239. Tangled Stars

Three of the four tangled stars are identical. Which one is different?
Cell membrane acts like an RC filter

Figure 1.8
Circuit Representation

Resistor and capacitor in series \( \rightarrow \) RC time constant

Figure 3.6
- First solved by William Thomson (aka Lord Kelvin) in ~1855

- Motivated by Atlantic submarine cable for intercontinental telegraphy
Axon behaves in fashion similar to a leaky submarine cable.
Core Conductor Model

→ Combine together both “models”
For $\Delta V_m$ small:

$$K_m = 2\pi a J_m = 2\pi a C_m \frac{dV_m}{dt} + 2\pi a G_m (V_m - V_m^o) = c_m \frac{dV_m}{dt} + g_m (V_m - V_m^o)$$

Combine with core-conductor model:

$$\frac{\partial^2 V_m}{\partial z^2} = (r_o + r_i) K_m - r_o K_e = (r_o + r_i) \left[ c_m \frac{\partial V_m}{\partial t} + g_m (V_m - V_m^o) \right] - r_o K_e$$

$$V_m + \frac{c_m}{g_m} \frac{\partial V_m}{\partial t} - \frac{1}{g_m(r_o + r_i)} \frac{\partial^2 V_m}{\partial z^2} = V_m^o + \frac{r_o}{g_m(r_o + r_i)} K_e$$
Introduce two new constants \((\tau_M \text{ and } \lambda_C)\)

\[
V_m + \frac{c_m}{g_m} \frac{\partial V_m}{\partial t} - \frac{1}{g_m (r_o + r_i)} \frac{\partial^2 V_m}{\partial z^2} = V_m^o + \frac{r_o}{g_m (r_o + r_i)} K_e
\]

\[
V_m + \tau_M \frac{\partial V_m}{\partial t} - \lambda_C^2 \frac{\partial^2 V_m}{\partial z^2} = V_m^o + r_o \lambda_C^2 K_e
\]

Let \(V_m = v_m + V_m^o\) : \((\text{incremental change in memb. potential})\)

\[
v_m + \tau_M \frac{\partial v_m}{\partial t} - \lambda_C^2 \frac{\partial^2 v_m}{\partial z^2} = r_o \lambda_C^2 K_e \quad \text{(Cable Equation)}
\]
Cable Equation

Let $v_m(z, t) = V_m(z, t) - V_m^o$ and $|v_m(z, t)| \ll |V_m^o|$:

$$v_m(z, t) + \tau_M \frac{\partial v_m(z, t)}{\partial t} - \chi_C^2 \frac{\partial^2 v_m(z, t)}{\partial z^2} = r_o \chi_C^2 K_e(z, t)$$

**Note:**
Somewhat similar to the diffusion equation (but not exactly due to extra $v_m$ term)

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2}$$
Constants: $\tau_M$ and $\lambda_C$

\[ V_m + \tau_M \frac{\partial V_m}{\partial t} - \lambda_C^2 \frac{\partial^2 V_m}{\partial z^2} = V_m^o + r_o \lambda_C^2 K_e \]

Space constant ($\lambda_C$) - property of cell, not just membrane

\[ \lambda_C = \frac{1}{\sqrt{(r_i + r_o) g_m}} \approx \sqrt{\frac{a}{2 \rho_i G_m}} \] (assuming $r_o << r_i$)

Wider axons $\rightarrow$ Further propagation/less degradation

Time constant ($\tau_M$) – independent of cellular dimensions

\[ \tau_M = \frac{c_m}{g_m} \]
Cable Equation

Let $v_m(z, t) = V_m(z, t) - V_m^o$ and $|v_m(z, t)| \ll |V_m^o| :$

$$v_m(z, t) + \tau_M \frac{\partial v_m(z, t)}{\partial t} - \lambda_C^2 \frac{\partial^2 v_m(z, t)}{\partial z^2} = r_o \lambda_C^2 K_e(z, t)$$

![Diagram](image)

Figure 3.8

Axon $\leftrightarrow$ Leaky submarine ‘cable’
Cable Model – Solution for spatial impulse

Amplitude falls off (re space const.)

Time-independent solution
→ constantly applied current

Figure 3.9

“match boundary conditions” at z=0

Figure 3.10

Space constant $\lambda_C$

Figure 3.11

→ Amplitude falls off (re space const.)
→ Space constant \((\lambda_c)\) typically on order of mm (even less for small unmyelinated fibers)

→ Solutions allow for propagation, but in a decremental fashion

→ Axons alone are not good ‘cables’ for sending signals long-ish distances!
Cable Model – Solution for temporal & spatial impulse

Assume infinitesimal electrode and $i_e(t)$ brief so that

$$k_e(z, t) = 0; \quad \text{if } z \neq 0 \text{ or } t \neq 0$$

For $t \neq 0$ or $z \neq 0$

$$v_m(z, t) + \tau_M \frac{\partial v_m}{\partial t} - \lambda_c^2 \frac{\partial^2 v_m}{\partial z^2} = 0$$

Let

$$v_m(z, t) = w(z, t)e^{-t/\tau_M}$$

Then

$$\frac{\partial v_m}{\partial t} = -\frac{1}{\tau_M}w(z, t)e^{-t/\tau_M} + \frac{\partial w}{\partial t}e^{-t/\tau_M}$$

$$\frac{\partial^2 v_m}{\partial z^2} = \frac{\partial^2 w}{\partial z^2}e^{-t/\tau_M}$$
Cable Model – (A) Solution

Substituting,

\[ w(z, t)e^{-t/\tau_M} - w(z, t)e^{-t/\tau_M} + \tau_M \frac{\partial w}{\partial t} e^{-t/\tau_M} - \lambda_C^2 \frac{\partial^2 w}{\partial z^2} e^{-t/\tau_M} = 0 \]

\[ \tau_M \frac{\partial w}{\partial t} = \lambda_C^2 \frac{\partial^2 w}{\partial z^2} \]

Solving cable equation (here w/ change of variable) is like diffusion equation!

Figure 3.23

Figure 3.24
Cable Model – (A) Solution

Figure 3.25
Cable Model – General Properties

→ Solutions allow for propagation, but in a decremental fashion
Cable Model – General Properties

→ Cellular dimensions re space constant ($\lambda_C$) determine whether a cell is electrically *small* or *large*.
Electrode 3 likely closest to end-plate.
Linearity \rightarrow \text{Superposition}

Figure 3.19
“Electronic distance”
“Temporal integration”

→ Key considerations with regard to synapses (i.e., inter-neuron communication)
Spatial integration

Figure 3.36
Looking Ahead: Hodgkin-Huxley

Decremental conduction

\[ I_e(t) \]

\[ V_{m1} \quad V_{m2} \quad V_{m3} \quad V_{m4} \]

Decrement-free conduction

\[ I_e(t) \]

\[ V_{m1} \quad V_{m2} \quad V_{m3} \quad V_{m4} \]

Electrically inexcitable cell

Electrically excitable cell
Hodgkin Huxley model

Figure 4.7
\[ G_K(V_m, t) = \bar{G}_K n^4(V_m, t) \]
\[ G_{Na}(V_m, t) = \bar{G}_{Na} m^3(V_m, t) h(V_m, t) \]
\[ n(V_m, t) + \tau_n(V_m) \frac{dn(V_m, t)}{dt} = n_\infty(V_m) \]
\[ m(V_m, t) + \tau_m(V_m) \frac{dm(V_m, t)}{dt} = m_\infty(V_m) \]
\[ h(V_m, t) + \tau_h(V_m) \frac{dh(V_m, t)}{dt} = h_\infty(V_m) \]