Synchronization of chaotic circuits

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Abstract

Using the Rossler equations as our model, we constructed an electronic circuit and demonstrated its chaotic output. Building an identical circuit, we weakly coupled the two together and showed that they were able to synchronize. From these results, we varied one circuit with respect to the other in an effort to understand how different the two can be and still synchronize.

1 INTRODUCTION

1.1 History of Chaos

Chaos is a robust phenomenon which appears in many aspects of modern science. While the study of it is relatively young, many important results have been found which yield vast insight into the nature of complex nonlinear dynamical systems. The techniques developed from these findings offer many exciting applications into future technology such as communication and neural networking.

Until recently, chaos was considered more of a nuisance than a valid scientifical aspect. Whether it was turbulent fluid flow, fibrillation of the heart, the irregularity at which a tap dripped or the complexity of the weather, there was little attempt to come to terms with these phenomenon from a scientific point of view.

This changed when a meteorologist from MIT named Edward Lorenz noticed some very interesting behavior in some equations he derived in an attempt to model the weather in 1961. He observed that in this set of nonlinear equations, making very small changes of the parameters had a huge effect on their solutions. In addition, he found that a very interesting symmetry to the system, what we now call the *butterfly attractor*.

These results paved the way for a rigorous mathematical study of chaos. While there is no formal definition of the term, chaos can be most simply defined as the observable pattern of making a small change in a complex, nonlinear system which produces a huge change in the behavior of the system. This is often called a *sensitive dependence upon initial conditions*.

Over the last 15 years, scientists have seen the values of these studies as many subtle behaviors of physical systems have been shown to stem from a chaotic origin. As a result, there has been a tremendous increase of interest in chaotic behavior in such diverse fields as nuclear physics, biology, socioeconomics, electrical engineering and solid state physics. The purpose of this paper is to address this concept of chaos and how it can be observed through electrical circuits. Of particular interest, we focus on the concept of synchronization between two chaotic systems as a means of controlling the chaos.

1.2 Synchronization

Synchronization can be thought of as the simplest kind of cooperation between chaotic systems. Through a weak coupling of the systems, they can behave in a similar way such that the differences between their behavior in the long run goes to zero. By weak, we mean that the two systems are able to act independently for the most part, not one system dominating the other. The possibility of two or more chaotic systems acting in a coherent synchronized way is not obvious. In real systems, it is not possible to reproduce exactly the same starting conditions or parameters for different systems. Thus from the sensitive dependence upon initial conditions, it does not seem possible that synchronization could occur.

While it is possible to couple many systems together, we deal with only two systems here and show indeed that synchronization between the two can be achieved. This is a very important result since the concept of synchronization has many important applications, such as in the field of communications. It is also believed that synchronization plays an important role in information processing in living organisms.

2 THEORY

2.1 Introductory Concepts

Here, we develop the basic theory neccessary in order to understand the concepts of chaos and synchronization. We start with the notion of a map, which is analogous to a discretized differential equation. Dealing with maps is similar to how we treat differential equations, the fundamental difference now being that we no longer move in a continuous fashion, but in discrete steps. You start with some initial condition and iterate the map much in the same way that you solve a differential equation. For example, consider the simple map

$$x_{n+1} = x_n + 1 \tag{1}$$

where $n \in \aleph$. Starting with the initial value $x_0 = 1$ and iterating the map four times, we obtain x_4 which has the value 5.

Another important concept is that of a fixed point. These are points which do not change under subsequent iterations of the map. We write this as $x_{n+k} = x_n$, where k is an integer and we define

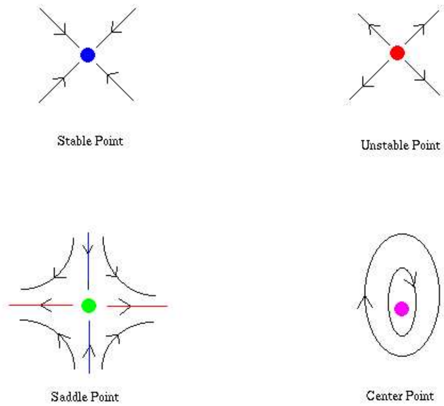

Figure 1: Stability of fixed points.

the fixed point to be of period k. These are very important because when dealing with highly nonlinear systems, we are able to linearize about fixed points and determine the behavior in the neighborhood of that point. Fixed points are generally classified into four categories as shown in Fig. 1, the arrows indicating what happens to a point as the map is iterated. There is a well-developed method for determining the behavior of a map near a fixed point as we outline here:

- First, we linearize the equations of the map about the fixed point. We write these in matrix form which is called the Jacobian (we denote this as matrix A).
- We then find the eigenvalues λ_i by calculating $det(A-\lambda I)$. There is one eigenvalue for every dimension of the system.
- These eigenvalues tell you the stability of the fixed point as follows: $|\lambda| < 1$ - stable (things move in), $|\lambda| > 1$ - unstable, $|\lambda| = 1$ - a center. Fixed points have more than one eigenvalue (assuming we are not looking at a one dimensional map) and if there is a mix of those prescribed above, that point is a saddle.

Another important concept of nonlinear systems is that of a *bifurcation*. Consider the quadratic map

$$x_{n+1} = a - {x_n}^2 \tag{2}$$

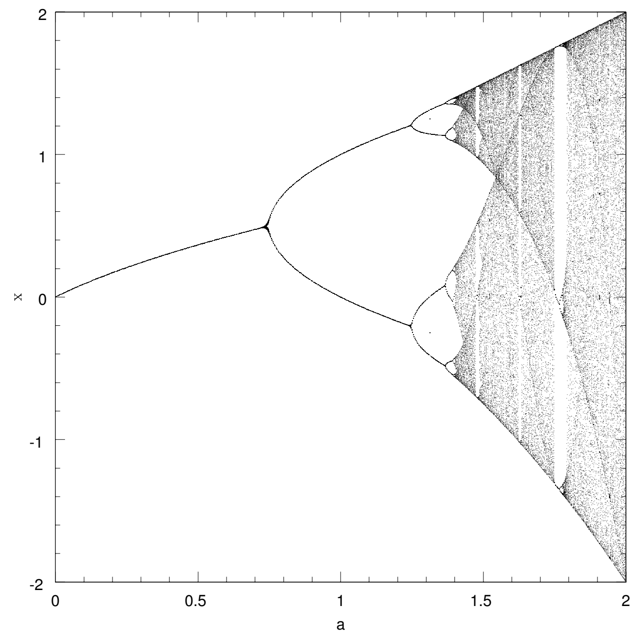

Figure 2: Period doubling cascade for the quadratic map.

where $a \in \Re$ is some constant. Changing its value will change the fixed points and their eigenvalues, thus changing the nature of the system. We thus call a a bifurcation parameter. We can easily write a program which iterates the map for some value of a. Given some appropiate initial condition x_0 (so that things do not diverge off to ∞), if we plot the iterates x_n versus a, we get an interesting pattern called the *period doubling cascade*, as shown in Fig. 2.

For small values of a, solutions tend towards a single, stable solution (in making the figure, the values of the first 30 iterations were thrown out so to see the long term behavior). But as a increases, more and more periodic fixed points start to appear. The iterates stop moving towards a single stable orbit and start getting pushed around by unstable high-period fixed points. The behavior of a single solution becomes extremely unpredictable due to this waterfall of unstable fixed points pushing the iterates around. This is a signature of *chaos*.

What does it mean for a system to be chaotic? While there is not an exact definition, there are many formal mathematical characterizations which classify a system as being chaotic. As mentioned earlier, a fairly straight-forward one is a sensitive dependence upon initial conditions. This means that if we choose two initial conditions arbitrarily close, after some number of iterations of the map, the two will have greatly diverged. This is measured by the *Lyapunov exponents*, which are average parameters of an exponential function that tells us how quickly the two will diverge. They are given by

$$\lambda = \lim_{n\to\infty} \left[\frac{1}{N} \ln \left| \frac{dF(x_0)^N}{dx_0} \right| \right] \tag{3}$$

where F is the map in question (note that these are different from the eigenvalues). A positive Lyapunov exponent means divergence while a negative one means convergence, the magnitude describing how quickly. All chaotic systems must have at least one positve Lyapunov exponent. In general, these must be computed numerically.

2.2 Synchronization

The general approach to synchronization is to take two or more identical chaotic systems and couple them together in such a way that the chaotic behavior of all the systems is the same. We consider the general coupled system

$$x_{n+1} = f(x_n) \tag{4}$$

$$y_{n+1} = g(x_n, y_n) \tag{5}$$

With this as our basis, it is insightful to stick with our example of the quadratic map. Our desire is to effectively combine two one-dimensional maps into a two-dimensional map. We introduce the coupling parameter α and write the combined system as

$$x_{n+1} = a - {x_n}^2 \tag{6}$$

$$y_{n+1} = a - [\alpha y_n + (1-\alpha)x_n]^2 \tag{7}$$

Here $0 \le \alpha \le 1$ and when $\alpha = 1$, the two systems are uncoupled and the two act independently. Our goal is to have the two coupled as loosely as possible while still having the two synchronize, which we define here to mean

$$\lim_{n\to\infty} (x_n - y_n) = 0 \tag{8}$$

Whether the two systems synchronize depends on a number of conditions such as parameter regions of the systems, coupling strength and how much the two systems differ with respect to the other. This is an active area of research generating a lot of excitement as new discoveries are being made everyday.

2.3 Coupled Rossler System

Our purpose is to now describe how we use electronic circuits to model the behavior of chaotic dynamical systems. The circuits we use are based on the well known chaotic Rossler equations [6]

$$\frac{dx}{dt} = -\alpha[\Gamma x + \beta y + \lambda z] \tag{9}$$

$$\frac{dy}{dt} = -\alpha[-x - \gamma y + 0.02y] \tag{10}$$

$$\frac{dz}{dt} = -\alpha[-g(x) + z] \tag{11}$$

where $g(x)$ is zero for $x \leq 3$ and $g(x) = \mu(x-3)$ when $x > 3$. When the parameters are set to the following values, the resulting behavior is chaotic: $\alpha = 10^4 s^{-1}$ (a time factor), $\Gamma = 0.05$, $\beta = 0.5$, $\lambda = 1.0$, $\gamma = 0.133$ and $\mu = 15$.

Using this as our *drive system*, we now want to create a secondary system which responds directly to changes in the drive system. We take some of the variables of the first system and use them to form a *response system*. Carrying this through for the Rossler system, we obtain the following cascaded response system [2]

$$\frac{dx'}{dt} = -\alpha[\Gamma x' + \beta y + \lambda z'] \tag{12}$$

$$\frac{dy''}{dt} = -\alpha[-x' - \gamma y + 0.02y''] \tag{13}$$

$$\frac{dz'}{dt} = -\alpha[-g(x') + z'] \tag{14}$$

These two systems, being chaotic, are coupled together in such a way that they will be synchronized. This can be shown through numerical techniques.

2.4 Chaotic Electronic Circuits

It is important that these results be tested on a real physical system. Our vehicle for this exploration will be coupled electronic circuits. These circuits must must display the following properties. First, the circuit needs to have a single attractor within the parameter region we wish to use. Next, it has to be simple enough so that an identical circuit can be built. Most importantly, it must exhibit chaos over a large region of parameters so that weak coupling to other circuits does not destroy the chaotic behavior.

Unfortunately, it is not entirely intuitive how we can use circuits to model the behavior of our differential equations. When a voltage crosses a certain electronic element, it changes in a certain way. These change can be combined in the form of a circuit which behaves in some systematic manner. Using certain electronic elements, one can model the voltage drop through the circuit using Kirchhoff's to form a set of differential equations which act in the same way as the desired system.

The parameters of these elements can than be adjusted so to be able to fine tune the behavior of the circuit. The exact method of how the modeled circuit is derived from the Rossler system is beyond the scope of this paper, but is described extensively in other texts [7,9].

3 EXPERIMENTAL SETUP

The drive circuit is shown in Fig. 3. The circuits consist of resistors, capacitors, diodes, SPDT switches, operational amplifiers (A1-A5) and potentiometers (variable resistors). Both circuits were constructed using a solderless breadboard and jumper wires to make all the neccessary connections. A wire was used to couple the two circuits together.

All op-amps were type 741 and were powered by a $\pm 12V$ power source. The positive input of the op-amps was grounded to the power source. Seperate power sources were needed for the two circuits. A third power source was used to supply the $-15V$ for the circuits. The diode type does not really matter since it onlys acts as a switch so that op-amp A4 only turns on when the voltage x exceeds $3V$. These two elements combined with the three resistors around them form the one nonlinearity in the circuit, which models the function $g(x)$. The switches are used for different purposes in both circuits. In the drive

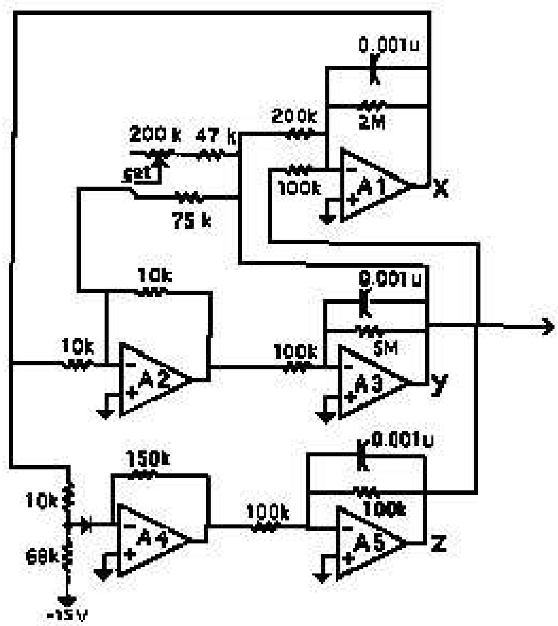

Figure 3: Schematic of Rossler drive cicuit.

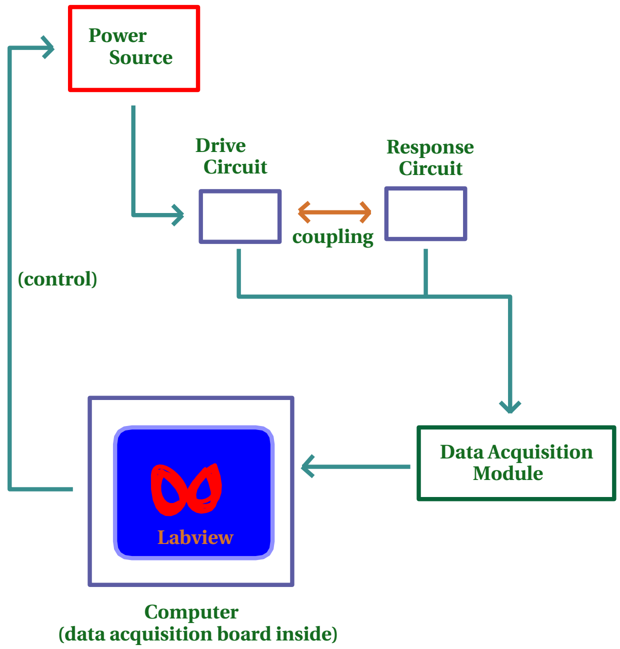

Figure 4: Setup for acquiring voltage data from circuits.

circuit, it allows for a change between a fixed resistor and a potentiometer. Thus it acts as a means for changing a parameter in the circuit so that it differs with respect to the other circuit (giving us the ability to observe how changing the circuits affects the synchronization between the two). In the response circuit, it allows a means or changing the circuit from a free running chaotic oscillator to a stable response circuit. Voltage data was measured at the locations labeled in Fig. 3. An oscilliscope was used to observe the behavior, but the actual data was recorded using a computer. The voltages were fed through a data acquisition module to an on-board card in the computer. Labview software was then used to acquire and analyze the data.

Each circuit was constructed independently, then tested to make sure that they were indeed chaotic. To be certain that both circuits were 'identical', the same type of electronic components were used in both circuits. Once both were completed, they were coupled together and all the connections made so that measurements could be taken.

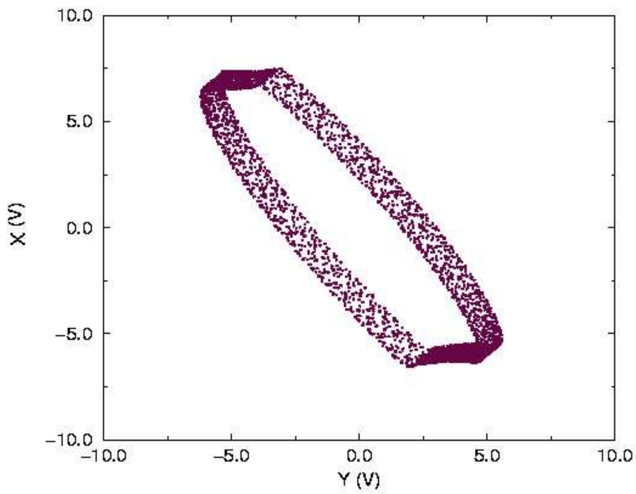

Figure 5: Chaotic attractor for Rossler circuit.

4 DATA AND ANALYSIS

Turning on the power sources, the systems fell immediantly into their nonperiodic steady-state behavior. Typical data sets involved about 2000 measurements sampled at a rate of 250 scans/second.

Measurements of the voltage across x and y in the drive circuit are plotted in Fig. 4. This figure shows the behavior of the voltages in phase space and we see some very interesting dynamics. This observed symmetry is the chaotic attractor of the circuit. What we see here is actually a trajectory being traced out, guided by the *strange attractor* of the system. The term attractor is applied since all trajectories are sucked into this path. When a trajectory passes through a point, it never passes through it again, but it comes infinitely close to it an infinite number of times, hence the term strange. This is characteristic of a chaotic system.

When we couple the two circuits together, we have the ability via a switch to control whether or not the reponse circuit synchronizes with the drive circuit or not. We see in Fig. 5 the unsynchronized signal between the two. Here, both are acting in a completely unrelated chaotic manner and the behavior looks more or less random. Though, it should be emphasized that while chaotic behavior might look ran-

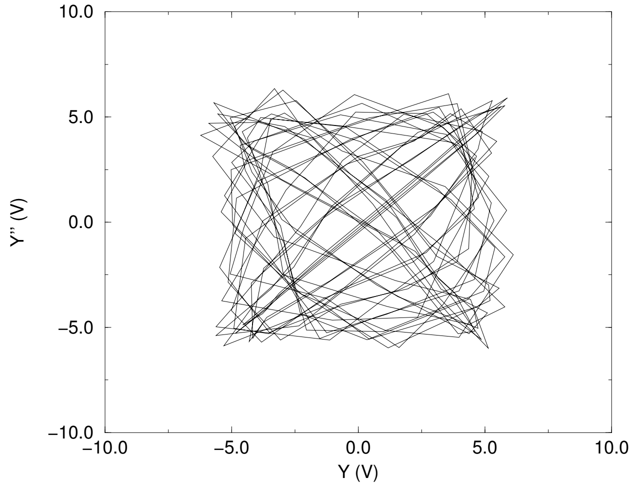

Figure 6: Unsynchronized signal between circuits.

dom, it is fundamentally different. While being quite unpredictable, there are symmetries of a chaotic system which allow it to be much more of a deterministic nature.

Flipping the switch, we see the synchronized behavior between the two circuits in Fig. 6. Here, the 45^o line represents the matching of the two signals and we see that the differences between the two are zero. We notice that there is a thickness to the line, which ideally should be flat. This is due to a small amount of noise in the system deriving from the power source and the tolerances in the electronic elements of the circuit.

Having gotten the two circuits to synchronize, we next made use of the switch is the drive circuit which allowed us to change the circuit with respect to the drive circuit. It was observed that if the resistance on the potentiometer was below a certain threshold value (about $68k\Omega$), the drive circuit lost its chaotic behavior and synchronization was lost. But above this value, the resistance could be maximized ($302k\Omega$) with almost no observable change to the synchronized signal. Thus the two circuits need not be identical, but only similar. How large the resistance could be made before synchronization was lost was around $400k\Omega$, as the drive system again lost its chaotic

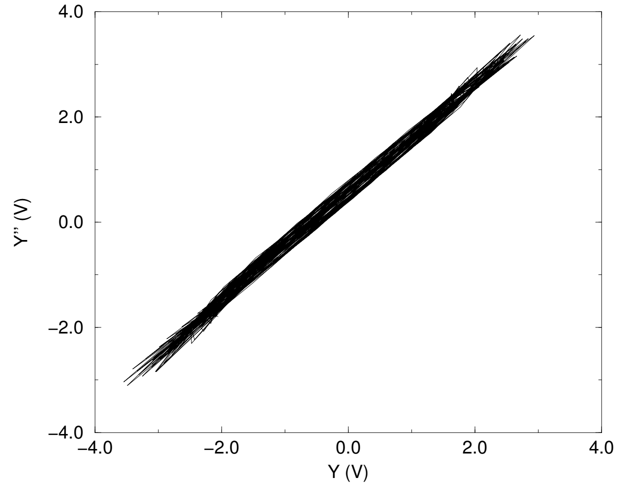

Figure 7: Synchronized signal between both circuits.

behavior.

Also observed was some very strange behavior when the $75k\Omega$ resistor in the drive circuit was replaced by a $12k\Omega$ resistor. In general, the time scales at which things happened were so short that they were not directly observable. But when this change was made, a very interesting symmetry emerged in the synchronized signal which acted on a dynamic timescale ranging from 0.25 to 10 seconds. The pattern was still along the $y = y''$ line, but the signal was observed to snake in close to the origin, stay there for a bit, then loop out away from the origin. This happened in a symmetric manner on both sides going away and back in towards the origin.

This resistor was replaced with a potentiometer to see how changing the resistance around this region would affect the dynamics. Below $10k\Omega$, the drive circuit was not chaotic. When the resistance was slightly increased, this behavior was seen. As the resistance was then slowly raised, the timescale was seen to speed up while the loops flattened out, smearing out along the $y = y''$ line and becoming the observed signal as shown in Fig. 6. A suggestion for this behavior is that a high period fixed point near the origin appeared in this parameter region which caused the observed behavior by pushing orbits away from the origin. As the resistance was increased, it got weaker and weaker until it disappeared.

Our approach in this experiment was one of a qualitative nature. We set out to construct a physical system which displayed the characteristics predicted by theory. For our purposes, error did not play a signifigant role in the broad picture we have painted. Nonetheless, for other possible scenarios we might face down the road (as discussed in the conclusions), it might be neccessary to take all possible sources of error into account. The major sources were from the circuit elements, the resistors having a 2% tolerance while the capacitors and the potentiometer had a tolerance of 5%. Other possible sources of error might have come from irregular output from the power source (though this would be negligible as very clean power sources were used), leakage of charge in the breadboard, and a bad circuit element (such as a faulty diode).

5 CONCLUSIONS

Chaos is sometimes described as a situation in which a system gets out of synchronization with itself, resulting a complex, nonperiodic behavior. What we have seen here is a means of physically observing chaotic behavior predicted by mathematical theory. Also, more importantly, we showed a means of controlling chaos through synchronization. There are many interesting applications for this which has generated a lot of active research. A very interesting step forward is incorporating this concept of synchronization into control of chaos in laser ring cavities, an exciting area of interest which offers many practical applications in communications.

But let us keep our heads out of the clouds for a minute and look at what possibilities and results we can still obtain from our circuit design. Some interesting avenues of exploration include:

- incorporating a greater diversity of circuit elements into the experiment to better understand how much we can change one circuit with respect to the other and still achieve synchronization between the two
- try to measure the Lyapunov exponents from our data and see how they compare to those predicted from theory (here, taking all sources of error into account would be vital)

- using some additional elements such as a function generator to produce the period doubling cascade for our circuit
- explore using the synchronization as a means for transmitting a signal, it would be fascinating to send a signal into the circuit as your input voltage and somehow retieve it back through the synchronized signal

The study of chaos and its control is certainly an interesting one. We live in a technological age where a lot is happening very quickly and we thus have the privelage of watching the mysteries of chaos unfold before our eyes.

6 Acknowledgements

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