven HO

\[ m\ddot{x} = -kx - m\gamma \dot{x} + F_0 \cos \omega t \quad \ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t \]

General (strictly real) solution:

\[ x(t) = x_1(t) + x_2(t) \]

\[ x(t) = x_0 \exp \left( -\frac{\gamma t}{2} \right) \cos(\omega_d t + \varphi) + A \cos(\omega t - \theta) \]

First part (i.e., \( x_1 \)) is just a decaying oscillation at the "free" (i.e., undriven) frequency:

\[ \omega_d = \omega_0 \left[ 1 - \left( \frac{\gamma}{2\omega_0} \right)^2 \right]^{1/2} \]

Second part (i.e., \( x_2 \)) is the steady-state response and a bit easier to derive via complex exponentials:

\[ x_2(t) = A \cos \omega t \cos \theta + A \sin \omega t \sin \theta \]
Case 4: Damped driven HO

\[ \ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{F_0}{m} e^{i\omega t} \]

**Sinusoidal driving force at frequency** \( \omega \)

**Assumption:** Ignore onset behavior and that system oscillates at frequency \( \omega \)

\[ x(t) = A e^{-i(\omega t + \delta)} \]

**Assumed form of solution**

\[
A(\omega) = \frac{F_0/m}{\left[ (\omega_0^2 - \omega^2)^2 + (\gamma \omega)^2 \right]^{1/2}} \\
\delta(\omega) = \arctan \left( \frac{\gamma \omega}{\omega^2 - \omega_0^2} \right)
\]

(magnitude)  
(phase)
Case 4: Damped driven HO

\[ A(\omega) = \frac{F_0}{m} \frac{1}{\left[ (\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2 \right]^{1/2}} \]

\[ \delta(\omega) = \arctan \left( \frac{\gamma\omega}{\omega^2 - \omega_0^2} \right) \]

⇒ Second-order oscillator behaves as a band-pass filter
Case 4: Damped driven HO

Three different key frequencies at play:

1. driving frequency \( (\omega) \)

2. free damped frequency \( (\omega_d) \)

3. resonant frequency \( (\omega_m) \)

\[
\begin{align*}
m\ddot{x} &= -kx - m\gamma \dot{x} + F_0 \cos \omega t \\
\omega_d &= \omega_0 \sqrt{1 - \frac{\gamma^2}{4\omega_0^2}} \\
\omega_m &= \omega_0 \sqrt{1 - \frac{\gamma^2}{2\omega_0^2}}
\end{align*}
\]
Key Idea: Tuned Responses Take Time

Second Order System
(resonant frequency $\omega_m$)

$\Rightarrow$ External driving force at frequency $\omega$

$$x(t) = A(\infty) \left[ 1 - e^{(-t/\tau)} \right]$$

$$\tau = 1/\gamma = Q / \omega_o$$
Steady-state frequency response

\[ Q \text{ is the ‘quality factor’} \]

\[ \ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{F_0}{m} e^{i\omega t} \]

\[ Q = \omega_0 / \gamma \]

→ Phase information tells us something about the damping
\[ \ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{F_o}{m} e^{i\omega t} \]

\[ \delta(\omega) = \arctan \left( \frac{\gamma \omega}{\omega^2 - \omega_0^2} \right) \]

\[ N \equiv f_0 \times \text{phase slope} \]
\[ N \propto 1/\gamma \]

\[ \Rightarrow \text{Characterizing phase slope near resonance provides measure of damping} \]
Impulse response

- Intuitively defined in two different (but equivalent) ways:

1. **Time response of ‘system’ when subjected to an impulse**
   (e.g., striking a bell w/ a hammer)

2. **Fourier transform of resulting response**
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(Important) Note: The Fourier transform of the impulse response is called the **transfer function**