PHYS 2010 (W20)
Classical Mechanics

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Review Material (re PHYS 1420/1010)

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Material contained in these slides comes from PHYS 1420 F19 as taught by CB (using the Kesten & Tauck text). This material is provided here primarily for the sake of review, and also contains problems sprinkled throughout (as well as at the end) that are likely worth trying to solve as a sort of "refresher". The salient topics covered here include:

- Notion of a "system"
- Definition of impulse
- Momentum (and associated conservation of such)
- Types of collisions
- Notion of center-of-mass (and associated motion of such)
- Rotational kinematics
- Rotational kinetic energy
- Inertia (and associated notion of inertial frames of reference)
- Moments of inertia
- Parallel-axis theorem
- Conservation of (rotational) energy
- Torque
- Rotational dynamics
- Angular momentum (and associated conservation of such)
Motion of things that are not "point masses"....

Various concepts at play here:

- Projectile motion
- Center of Mass
- Rotation
- Moment of inertia
System?

- Note that many of our “definitions” thus far included the notion of a “system”

This states that the center of mass of a system of particles moves as though all the mass of the system were concentrated at the center of mass and all the external forces were applied at that point.

Suppose that the sum of the external forces acting on a system is zero. Then, from Eq. 9–17,

\[
\frac{d\mathbf{P}}{dt} = 0 \quad \text{or} \quad \mathbf{P} = \text{constant}.
\]

**Conservation of linear momentum:** When the net external force on a system is zero, the total momentum \( \mathbf{P} \) of the system—the vector sum of the individual momenta \( m\mathbf{v} \) of its constituent particles—remains constant.

⇒ But what is a system!?!
“In physics, a physical system is a portion of the physical universe chosen for analysis. Everything outside the system is known as the environment.”

https://en.wikipedia.org/wiki/Physical_system
The first step in the problem-solving strategy asks you to clearly define the system. This is worth emphasizing because many problem-solving errors arise from trying to apply momentum conservation to an inappropriate system. The goal is to choose a system whose momentum will be conserved. Even then, it is the total momentum of the system that is conserved, not the momenta of the individual particles within the system. Whether or not momentum is conserved as a ball falls to earth depends on your choice of the system.

(a) System = ball

External force. Impulse changes the ball’s momentum.

(b) System = ball + earth

Interaction forces within an isolated system. The system’s total momentum is conserved.
Recall we showed how a change in momentum, since requiring a change in velocity, required a force to act for a certain time.

\[ \int_{t_i}^{t_f} \vec{F} \, dt = \vec{p}_f - \vec{p}_i \]

We call this change in momentum an Impulse, \( \vec{J} \).

\[ \vec{J} = \vec{p}_f - \vec{p}_i \]
**Impulse**

- Impulse is important in the sense that to change the velocity of an object (say a cliff jumper) from a very fast speed to zero, requires the same Impulse whether or not he lands on water or a hard surface.

- What is different is the **time** over which the Impulse acts.

- The average force exerted on the object during the impulse is:
  \[ F_{avg} = \frac{\Delta p}{\Delta t} = \frac{J}{\Delta t} \]

This quantity has a broad range of physical and physiological implications.
Impulse

\[ F_{avg} = \frac{\Delta p}{\Delta t} \]

- Therefore, the shorter time of the impulse, the greater the force on the object.
- To minimize the force exerted during a impulse, it is a good idea to increase the time.
- Air bags in cars, padding inside protective equipment, soft foam in your shoes, all serve to increase the time over which an impulse acts, and hence reduce the average force.
Since Impulse = change in momentum = integral of force with time

\[ \vec{J} = \int_{t_i}^{t_f} \vec{F} \, dt \]

And since we know that the area under a curve is like an integral, we can find the Impulse delivered to an object from a Force vs Time graph*.

Note that for a given Impulse, (cliff jumper into water or onto a hard surface) the area under the force vs time graph must be a constant.

*not to be confused with finding the work done from the area under a force vs distance graph
"Force vs Time Graphs"

\[ \Delta p_x = p_{fx} - p_{ix} = \int_{t_i}^{t_f} F_x(t) \, dt \]

- **force vs time graph for jump onto hard surface.**
  - \( F_{max} \) causes harm

- **force vs time graph for jump into water.** \( F_{max} \) can be tolerated

The impulse equals the area under the force vs. time curve.

The impulse of the average force equals the impulse of the actual time-varying force.
Impulse

A large force exerted for a small interval of time is called an impulsive force.

A particle undergoes a collision.

Before:

The particle approaches...

During:

... collides for $\Delta t$...

After:

... and recedes.

The impulsive force is a function of time. It grows to a maximum, then returns to zero.


A tennis ball collides with a racket. Notice that the right side of the ball is flattened.

Newton’s 2nd:

$$ma_x = m\frac{dv_x}{dt} = F_x(t)$$

$$m \, dv_x = F_x(t) \, dt$$

Now integrate over the collision interval:

$$m \int_{v_i}^{v_f} dv_x = m(v_f - v_i) = \int_{t_i}^{t_f} F_x(t) \, dt$$
Impulse

\[ m \int_{v_i}^{v_f} dv_x = mv_{fx} - mv_{ix} = \int_{t_i}^{t_f} F_x(t) \, dt \]

Now we bring in momentum:

\[ \Delta p_x = p_{fx} - p_{ix} = \int_{t_i}^{t_f} F_x(t) \, dt \]

impulse \( = J_x \equiv \int_{t_i}^{t_f} F_x(t) \, dt \)

= area under the \( F_x(t) \) curve between \( t_i \) and \( t_f \)

\[ \Delta p_x = J_x \quad \text{(impulse-momentum theorem)} \]

Looking at the impulse graphically.

(a)

Impulse \( J_x \) is the area under the force curve.

(b)

The area under the rectangle of height \( F_{avg} \) is the same as the area in part (a).
Impulse

\[ \Delta p_x = J_x \quad \text{(impulse-momentum theorem)} \]

The impulse-momentum theorem helps us understand a rubber ball bouncing off a wall.

The wall applies an impulse to the ball.

Before:

\[ v_{ix} > 0 \]

After:

\[ v_{fx} < 0 \]

\[ F_x \]

\[ t \]

\[ J_x = \text{area under curve} \]

Maximum compression

Contact begins

Contact ends

The impulse changes the ball’s momentum.
Collisions

• **Elastic**: objects collide and bounce sharply off one another with no permanent deformation.
  - momentum is conserved
  - mechanical energy is conserved (kinetic+potential)

• **Inelastic**: objects collide and bounce off each other but there is some permanent deformation of the object.
  - momentum is conserved
  - mechanical energy is not conserved (lost to deformation and heat.)

• **Completely Inelastic**: objects collide and stick together, and travel along a common path after the collision.
  - momentum is conserved
  - mechanical energy is not conserved (lost to deformation and heat.)
Inelastic Collisions

Here the two particles "stick together"....

\[ m_1 v_1 = (m_1 + m_2) \hat{V} \]

\[ \hat{V} = \frac{m_1}{m_1 + m_2} v_1 \]

Energy is not conserved

\[ K_i = \frac{1}{2} m_1 v_1^2 \]
\[ K_f = \frac{1}{2} (m_1 + m_2) V^2 \]
\[ K_f = \left( \frac{m_1}{m_1 + m_2} \right) K_i \]

Useful, but be careful.....

Special Case: The Objects Have the Same Mass

Special Case: The Moving Object Is More Massive than the One at Rest

Special Case: The Moving Object Is Less Massive than the One at Rest

\[ V = \frac{m_1}{m_1 + m_2} v_1 \approx \frac{m_1}{m_2} v_1 \approx 0 \quad v_1 \approx 0 \]

e.g., when precisely is this approximation valid?

Kesten & Tauck
Elastic Collisions

For elastic case, consider 2\textsuperscript{nd} particle "at rest"...

Here, energy is conserved

K&T go on to derive expressions for the resulting velocities:

\[ v_{1,f} = v_{1,i} \left( \frac{m_1 - m_2}{m_1 + m_2} \right) \]
\[ v_{2,f} = v_{1,i} \left( \frac{2m_1}{m_1 + m_2} \right) \]

Be careful though (e.g., what precisely is a "trick"?)

After you’ve seen expressions such as Equation 7-27 a few times, you’ll come to recognize that \( v_{1,i}^2 - v_{1,f}^2 \) can be factored into \((v_{1,i} + v_{1,f})(v_{1,i} - v_{1,f})\), a “trick” that is often useful in simplifying equations.
Elastic Collisions

Those "special cases" again.....

Special Case: The Objects Have the Same Mass
Special Case: The Moving Object Is More Massive than the One at Rest
Special Case: The Moving Object Is Less Massive than the One at Rest

Note also that it is energy that is conserved, not just kinetic energy....

$H = \frac{2m_1^2v_1^2}{(m_1 + m_2)^2g}$

$\Rightarrow$ There is an untrue statement here!
STOP TO THINK 9.3  Objects A and C are made of different materials, with different “springiness,” but they have the same mass and are initially at rest. When ball B collides with object A, the ball ends up at rest. When ball B is thrown with the same speed and collides with object C, the ball rebounds to the left. Compare the velocities of A and C after the collisions. Is $v_A$ greater than, equal to, or less than $v_C$?
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Less than. The ball’s momentum $m_Bv_B$ is the same in both cases. Momentum is conserved, so the total momentum is the same after both collisions. The ball that rebounds from C has negative momentum, so C must have a larger momentum than A.
The cart’s change of momentum is

a. $-30 \text{ kg m/s}$
b. $-20 \text{ kg m/s}$
c. $0 \text{ kg m/s}$
d. $10 \text{ kg m/s}$
e. $20 \text{ kg m/s}$
f. $30 \text{ kg m/s}$
STOP TO THINK 9.1

The cart’s change of momentum is

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d. $10 \text{ kg m/s}$
e. $20 \text{ kg m/s}$
f. $30 \text{ kg m/s}$
More complicated motions....

- Several important points are conveyed here:
  - center-of-mass
  - rotational motion
  - angular momentum
Motion of the Center of Mass
The center of mass of a system of \( n \) point masses \( m_1, m_2, \ldots, m_n \) located at positions \( x_1, x_2, \ldots, x_n \) along the \( x \)-axis is given by

\[
\bar{x} = \frac{\sum x_i m_i}{\sum m_i}.
\]

Fig. 9–6 Example 3. Finding the motion of the center of mass of three masses, each subjected to a different force. The forces all lie in the plane defined by the particles. The distances indicated along the axes are in meters.
Finding the center of mass (CM)

Easy case (discrete)

The center of mass of a system of $n$ point masses $m_1, m_2, \ldots, m_n$ located at positions $x_1, x_2, \ldots, x_n$ along the $x$-axis is given by

$$\bar{x} = \frac{\sum x_i m_i}{\sum m_i}.$$ 

The numerator is the sum of the moments of the masses about the origin; the denominator is the total mass of the system.

> Left-hand term is the vector indicating the center of mass relative to your chosen coordinate system

Wolfson notation

$$\vec{r}_{cm} = \frac{\sum m_i \vec{r}_i}{M}$$

Kesten & Tauck notation

$$x_{CM} = \frac{1}{M_{tot}} \sum_{i=1}^{N} m_i x_i$$

**TIP** Choosing the Origin

Choosing the origin at one of the masses here conveniently makes one of the terms in the sum $\sum m_i x_i$ zero. But, as always, the choice of origin is purely for convenience and doesn’t influence the actual physical location of the center of mass. **Exercise 16** demonstrates this point, repeating **Example 9.1** with a different origin.

**TIP** Exploit Symmetries

It’s no accident that $x_{cm}$ here lies on the vertical line that bisects the triangle; after all, the triangle is symmetric about that line, so its mass is distributed evenly on either side. Exploit symmetry whenever you can; that can save you a lot of computation throughout physics!
Example 2. Find the center of mass of the triangular plate of Fig. 9–5.
Be smart! Sometimes a simple graphical approach is all you need...

If a body can be divided into parts such that the center of mass of each part is known, the center of mass of the body can usually be found simply. The triangular plate may be divided into narrow strips parallel to one side. The center of mass of each strip lies on the line which joins the middle of that side to the opposite vertex. But we can divide up the triangle in three different ways, using this process for each of three sides. Hence the center of mass lies at the intersection of the three lines which join the middle of each side with the opposite vertices. This is the only point that is common to the three lines.
A sculptor decides to portray a bird (Fig. 9-13). Luckily the final model is actually able to stand upright. The model is formed of a single sheet of metal of uniform thickness. Of the points shown, which is most likely to be the center of mass?
Review: Finding the center of mass (CM)

Easy case (discrete)

The center of mass of a system of \( n \) point masses \( m_1, m_2, \ldots, m_n \) located at positions \( x_1, x_2, \ldots, x_n \) along the \( x \)-axis is given by

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\bar{x} = \frac{\sum x_i m_i}{\sum m_i}.
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The numerator is the sum of the moments of the masses about the origin; the denominator is the total mass of the system.

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Review: Finding the center of mass (CM)

Harder case (1-D continuous mass distribution)

Continuous Mass Density

Instead of discrete masses arranged along the $x$-axis, suppose we have an object lying on the $x$-axis between $x = a$ and $x = b$. At point $x$, suppose the object has mass density (mass per unit length) of $\delta(x)$. To calculate the center of mass of such an object, divide it into $n$ pieces, each of length $\Delta x$. On each piece, the density is nearly constant, so the mass of the piece is given by density times length. See Figure 8.51. Thus, if $x_i$ is a point in the $i^{th}$ piece,

$$\text{Mass of the } i^{th} \text{ piece, } m_i \approx \delta(x_i) \Delta x.$$  

Then the formula for the center of mass, $\bar{x} = \sum x_i m_i / \sum m_i$, applied to the $n$ pieces of the object gives

$$\bar{x} = \frac{\sum x_i \delta(x_i) \Delta x}{\sum \delta(x_i) \Delta x}.$$  

In the limit as $n \to \infty$ we have the following formula:

The center of mass $\bar{x}$ of an object lying along the $x$-axis between $x = a$ and $x = b$ is

$$\bar{x} = \frac{\int_a^b x \delta(x) \, dx}{\int_a^b \delta(x) \, dx},$$

where $\delta(x)$ is the density (mass per unit length) of the object.

Interdisciplinary connection: Riemann sums and integrals!
Review: Finding the center of mass (CM)

Harder case (1-D continuous mass distribution)

In the limit as \( n \to \infty \) we have the following formula:

The **center of mass** \( \bar{x} \) of an object lying along the \( x \)-axis between \( x = a \) and \( x = b \) is

\[
\bar{x} = \frac{\int_a^b x \delta(x) \, dx}{\int_a^b \delta(x) \, dx},
\]

where \( \delta(x) \) is the density (mass per unit length) of the object.

As in the discrete case, the denominator is the total mass of the object.

Wolfson notation

\[
\vec{r}_{cm} = \lim_{\Delta m_i \to 0} \frac{\sum \Delta m_i \vec{r}_i}{M} = \frac{\int \vec{r} \, dm}{M} \quad (\text{center of mass, continuous matter})
\]
TACTICS 9.1 Setting Up an Integral

An integral like \( \int x \, dm \) can be confusing because you see both \( x \) and \( dm \) after the integral sign and they don’t seem related. But they are, and here’s how to proceed:

1. Find a suitable shape for your mass elements, preferably one that exploits any symmetry in the situation. One dimension of the elements should involve an infinitesimal interval in one of the coordinates \( x, y, \) or \( z \). In Example 9.3, the mass elements were strips, symmetric about the wing’s centerline and with width \( dx \).

2. Find an expression for the infinitesimal area of your mass elements (in a one-dimensional problem it would be the length; in a three-dimensional problem, the volume). In Example 9.3, the infinitesimal area of each mass element was the strip height \( h \) multiplied by the width \( dx \).

3. Form ratios that relate the infinitesimal coordinate interval to the physical quantity in the integral—which in Example 9.3 is the mass element \( dm \). Here we formed the ratio of the area of a mass element to the total area, and equated that to the ratio of \( dm \) to the total mass \( M \).

4. Solve your ratio statement for the infinitesimal quantity, in this case \( dm \), that appears in your integral. Then you’re ready to evaluate the integral.

Sometimes you’ll be given a density—mass per volume, per area, or per length—and then in place of steps 3 and 4 you find \( dm \) by multiplying the density by the infinitesimal volume, area, or length you identified in step 2.

Although we described this procedure in the context of Example 9.3, it also applies to other integrals you’ll encounter in different areas of physics.
Area of a circle via Riemann sums

\[ \text{area of "strip" } (\equiv A_s) \approx 2 \cdot y \cdot \Delta x = 2 \sqrt{1-x^2} \Delta x \]
(since \( x^2 + y^2 = 1 \))

\[ \Rightarrow \text{area of circle } (A_c) \approx \sum_{x=-1}^{x=1} A_s = \sum_{x=-1}^{x=1} 2 \sqrt{1-x^2} \Delta x \]

\[ \text{taking the limit where our strip get infinitesimally small:} \]
\[ A_c = \lim_{\Delta x \to 0} \frac{1}{2} \sum_{x=-1}^{x=1} 2 \sqrt{1-x^2} \Delta x = \int_{-1}^{1} 2 \sqrt{1-x^2} \, dx \]

\[ \text{Using a trusty "table of integrals", we find:} \]
\[ \int \sqrt{a^2-x^2} \, dx = \frac{x \sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1} \left( \frac{x}{a} \right) \]

\[ \text{Back to our problem:} \]
\[ A_c = \int_{-1}^{1} 2 \sqrt{1-x^2} \, dx = 2 \left[ \frac{x \sqrt{1-x^2}}{2} + \frac{1}{2} \sin^{-1} x \right]_{-1}^{1} \]
\[ = \left( \sqrt{1-1} + \sin^{-1}(1) \right) - \left( -\sqrt{1-1} - \sin^{-1}(-1) \right) = \sin^{-1}(1) - \sin^{-1}(-1) \]
\[ = \frac{\pi}{2} - (-\frac{\pi}{2}) = \pi = \pi \cdot 1^2 \]

\[ \text{Note: these are the "inverse-trig functions" (if } x = \sin y, \text{ then } y = \sin^{-1} x, \text{ i.e. the angle whose sine is } x; \sin^{-1}(1) = \frac{\pi}{2}, \text{ and } \sin^{-1}(-1) = -\sin^{-1} x) \]
Finding the center of mass

Harder-er case (2ff-D continuous mass distribution)

For a system of masses that lies in the plane, the center of mass is a point with coordinates \((\bar{x}, \bar{y})\). In three dimensions, the center of mass is a point with coordinates \((\bar{x}, \bar{y}, \bar{z})\). To compute the center of mass in three dimensions, we use the following formulas in which \(A_x(x)\) is the area of a slice perpendicular to the \(x\)-axis at \(x\), and \(A_y(y)\) and \(A_z(z)\) are defined similarly. In two dimensions, we use the same formulas for \(\bar{x}\) and \(\bar{y}\), but we interpret \(A_x(x)\) and \(A_y(y)\) as the lengths of strips perpendicular to the \(x\)- and \(y\)-axes, respectively.

\[
\bar{x} = \frac{\int x\delta A_x(x) \, dx}{\text{Mass}} \quad \bar{y} = \frac{\int y\delta A_y(y) \, dy}{\text{Mass}} \quad \bar{z} = \frac{\int z\delta A_z(z) \, dz}{\text{Mass}}.
\]

Note: If the density is not constant, finding the CM may require double/triple integrals and multivariable calculus (i.e., beyond the scope of 1st year PHYS 1420!)
Ex.

Find the coordinates of the center of mass of the isosceles triangle in Figure 8.52. The triangle has constant density and mass $m$.

For a region of constant density $\delta$, the center of mass is given by

$$
\bar{x} = \frac{\int x \delta A_x(x) \, dx}{\text{Mass}} \quad \bar{y} = \frac{\int y \delta A_y(y) \, dy}{\text{Mass}} \quad \bar{z} = \frac{\int z \delta A_z(z) \, dz}{\text{Mass}}.
$$

- Need to determine the density $\delta \text{ [kg/m}^2\text{]}$
- Be careful w/ the units (e.g., $[A_x] = m$, meaning it is a length and not an area!)
For a region of constant density $\delta$, the center of mass is given by

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4. Solve your ratio statement for the infinitesimal quantity, in this case $dm$, that appears in your integral. Then you’re ready to evaluate the integral.

Sometimes you’ll be given a density—mass per volume, per area, or per length—and then in place of steps 3 and 4 you find $dm$ by multiplying the density by the infinitesimal volume, area, or length you identified in step 2.

Although we described this procedure in the context of Example 9.3, it also applies to other integrals you’ll encounter in different areas of physics.
Because the mass of the triangle is symmetrically distributed with respect to the $x$-axis, $\bar{y} = 0$. We expect $\bar{x}$ to be closer to $x = 0$ than to $x = 1$, since the triangle is wider near the origin.

The area of the triangle is $\frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$. Thus Density = Mass/Area = $2m$. If we slice the triangle into strips of width $\Delta x$, then the strip at position $x$ has length $A_x(x) = 2 \cdot \frac{1}{2} (1 - x) = (1 - x)$. (See Figure 8.53.) So

$$\text{Area of strip} = A_x(x) \Delta x \approx (1 - x) \Delta x.$$  

Since the density is $2m$, the center of mass is given by

$$\bar{x} = \frac{\int x \delta A_x(x) \, dx}{\text{Mass}} = \frac{\int_0^1 2mx(1 - x) \, dx}{m} = 2 \left( \frac{x^2}{2} - \frac{x^3}{3} \right) \bigg|_0^1 = \frac{1}{3}.$$  

So the center of mass of this triangle is at the point $(\bar{x}, \bar{y}) = (1/3, 0)$.
Ex.

Find the volume of the sphere of radius $r$ centered at the origin.
The cross section at $x$ is perpendicular to the $x$-axis (see Figure 6.39). It is a disk of radius $y = \sqrt{r^2 - x^2}$ whose area is

$$A(x) = \pi y^2 = \pi (r^2 - x^2)$$

Since the solid is between $-r$ and $r$, we find

$$\text{Volume} = \int_{-r}^{r} \pi (r^2 - x^2) \, dx$$

The integrand is continuous on $[-r, r]$. Evaluating the integral yields

$$= \pi \left[ r^2x - \frac{1}{3}x^3 \right]_{-r}^{r}$$

$$= \pi \left[ (r^3 - \frac{1}{3}r^3) - (-r^3 + \frac{1}{3}r^3) \right] = \pi \left( \frac{2}{3}r^3 + \frac{2}{3}r^3 \right) = \frac{4}{3}\pi r^3$$
Ex.

Find the center of mass of a hemisphere of radius 7 cm and constant density \( \delta \).

Kesten & Tauck ch.7 problem

98. \( \bullet \bullet \bullet \text{Calc} \) Determine the center of mass of a solid hemisphere of mass \( M \) and radius \( R \), relative to the center of the base of the hemisphere.
Ex. (SOL)

Stand the hemisphere with its base horizontal in the $xy$-plane, with the center at the origin. Symmetry tells us that its center of mass lies directly above the center of the base, so $\bar{x} = \bar{y} = 0$. Since the hemisphere is wider near its base, we expect the center of mass to be nearer to the base than the top.

To calculate the center of mass, slice the hemisphere into horizontal disks as in Figure 8.9 on page 375. A disk of thickness $\Delta z$ at height $z$ above the base has

$$\text{Volume of disk} = A_z(z)\Delta z \approx \pi(7^2 - z^2)\Delta z \text{ cm}^3.$$ 

So, since the density is $\delta$,

$$\bar{z} = \frac{\int z\delta A_z(z)\,dz}{\text{Mass}} = \frac{\int_0^7 z\delta\pi(7^2 - z^2)\,dz}{\text{Mass}}.$$ 

Since the total mass of the hemisphere is $(\frac{2}{3}\pi7^3)\delta$, we get

$$\bar{z} = \frac{\delta\pi \int_0^7 (7^2 z - z^3)\,dz}{\text{Mass}} = \frac{\delta\pi (7^2z^2/2 - z^4/4)\big|_0^7}{\text{Mass}} = \frac{\frac{7^4}{4}\delta\pi}{\frac{2}{3}\pi7^3\delta} = \frac{21}{8} = 2.625 \text{ cm}.$$ 

The center of mass of the hemisphere is 2.625 cm above the center of its base. As expected, it is closer to the base of the hemisphere than its top.

Figure 8.54: Slicing to find the center of mass of a hemisphere
Stepping back a moment...

The weight of Newton’s contribution should now be a bit more apparent...

\[ F_{12} = -G \frac{m_1 m_2}{|\mathbf{r}_{12}|^2} \hat{\mathbf{r}}_{12} \]

where

- \( F_{12} \) is the force applied on object 2 due to object 1,
- \( G \) is the **gravitational constant**,
- \( m_1 \) and \( m_2 \) are respectively the masses of objects 1 and 2,
- \(|\mathbf{r}_{12}| = |\mathbf{r}_2 - \mathbf{r}_1|\) is the distance between objects 1 and 2, and
- \( \hat{\mathbf{r}}_{12} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|} \) is the **unit vector** from object 1 to 2.


---

**Fig. 16–6** Gravitational attraction of a section \( dS \) of a spherical shell of matter on a particle of mass \( m \).
Area of a circle via Riemann sums

\[ \text{area of strip } \Delta A = 2 \cdot y \cdot \Delta x = 2 \sqrt{1-x^2} \Delta x \]
(since \( x^2 + y^2 = 1 \))

\[ \sum_{x=0}^{x=1} A_x = \sum_{x=0}^{x=1} 2 \sqrt{1-x^2} \Delta x \]

Taking the limit where our strip get indefinitely small:

\[ A_c = \lim_{\Delta x \to 0} \sum_{x=0}^{x=1} 2 \sqrt{1-x^2} \Delta x = \int_0^1 2 \sqrt{1-x^2} \, dx \]

Using a trusty "table of integrals", we find:

\[ \int \sqrt{a^2-x^2} \, dx = \frac{x \sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1} \left( \frac{x}{a} \right) \]

Back to our problem:

\[ A_c = \int_0^1 2 \sqrt{1-x^2} \, dx = 2 \left[ \frac{x \sqrt{1-x^2}}{2} + \frac{1}{2} \sin^{-1} x \right]_0^1 \]

\[ = \sqrt{1-1} + \sin^{-1}(1) - \left( \frac{1}{2} \right) = \frac{\pi}{2} - \frac{1}{2} \]

Because the mass of the triangle is symmetrically distributed with respect to the x-axis, \( \bar{y} = 0 \). We expect \( \bar{x} \) to be closer to \( x = 0 \) than to \( x = 1 \), since the triangle is wider near the origin.

The area of the triangle is \( \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2} \). Thus Density = Mass/Area = 2m. If we slice the triangle into strips of width \( \Delta x \), then the strip at position \( x \) has length \( A_x(x) = 2 \cdot \frac{1}{2} (1-x) = (1-x) \). (See Figure 8.53.) So

\[ \text{Area of strip } = A_x(x) \Delta x \approx (1-x) \Delta x. \]

Since the density is \( 2m \), the center of mass is given by

\[ \bar{x} = \frac{\int x \delta A_x(x) \, dx}{\text{Mass}} = \frac{\int_0^1 2mx(1-x) \, dx}{m} = 2 \left( \frac{x^2}{2} - \frac{x^3}{3} \right) \bigg|_0^1 = \frac{1}{3}. \]

So the center of mass of this triangle is at the point \( (\bar{x}, \bar{y}) = (1/3, 0) \).
Motion of the Center of Mass

Note: If the net force on a system is zero, then the CM does not move, leading to a redistribution of the "particles" inside so to maintain conservation of (linear) momentum.
EXAMPLE 9.4 CM Motion: Circus Train

Jumbo, a 4.8-t elephant, stands near one end of a 15-t railcar at rest on a frictionless horizontal track. (Here t is for tonne, or metric ton, equal to 1000 kg.) Jumbo walks 19 m toward the other end of the car. How far does the car move?

INTERPRET We’re asked about the car’s motion, but we can interpret this problem as being fundamentally about the center of mass. We identify the relevant system as comprising Jumbo and the car. Because there’s no net external force acting on the system, its center of mass can’t move.

DEVELOP Figure 9.8a shows the initial situation. The symmetric car has its CM at its center (here we care only about the x-component). Let’s take a coordinate system that’s fixed to the ground and that has \( x = 0 \) at this initial location of the car’s center. After the car moves, its center will be somewhere else! Equation 9.2 applies—here in the simpler one-dimensional, two-object form we used in Example 9.1:

\[
x_{cm} = \frac{(m J x_J + m_c x_c)}{M},
\]

where we use the subscripts J and c for Jumbo and the car, respectively, and where \( M = m J + m_c \) is the total mass. We have a before/after situation in which the CM position can’t change, so we’ll write two versions of this expression, before and after Jumbo’s walk. We’ll then set them equal to state mathematically that the CM itself doesn’t move; that is, we’ll write \( x_{cm, i} = x_{cm, f} \) where the subscripts i and f designate quantities associated with the initial and final states, respectively.

![Diagram of Jumbo and railcar](image)

We chose our coordinate system so that the car’s initial position was \( x_{ci} = 0 \), so our expression for the initial position of the system’s center of mass becomes

\[
x_{cm, i} = \frac{m J x_J}{M}
\]

Our expression for the final center-of-mass position, after Jumbo’s walk, is \( x_{cm, f} = \frac{(m J x_J + m_c x_c)}{M} \). We don’t know either of the final coordinates \( x_{Jf} \) or \( x_{cf} \) here, but we do know that Jumbo walks 19 m with respect to the car. The elephant’s final position \( x_{Jf} \) is therefore 19 m to the right of \( x_{cm, f} \), adjusted for the car’s displacement. Therefore Jumbo ends up at \( x_{cf} = x_{Ji} + 19 m + x_{cf} \). You might think we need a minus sign because the car moves to the left. That’s true, but the sign of \( x_{cf} \) will take care of that. Trust algebra! So our expression for the final center-of-mass position is

\[
x_{cm, f} = \frac{m J x_{Ji} + m_c x_{cf}}{M} = \frac{m J (x_{Ji} + 19 m + x_{cf}) + m_c x_{cf}}{M}
\]

EVALUATE Finally, we equate our expressions for the initial and final positions of the center of mass. Again, that’s because there are no forces external to the elephant–car system acting in the horizontal direction, so the center-of-mass position can’t change. Thus we have \( x_{cm, i} = x_{cm, f} \), or

\[
\frac{m J x_{Ji}}{M} = \frac{m J (x_{Ji} + 19 m + x_{cf}) + m_c x_{cf}}{M}
\]

The total mass \( M \) cancels, so we’re left with the equation \( m J x_{Ji} = m J (x_{Ji} + 19 m + x_{cf}) + m_c x_{cf} \). We aren’t given \( x_{Ji} \), but the term \( m J x_{Ji} \) is on both sides of this equation, so it cancels, leaving \( 0 = m (19 m + x_{cf}) + m_c x_{cf} \). We solve for the unknown \( x_{cf} \) to get

\[
x_{cf} = \frac{(19 m) m J}{(m J + m_c)} = \frac{(19 m)(4.8 t)}{(4.8 t + 15 t)} = -4.6 m
\]

The minus sign here indicates a displacement to the left, as we anticipated (Fig. 9.8b). Because the masses appear only in ratios, we didn’t need to convert to kilograms.

ASSESS The car’s 4.6-m displacement is quite a bit less than Jumbo’s (which is 19 m – 4.6 m, or 14.4 m relative to the ground). That makes sense because Jumbo is considerably less massive than the car.

Note that there are a few salient steps here:

- Determine the CM
- Realize the CM does not change
- Figure out how Jumbo’s position changes relative to the CM and the railcar
Motion of the Center of Mass

Be careful: What K&T state here, as it is incorrect. The boat doesn't *approximately* move, but move it does (as it must given the stated conservation law!)

(a) Massive boat

The center of mass of the system is close to the center of mass of the boat, because the boat is so massive relative to the boy and the ball.

Compare the center of mass position to the fixed position of the tree. The center of mass doesn’t move, even after the ball is thrown.

Kesten & Tauck
A dog, weighing 10.0 lb is standing on a flatboat so that he is 20 ft from the shore. He walks 8.0 ft on the boat toward shore and then halts. The boat weighs 40 lb, and one can assume there is no friction between it and the water. How far is he from the shore at the end of this time? (*Hint:* The center of mass of boat + dog does not move. Why?) The shoreline is also to the left in Fig. 9-15.

**Bonus:** What breed of dog is Snoopy?
Motion of the Center of Mass

- From our definition of center of mass:

\[ M \mathbf{r}_{cm} = m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 + \cdots + m_n \mathbf{r}_n, \]

- Differentiating w/ respect to \( t \):
  (assuming mass stays const.)

\[ M \frac{d \mathbf{r}_{cm}}{dt} = m_1 \frac{d \mathbf{r}_1}{dt} + m_2 \frac{d \mathbf{r}_2}{dt} + \cdots + m_n \frac{d \mathbf{r}_n}{dt} \]

\[ M \mathbf{v}_{cm} = m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 + \cdots + m_n \mathbf{v}_n, \]

- Differentiating again w/ respect to \( t \):

\[ M \frac{d \mathbf{v}_{cm}}{dt} = m_1 \frac{d \mathbf{v}_1}{dt} + m_2 \frac{d \mathbf{v}_2}{dt} + \cdots + m_n \frac{d \mathbf{v}_n}{dt} \]

\[ = m_1 \mathbf{a}_1 + m_2 \mathbf{a}_2 + \cdots + m_n \mathbf{a}_n, \]

\[ M \mathbf{a}_{cm} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n. \]

Hence the total mass of the group of particles times the acceleration of its center of mass is equal to the vector sum of all the forces acting on the group of particles.
Included amongst these forces are *internal* ones, in that from Newton’s 3rd Law, they will occur in (equal but opposite) pairs and thereby cancel.

\[ M \mathbf{a}_{cm} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n. \]

\[ M \mathbf{a}_{cm} = \mathbf{F}_{ext}. \]

⇒ So only *external* forces effectively contribute.

This states that the center of mass of a system of particles moves as though all the mass of the system were concentrated at the center of mass and all the external forces were applied at that point.
Then, in 1969, David Egger and Paul Davidovits of Yale University combined confocal microscopy with lasers to increase the resolution still further. Now, laser confocal microscopy, combined with computers, can produce exquisite three-dimensional images of complex biological material, including the highly amorphous structure of dendritic cells.
From the midterm exam:

3. (35 points) Consider the scanning part of a the drive motor and spinning disk of a confocal microscope. Suppose the disk has radius $R$ and is initially at rest. It then speeds up with angular acceleration $\alpha$.

a. Determine an expression for the tangential velocity after the disk has rotated through an angle $\Delta \phi$.

b. Similarly, determine an expression (in terms of $\Delta \phi$ and any other relevant quantities) for the centripetal acceleration.
3. (35 points) Consider the scanning part of a drive motor and spinning disk of a confocal microscope. Suppose the disk has radius $R$ and is initially at rest. It then speeds up with angular acceleration $\alpha$.

a. Determine an expression for the tangential velocity after the disk has rotated through an angle $\Delta\phi$.

- We know: disk radius ($R$), const. angular accel. ($\alpha$) and initially at rest ($\omega_0 = 0$)

- Angular velocity: $\omega = \int \alpha \, dt = \alpha t + \omega_0 = \alpha t$

- Angular position: $\theta = \int \omega \, dt = \int \alpha t \, dt = \frac{1}{2} \alpha t^2 + \theta_0 = \Delta\phi$

- So $\Delta \phi = \frac{1}{2} \alpha t^2 \implies t = \sqrt{\frac{2 \Delta \phi}{\alpha}}$

- Now $\omega = \alpha t = \alpha \sqrt{\frac{2 \Delta \phi}{\alpha}} = \sqrt{2 \alpha \Delta \phi}$

- $\omega = \frac{V}{R} \implies V = R \sqrt{2 \alpha \Delta \phi}$

b. Similarly, determine an expression (in terms of $\Delta \phi$ and any other relevant quantities) for the centripetal acceleration.

\[ a = \frac{V^2}{R} = 2R \alpha \Delta \phi \]
Review (re Circular Motion)

- 1-D kinematics translates directly to circular motion (in polar coords.)

**TABLE 4.1** Rotational and linear kinematics for constant acceleration

<table>
<thead>
<tr>
<th>Rotational kinematics</th>
<th>Linear kinematics</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_f = \omega_i + \alpha \Delta t$</td>
<td>$v_{fs} = v_{is} + a_s \Delta t$</td>
</tr>
<tr>
<td>$\theta_f = \theta_i + \omega_i \Delta t + \frac{1}{2} \alpha (\Delta t)^2$</td>
<td>$s_f = s_i + v_{is} \Delta t + \frac{1}{2} a_s (\Delta t)^2$</td>
</tr>
<tr>
<td>$\omega_f^2 = \omega_i^2 + 2\alpha \Delta \theta$</td>
<td>$v_{fs}^2 = v_{is}^2 + 2a_s \Delta s$</td>
</tr>
</tbody>
</table>

Knight (2013)

→ However, things tend to otherwise get a bit more *complicated* as we will see...
Rotational Kinetic Energy

• Imagine only a small section of the wheel. It has some mass, and it is moving with some velocity. That piece has some kinetic energy.
• Since kinetic energy is a scalar, the KE of all such pieces should add.
• Something spinning may not have the location of its centre of mass moving. So it would not have any translational kinetic energy.

• But it would have some motional energy, since it takes work to spin an object!
Rotational Kinetic Energy

Consider an element of a rod (with mass $m_i$) undergoing circular motion

A small element of the blade a distance $r_i$ from the rotation point and having mass $m_i$ travels a distance $r_i \theta$ in a time $t$.

The velocity of this element of the blade is $v_i = \frac{\text{change in position}}{\text{change in time}} = \frac{r_i \theta}{t}$

The element of the blade has kinetic energy $K_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i \left( \frac{r_i \theta}{t} \right)^2$
Rotational Kinetic Energy

The blade rotates through this angle $\theta$ in a time $t$.

A small element of the blade a distance $r_i$ from the rotation point and having mass $m_i$ travels a distance $r_i \theta$ in a time $t$.

The velocity of this element of the blade is $v_i = \frac{\text{change in position}}{\text{change in time}} = \frac{r_i \theta}{t}$

The element of the blade has kinetic energy $K_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i \left( \frac{r_i \theta}{t} \right)^2$

Angular velocity

$\omega = \frac{\Delta \theta}{\Delta t}$

$\omega = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} = \frac{d \theta}{dt}$

$K_i = \frac{1}{2} m_i r_i^2 \omega^2$

Velocity of mass element $m_i$

Note: All points on the bar have the same $\omega$

What about the bar as a whole?

$K = \sum K_i = \sum \frac{1}{2} m_i r_i^2 \omega^2$

$K = \frac{1}{2} \left( \sum m_i r_i^2 \right) \omega^2$
Rotational Kinetic Energy

The element of the blade has kinetic energy $K_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i \left( \frac{r_i \theta}{t} \right)^2$

The velocity of this element of the blade is $v_i = \frac{\text{change in position}}{\text{change in time}} = \frac{r_i \theta}{t}$

Whereas we interpreted mass as a property of matter that represents the resistance of an object to a change in translational velocity, the moment of inertia represents the resistance of an object to a change in rotational or angular velocity. In the same way that we defined inertia as the tendency of an object to resist a change in translational motion, we can define rotational inertia as the tendency of an object to resist a change in rotational motion.

Moment of Inertia

\[ K = \frac{1}{2} \left( \sum m_i r_i^2 \right) \omega^2 \]

\[ I = \sum m_i r_i^2 \]

\[ K_{\text{rotational}} = \frac{1}{2} I \omega^2 \]
Rotational vs "Linear" Kinetic Energy

These two wheels have the same mass and size, but the first is a ring with no mass at the center while the other is a uniform disk.

Sliding at the same linear velocity, both have the same linear kinetic energy.

When both rotate at the same angular velocity, the ring has greater rotational kinetic energy. More of its mass is farther from the rotation axis, resulting in a larger moment of inertia around that axis.
Rotational Kinetic Energy

The kinetic energy of a rod rotating around an axis perpendicular to the rod and through its end is four times larger...

...than the kinetic energy of the rod when it rotates at the same angular velocity around an axis perpendicular to the rod and through its center.

The more mass there is farther from the rotation axis, the larger the moment of inertia, and the larger the rotational kinetic energy for any given angular velocity.
**Inertia**

**Newton’s first law of motion:** A body in uniform motion remains in uniform motion, and a body at rest remains at rest, unless acted on by a nonzero net force.

Inertia is the resistance of any physical object to any change in its velocity. This includes changes to the object’s speed, or direction of motion. An aspect of this property is the tendency of objects to keep moving in a straight line at a constant speed, when no forces act upon them.

Inertia comes from the Latin word, *inertis*, meaning idle, sluggish. Inertia is one of the primary manifestations of mass, which is a quantitative property of physical systems. Isaac Newton defined inertia as his first law in his *Philosophiae Naturalis Principia Mathematica*, which states:

The *vis insita*, or innate force of matter, is a power of resisting by which every body, as much as in it lies, endeavours to preserve its present state, whether it be of rest or of moving uniformly forward in a straight line.\(^1\)

Wikipedia (re inertia)
Reminder: Inertial Reference Frames

- Similar to walking along on a (moving) train

- Frame of reference is inertial if it is in uniform motion (which also includes being at rest). That is, the frame itself is not accelerating

- Strictly speaking, “Earth” is not an inertial frame (due to the rotation of earth)

→ Newton’s laws only apply in inertial reference frames
• The moment of inertia, $I$, tells us how difficult it is to change the angular speed of a rotating (or rotatable) object.

\[ K_{\text{rotational}} = \frac{1}{2}I\omega^2 \]

• This is the same idea as mass quantifying how difficult it is to change the linear speed of an object.

• Moment of inertia depends on the particular sum of $m_i r_i^2$ terms, which depends on the shape of the object, and about which axis the object is rotating.

• In principle, one requires a volume integral to find this sum.

• In practice, for some regularly shaped objects rotating about certain axes, these sums are well known.
Moment of Inertia: Multiple masses

\[ I = \sum m_i r_i^2 \]

Note the similarity here w/ respect to CM calculations...

\[ x_{CM} = \frac{1}{M_{tot}} \sum_{i=1}^{N} m_i x_i \]

... a key difference being that here things are moving (or more specifically, rotating)
Moment of Inertia: Uniform bar rotating about one end

Now in integral form, but...

\[ I = \int r^2 \, dm \]

... how are \( r \) and \( m \) functionally-related?

Uniform density! (equal proportions) \[ \frac{\Delta m}{\Delta r} = \frac{M}{L} \]

Now in the limit:

\[ dm = \frac{M}{L} \, dr \]

\[ I = \frac{M \, r^3}{3} \bigg|_0^L = \frac{M}{L} \left( \frac{L^3}{3} - \frac{0^3}{3} \right) = \frac{ML^2}{3} \]
Moment of Inertia: Uniform bar rotating about one end

\[ I = \frac{1}{3}ML^2 \]

\[ I = \frac{M r^3}{3L} \bigg|_0^L = \frac{M}{L} \left( \frac{L^3}{3} - \frac{0^3}{3} \right) = \frac{ML^2}{3} \]
Moment of Inertia: Uniform bar rotating about its center

An object does not have “a” moment of inertia. Rather, it has a moment of inertia defined for rotation around each specific choice of rotation axis.

Recall: Note the similarity here w/ respect to CM calculations...

... a key difference being that here things are moving (or more specifically, rotating)

Just change the limits of integration!

Question: What is a thin rod and why is that needed?
The kinetic energy of a rod rotating around an axis perpendicular to the rod and through its end is four times larger...

...than the kinetic energy of the rod when it rotates at the same angular velocity around an axis perpendicular to the rod and through its center.

The more mass there is farther from the rotation axis, the larger the moment of inertia, and the larger the rotational kinetic energy for any given angular velocity.
Moments of Inertia of Uniform Bodies of Various Shapes

- Thin cylindrical shell about axis
  \[ I = MR^2 \]
  - Solid cylinder about axis
    \[ I = \frac{1}{2} MR^2 + \frac{1}{12} ML^2 \]
  - Thin rod about perpendicular line through center
    \[ I = \frac{1}{12} ML^2 \]
  - Thin spherical shell about diameter
    \[ I = \frac{1}{2} MR^2 \]

- Hollow cylindrical shell about axis
  \[ I = \frac{1}{2} MR^2 \]
- Hollow cylindrical shell about diameter through center
  \[ I = \frac{1}{4} MR^2 + \frac{1}{12} ML^2 \]

- Solid rectangular parallelepiped about axis through center perpendicular to face
  \[ I = \frac{1}{12} M(a^2 + b^2) \]

* A disk is a cylinder whose length \( L \) is negligible. By setting \( L = 0 \), the above formulas for cylinders hold for disks.
Moments of Inertia

- Solid cylinder or disc, symmetry axis: \( I = \frac{1}{2} MR^2 \)
- Hoop about symmetry axis: \( I = MR^2 \)
- Solid sphere: \( I = \frac{2}{5} MR^2 \)
- Rod about center: \( I = \frac{1}{12} ML^2 \)

- Solid cylinder, central diameter: \( I = \frac{1}{4} MR^2 + \frac{1}{12} ML^2 \)
- Hoop about diameter: \( I = \frac{1}{2} MR^2 \)
- Thin spherical shell: \( I = \frac{2}{3} MR^2 \)
- Rod about end: \( I = \frac{1}{3} ML^2 \)
Aside: Some of those other shapes....

Thin hoop rotating about center

Hoop about symmetry axis

$I = MR^2$

→ When does a "thin hoop" become a "thin cylindrical shell"?

Kesten & Tauck
Aside: Some of those other shapes....

Thin hoop rotating about center

Similar argument before re
density & proportionality

\[
\frac{\text{angle of slice}}{\text{angle of ring}} = \frac{\text{mass of slice}}{\text{mass of ring}}
\]
Reminder (re proportionality)

**Mathematical Aside: Proportionality and Proportional Reasoning**

The concept of **proportionality** arises frequently in physics. A quantity symbolized by \( u \) is proportional to another quantity symbolized by \( v \) if

\[
  u = cv
\]

where \( c \) (which might have units) is called the **proportionality constant**. This relationship between \( u \) and \( v \) is often written

\[
  u \propto v
\]

where the symbol \( \propto \) means “is proportional to.”

If \( v \) is doubled to \( 2v \), then \( u \) is doubled to \( c(2v) = 2(cv) = 2u \).

In general, if \( v \) is changed by any factor \( f \), then \( u \) changes by the same factor. This is the essence of what we mean by proportionality.

A graph of \( u \) versus \( v \) is a straight line **passing through the origin** (i.e., the \( y \)-intercept is zero) with slope \( c \). Notice that proportionality is a much more specific relationship between \( u \) and \( v \) than mere linearity. The linear equation \( u = cv + b \) has a straight-line graph, but it doesn’t pass through the origin (unless \( b \) happens to be zero) and doubling \( v \) does not double \( u \).

If \( u \propto v \), then \( u_1 = cv_1 \) and \( u_2 = cv_2 \). Dividing the second equation by the first, we find

\[
  \frac{u_2}{u_1} = \frac{v_2}{v_1}
\]

By working with **ratios**, we can deduce information about \( u \) without needing to know the value of \( c \). (This would not be true if the relationship were merely linear.) This is called **proportional reasoning**.

Proportionality is not limited to being linearly proportional. The graph on the left below shows that \( u \) is clearly not proportional to \( w \).

But a graph of \( u \) versus \( 1/w^2 \) is a straight line passing through the origin, thus, in this case, \( u \) is proportional to \( 1/w^2 \), or \( u \propto 1/w^2 \). We would say that “\( u \) is proportional to the inverse square of \( w \).”

\[ u \propto \frac{1}{w^2} \]

**Example** \( u \) is proportional to the inverse square of \( w \). By what factor does \( u \) change if \( w \) is tripled?

**Solution** This is an opportunity for proportional reasoning; we don’t need to know the proportionality constant. If \( u \) is proportional to \( 1/w^2 \), then

\[
  \frac{u_2}{u_1} = \frac{1/w_2^2}{1/w_1^2} = \frac{w_1^2}{w_2^2} = \left( \frac{w_1}{w_2} \right)^2
\]

Tripling \( w \), with \( w_2/w_1 = 3 \), and thus \( w_1/w_2 = \frac{1}{3} \), changes \( u \) to

\[
  u_2 = \left( \frac{w_1}{w_2} \right)^2 u_1 = \left( \frac{1}{3} \right)^2 u_1 = \frac{1}{9} u_1
\]

Tripling \( w \) causes \( u \) to become \( \frac{1}{9} \) of its original value.

Many **Student Workbook** and end-of-chapter homework questions will require proportional reasoning. It’s an important skill to learn.

→ Because the density is uniform (i.e., constant), **proportionality** reasoning applies here!
Aside: Some of those other shapes....

Thin hoop rotating about center

\[ I = \int r^2 \, dm \]

\[ \frac{d\theta}{2\pi} = \frac{dm}{M} \]

\[ dm = \frac{M}{2\pi} \, d\theta \]

\[ I = \int r^2 \frac{M}{2\pi} \, d\theta = \frac{M}{2\pi} \int r^2 \, d\theta \]

\[ I = \frac{MR^2}{2\pi} \int_{0}^{2\pi} d\theta \]

\[ I = \frac{MR^2}{2\pi} \theta \bigg|_{0}^{2\pi} = \frac{MR^2}{2\pi} (2\pi - 0) = MR^2 \]

Similar argument before re density & proportionality

\[ \frac{\text{angle of slice}}{\text{angle of ring}} = \frac{\text{mass of slice}}{\text{mass of ring}} \]
Aside: Some of those other shapes....

Thin hoop rotating about center

\[ I = MR^2 \]

\[ \rightarrow \text{Remarkably} \; L \text{ does not factor in here!} \]
Aside: Some of those other shapes....

Thin disk rotating about center

Similar argument before re density & proportionality

Double integrals! (beyond our scope)
Moments of Inertia of Uniform Bodies of Various Shapes*

- Thin cylindrical shell about axis: $I = MR^2$
- Thin cylindrical shell about diameter through center: $I = \frac{1}{2}MR^2 + \frac{1}{12}ML^2$
- Thin rod about perpendicular line through center: $I = \frac{1}{12}ML^2$
- Thin spherical shell about diameter: $I = \frac{2}{3}MR^2$

- Solid cylinder about axis: $I = \frac{1}{2}MR^2$
- Solid cylinder about diameter through center: $I = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$
- Thin rod about perpendicular line through one end: $I = \frac{1}{3}ML^2$
- Solid sphere about diameter: $I = \frac{2}{3}MR^2$

- Hollow cylinder about axis: $I = \frac{1}{4}MR^2$
- Hollow cylinder about diameter through center: $I = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$

- Solid rectangular parallelepiped about axis through center perpendicular to face: $I = \frac{1}{12}M(a^2 + b^2)$

* A disk is a cylinder whose length $L$ is negligible. By setting $L = 0$, the above formulas for cylinders hold for disks.
• Tells you how the moment of inertia would change for one of those regular objects it we chose to rotate the object about some other axis.

\[ I = I_{CM} + Mh^2 \]

where \( I_{CM} \) is the moment of inertia through the centre of mass as indicated in some table. 

\( M \) is the mass of the object. 

\( h \) is the distance of from the centre of mass to the new axis of rotation.
Parallel-Axis Theorem

\[ I = I_{CM} + Mh^2 \]
Parallel-Axis Theorem: (quasi-Proof)

The kinetic energy of a rod rotating around an axis perpendicular to the rod and through its end is four times larger...

...than the kinetic energy of the rod when it rotates at the same angular velocity around an axis perpendicular to the rod and through its center.

The more mass there is farther from the rotation axis, the larger the moment of inertia, and the larger the rotational kinetic energy for any given angular velocity.

\[ I = I_{CM} + Mh^2 \]

\[ I = \frac{ML^2}{12} + M\left(\frac{L}{2}\right)^2 \]

\[ = \frac{ML^2}{12} + \frac{ML^2}{4} \]

\[ = \frac{ML^2}{12} + \frac{3ML^2}{12} \]

\[ = \frac{4ML^2}{12} \]

\[ = \frac{ML^2}{3} \]
Conservation of Energy (Revisited)

Both disk and hoop have radius $R$ and mass $M$.

$$K_i + U_i = K_f + U_f + |W_{nc}|$$

Now we factor rotation into the kinetic energy consideration:

$$K_{\text{translational, i}} + K_{\text{rotational, i}} + U_i = K_{\text{translational, f}} + K_{\text{rotational, f}} + U_f + |W_{nc}|$$
Conservation of Energy (Revisited)

- Ignore losses
- Consider one object at a time
- Assume it is initially at rest

\[ K_{\text{translational}, i} + K_{\text{rotational}, i} + U_i = K_{\text{translational}, f} + K_{\text{rotational}, f} + U_f \]

\[ Mgh_i = \frac{1}{2} Mv_f^2 + \frac{1}{2} I\omega_f^2 + Mgh_f \]

\[ \frac{1}{2} Mv_f^2 = Mgh_i - Mgh_f - \frac{1}{2} I\omega_f^2 = MgH - \frac{1}{2} I\omega_f^2 \]

\[ v = \left( \frac{2\pi}{\Delta t} \right) R = \omega R \]
Conservation of Energy (Revisited)

When the smoke clears....

\[ v_{\text{disk}}, f = \sqrt{\frac{4}{3}gH} \]

\[ v_{\text{hoop}}, f = \sqrt{gH} \]

The "disk" wins the race. Perhaps not very intuitive at first....

... until you keep in mind a key principle at play here: *Energy can be stored in a variety of ways*
The kinetic energy of a rod rotating around an axis perpendicular to the rod and through its end is four times larger...

...than the kinetic energy of the rod when it rotates at the same angular velocity around an axis perpendicular to the rod and through its center.

The more mass there is farther from the rotation axis, the larger the moment of inertia, and the larger the rotational kinetic energy for any given angular velocity.
Vector algebra follows familiar rules (cont)

FIGURE 3.7 Working with vectors.

The length of $\vec{B}$ is “stretched” by the factor $c$. That is, $\vec{B} = c\vec{A}$.

$\vec{A} = (A, \theta)$

$\vec{B} = c\vec{A} = (cA, \theta)$

$\vec{B}$ points in the same direction as $\vec{A}$.

Multiplication by a scalar

The negative of a vector

Vector subtraction: What is $\vec{A} - \vec{C}$? Write it as $\vec{A} + (-\vec{C})$ and add!

Tip-to-tail method using $-\vec{C}$

Parallelogram method using $-\vec{C}$

$\vec{A} + (-\vec{A}) = \vec{0}$. The tip of $-\vec{A}$ returns to the starting point.

Vector $-\vec{A}$ is equal in magnitude but opposite in direction to $\vec{A}$.

The zero vector $\vec{0}$ has zero length.

Note: Multiplication of two vectors is a bit trickier (we’ll get there later in the semester)
The "dot product" (re work)

\[ W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} \]

\[ \vec{A} \cdot \vec{B} = AB \cos \theta \]

→ Another means to multiply vectors (the "cross product") now arises as we head into the notion of torque...

\[ \vec{r} = \vec{r} \times \vec{F} = |\vec{r}| \ |\vec{F}| \sin \phi \]

where \( \vec{r} \) is the distance from the axis of rotation

\( \vec{F} \) is the applied force

\( \phi \) is the angle between \( \vec{r} \) and \( \vec{F} \)
• But how does one achieve a particular angular speed? What cause angular acceleration?

• For translational motion, it was a force that caused changes in velocity.

• For rotational motion, it is the torque, \( \tau \), (tau) that causes changes in angular speed. Torque can be thought of as the angular force.
Torque

Figure 8-24 Torque $\tau$ is the rotational analog of force and takes into account the distance $r$ between where a force $F$ is applied and the rotation axis. Torque also takes into account the angle $\varphi$ between the force vector $\vec{F}$ and the $\vec{r}$ vector that points from the rotation axis to the point at which the force is applied.

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Units:

$$[\tau] = [r][F][\sin \varphi] = m \cdot N$$

Scalar version (figure above motivates where this comes from...)
Review: Vector Algebra

The "dot product" (re work)

\[ W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} \]

\[ \vec{A} \cdot \vec{B} = AB \cos \theta \]

\[ \vec{r} = \vec{r} \times \vec{F} = |\vec{r}| |\vec{F}| \sin \phi \]

where \( \vec{r} \) is the distance from the axis of rotation

\( \vec{F} \) is the applied force

\( \phi \) is the angle between \( \vec{r} \) and \( \vec{F} \)

→ Another means to multiply vectors (the "cross product") arises as we deal w/ torque...

Force and displacement are in the same direction, so work \( W = F \Delta x \).

We'll come back to the vector aspect of things in a bit....
Biomechanical example

Let us consider a "rotational" context, but under equilibrium conditions (i.e., things are "balanced" so they do not actually move)

\[ \sum_{i} \tau_{i} = \sum_{i} r_{i} F_{i} = 0. \]

Assuming no net torques, you don't "tip over" as you stand

**FIGURE 1.4.** A person standing. (a) The forces on the person. (b) A free-body or force diagram.
Biomechanical example

Now consider standing on the "balls" of your feet. We can estimate the force associated with your Achille's tendon...

"The Achilles tendon connects the calf muscles (the gastrocnemius and the soleus) to the calcaneus at the back of the heel"

from Homer's *The Iliad*
Biomechanical example

Now consider standing on the "balls" of your feet. We can estimate the force associated with your Achille's tendon...

- \( F_T \) – force exerted by tendon on foot
- \( F_B \) – force of leg bones (tibia & fibula) on foot
- \( W \) – normal force re weight of body

Assumptions
- At equilibrium
- Ignore mass of foot
- Treat foot as rigid body
- Ignore horizontal torque contributions
Biomechanical example

Now consider standing on the "balls" of your feet. We can estimate the force associated with your Achille's tendon...

- $F_T$ – force exerted by tendon on foot
- $F_B$ – force of leg bones (tibia & fibula) on foot
- $W$ – normal force re weight of body

Note the change in how the angle is defined here!
Biomechanical example

Now consider standing on the "balls" of your feet. We can estimate the force associated with your Achille's tendon...

Translational equilibrium requires:

\[ F_T \cos(7^\circ) + W - F_B \cos \theta = 0, \]

\[ F_T \sin(7^\circ) - F_B \sin \theta = 0. \]

Now consider torques:

\[ \tau = rF \sin \varphi \]

\[ \sum_i \tau_i = \sum_i r_i F_i = 0. \]

Rotational equilibrium requires:

\[ 10W - 5.6F_T \cos 7^\circ = 0. \]

\[ F_T = \frac{10W}{5.6 \cos 7^\circ} = 1.8W. \]

Solving for the other parts:

\[ (1.8)(W)(0.993) + W = F_B \cos \theta, \]

\[ 2.8W = F_B \cos \theta. \]

\[ (1.8)(W)(0.122) = F_B \sin \theta, \]

\[ 0.22W = F_B \sin \theta. \]

\[ \tan \theta = \frac{0.22}{2.8} = 0.079, \]

\[ \theta = 4.5^\circ. \]
Now consider standing on the "balls" of your feet. We can estimate the force associated with your Achille's tendon...

"The tension in the Achilles tendon is nearly twice the person’s weight, while the force exerted on the leg by the talus is nearly three times the body weight. One can understand why the tendon might rupture."

Hobbie & Roth
Biomechanical considerations....

This sort of analysis can be extended to a wide variety of problems to get a 1st order estimate of things....
Reminder: Circular "kinematics"

- 1-D kinematics translates directly to circular motion (in polar coords.)

### TABLE 4.1  Rotational and linear kinematics for constant acceleration

<table>
<thead>
<tr>
<th>Rotational kinematics</th>
<th>Linear kinematics</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega_f = \omega_i + \alpha \Delta t )</td>
<td>( v_{fs} = v_{is} + a_s \Delta t )</td>
</tr>
<tr>
<td>( \theta_f = \theta_i + \omega_i \Delta t + \frac{1}{2} \alpha (\Delta t)^2 )</td>
<td>( s_f = s_i + v_{is} \Delta t + \frac{1}{2} a_s (\Delta t)^2 )</td>
</tr>
<tr>
<td>( \omega_f^2 = \omega_i^2 + 2\alpha \Delta \theta )</td>
<td>( v_{fs}^2 = v_{is}^2 + 2a_s \Delta s )</td>
</tr>
</tbody>
</table>

Knight (2013)

→ But wait, if the quantities on the right are all vectors, shouldn't the ones on the left also be too?
Consider rotating a book 90° in 3-D

The order and direction you do it in matters re the final orientation of the book

Motivates that angular quantities need to be treated like vectors as well...
Vector nature of angular quantities

Figure 8-35 (a) No single vector that lies in the plane of rotation indicates the direction of rotation. (b) The angular momentum vector $\vec{L}$ points in a direction perpendicular to the plane of rotation.

(b) The angular momentum vector points in a direction perpendicular to the rotation plane.

Is angular momentum up or down? The direction is up in a right-handed sense. Curl the fingers on your right hand in the direction of motion and stick your thumb straight out; your thumb points in the direction of the angular momentum vector.
Vector nature of angular quantities

We choose a out-of-plane convention

Which leads us back to the cross product

\[ \vec{\tau} = \vec{r} \times \vec{F} \]

\[ \tau = rF \sin \varphi \]
Cross Product

Order matters!

Ultimately, this is a convention....
Right-hand Rule (RHR)

Key aspect here to get correct is that your fingers turn \textbf{a} towards \textbf{b} through the \textit{smaller angle}.
Right-hand Rule (RHR)

Right hand rule: Curl the fingers on your right hand from $\vec{A}$ to $\vec{B}$ along the closest path. Stick out your thumb; it points in the direction of $\vec{C}$, the result of the cross product $\vec{A} \times \vec{B}$.

$\vec{C} = \vec{A} \times \vec{B}$

$\vec{A}$ and $\vec{B}$ lie in the $xy$ plane in this example. $\vec{C}$ points in the positive $z$ direction.

Rotation axis

Kesten & Tauck
To apply the right hand rule correctly, curl your fingers from the direction of $\vec{r}$ to $\vec{F}$ along the shortest path.

Angle $\varphi$ is defined counterclockwise from the direction of $\vec{r}$ to $\vec{F}_1$. In this case $\varphi$ corresponds to the shortest path from the direction of $\vec{r}$ to $\vec{F}_1$.

In this case the counterclockwise definition of $\varphi$ does not correspond to the shortest path from the direction of $\vec{r}$ to $\vec{F}_1$. To apply the right hand rule, use the shortest path from $\vec{r}$ to $\vec{F}_1$.

\[ \vec{r} = \vec{r} \times \vec{F} \]

\[ \tau = rF \sin \varphi \]
Note the visual convention re "tip of an arrow" versus "tail of an arrow" when plotting in 2-D (located at the point of rotation axis)

Fig. 12-2 The plane shown is that defined by \( \mathbf{r} \) and \( \mathbf{F} \) in Fig. 12-1. (a) The magnitude of \( \tau \) is given by \( F r_{\perp} \) (Eq. 12-2b) or by \( r F_{\perp} \) (Eq. 12-2c). (b) Reversing \( \mathbf{F} \) reverses the direction of \( \tau \). (c) Reversing \( \mathbf{r} \) reverses the direction of \( \tau \). (d) Reversing \( \mathbf{F} \) and \( \mathbf{r} \) leaves the direction of \( \tau \) unchanged. The directions of \( \tau \) are represented by \( \bigcirc \) (perpendicularly out of the figure, the symbol representing the tip of an arrow) and by \( \bigotimes \) (perpendicularly into the figure, the symbol representing the tail of an arrow).
Wait a sec....

What about that centripetal acceleration we derived awhile back?! 

\[ a = \frac{v^2}{r} \] (uniform circular motion)

- \( v \) is the tangential velocity
- \( a \) is the radial (or centripetal) acceleration

Things are getting a tad complicated. So let's slow down for a moment (pun!) and take a step back to see how various quantities of interest (inter-)relate...
Interrelationships: Linear vs Angular Kinematics

Fig. 11–9  (a) A rigid body rotates about a fixed axis through $O$ perpendicular to the page. The point $P$ sweeps out an arc $s$ which subtends an angle $\theta$.  (b) The acceleration $\mathbf{a}$ of point $P$ has components $\mathbf{a}_T$ (tangential) where $a_T = \alpha r$ and $\mathbf{a}_R$ (radial) where $a_R = v^2/r = \omega^2 r$ ($\omega =$ angular speed).

Resnick & Halliday
Interrelationships: Linear vs Angular Kinematics

Fig. 11-11 The directions of the vectors $\mathbf{r}$, $\mathbf{v}$, $\mathbf{a}_T$, $\mathbf{a}_R$, $\omega$ and $\alpha$ for a particle rotating in a circle about the z-axis.

$v = \omega \times r$

$a = a_T + a_R$,
Interrelationships: Linear vs Angular Kinematics

\[ a = a_T + a_R, \]

Tangential accel.

\[ a = u_\theta \alpha r - u_r \omega^2 r. \]

Radial accel.

Note that \( a_R \) IS NOT \( \alpha \)
(the angular accel. \( \alpha = d\omega/dt \))
### Rectilinear Motion

<table>
<thead>
<tr>
<th>Property</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Displacement</td>
<td>$x$</td>
</tr>
<tr>
<td>Velocity</td>
<td>$v = \frac{dx}{dt}$</td>
</tr>
<tr>
<td>Acceleration</td>
<td>$a = \frac{dv}{dt}$</td>
</tr>
<tr>
<td>Mass</td>
<td>$M$</td>
</tr>
<tr>
<td>Force</td>
<td>$F = Ma$</td>
</tr>
<tr>
<td>Work</td>
<td>$W = \int F , dx$</td>
</tr>
<tr>
<td>Kinetic energy</td>
<td>$\frac{1}{2}Mv^2$</td>
</tr>
<tr>
<td>Power</td>
<td>$P = Fv$</td>
</tr>
<tr>
<td>Linear momentum</td>
<td>$Mv$</td>
</tr>
</tbody>
</table>

### Rotation about a Fixed Axis

<table>
<thead>
<tr>
<th>Property</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angular displacement</td>
<td>$\theta$</td>
</tr>
<tr>
<td>Angular velocity</td>
<td>$\omega = \frac{d\theta}{dt}$</td>
</tr>
<tr>
<td>Angular acceleration</td>
<td>$\alpha = \frac{d\omega}{dt}$</td>
</tr>
<tr>
<td>Rotational inertia</td>
<td>$I$</td>
</tr>
<tr>
<td>Torque</td>
<td>$\tau = I\alpha$</td>
</tr>
<tr>
<td>Work</td>
<td>$W = \int \tau , d\theta$</td>
</tr>
<tr>
<td>Kinetic energy</td>
<td>$\frac{1}{2}I\omega^2$</td>
</tr>
<tr>
<td>Power</td>
<td>$P = \tau\omega$</td>
</tr>
<tr>
<td>Angular momentum</td>
<td>$I\omega$</td>
</tr>
</tbody>
</table>

---

Now we come to that last bit: **Angular momentum**
Rotational Dynamics

Torque = "Force analog"

\[ \vec{\tau} = \vec{r} \times \vec{F} \]

Now what if all the torques don't add up to zero?

\[ \sum_{i} \tau_i = \sum_{i} r_i F_i = 0. \]

Think along the lines of rotational analog of Newton's 2\textsuperscript{nd}

\[ \vec{F}_{\text{net}} = m\vec{a} \]

Product of object’s mass and its acceleration; not a force.

Net force: the vector sum of all real, physical forces acting on an object.

Equal sign indicates that the two sides are mathematically equal — but that doesn’t mean they’re the same physically. Only \( \vec{F}_{\text{net}} \) involves physical forces.

Fig. 12–1 A force \( \vec{F} \) is applied to a particle \( P \), displaced \( \vec{r} \) relative to the origin. The force vector makes an angle \( \theta \) with the radius vector \( \vec{r} \). The torque \( \tau \) about \( O \) is shown. Its direction is perpendicular to the plane formed by \( \vec{r} \) and \( \vec{F} \) with the sense given by the right-hand rule.
Rotational Dynamics

→ All the bits and pieces we have developed thus far will come into play (e.g., conservation of momentum, moments of inertia, etc...)

Reminder:

Law of conservation of momentum The total momentum \( \vec{P} \) of an isolated system is a constant. Interactions within the system do not change the system’s total momentum.

Linear momentum

\[ \vec{p} = m \vec{v} \]

Connection to Newton's 2nd

\[ \vec{F} = \frac{d\vec{p}}{dt} \]

Angular momentum (defined as)

\[ \vec{L} = \vec{r} \times \vec{p} \]

"Rotational Newton's 2nd"

\[ \vec{\tau} = \frac{d\vec{L}}{dt} \]

Angular momentum is conserved when all torques add up to zero

\[ \vec{F}_{\text{net}} = ma \]

Net force: the vector sum of all real, physical forces acting on an object

Equal sign indicates that the two sides are mathematically equal — but that doesn’t mean they’re the same physically. Only \( \vec{F}_{\text{net}} \) involves physical forces.

Product of object’s mass and its acceleration; not a force.
Rotational Dynamics

Let us derive this relationship

Definition of torque

\[ \vec{F} = \frac{d\vec{p}}{dt} \]
\[ \vec{\tau} = \vec{r} \times \vec{F} \]

\[ r \times F = r \times \frac{d\vec{p}}{dt} \]

\[ \tau = r \times \frac{d\vec{p}}{dt} \]

Now we differentiate (definition of angular momentum re time)

\[ \dot{L} = \dot{r} \times \dot{p} \]

\[ \frac{dL}{dt} = \frac{dr}{dt} \times p + r \times \frac{dp}{dt} \]

Chain rule applies!

\[ \frac{dL}{dt} = (v \times mv) + r \times \frac{dp}{dt} \]

Now \( \frac{dr}{dt} \) is just the inst. velocity \( v \)

The velocity \( v \) is in the same direction as the momentum \( mv \)
(Thus the cross product is zero)

Resnick & Halliday
Rotational Dynamics

\[ \frac{dl}{dt} = (v \times m v) + r \times \frac{dp}{dt} \]

Let us derive this relationship

\[ \dot{\tau} = \frac{d\vec{L}}{dt} \]

\[ \frac{dl}{dt} = r \times \frac{dp}{dt} \]

\[ \tau = \frac{dl}{dt}, \]

(or in English)

The time rate of change of angular momentum of a particle is equal to the torque acting on it

\[ \tau_x = (dl/dt)_x \quad \tau_y = (dl/dt)_y, \quad \tau_z = (dl/dt)_z. \]

Scalar equivalents...
A familiar (non-rotational) example...

A particle of mass $m$ is released from rest at point $a$ in Fig. 12–4, falling parallel to the (vertical) $y$-axis. (a) Find the torque acting on $m$ at any time $t$, with respect to origin $O$. (b) Find the angular momentum of $m$ at any time $t$, with respect to this same origin. (c) Show that the relation $\tau = dL/dt$ (Eq. 12–7) yields a correct result when applied to this familiar problem.

**Note:** This is not a rotational problem per se, but that doesn't mean we can not treat it using our tools thus far developed...

Fig. 12–4 A particle of mass $m$ drops vertically from point $a$. The torque and the angular momentum about $O$ are directed perpendicularly into the figure, as shown by the symbol $\otimes$ at $O$. 

Resnick & Halliday
A familiar (non-rotational) example...

Torque:
\[ \tau = \vec{r} \times \vec{F} \]
\[ \tau = rF \sin \varphi \]

Here:
\[ r \sin \theta = b \text{ and } F = mg \]

\[ \tau = mgb = \text{a constant.} \]

Torque goes into the page

Note that torque is simply the product of \( mg \) and the "moment arm" \( (b) \)

Angular momentum:
\[ l = \vec{r} \times \vec{p}, \]
\[ l = rp \sin \theta. \]

Here:
\[ r \sin \theta = b \text{ and } p = mv = m(gt) \]

\[ l = mgbt. \]

Ang. momentum also goes into the page, but magnitude increases w/ time

As expected!

Note that if we drop the \( b \) terms on both sides...

\[ mg = \frac{d}{dt} (mv) \]

\[ \vec{F} = \frac{d\vec{p}}{dt} \]

... which just leads back to Newton's 2\textsuperscript{nd}
A familiar (non-rotational) example...

Thus, as we indicated earlier, relations such as $\tau = dl/dt$, though often vastly useful, are not new basic postulates of classical mechanics but are rather the reformulation of the Newtonian laws for rotational motion.

Note that the values of $\tau$ and $l$ depends on our choice of origin, that is, on $b$. In particular, if $b = 0$, then $\tau = 0$ and $l = 0$. 

Resnick & Halliday
Conservation of Angular Momentum

Assumption: We are talking about an inertial reference frame

Time rate of change of total angular momentum of a system of particles about a fixed point is equal to the sum of 
*external* torques acting on it

\[ \tau_{\text{ext}} = \frac{dL}{dt}. \]

Suppose now that \( \tau_{\text{ext}} = 0 \); then \( \frac{dL}{dt} = 0 \) so that \( L = \) a constant.

*When the resultant external torque acting on a system is zero, the total vector angular momentum of the system remains constant. This is the principle of the conservation of angular momentum.*

When the resultant external torque on the system is zero, we have

\[ L = \text{a constant} = L_0, \]
Conservation of Angular Momentum

Consider a "rigid body" rotating about a fixed axis. Then:

Conservation of angular momentum then implies:

\[ \dot{L} = I \ddot{\omega} \]

\[ I \omega = I_0 \omega_0 = \text{a constant.} \]
A spinning figure skater starts from an initial angular velocity of $\omega_i = 12[\text{rad/s}]$ with her arms extending away from her body. In this position, her body's moment of inertia is $I_i = 3[\text{kg m}^2]$. The skater then brings her arms close to her body, and in the process her moment of inertia changes to $I_f = 0.5[\text{kg m}^2]$. What is her new angular velocity?

This is a job for the law of conservation of angular momentum:

$$L_i = L_f \Rightarrow I_i \omega_i = I_f \omega_f.$$  

We know $I_i$, $\omega_i$, and $I_f$, so we can solve for the final angular velocity $\omega_f$. The answer is $\omega_f = I_i \omega_i / I_f = 3 \times 12 / 0.5 = 72[\text{rad/s}]$, which corresponds to 11.46 turns per second.
Conservation of Angular Momentum

Fig. 13–6 A diver leaves the diving board with arms and legs outstretched and with some initial angular velocity. Since no torques are exerted on him about his center of mass, \( L = I \omega \) is constant while he is in the air. When he pulls his arms and legs in, since \( I \) decreases, \( \omega \) increases. When he again extends his limbs, his angular velocity drops back to its initial value. Notice the parabolic motion of his center of mass, common to all two-dimensional motion under the influence of gravity.

→ There is a bit more here though...
Conservation of Angular Momentum

By "tucking in", there is less mass further away re the CM, so therefore $I$ gets smaller.

$I = \sum m_i r_i^2$

As a result, $\omega$ goes up (and back down as he "extends out" later on).

$I \omega = I_0 \omega_0 = \text{a constant.}$

However, kinetic energy is not constant. Even though the moment of inertia decreases...

$\frac{1}{2} I \omega^2 > \frac{1}{2} I_0 \omega_0^2$

... it follows that the kinetic energy increases!

→ So while momentum is conserved here (i.e., no external torques), the "system" increases in energy by virtue of the diver doing work to extend/pull in body parts to change $I$.

Note parabolic trajectory of the center-of-mass (CM)
Fig. 13-1  (a) A precessing top, showing the angular momentum $\mathbf{L}$, the weight $mg$ and the vector $\mathbf{r}$ which locates the center of mass.  (b) The cone swept out by the precessing axis of the top.  The angular velocity of precession is shown pointing vertically upward.
Spinning Top

Spinning top that is precessing → Top does not fall over because of its momentum

→ Note directional conventions (i.e., longitudinal vs transverse)

Figure 39.12 The axis of rotation of a spinning top precesses about the vertical. The angular momentum vector of the top has a vertical longitudinal component which is constant and a horizontal transverse component which rotates about the vertical.
Earth's motion – Three different rotational axes:
1. Around sun
2. About central axis
3. Nutation of axis

Main cause: gravitational torque from the Moon, the Sun and the planets
We could use the pieces we now have in our toolbox to derive the precession frequency

$$\omega_p = \frac{mgr}{L}.$$
What are the basic ingredients for NMR/MRI?

- static field
- particles (e.g., protons as spinning tops)
- coil to perturb particles from static field and measure resulting dynamics (via ‘pulse sequences’ of RF photons*)
- Fourier transforms

* RF – Radio Frequency
Fig. 3.147a–d. Classical representation of the NMR experiment. a In equilibrium the nuclear spins are distributed in the states $\alpha$ and $\beta$ according to the Boltzmann distribution. b At resonance and with a sufficiently strong RF field, the populations of $\alpha$ and $\beta$ are equalized and the spins precess in phase at the Larmor frequency $\omega_L$. c Longitudinal relaxation restores the equilibrium distribution of the spins. d The phase coherence of the spins is lost by transverse relaxation. In reality the processes c and d proceed simultaneously.
Universality of conservation of angular momentum

The conservation of angular momentum principle holds in atomic and nuclear physics as well as in celestial and macroscopic regions. Since Newtonian mechanics does not hold in the atomic and nuclear domain, this conservation law must be more fundamental than Newtonian principles.

→ Very deep idea here that connects back to the start of the course...

- What does “physics” even mean?

(from wikipedia)

“Physics (from Ancient Greek: φυσική (ἐπιστήμη) phusikē (epistēmē) ’knowledge of nature’, from φύσις phúsis "nature)....

... is the natural science that involves the study of matter and its motion and behavior through space and time, along with related concepts such as energy and force. One of the most fundamental scientific disciplines, the main goal of physics is to understand how the universe behaves.”
Universality of conservation of angular momentum

Further reading:

The conservation of angular momentum principle holds in atomic and nuclear physics as well as in celestial and macroscopic regions. Since Newtonian mechanics does not hold in the atomic and nuclear domain, this conservation law must be more fundamental than Newtonian principles. In our derivation of this principle we must have made more rigid assumptions than we needed to. This is true even in the framework of classical mechanics. The student should note the key role played by Newton’s third law in our deduction of this conservation principle. This law was used to justify the assumption that the sum of the internal torques was zero. It was necessary to assert not only that the action and reaction forces were equal and opposite (the “weak” form of the third law) but also that these forces were directed along the line joining the two particles (the “strong” form of the third law). The strong form is known to be violated in some electromagnetic interactions. However, the assumption that the sum of the internal torques in a system of particles is zero can be proven on the basis of a much less stringent requirement than that the third law should hold.*

The law of conservation of angular momentum, as we have formulated it, holds for a system of bodies whenever the bodies can be treated as particles, that is, whenever effects due to the rotation of the individual bodies can be neglected. When the individual bodies have rotation, the conservation of angular momentum principle is still valid, providing we include the angular momentum associated with this rotation. However, the bodies then are no longer simple particles whose motion can be described by particle dynamics.
Universality of conservation of angular momentum

Further reading:

In atomic and nuclear physics we find that the “elementary particles” such as electrons, protons, mesons, and neutrons have angular momentum associated with an intrinsic spinning motion, as well as with orbital motion about some external point. When we use the law of conservation of total angular momentum we must include this spin angular momentum in the total. A fundamental aspect of atomic, molecular, and nuclear systems is that their angular momenta can take on only definite discrete values, rather than a continuum of values. Angular momentum is said to be quantized. Hence, angular momentum plays a central role in the description of the behavior of such systems (see Problems 4 and 6). These ideas will be developed in later chapters.

If we were to regard the sun, planets, and satellites as particles having no intrinsic spinning motion, the angular momentum of the solar system would turn out not to be constant. But these bodies do have intrinsic rotations; in fact, tidal forces convert some of the intrinsic spinning angular momentum into orbital angular momentum of the planets and satellites. When we use the law of conservation of the angular momentum, we must include this spin angular momentum in the total. The conservation of angular momentum plays a key role in the evaluation of theories of the origin of the solar system, the contraction of giant

Universality of conservation of angular momentum

Further reading:

stars, and other problems in astronomy.* Some astronomical applications will be considered in Chapter 16.

The basis for this rather simple way of analyzing the total angular momentum of atomic or astronomical systems is a theorem (see Problem 10) that the total angular momentum \( \mathbf{L} \) of any system with respect to the origin of an inertial reference frame may be computed by adding the angular momentum with respect to its center of mass (\textit{spin} angular momentum) to the angular momentum arising from the motion of the center of mass with respect to the origin (\textit{orbital} angular momentum).

The conservation laws of total energy and of linear momentum and angular momentum are fundamental to physics, being valid in all modern physical theories. We shall have occasion to use them many times in later chapters.
Additional examples for study...... (some w/ solutions, some w/o)
Find the center of mass of a 2-meter rod lying on the $x$-axis with its left end at the origin if:
(a) The density is constant and the total mass is 5 kg.  
(b) The density is $\delta(x) = 15x^2$ kg/m.

(a) Since the density is constant along the rod, we expect the balance point to be in the middle, that is, $\bar{x} = 1$. To check this, we compute $\bar{x}$. The density is the total mass divided by the length, so $\delta(x) = 5/2$ kg/m. Then

$$\bar{x} = \frac{\text{Moment}}{\text{Mass}} = \frac{\int_0^2 x \cdot \frac{5}{2} \, dx}{5} = \frac{1}{5} \cdot \frac{5}{2} \cdot \frac{x^2}{2} \bigg|_{0}^{2} = 1 \text{ meter}.$$  

(b) Since more of the mass of the rod is closer to its right end (the density is greatest there), we expect the center of mass to be in the right half of the rod, that is, between $x = 1$ and $x = 2$. We have

$$\text{Total mass} = \int_0^2 15x^2 \, dx = 5x^3 \bigg|_{0}^{2} = 40 \text{ kg}.$$  

Thus,

$$\bar{x} = \frac{\text{Moment}}{\text{Mass}} = \frac{\int_0^2 x \cdot 15x^2 \, dx}{40} = \frac{15}{40} \cdot \frac{x^4}{4} \bigg|_{0}^{2} = 1.5 \text{ meter}.$$
First we find the coordinates of the center of mass. From Eq. 9–3,

\[ x_{cm} = \frac{(8.0 \times 4) + (4.0 \times -2) + (4.0 \times 1)}{16} \text{ meters} = 1.8 \text{ meters}, \]

\[ y_{cm} = \frac{(8.0 \times 1) + (4.0 \times 2) + (4.0 \times -3)}{16} \text{ meters} = 0.25 \text{ meter}. \]

These are shown as C in Fig. 9–6.

To obtain the acceleration of the center of mass, we first determine the resultant external force acting on the system consisting of the three particles. The x-component of this force is

\[ F_x = 14 \text{ nt} - 6.0 \text{ nt} = 8.0 \text{ nt}, \]

and the y-component is

\[ F_y = 16 \text{ nt}. \]

Hence the resultant external force has a magnitude

\[ F = \sqrt{(8.0)^2 + (16)^2} \text{ nt} = 18 \text{ nt}, \]

and makes an angle \( \theta \) with the x-axis given by

\[ \tan \theta = \frac{16 \text{ nt}}{8.0 \text{ nt}} = 2.0 \quad \text{or} \quad \theta = 63^\circ. \]

Then, from Eq. 9–10, the acceleration of the center of mass is

\[ a_{cm} = \frac{F}{M} = \frac{18 \text{ nt}}{16 \text{ kg}} = 1.1 \text{ meters/sec}^2, \]

making an angle of 63° with the x-axis.

Although the three particles will change their relative positions as time goes on, the center of mass will move, as shown, with this constant acceleration.