The permutation group S_3

Consider the permutations of 3 objects. There are $3! = 6$ permutations. We can label a permutation by its action on a set of objects $\{O_1, O_2, O_3\}$ to locations $\{L_1, L_2, L_3\}$.

For example, the permutation $(2,3,1)$ corresponds to moving object O_2 to L_1, O_3 to L_2 and O_1 to L_3.

So the set of permutations are:

$$\begin{aligned}
e &\equiv (1,2,3) \\
a &\equiv (2,3,1) \\
b &\equiv (3,1,2) \\
x &\equiv (1,3,2) \\
y &\equiv (3,2,1) \\
z &\equiv (2,1,3)
\end{aligned}$$

The group multiplication table is:

	e	a	b	x	y	z
e	e	a	b	x	y	z
a	a	b	e	z	x	y
b	b	e	a	y	z	x
x	x	y	z	e	a	b
y	y	z	x	b	e	a
z	z	x	y	a	b	e

Alternatively, one can think in terms of rotations $\{a, b\}$ and reflections $\{x, y, z\}$ of an equilateral triangle.

We see that the set $\{e, a, b\}$ forms a subgroup. We see that this subgroup is the same as Z_3. Groups with different physical origin, and yet the same group structure are isomorphic.

A two dimensional irreducible representation of S_3 is:

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(a) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad D(b) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$$D(z) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(x) = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad D(y) = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

It is interesting that this irreducible representation is more that 1 dimensional. It is necessary that at least some of the representations of a <u>non-Abelian</u> group must be matrices rather than numbers. Only matrices can reproduce the non-Abelian multiplication laws.

Here's a 3 dimensional representation of S_3

$$D(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D(a) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad D(b) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$D(z) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad D(y) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

This particular representation is important because it is the defining representation for the group - it actually implements the permutations on the states.

$$\begin{aligned} D_3(a)|1\rangle &= \sum_k |k\rangle \, [D_3(a)]_{k1} = |3\rangle \\ D_3(a)|2\rangle &= \sum_k |k\rangle \, [D_3(a)]_{k2} = |1\rangle \\ D_3(a)|3\rangle &= \sum_k |k\rangle \, [D_3(a)]_{k3} = |2\rangle \end{aligned}$$

This 3 dimensional representation decomposes into a direct sum of the irreducible representations,

$$D_3 = D_1 \oplus D_2$$