Universal Gravity*

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Abstract

What is the best way to reduce trade frictions when resources are scarce? To answer this question, we develop a framework that nests previous general equilibrium gravity models and show that the macro-economic implications of these various models depend crucially on two key model parameters, which we term the "gravity constants." Based only on the value of the gravity constants, we derive sufficient conditions for the existence and uniqueness of the trade equilibrium and, given observed trade flows, completely characterize all comparative statics for any change in bilateral trade frictions. We then develop a methodology for estimating these gravity constants without needing to assume a particular micro-foundation of the gravity trade model. Finally, we use these results to derive the set of trade friction reductions that (to a first-order) maximize welfare gains given an arbitrary constraint.

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1 Introduction

What is the best way to reduce trade frictions when resources are scarce? This question is of first-order importance to policy makers, but trade economists have made little progress finding an answer. A primary difficulty is that the numerous and varied general equilibrium forces at play in trade models imply that changing any particular bilateral trade friction will affect not only those trading partners but also all other locations in the world. This makes it difficult to predict the effect of any particular change in trade frictions, let alone determine the optimal set of trade friction reductions. Adding to the difficulty is that while some work has been done understanding general equilibrium forces in particular models, little is known about the extent to which these forces differ across models. At a first glance, characterizing the optimal set of bilateral trade friction reductions while accounting for general equilibrium forces without relying on a particular model seems a daunting task.

In this paper, we attempt this daunting task. To do so, we develop a framework that nests previous gravity trade models in order to provide a "universal" characterization of their general equilibrium forces.[1] We show that the main theoretical properties and, given observed trade flows, all comparative statics of gravity trade models depend solely on two key model parameters that we call "gravity constants." We then provide closed form expressions for the complete set of (local) comparative statics and use these expressions to develop a method of estimating the gravity constants without needing to choose a particular trade model. Finally, we use the resulting estimates to derive the set of unilateral and multilateral trade friction reductions that (to a first-order) maximize welfare gains given an arbitrary constraint.

The universal gravity framework we develop is based on four restrictions: (i) a "modern" version of gravity, whereby bilateral trade flows depend on (endogenous) exporter and importer shifters and (exogenous) bilateral trade frictions;[2] (ii) aggregate output equals total sales; (iii) trade is balanced; and (iv) (gross) income is a log-linear function of the exporting and importing shifters (which practically translates to the condition that gross income is proportional to labor income). The aforementioned gravity constants are simply the coefficients of this log-linear function. It turns out that these assumptions – which are ubiquitous

[1]Examples of gravity models covered under our specification is perfect competition models such as Anderson (1979), Anderson and Van Wincoop (2003), Eaton and Kortum (2002), Caliendo and Parro (2010) monopolistic competition models such as Krugman (1980), Melitz (2003) as specified by Chaney (2008), Arkolakis, Demidova, Klenow, and Rodríguez-Clare (2008), Di Giovanni and Levchenko (2008), Dekle, Eaton, and Kortum (2008), and the Bertrand competition model of Bernard, Eaton, Jensen, and Kortum (2003); see Table 1 for details.

[2]This version of the gravity model was first introduced by Eaton and Kortum (2002), Anderson and Van Wincoop (2003), and Redding and Venables (2004). Baldwin and Taglioni (2006) and Head and Mayer (2013) carefully discuss the econometric issues arising from the use of this specification.

throughout the trade literature – impose sufficient structure on aggregate trade flows to completely characterize all general equilibrium interactions.

We derive sufficient conditions for the existence and uniqueness of the equilibrium of the model that depend solely on the gravity constants.Given the simple mapping of different gravity models to gravity constants, these sufficient conditions are straightforward to check and relax the sufficient conditions presented by Alvarez and Lucas (2007). This methodology can also be extended to consider multiple sectors of production, as in Chor (2010), Costinot, Donaldson, and Komunjer (2010), and Caliendo and Parro (2010) and economic geography models with mobile factors of production as in Helpman (1998) and Allen and Arkolakis (2014).

When trade frictions are "quasi-symmetric" (as is assumed in much of the empirical gravity literature, e.g. Eaton and Kortum (2002) and Waugh (2010)), we further show that the unique way of satisfying trade balance is for trade flows to be bilaterally balanced. This implies an equilibrium relationship between the exporting and importing shifters, a result (implicitly) used by Anderson and Van Wincoop (2003), Allen and Arkolakis (2014) and others to simplify the equilibrium system of equations for particular models. Given this result, we show that quasi-symmetric trade costs extend the range of gravity constants for which uniqueness can be ensured.

Our second theoretical result is to show that there exist two "dual" interpretations of the general equilibrium gravity model. In the first interpretation, a planner maximizes world income subject to trade remaining balanced and an aggregate world resource constraint. In the second interpretation, a planner maximizes a weighted average of world welfare subject to only the aggregate world resource constraint, where welfare is assumed to be written as a function of trade openness (as in the class of trade models considered by Arkolakis, Costinot, and Rodríguez-Clare (2012)). Using these dual interpretations, we apply the envelope theorem to derive the elasticity of both world income and world welfare to any bilateral trade costs, which can both be expressed solely as a function of observed trade flows and the gravity constants. While the expression for world income is, to the best of our knowledge, novel, the expression for world welfare has been derived previously for gravity models with CES demand by Atkeson and Burstein (2010), Burstein and Cravino (2012), and Fan, Lai, and Qi (2013); our derivation extends this result to any gravity trade model where welfare can be expressed as a function of trade openness (which Arkolakis, Costinot, Donaldson, and Rodríguez-Clare (2012) show holds for a large class of homothetic utility functions). This latter results will prove useful for determining the welfare-maximizing trade friction reductions.

We then turn to the empirical properties of the model by asking what can be said using our framework given observed trade flows. We first characterize the extent to which model fundamentals can be recovered from the trade data. We show that trade models are intrinsically under-identified: the same trade data can be perfectly matched by different combinations of model fundamentals. Notably, the gravity constants cannot be identified from observed trade flows alone. This result provides a general characterization of the non-identification inherent to gravity models discussed for particular models by Waugh (2010), Eaton, Kortum, Neiman, and Romalis (2011), Burstein and Vogel (2012), Ramondo, Rodríguez-Clare, and Saborio-Rodriguez (2012) and Arkolakis, Ramondo, Rodríguez-Clare, and Yeaple (2013).

To examine how changes in bilateral trade frictions affect equilibrium trade flows and incomes we first derive an analytical expression for the (large) matrix of elasticities of all bilateral trade flows to changes in all bilateral trade frictions. As with the aggregate elasticities, this expression depends only on observed trade flows and the gravity constants, indicating that apart from these two model parameters, all macro-economic implications – i.e. the changes in trade flows, gross incomes, and, given the assumption above, welfare – for all gravity models are the same. We then derive a system of equations that show how arbitrary (possibly non-infinitesimal) changes to the trade friction matrix affect macro-economic variables; this expression also depends only on the gravity constants and observed trade flows. While the non-infinitesimal results generalize those developed by Dekle, Eaton, and Kortum (2008) and Arkolakis, Costinot, and Rodríguez-Clare (2012), the closed form solution for the trade elasticities is, the the best of our knowledge, the first in the literature.

We then turn to the question of welfare-maximizing trade friction reductions. Given the closed form expressions of the comparative statics, we show how to calculate the set of trade friction changes that maximize (to a first-order) the change in welfare for an arbitrary constraint on the total change in trade frictions. In special cases, this expression also takes a closed form: for example, the eigenvector corresponding to the largest eigenvalue of the observed trade flow matrix is the set of trade friction reductions that maximizes the increase in world welfare, countries are constrained to reduce their trade frictions non-discriminately, and the costs of trade friction reductions are equal for all countries. While Ossa (2014) uses a computational approach to calculate optimal tariffs, we offer the first closed form characterization of the welfare-maximizing change in trade frictions in a many location general equilibrium framework.

More generally, the welfare-maximizing set of trade friction reductions will depend on the value of the gravity constants, so we show how the general equilibrium gravity estimator can be combined with the observed trade flows and an observed trade cost shock – in our case, a number of countries joining the WTO between 1995 and 2005 – to estimate the gravity constants without needing to specify a micro-economic foundation. Using the analytical expression for comparative statics we derive, we develop a new gravity estimator that explicitly incorporates the general equilibrium effects that a change in the bilateral trade friction between any two countries has on all other bilateral trade flows. This estimator is in the spirit of the structural estimation done by Anderson and Van Wincoop (2003); unlike Anderson and Van Wincoop (2003), however, we derive a closed form solution for estimator, and show that it can be interpreted as an ordinary least squares regression where the typical gravity regressors have undergone a transformation to account for general equilibrium effects.

Finally, we use our estimates of the gravity constants to determine the welfare-maximizing unilateral and multilateral set of trade friction reductions. We assume for simplicity that all trade frictions are equally costly to reduce, but emphasize that our methodology can be applied for any arbitrary cost of reducing trade frictions. We find that countries like Cuba and North Korea have the most to gain from unilaterally reducing their import frictions. The welfare-maximizing multilateral trade friction reductions are concentrated amongst the wealthiest countries, which causes the welfare in those countries to disproportionately increase; indeed, the welfare of the poorest countries are actually falls slightly.

Our work is related to a small but growing literature analyzing the structure of general equilibrium models of trade. Notably, Arkolakis, Costinot, and Rodríguez-Clare (2012) derive a closed form expression for changes in welfare as a function of changes in openness, allowing for ex-post evaluations of the welfare effects of changes in trade frictions. In contrast, we derive analytical expressions for all macroeconomic outcomes of interest using only the initial level of trade flows, thereby allowing for ex-ante predictions of the change in welfare for possible changes in trade frictions. Notice that in all the models we consider the elasticity of trade does not vary with the level of trade (for models with variable elasticity see Novy (2010), Head, Mayer, and Thoenig (2014),Melitz and Redding (2014), and Fajgelbaum and Khandelwal (2013)). While we assume that the gravity constants are fixed in this paper, our derivations show how changes to those gravity constants affect the general equilibrium forces of the model.

Our paper is also related to Costinot (2009), who examines the patterns of trade that hold true across many models. His primary focus, however, is on the specialization of countries in particular sectors, whereas we are concerned with the pattern of aggregate trade flows in a gravity framework. More broadly, our paper shows that the macroeconomic conditions inherent in gravity trade models impose sufficient structure so that its particular microeconomic details do not pose a problem in its characterization. Given the difficulties arising in guaranteeing the uniqueness of equilibrium and characterizing comparative statics in general equilibrium models, this constitutes a significant benefit.[3]

[3]See Kehoe (1985) and Shafer and Sonnenschein (1993) discussing issues arising from aggregation or the

The paper is organized as follows. In the next section, we present the universal framework and discuss how it nests existing general equilibrium gravity models. In Section 3, we present the theoretical results for existence and uniqueness, as well as the dual interpretations of the problem. In Section 4, we present the empirical results for identification and comparative statics given observed trade flows. In Section 5, we apply these results study optimal trade friction reductions. Section 6 concludes.

2 The universal gravity framework

Consider a world comprised of a set $S \equiv \{1, ..., N\}$ of locations.[4] Let Y_i denote the gross income of location i, X_{ij} the total value of location j's imports from location i, and $K_{ij} > 0$ the associated bilateral trade frictions. In our universal gravity framework, the endogenous outcomes of the model are summarized by the variables γ_i and δ_i, which capture the exporting and importing "capacity" of each location, respectively. In the context of a particular micro-founded model, these variables will map to a combination of endogenous model outcomes – such as wages or the price index of a location – and model fundamental parameters – such as productivities or labor endowments.

We focus our attention on models satisfying several equilibrium conditions. These conditions are sufficient to fully characterize the general equilibrium structure of the models yet general enough to nest a number of seminal gravity trade models.

Gravity. Our first condition restricts our attention to trade models which yield a generalized form of the gravity equation pioneered by Tinbergen (1962).

Condition 1. For any countries $i \in S$ and $j \in S$, the value of aggregate bilateral trade flows is given by $X_{ij} = K_{ij}\gamma_i\delta_j$.

The two endogenous variables have an explicit role in this equation as exporting and importing shifters. The exogenous bilateral trade frictions capture the effects of bilateral trade costs; they could be inverse functions of bilateral distance, various exporting barriers faced by exporting countries, etc. Note that larger values of K_{ij} indicate lower bilateral trade frictions.

Goods market clearing and trade balance. We proceed by defining two equilibrium conditions that are standard assumptions for modern general equilibrium gravity models: goods market clearing and trade balance. We say that *goods markets clear* if the output

consideration of production.

[4]The choice of a finite number of locations is not necessary for the the results that follow, but it saves on notation, avoids several thorny technical issues, and is consistent with the majority of the trade literature.

for all $i \in S$ is equal to the value of the good sold to all destinations. This condition is practically an accounting identity. Formally:

Condition 2. For any location $i \in S$, $Y_i = \sum_{j \in S} X_{ij}$.

Furthermore we assume that *trade is balanced,* i.e. that output for all $i \in S$ is equal to the amount spent on good purchased from all other destinations:

Condition 3. For any location $i \in S$, $Y_i = \sum_{j \in S} X_{ji}$.

While balanced trade is a standard equilibrium condition in general equilibrium gravity models, it is important to note that trade is not balanced empirically. This empirical discrepancy is an inherent limitation arising from the use of a static model to explain an empirical phenomenon with dynamic aspects. However, given both its ubiquity in the literature and the necessarily *ad hoc* nature of any alternative assumption (e.g. exogenous trade deficits), balanced trade seems the natural assumption on which to focus. We relax this assumption in the characterization of the empirical properties of the model in Section 4 when we introduce exogenous deficits across countries, following Dekle, Eaton, and Kortum (2008).[5]

Relationship between income and the shifters. Our last condition postulates a log-linear parametric relationship between gross income and the exporting and importing shifters:

Condition 4. For any location $i \in S$, $Y_i = B_i \gamma_i^{\alpha} \delta_i^{\beta}$, where we define $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ to be the *gravity constants* and $B_i > 0$ is an (exogenous) location specific shifter.

Condition C.4 regulates the extent to which income responds to changes in the two endogenous shifters. The gravity constants determine the importance of the exporting and importing shifters in determining a location's income, which will prove crucial in determining the general equilibrium forces of the system. In practice, C.4 is analogous to the standard condition that the income in a location is proportional to the income earned by the factors of production in that location (e.g. local labor). As a result, models in which income is not proportional to returns to factors will not generally satisfy C.4: for example, by including tariff revenue as an additive source of income, Ossa (2014) does not satisfy this condition.

[5] In fact, total income may differ from total spending even in a static setup, e.g. if fixed exporting costs are partially paid in labor of the exporting location (see e.g. Arkolakis (2010) and Di Giovanni and Levchenko (2009)). Even in such cases, however, for the trade balance to hold it is sufficient that the income generated from the fixed exporting cost in each bilateral trading relationship is a constant fraction of bilateral sales, as pointed out by Arkolakis, Costinot, and Rodríguez-Clare (2012).

Our final condition is a normalization. Note that the system of equations defined by C.2-C.3 already implies Warlas' law.[6] As a result, we make the following assumption to pin down the equilibrium level of trade flows.

Condition 5. World income equals to one:[7]

$$\sum_i Y_i = 1. \tag{1}$$

In what follows, for any given gravity constants α and β, income shifters $\{B_i\}$ and bilateral trade frictions $\{K_{ij}\}$, we define a *general equilibrium gravity model* to be a set of export shifters $\{\gamma_i\}$ and import shifters $\{\delta_i\}$ that satisfy gravity, goods market clearing, trade balance, factor market clearing, and the normalization (1), i.e. a general equilibrium gravity model is any trade model where conditions C.1-C.5 are satisfied.

Example: the Armington model To make things concrete, we will provide a simple example of a general equilibrium trade model. In the Armington (1969) model, first formulated in general equilibrium by Anderson (1979), each location produces a differentiated variety (which is sold at marginal cost) and consumers have constant elasticity of substitution (CES) preferences with elasticity of substitution σ and where we denote by P_i the Dixit-Stiglitz CES price index across all varieties. We assume that production combines labor and an intermediate input in a Cobb-Douglas fashion, where the share of labor is given by $\zeta \in (0, 1]$, and the intermediate input uses the same CES aggregator of goods from all countries as the final consumption good. Thus, with productivity A_i, the unit cost of production in location i is simply $w_i^{\zeta} P_i^{1-\zeta}/A_i$.

In this model, the value of bilateral trade from $i \in S$ to $j \in S$ is:

$$X_{ij} = \tau_{ij}^{1-\sigma} \left(\frac{w_i^{\zeta} P_i^{1-\zeta}}{A_i} \right)^{1-\sigma} P_j^{\sigma-1} Y_j \tag{2}$$

where w_i is location's i wage, A_i is the location's productivity and the marginal production cost is $\frac{w_i}{A_i}$, τ_{ij} is the iceberg cost of delivering i's good in destination j, and Y_i is again its income. It is also straightforward to show that output is proportional to wage income and is given by

$$Y_i = w_i L_i / \zeta \tag{3}$$

[6] To see this note that summing these two equations over all $i \neq N$ and equating them we obtain $\sum_{i \neq N} \sum_j X_{ij} = \sum_{i \neq N} \sum_j X_{ji}$. By the definition of gross world income being total trade across all markets we obtain trade balance for the Nth location which implies Warlas' law.

[7] This is a valid normalization as long as $\alpha \neq \beta$. When $\alpha = \beta$, a suitable alternative normalization is $\sum_{i \in S} \gamma_i = 1$. None of the following results, unless explicitly noted, depend on the normalization chosen.

where L_i is the population in location i. According to the definition of gravity, C.1, we have

$$\gamma_i \equiv \left(\frac{w_i^{\zeta} P_i^{1-\zeta}}{A_i}\right)^{1-\sigma}, \ \delta_i \equiv P_i^{\sigma-1} Y_i,$$

which allows us to write C.4 as

$$Y_i = \gamma_i^{\frac{1}{1-\sigma\zeta}} \delta_i^{\frac{1-\zeta}{1-\sigma\zeta}} A_i^{\frac{\sigma-1}{\sigma\zeta-1}} L_i^{\frac{\zeta(\sigma-1)}{\sigma\zeta-1}},$$

so that $\alpha \equiv \frac{1}{1-\sigma\zeta}$, $\beta \equiv \frac{1-\zeta}{1-\sigma\zeta}$, and $B_i = A_i^{\frac{\sigma-1}{\sigma\zeta-1}} L_i^{\frac{\zeta(\sigma-1)}{\sigma\zeta-1}}$. Note that if $\sigma > 1$ and $\sigma\zeta > 1$, then $\alpha, \beta < 0$ and a higher productivity A_i will increase both the exporting ability and the income of the location. At the same time increases in wages increase exports but decrease income.

Table 1 lists a number of seminal trade models that fall into the universal gravity framework and show how the micro-founded model fundamentals map to the gravity constants α and β.[8] There are several things to highlight from this table: first, while in some models the elasticity of trade flows to the variable trade cost (often called the "trade elasticity") maps one-to-one to a gravity constant, in others it does not. Hence, the trade elasticity and the gravity constants are distinct parameters in general equilibrium gravity models. Second, the interpretation of the gravity constants depends on the particular model; as a result, different micro-foundations for the gravity model may imply different preferred values for the gravity constants. As we will see in the next two sections, this will prove important, as the gravity constants determine the strength of the general equilibrium forces in the model.

3 Theoretical properties

We first consider the theoretical properties of the general equilibrium gravity framework.

3.1 Existence and Uniqueness

In this section, we provide sufficient conditions for establishing existence and uniqueness in a general equilibrium gravity model. We start by formulating the equilibrium system implied by our assumptions. Using C.2 and C.3 and substituting out X_{ij} and Y_i with the definitions C.1 and C.4, respectively, yields:

[8] While these models all feature constant elasticity demand, our results hold for all models that satisfy the conditions below even if they depart from this assumption. See for example the class of homothetic and non-homothetic demand functions considered by Arkolakis, Costinot, Donaldson, and Rodríguez-Clare (2012).

$$B_i \gamma_i^{\alpha-1} \delta_i^{\beta} = \sum_j K_{ij} \delta_j \tag{4}$$

and

$$B_i \gamma_i^{\alpha} \delta_i^{\beta-1} = \sum_j K_{ji} \gamma_j \tag{5}$$

and using C.4, C.5 becomes:

$$\sum_i B_i \gamma_i^{\alpha} \delta_i^{\beta} = 1. \tag{6}$$

Thus, given model fundamentals B_i, K_{ij} and gravity constants α, β, equilibrium is defined as γ_i and δ_i for all $i \in S$ such that equations (4), (5) and (6) are satisfied. In the special case where $\alpha = \beta = 1$, it is immediately evident from equations (4) that (5) have a solution only if the matrices with elements $\frac{K_{ij}}{B_i}$ and $\frac{K_{ji}}{B_i}$ both have a largest eigenvalue equal to one. Since this will not generally be true, in what follows we exclude this case.

To proceed, we define $x_i \equiv B_i \gamma_i^{\alpha-1} \delta_i^{\beta}$ and $y_i \equiv B_i \gamma_i^{\alpha} \delta_i^{\beta-1}$. By reformulating the system in terms of x_i, y_i (see Appendix A.1 for details), equations (4) and (5) take the form of a standard system of non-linear equations. It turns out that this reformulation of the problem provides a method of solving for the trade equilibrium system using functions that map a compact space onto itself. This has two advantages over the standard formulation given in equations (4) and (5): first, by restricting the potential solution space, it facilitates the calculation of the equilibrium; second, it allows us to generalize results used in the study of integral equations to prove the following theorem:

Theorem 1. *Consider any general equilibrium gravity model. If $\alpha + \beta \neq 1$, then:*

i) The model has a positive solution and all possible solutions are positive;

ii) If $\alpha, \beta \leq 0$ or $\alpha, \beta \geq 1$, then the solution is unique.

Proof. See Appendix A.1. □

Note that condition (ii) of Theorem 1 provides sufficient conditions for uniqueness; for certain parameter constellations (e.g. particular geographies of trade costs), equilibria may be unique even if the conditions are not satisfied. In practice, however, we have found that there exist multiple equilibrium for particular geographies when condition (ii) is violated.

The methodology to prove Theorem 1 turns out to be quite general. In what follows, we provide two useful extensions.

Multiple sectors

Our approach also can be naturally extended to the cases where there are multiple sectors. Suppose there are a set $s \in \left\{1, ..., \tilde{S}\right\}$ of sectors and that the bilateral trade flow between location i and location j in sector s is

$$X_{ij}^s = K_{ij}^s \left(\gamma_i\right) \left(\delta_j^s\right),$$

where K_{ij}^s can include sector-specific trade frictions or productivities. With multi-sector gravity models, we implicitly assume that there are no frictions on labor markets so that the wages in location i is equalized across sectors. That is why we can assume that the origin effect, γ_i, is independent of the sector s. Condition C.4 becomes:

$$Y_i = B_i \left(\gamma_i\right)^\alpha \left(\prod_s \left(\delta_i^s\right)^{\theta^t}\right)^\beta.$$

The first two terms are the same as before, but the last term is slightly different from what we have in a single-sector economy.

The other two equilibrium conditions are:

$$\begin{aligned} \sum_j X_{j,i}^s &= B_i^s Y_i \\ \sum_s \sum_j X_{i,j}^s &= Y_i. \end{aligned}$$

The first equation assumes that location i's expenditure in each sector is a constant fraction of its total income. The second equation is the extension of the good market clearing condition we have in a single-sector case.

It turns out that the conditions for uniqueness with multiple sectors are the same as with a single sector, which we formalize in the following proposition:

Corollary 1. *(1) There exists a solution to the multi-sector gravity model if $\alpha, \beta \leq 0$ or $\alpha, \beta > 1$. (2) That solution is unique if $\alpha, \beta \leq 0$ or $\alpha, \beta > 1$.*

Proof. See Online Appendix B.1. □

Note that unlike the single sector case, we cannot prove the existence of a solution when it is not unique; this is due to the presence of cross-sectoral linkages.

Economic geography

Our approach can also be naturally extended to allow for labor mobility as in economic geography models. To do so, we slightly alter condition C.4 to allow for the gross income in a location to depend additionally on an endogenous constant λ, which can be interpreted as a monotonic transformation of welfare (which is equalized across locations in economic geography models). The level of this endogenous constant is then determined by a labor market clearing condition that can be written as a sum across locations of a log-linear function of endogenous variables.

Condition 4'. For any location $i \in S$, $Y_i = \frac{1}{\lambda} B_i \gamma_i^{\alpha} \delta_i^{\beta}$, where $\lambda > 0$ is an endogenous variable and all other variables are as above. Furthermore, we require that $\lambda^c = \sum_i C_i \gamma_i^d \delta_i^e$ for some $c, d, e \in \mathbb{R}$.

It is straightforward to show that the economic geography model of Allen and Arkolakis (2014) (which under certain parametric configurations is isomorphic to the economic geography models of Helpman (1998), Redding (2014), and Bartelme (2014)) satisfies Condition C.4'; see Online Appendix B.3 for details.

Given this alternative condition, we modify part (ii) of Theorem 1 slightly to prove the following Corollary:

Corollary 2. *Consider any economic geography model that satisfies conditions C.1, C.2, C.3, C.4', and C.5. Then (i) there exists a solution as long as $\alpha + \beta \neq 1$; and (ii) the equilibrium is unique if $\alpha, \beta \leq 0$ or $\alpha, \beta > 1$.*

Proof. See Online Appendix B.2. ☐

We should note that Corollary 2 shows that the uniqueness condition presented in Allen and Arkolakis (2014) holds for an arbitrary set of trade costs, i.e. the assumption in that paper of symmetric trade costs is not necessary. Indeed, as we now show, assuming a (generalized) form of trade cost symmetry has important implications for the equilibrium.

3.2 Quasi-symmetry

It turns out that we can extend the range in which uniqueness is guaranteed if we constrain our analysis to a particular class of trade frictions which are the focus of a large empirical literature on estimating gravity trade models. We call these trade frictions quasi-symmetric.

Definition 1. Quasi Symmetry: We say the trade frictions matrix $\mathbf{K}$ is *quasi-symmetric* if there exists a symmetric $N \times N$ matrix $\tilde{\mathbf{K}}$ and $N \times 1$ vectors K^A and K^B such that for all

$i, j \in S$ we have:

$$K_{ij} = \tilde{K}_{ij} K_i^A K_j^B, \text{ where } \tilde{K}_{ij} = \tilde{K}_{ji}$$

Loosely speaking, quasi-symmetric trade frictions are those that are reducible to a symmetric component and exporter- and importer-specific components. While restrictive, it is important to note that the vast majority of papers which estimate gravity equations assume that trade frictions are quasi-symmetric; for example Eaton and Kortum (2002) and Waugh (2010) assume that trade costs are composed by a symmetric component that depends on bilateral distance and on a destination or origin fixed effect.

When trade frictions are quasi-symmetric we can show that the system of equations (31) and (32) can be dramatically simplified, and the uniqueness more sharply characterized.

Theorem 2. *Consider any general equilibrium gravity model with quasi-symmetric trade costs. Then:*

i) The balanced trade condition is equivalent to the origin and destination shifters being equal up to scale, i.e.

$$\gamma_i K_i^A = \kappa \delta_i K_i^B \tag{7}$$

for some $\kappa > 0$ that is part of the solution of the equilibrium.

ii) If $\alpha + \beta \leq 0$ or $\alpha + \beta \geq 2$, the model has a unique positive solution.

Proof. See Appendix A.2. □

Part i) of the Theorem 2 is particularly useful since it allows to simplify the equilibrium system into a single non-linear equation:

$$\gamma_i^{\alpha+\beta-1} = \kappa^{\beta-1} \sum_j \widetilde{K}_{ij} B_i^{-1} \left(K_i^A\right)^{1-\beta} \left(K_i^B\right)^{\beta} \gamma_j. \tag{8}$$

In addition, because the exporter and importer shifters in gravity models will (generally) be composites of exogenous and endogenous variables, by showing that the two shifters are equal up to scale, Theorem 2 provides a more precise analytical characterization of the equilibrium. We should note that the results of Theorem 2 have already been used in the literature for particular models, albeit implicitly. The most prominent example is Anderson and Van Wincoop (2003), who use the result to show the bilateral resistance is equal to the price index.[9] To our knowledge, Head and Mayer (2013) are the first to recognize the importance of balanced trade and market clearing in generating the result for the Armington model; however, Theorem 2 shows that the result applies more generally to any general

[9]The result is also used in economic geography by Allen and Arkolakis (2014) to simplify a set on non-linear integral equations into a single integral equation.

equilibrium gravity model with quasi-symmetrical trade costs and balanced trade. Note, however, that part (i) of Theorem 2 also implies that for all $i, j \in S$, we have $X_{ij} = X_{ji}$, i.e. trade flows are bilaterally balanced, which is typically rejected empirically.

Figure 1 illustrates the range of α and β for which uniqueness of the model can be guaranteed. It should be noted that while most of the examination of existence and uniqueness of trade equilibria has proceeded on a model-by-model case, the gross substitute methodology used by Alvarez and Lucas (2007) has proven enormously helpful in establishing conditions for existence and uniqueness. It can be shown (see Online Appendix B.4) that an application of the gross-substitutes methodology works only when $\alpha \leq 0$ and $\beta \leq 0$; hence, the tools used in Theorems 1 and 2 extend the range of trade models for which uniqueness can be proven, including, for example, the Armington model with intermediate inputs.

Example: Armington model with quasi-symmetry Consider again an Armington model with intermediate inputs, but now assume that trade costs are quasi-symmetric. From part (i) of Theorem 2, we have $\gamma_i = \kappa \delta_i$, which implies:

$$\left(\frac{w_i^{\delta} P_i^{1-\delta}}{A_i}\right)^{1-\sigma} K_i^A = \kappa P_i^{\sigma-1} w_i L_i K_i^B,$$

or equivalently:

$$P_i = w_i^{\frac{1+(\sigma-1)\delta}{(1-\sigma)(2-\delta)}} \left(\kappa L_i A_i^{1-\sigma} \frac{K_i^B}{K_i^A}\right)^{\frac{1}{(1-\sigma)(2-\delta)}}. \tag{9}$$

Equation (9) provides some intuition for the uniqueness condition presented in Theorem 2: when $\sigma < \frac{1}{2}$, it is straightforward to show that the elasticity of the price index with respect to the wage is less than one. This implies that the wealth effect may dominate the substitution effect, so that the excess demand function need not be downward sloping.

In addition, combining equation (9) with equation (8), assuming $\delta = 1$, and noting that welfare $W_i = \frac{w_i}{P_i}$ yields the following equation:

$$\kappa W_i^{\sigma\tilde{\sigma}} L_i^{\tilde{\sigma}} = \sum_j K_{ij} A_i^{(\sigma-1)\tilde{\sigma}} A_j^{\sigma\tilde{\sigma}} L_j^{\tilde{\sigma}} W_j^{-(\sigma-1)\tilde{\sigma}}, \tag{10}$$

where $\tilde{\sigma} \equiv \frac{\sigma-1}{2\sigma-1}$.[10] Equation (10) holds for both trade models (where labor is fixed) and

[10] When there are only two countries (so that trade costs are necessarily quasi-symmetric), we can use equation (10) to derive a single non-linear equation that yields the relative welfare in the two countries

$$K_{22}\left(\frac{W_1}{W_2}\right)^{\sigma\tilde{\sigma}} - K_{11}\left(\frac{W_1}{W_2}\right)^{(1-\sigma)\tilde{\sigma}} + K_{21}\left(\frac{W_1}{W_2}\right)^{\tilde{\sigma}} = K_{12}.$$

Comparative statics for welfare with respect to changes in K_{ij} can be characterized using the implicit function theorem in this case.

economic geography models (where labor is mobile); in the former case, L_i is treated as exogenous parameter and W_i solved for; in the latter case L_i is treated as endogenous and W_i is assumed to be constant across locations. Hence, Theorem 2 highlights the fundamental similarity between trade and economic geography models.

3.3 Two dual representations

In this section, we show that the solution of the general equilibrium gravity model can be equivalently expressed as the solution to two distinct maximization problems: one for world income and one for world welfare. These dual interpretations allow us to apply the envelope theorem to derive expressions for the elasticity of world income and world welfare, respectively, to any change in bilateral trade frictions.

Consider first the problem of choosing the set of origin and destination shifters to maximize world income subject to trade remaining balanced and the aggregate feasibility constraint that world income can be equivalently calculated by summing over trade flows or using condition C.4:

$$\max_{\{\gamma\},\{\delta\}} \sum_{i\in S}\sum_{j\in S} K_{ij}\gamma_i\delta_j$$

$$\text{s.t. } \sum_j K_{ij}\gamma_i\delta_j = \sum_j K_{ji}\gamma_j\delta_i \ \forall i \in S \text{ and } \sum_{i\in S}\sum_{j\in S} K_{ij}\gamma_i\delta_j = \sum_{i\in S} B_i\gamma_i^{\alpha}\delta_i^{\beta}, \tag{11}$$

where we now choose as a numeraire that $\gamma_1 = 1$ rather than choosing world income as a numeraire (since maximizing the numeraire is not a well defined problem).

Alternatively, consider the problem of maximizing a weighted average of world welfare subject to only the aggregate feasibility constraint. Of course, in the absence of a micro-foundation of the gravity trade model nothing can be directly said about the welfare of the equilibrium (as we have not specified preferences). However, Arkolakis, Costinot, and Rodríguez-Clare (2012) show that for a large class of trade models, the welfare of a location can be written solely as an increasing function of its openness to trade and an exogenous parameter, i.e. for all $i \in S$, welfare in location i, can be written as:[11]

$$W_i = C_i^W \lambda_{ii}^{-1/\rho} = C_i^W \left(B_i\gamma_i^{\alpha-1}\delta_i^{\beta-1}\right)^{1/\rho}, \tag{12}$$

where $C_i^W > 0$ is an (exogenous) parameter and $\rho > 0$ is an exogenous scalar. If welfare can be written as in equation (12), we can define world welfare as a weighted average of the

[11] In addition to CES preferences, this includes a larger class of homothetic demand functions including the symmetric translog demand function (see also Feenstra (2003b)) and the Kimball demand function (see Kimball (1995)); see Arkolakis, Costinot, Donaldson, and Rodríguez-Clare (2012).

welfare in each location:

$$W \equiv \sum_{i \in S} \omega_i W_i = \sum_{i \in S} \omega_i C_i^W \left(B_i \gamma_i^{\alpha-1} \delta_i^{\beta-1} \right)^{1/\rho},$$

where $\omega_i > 0$ are the weights placed on the welfare in each location. Then the following world welfare maximization problem is well defined:

$$\max_{\{\gamma\},\{\delta\}} W$$

$$\text{s.t. } \sum_{i \in S} \sum_{j \in S} K_{ij} \gamma_i \delta_j = \sum_{i \in S} B_i \gamma_i^{\alpha} \delta_i^{\beta}. \tag{13}$$

It turns out that the solution to both the world income maximization problem (11) and the world welfare maximization problem (13) is the solution to the general equilibrium gravity model, which we prove in the following proposition:

Proposition 1. *Consider any general equilibrium gravity model. If $\alpha + \beta > 2$ or $\alpha + \beta < 0$ (which by part (ii) of Theorem 2 guarantees uniqueness), then:*

(i) The solution of the general equilibrium gravity model is equivalent to the solution of the world income maximization problem (11).

(ii) If welfare can be expressed as in equation (12), then there exists a set of weights $\{\omega_i\}$ such that the solution of the general equilibrium trade model is equivalent to the solution of the world welfare maximization problem (13).

Proof. See Appendix A.3. □

An advantage of the dual approach is that it allows us to apply the envelope theorem to derive an expression for how any change in bilateral trade frictions affects world income and world welfare. Using the world income maximization dual interpretation, the elasticity of world income to K_{ij} is:

$$\frac{\partial \ln Y^W}{\partial \ln K_{ij}} = \left[(\kappa_i - \kappa_j) + \frac{\alpha + \beta}{\alpha + \beta - 2} \right] \frac{X_{ij}}{Y^W}, \tag{14}$$

where κ_i is the Lagrange multiplier on the balanced trade constraint and can be shown to be the solution to the following linear system:

$$\frac{\beta - \alpha}{\alpha + \beta - 2} + \kappa_i = \sum_{j \in S} \frac{X_{ij}}{Y_i} \kappa_j.$$

When trade costs are quasi-symmetric, part (i) of Theorem 2 implies that $X_{ij} = X_{ji}$ so that expression (14) becomes even more straightforward:

$$\frac{1}{2}\left(\frac{\partial \ln Y^W}{\partial \ln K_{ij}} + \frac{\partial \ln Y^W}{\partial \ln K_{ji}}\right) = \frac{\alpha+\beta}{\alpha+\beta-2}\frac{X_{ij}}{Y^W}, \tag{15}$$

i.e. a symmetric increase in any pair of K_{ij} (i.e. a symmetric reduction of bilateral trade frictions) increases world income by an amount proportional to the importance of those bilateral trade flows, where the proportion is a function of the gravity constants.[12]

Applying the envelope theorem to the world welfare maximization interpretation, the elasticity of world welfare to K_{ij} is even simpler:

$$\frac{\partial \ln W}{\partial \ln K_{ij}} = \frac{1}{\rho}\frac{X_{ij}}{Y^W}. \tag{16}$$

This expression has been derived for gravity models with CES demand by Atkeson and Burstein (2010), Burstein and Cravino (2012), and Fan, Lai, and Qi (2013); our derivation extends this result to any gravity trade model where welfare can be expressed as in equation (12). This expression will prove useful when examining the set of welfare-maximizing trade frictions reductions in Section 5.

4 Empirical implications

Thus far, we have examined the theoretical properties of the general equilibrium gravity framework. We now ask in what ways can the general equilibrium gravity framework be used in conjunction with an observed set of bilateral trade flows. In particular, given any set of observed trade flows $\{X_{ij}\}$ and gravity constants α and β, we show to what extent model fundamentals such as bilateral trade frictions can be recovered and derive expressions for how the model equilibrium will change with any change in the underlying bilateral trade flows.

Before proceeding to these results, however, we must address an issue familiar to trade empiricists: in contrast to assumption C.3, trade data is usually not balanced. It is not obvious how one ought to address unbalanced trade (which we view as a dynamic phenomenon) in the context of a static model. In what follows, we treat the trade deficits as exogenous, as in Dekle, Eaton, and Kortum (2008). Define $E_i \equiv \sum_{j \in S} X_{ji}$ to be the expenditure in location $i \in S$, $Y_i \equiv \sum_{j \in S} X_{ij}$ to be the output in location $i \in S$ and $\bar{D}_i \equiv E_i - Y_i$ to be the

[12] Note that if $\alpha + \beta > 2$ or $\alpha + \beta < 0$, then $\frac{\alpha+\beta}{\alpha+\beta-2} > 0$.

(exogenous) trade deficit. In this case, equation (5) becomes:

$$B_i \gamma_i^{\alpha} \delta_i^{\beta} + \bar{D}_i = \sum_j K_{ji} \gamma_j \delta_i \tag{17}$$

There are two disadvantages to allowing for exogenous deficits: first, the theoretical results presented above (in particular, the uniqueness of the equilibrium) do not necessarily hold; second, welfare cannot be expressed as in equation (12). Subject to these caveats, the empirical results below hold with (exogenous) trade deficits.

4.1 Identification

We first examine the extent to which one can recover model parameters given observed trade flows alone, which we summarize in the following proposition.

Proposition 2. *Take as given any (possibly unbalanced) set of observed trade flows* $\{X_{ij}\}$*. Choose any gravity constants* α *and* β*, set of income shifters* $\{B_i\}$ *and set of own trade flow frictions* $\{K_{ii}\}$*. Then there exists a unique set of* $\{K_{ij}\}_{i \neq j}$ *(and set of exogenous trade deficits* $\{\bar{D}_i\}$*) that, given the chosen parameters, yield equilibrium trade flows that equal to the observed trade flows.*

Proof. See Appendix A.4. □

Proposition 2 shows that general equilibrium gravity models are fundamentally under-identified in two ways. First, there exists an inability to determine which model parameter is responsible for the level of trade flows. In particular, the scale of the bilateral trade frictions and the income shifters cannot be separately identified: intuitively, a larger value of the income shifter can be counteracted with lower bilateral trade frictions without affecting the equilibrium. Second, the observed trade flows can be rationalized by the model for any chosen value of α and β (as long as $\alpha \neq \beta$). That is, the gravity constants cannot be identified using trade flow data alone. This result underpins why previous attempts to estimate (transformations of) these gravity constants have relied on additional sources of data such as prices (see e.g. Eaton and Kortum (2002), Simonovska and Waugh (2009), and Waugh (2010)). As we show below in Section 5.2, however, the gravity constants can be estimated without relying on a particular model if both trade flows and information regarding trade frictions are observed.

4.2 Comparative Statics

In this section, we consider how changes in model fundamentals affect trade flows and income. We first consider infinitesimal changes and derive a closed form expression that yields the elasticities of all exporter and importer shifters to all bilateral trade frictions. This expression depends only on observed trade flows and the gravity constants. We then show the same result holds for an arbitrary (non-infinitesimal) change to the trade friction matrix.

4.2.1 Local Comparative Statics

Consider a local change in any bilateral trade friction K_{ij}; how does this affect equilibrium trade flows and incomes? It turns out it is possible to provide an analytical expression for the elasticity of all exporter or importers shifters to all infinitesimal changes in bilateral trade frictions.

Define $\mathbf{X}$ to be the observed $N \times N$ trade flow matrix whose $\langle i,j \rangle^{th}$ element is X_{ij}, $\mathbf{Y}$ is the $N \times N$ diagonal income matrix whose i^{th} diagonal element is Y_i, and $\mathbf{E}$ is the $N \times N$ diagonal income matrix whose i^{th} diagonal element is E_i. Define the $2N \times 2N$ matrix $\mathbf{A}$ as follows:

$$\mathbf{A} \equiv \begin{pmatrix} (\alpha - 1)\,\mathbf{Y} & \beta\mathbf{Y} - \mathbf{X} \\ \alpha\mathbf{E} - X^T & (\beta - 1)\,\mathbf{Y} \end{pmatrix}.$$

Furthermore, define $\mathbf{A}^+$ to be the Moore-Penrose pseudo-inverse of $\mathbf{A}$ and A^+_{kl} to be the $\langle k,l \rangle^{th}$ element of $\mathbf{A}^+$. The following proposition shows how all elasticities can be immediately determined from matrix $\mathbf{A}^+$:

Proposition 3. *Consider any general equilibrium gravity model yielding the matrix of equilibrium trade, income and expenditure flows* $\mathbf{X}, \mathbf{Y}, \mathbf{E}$, *respectively. Then:*

i) If $\mathbf{A}$ *has rank* $2N - 1$, *then:*

$$\frac{\partial \ln \gamma_l}{\partial \ln K_{ij}} = X_{ij} \times \left(A^+_{l,i} + A^+_{N+l,j} + c\right) \text{ and } \frac{\partial \ln \delta_l}{\partial \ln K_{ij}} = X_{ij} \times \left(A^+_{N+l,i} + A^+_{l,j} + c\right), \tag{18}$$

where c *is a scalar that ensures the normalization C.5 holds.*

ii) If trade is balanced then $\mathbf{A}$ *is rank* $2N - 1$ *(1) when either* $\alpha, \beta \leq 0$ *or* $\alpha, \beta > 1$ *or* $|\alpha| > 1$ *and* $\beta = 1$ *or* $|\beta| > 1$ *and* $\alpha = 1$*; or (2) for all but a finite number of constellations of* (α, β) *if trade costs are quasi-symmetric.*

Proof. See Appendix A.5. □

We should note that the choice of the constant c (and hence the elasticities) will depend on the normalization chosen: given C.5, $c = \frac{1}{(\alpha+\beta)Y^W} X_{ij} \sum_l Y_l \left(\alpha\left(A^+_{l,i} + A^+_{N+l,j}\right) + \beta\left(A^+_{N+l,i} + A^+_{l,j}\right)\right)$, whereas the alternative normalization $\gamma_1 = 1$ implies $\frac{\partial \ln \gamma_1}{\partial \ln K_{ij}} = 0$, so that $c = X_{ij} \times (A_{1,i} + A_{N+1,j})$. We should also note that while the expression for the elasticities will hold whenever $\mathbf{A}$ has rank $2N - 1$, which apart from the sufficient conditions above can also be checked directly given observed trade flows and a chosen set of gravity constants.

Because all model outcomes (e.g. trade flows and location incomes) are functions of the exporter and importer shifters, Proposition 3 provides a closed form solution for the the complete set of model elasticities. In particular, it is straightforward to determine how changing the trade costs from i to j affects trade flows between any other bilateral trade pair k and l:[13]

$$\frac{\partial \ln X_{kl}}{\partial \ln K_{ij}} = \frac{\partial \ln \gamma_k}{\partial \ln K_{ij}} + \frac{\partial \ln \delta_l}{\partial \ln K_{ij}} = X_{ij} \times \left(A^+_{k,i} + A^+_{N+k,j} + A^+_{N+l,i} + A^+_{l,j} + 2c\right). \tag{19}$$

Similarly, Proposition 3 can be applied to determine how changing the trade costs from i to j affects income in any location l:

$$\frac{\partial \ln Y_l}{\partial \ln K_{ij}} = \alpha \frac{\partial \ln \gamma_l}{\partial \ln K_{ij}} + \beta \frac{\partial \ln \delta_l}{\partial \ln K_{ij}} = X_{ij} \times \left(\alpha \left(A^+_{l,i} + A^+_{N+l,j} + c\right) + \beta \left(A^+_{N+l,i} + A^+_{l,j} + c\right)\right). \tag{20}$$

If trade flows are balanced and welfare can be written as in equation (12), then we can also determine the elasticity of welfare in any location l to any change in trade costs from i to j:

$$\frac{\partial \ln W_l}{\partial \ln K_{ij}} = X_{ij} \times \frac{1}{\rho} \left((\alpha - 1) \left(A^+_{l,i} + A^+_{N+l,j} + c\right) + (\beta - 1) \left(A^+_{N+l,i} + A^+_{l,j} + c\right)\right) \tag{21}$$

Hence, given observed trade flows and the gravity constants α and β (and ρ in the context of welfare), all general equilibrium gravity models deliver identical predictions for all local comparative statics. We use this powerful result in Section 5 to characterize the welfare-maximizing set of trade friction reductions.

4.2.2 Global Comparative Statics

Now consider how an arbitrary change in the trade friction matrix $\mathbf{K}$ affects bilateral trade flows. In what follows, we denote with a hat the ratio of the new to old value of the variable, i.e. $\hat{x} \equiv \frac{x^{new}}{x^{old}}$. The following proposition, which generalizes the results of Dekle, Eaton, and Kortum (2008), provides an analytical expression relating the change in the exporter and importer shifters to the change in trade frictions and the initial exporting and importing shares:

[13] If $k = i$ and $l = j$, then $\frac{\partial \ln X_{kl}}{\partial \ln K_{ij}} = 1 + \frac{\partial \ln \gamma_k}{\partial \ln K_{ij}} + \frac{\partial \ln \delta_l}{\partial \ln K_{ij}}$, where the addition of one accounts for the direct effect on K_{kl}.

Proposition 4. *Consider any given set of observed trade flows* $\mathbf{X}$*, gravity constants* α *and* β*, and change in the trade friction matrix* $\hat{\mathbf{K}}$*. Then the percentage change in the exporter and importer shifters,* $\{\hat{\gamma}_i\}$ *and* $\left\{\hat{\delta}_i\right\}$*, if it exists, will solve the following system of equations:*

$$\hat{\gamma}_i^{\alpha-1}\hat{\delta}_i^{\beta} = \sum_j \left(\frac{X_{ij}}{Y_i}\right)\hat{K}_{ij}\hat{\delta}_j \text{ and } \hat{\gamma}_i^{\alpha}\hat{\delta}_i^{\beta}\frac{Y_i}{E_i} + \frac{\bar{D}_i}{E_i} = \sum_{j\in S}\left(\frac{X_{ji}}{E_i}\right)\hat{K}_{ji}\hat{\gamma}_j\hat{\delta}_i. \tag{22}$$

Proof. See Appendix A.6. □

Note that equation (22) inherits the same mathematical structure as equations (4) and (17). As a result, if trade is balanced (so that $\bar{D}_i = 0$ and $Y_i = E_i$ for all $i \in S$), then part (i) of Theorem 1 proves that there will exist a solution to equation (22) and part (ii) of Theorem 1 provides conditions for its uniqueness.

As with the local comparative statics, equation (22) only depends on trade data and parameters α and β; hence, for any given change in trade frictions, all the gravity trade models with the same α and β must imply the same change in the exporter and importer shifters γ_i and δ_i and hence trade flows and incomes. If welfare can be written as in equation (12), the change in location and global welfare will also be the same.

This proposition characterizes the comparative statics for a wide class of gravity trade models. In the case where $\beta = 0$, it can be shown (see Online Appendix B.5) that the comparative statics can be characterized using import shares alone. This special case (and its welfare implications) is discussed in Proposition 2 of Arkolakis, Costinot, and Rodríguez-Clare (2012).

5 Welfare maxizing trade friction changes

Armed with the theoretical results above, we finally turn to the question originally posed: what is the set of changes in bilateral trade frictions that maximize welfare subject to an arbitrary constraint?

5.1 The optimization problem

We first demonstrate how the local comparative static results from Proposition 3 can be used to inform the choice of optimal trade policy, as well as estimate the potential welfare gains from such a policy. As is well known, reducing trade frictions (e.g. increasing K_{ij}) increases trade openness, thereby increasing welfare. In general, this implies trade policies that reduce bilateral trade frictions are welfare enhancing. What is less understood, however, is how to quantify the *relative* benefit of reducing various bilateral trade frictions. In particular, we ask: how much should each bilateral trade friction be reduced in order to maximize the increase in welfare when reducing trade frictions between a certain bilateral pair comes at the expense of not being able to reduce trade frictions as much elsewhere? Having an answer to this question seems necessary in order to understand how policy makers with limited resources ought best allocate those resources.

Before proceeding, we should note that because of the particular form of condition C.4, our approach does not allow one to consider tariff revenue (as, e.g., in Ossa (2014)); instead, our approach is best suited to the study of other trade policies such as infrastructure development, non-tariff barriers, etc. that change trade frictions without directly affecting income. Empirical evidence suggests these non-tariff trade costs comprise a significant portion of trade policy restrictions (see e.g. Anderson and Van Wincoop (2004) and Looi Kee, Nicita, and Olarreaga (2009)).

Formally, consider a planner who seeks to choose a set of changes in trade frictions in order to maximize a weighted average of the (first-order) change in welfare across all locations:

$$\max_{\{z_{ij}\}} \sum_{l\in S}\sum_{i\in S}\sum_{j\in S}\omega_l \frac{\partial \ln W_l}{\partial \ln K_{ij}} z_{ij} \text{ s.t. } G(\mathbf{z}) = 0, \tag{23}$$

where ω_l is the weight the planner places on the change in welfare in location l, z_{ij} is the percentage change in the bilateral trade friction K_{ij}, $\mathbf{z}$ is the $N \times N$ matrix whose $\langle i, j\rangle$ element is z_{ij} and $G(\mathbf{z})$ is a function which specifies the constraint under which the planner operates. For example, $G(\mathbf{z}) = \sum_{i\in S}\sum_{j\in S} p_{ij} z_{ij} - B$ would reflect the fact that the total expenditure on bilateral trade friction reductions has a budget of B (where the price of a percentage increase in K_{ij} is p_{ij}). In general, $G(\cdot)$ captures whatever factors prevent the planner from simply reducing trade frictions (increasing K_{ij}) as much as possible, thereby making the problem economically interesting. Note that while our framework offers no guidance on what form those constraints take, our procedure is equally valid for any (differentiable) $G(\cdot)$.

From equation (21), once the matrix $\mathbf{A}^+$ has been calculated from observed trade flows and the gravity constants, the elasticity of welfare in any location $l \in S$ with respect to the change in trade costs between any two countries $i \in S$ and $j \in S$, i.e. $\frac{\partial \ln W_l}{\partial \ln K_{ij}}$, can be immediately determined from a linear combination of elements of the matrix. We should note that while it is possible to calculate the set of welfare elasticities $\left\{\frac{\partial \ln W_l}{\partial \ln K_{ij}}\right\}_{i,j}$ without the use of equation (21), doing so would require changing each bilateral trade cost separately by a small amount and recalculating the model equilibrium; since there are N^2 bilateral trade costs, such a process would be onerous.

Once the complete set of welfare elasticities is calculated equation (21), equation (23) is simply a constrained optimization problem with the familiar first order necessary conditions for all $i \in S$ and $j \in S$:

$$\omega_l \frac{\partial \ln W_l}{\partial \ln K_{ij}} = \lambda \frac{\partial G(\mathbf{z})}{\partial z_{ij}}. \tag{24}$$

The N^2 equations in equation (24) along with the constraint $G(\mathbf{z}) = 0$ can then be jointly solved to determine the welfare-maximizing trade cost changes $\mathbf{z}^*$ and the Lagrange multiplier λ.

In what follows, we consider two special cases of equation (23): (1) a unilateral trade policy governing import frictions; and (2) a multilateral trade policy where trade friction reductions are non-discriminatory.

Welfare-maximizing unilateral trade friction reductions

Consider a particular location $l \in S$ that only has the ability to reduce the trade frictions from its importers. For simplicity, assume that $G(\mathbf{z}) \equiv \frac{1}{2}\sum_{i \in S} z_{il}^2 - 1$, i.e. the cost of reducing (or increasing) location l's import frictions is convex. Then equation (23) becomes:

$$\max_{\{z_{il}\}_{i \in S}} \sum_{i \in S} \frac{\partial \ln W_l}{\partial \ln K_{il}} z_{il} \text{ s.t. } \sum_{i \in S} z_{il}^2 = 1 \tag{25}$$

The first order conditions (24) immediately imply that the welfare-maximizing unilateral trade friction reductions $\{z_{il}^{uni}\}$ are proportional to the welfare elasticities:

$$z_{il}^{uni} = \frac{\partial \ln W_l}{\partial \ln K_{ij}} / \lambda,$$

where $\lambda = \left(\sum_{i \in S} \left(\frac{\partial \ln W_l}{\partial \ln K_{ij}}\right)^2\right)^{\frac{1}{2}}$ is the Lagrange multiplier, which captures how much the elasticity of the change in welfare of a country to increasing the extent of trade friction changes in the welfare-maximizing way: i.e. the "potential unilateral gains for trade."

Welfare-maximizing multilateral "non-discriminatory" trade friction reductions

Consider now a planner who attempts to maximize a weighted average of the world-wide increase in welfare subject to a "non-discrimination" constraint where a location must equally reduce its trade frictions with all its exporting and importing partners. We assume that the weights attached to each location are those so that the welfare maximization problem corresponds to the competitive equilibrium (see part (ii) of Proposition 1). As in the previous example, we assume for simplicity that the trade friction reduction costs are convex. Then equation (23) becomes:

$$\max_{\{z_i\}_{i\in S}} \sum_{i\in S}\sum_{j\neq i} \frac{\partial \ln W}{\partial \ln K_{ij}} z_i z_j \text{ s.t. } \sum_{i\in S} z_i^2 = 1.$$

(We exclude the $i = j$ terms so that domestic trade costs remain unchanged). The first order conditions (24) now imply that the welfare-maximizing multilateral trade friction reductions $\{z_i^*\}$ solve the following system of equations:

$$\begin{aligned} 2\lambda z_i^* &= \sum_{j\neq i} \left(\frac{\partial \ln W}{\partial \ln K_{ij}} + \frac{\partial \ln W}{\partial \ln K_{ji}} \right) z_j^* \iff \\ \tilde{\lambda} z_i^* &= \sum_{j\neq i} \left(X_{ij} + X_{ji}\right) z_j^*, \end{aligned} \tag{26}$$

where λ is the Lagrange multiplier, $\tilde{\lambda} \equiv 2\rho^{-1}\lambda Y^W$, and the second line relied on the relationship between the world welfare elasticity and observed trade flows given in equation (16) $\frac{\partial \ln W}{\partial \ln K_{ij}} = \frac{1}{\rho}\frac{X_{ij}}{Y^W}$. Equation (26) shows that *the reduction in bilateral trade frictions which maximizes world welfare is simply the eigenvector of the observed trade flows corresponding to the largest eigenvalue* (when the trade matrix is added to its transpose and has zeros along the diagonal). Furthermore, that largest eigenvalue is proportional to the elasticity of world welfare to increasing the extent of the trade friction reductions in a welfare-maximizing way.

5.2 Estimating the gravity constants

As Proposition 3 shows, given observed trade flows and a set of gravity constants, the welfare elasticities can be calculated without specifying a particular micro-foundation of the gravity model. While trade flows are easily observed, it is not obvious how to choose the correct gravity constants, especially since Proposition 2 shows that observed trade flow data can be rationalized for any set of gravity constants.

One option would be to calibrate the gravity constants to values agreed upon by the literature. For example, in the context of an Armington trade model with intermediate inputs, we could choose to match a trade elasticity of negative four (consistent with Simonovska and Waugh (2009)) and a labor share in the production function of one-half (consistent with Alvarez and Lucas (2007)) – hereafter the "AL" parameter constellation – yielding gravity constants of $\alpha = -\frac{2}{3}$, $\beta = -\frac{1}{3}$, and $\rho = 2$.[14] Alternatively, we could choose to match Eaton

[14] Alvarez and Lucas (2007) also report the results of their simulations for two alternative values of the trade elasticity.

and Kortum (2002) – hereafter the "EK" parameter constellation – who find a trade elasticity of 8.28 and a labor share of 0.21, yielding gravity constants of $\alpha = -1.05$, $\beta = -0.83$, and $\rho = 1.73$. We instead opt to estimate the gravity constants in order to best match the observed general equilibrium forces present in the data.

Given Proposition 2, we know that trade data alone is insufficient to estimate the gravity constants; however, if both trade data and information about trade costs are observed, then the gravity constants can be recovered. Suppose, for example, that (the change in) trade costs is a function of a vector of observables $\hat{T}_{ij}$, i.e. $\ln \hat{K}_{ij} = \hat{T}'_{ij}\mu$, where the prime denotes a transpose. Then the gravity constants can be recovered in a two-stage estimation process. First, one estimates the (log) change in exporter and importer shifters using the observed (log) change in trade flows, $\ln \hat{X}^o_{ij}$:

$$\ln \hat{X}^o_{ij} = \hat{T}'_{ij}\mu + \ln \hat{\gamma}_i + \ln \hat{\delta}_j + \varepsilon_{ij},$$

where we interpret the residual ε_{ij} as classical measurement error. Second, one estimates the gravity constants by projecting the observed (log) change in income, $\ln \hat{Y}^o_i$, on the estimated change in exporter and importer shifters $\left\{\ln \hat{\gamma}^E_i\right\}$ and $\left\{\ln \hat{\delta}^E_j\right\}$:

$$\ln \hat{Y}^o_i = \alpha \ln \hat{\gamma}^E_i + \beta \ln \hat{\delta}^E_i + \nu_i.$$

While theoretically straightforward, this procedure is practically difficult, as the model predicts that the residual ν_i – unless it is pure measurement error – will be correlated with both $\ln \hat{\gamma}^E_i$ and $\ln \hat{\delta}^E_i$. This omitted variable bias arises because any unobserved change in the income shifter B_i (which causes the income of a location to be higher than observables would imply) will enter the residual and increase both the location's exports (through goods market clearing) and imports (through balanced trade). As a result, estimates of α and β will be biased upwards.

An alternative procedure is to rely on the general equilibrium structure of the model. By incorporating the general equilibrium effects within the estimator, there is no need for a two stage estimation procedure. In particular, we use the structure of the model – which incorporates both C.1 (corresponding to the first stage above) and C.4 (corresponding to the second stage above) – to calculate the change in the exporter and importer shifters directly. Formally, we can estimate the gravity constants α and β and the trade cost parameter μ by minimize the squared errors of the observed change in trade costs and the predicted change in trade costs:

$$(\alpha^*, \beta^*, \mu^*) \equiv \arg \min_{\alpha,\beta\in\mathbb{R},\mu\in\mathbb{R}^S} \sum_i \sum_j \left(\ln \hat{X}^o_{ij} - \hat{T}'_{ij}\mu - \ln \hat{\gamma}_i \left(\hat{\mathbf{T}}\mu; \alpha, \beta \right) - \ln \hat{\delta}_j \left(\hat{\mathbf{T}}\mu; \alpha, \beta \right) \right)^2, \tag{27}$$

where we emphasize that the change in the origin and destination shifters will be determined in general equilibrium and depend on both the gravity constants and the trade cost parameter.

It turns out that equation (27) is best solved by first estimating the μ given a set of gravity constants α and β and then solving for the α and β. Denote $\mu(\alpha, \beta)$ as the trade cost parameter which minimizes the squared error for a given α and β. Consider the following first order approximations of the log change in the exporter and importer shifters:

$$\ln \hat{\gamma}_i \left(\hat{\mathbf{T}}\mu \right) \approx \sum_k \sum_l \frac{\partial \ln \gamma_i}{\partial \ln K_{kl}} \hat{T}'_{kl}\mu \text{ and } \ln \hat{\delta}_j \left(\hat{\mathbf{T}}\mu \right) \approx \sum_k \sum_l \frac{\partial \ln \delta_j}{\partial \ln K_{kl}} \hat{T}'_{kl}\mu. \tag{28}$$

By taking first order conditions and applying these first order approximations, we can derive a straightforward closed form solution for $\mu(\alpha, \beta)$ (once we turn the $N \times N$ matrices into $N^2 \times 1$ vectors).[15] Let $\hat{\mathbf{T}}$ now denote the $N^2 \times S$ vector whose $\langle i + j(N-1) \rangle$ row is the $1 \times S$ vector $\hat{T}'_{ij}$, $\mathbf{D}(\alpha, \beta)$ is the $N^2 \times N^2$ matrix whose $\langle i + j(N-1), k + l(N-1) \rangle$ element is $\frac{\partial \ln X_{ij}}{\partial \ln K_{kl}}$ (which from Section 4.2 is a function only of the gravity constants and observed trade flows), and $\hat{\mathbf{y}}$ denote the $N^2 \times 1$ vector whose $\langle i + j(N-1) \rangle$ row is $\ln \hat{X}^o_{ij}$. Then the general equilibrium gravity estimator is:

$$\mu(a, \beta) = \left(\left(\mathbf{D}(\alpha, \beta) \hat{\mathbf{T}} \right)' \left(\mathbf{D}(\alpha, \beta) \hat{\mathbf{T}} \right) \right)^{-1} \left(\mathbf{D}(\alpha, \beta) \hat{\mathbf{T}} \right)' \hat{\mathbf{y}}. \tag{29}$$

Equation (29) says that, to a first order, the general equilibrium estimator is the coefficient one gets from of an ordinary squares regression of the observed hatted variables on a "general equilibrium transformed" explanatory variable $\hat{T}^{GE}_{ij}$:

$$\ln \hat{X}^o_{ij} = \left(\hat{T}^{GE}_{ij} \right)' \mu + \varepsilon_{ij},$$

where:

$$\hat{T}^{GE}_{ij} \equiv \sum_k \sum_l \frac{\partial \ln \hat{X}_{ij}}{\partial \ln \hat{K}_{kl}} \hat{T}_{kl}.$$

[15] In principal, the general equilibrium estimator could be calculated without applying a first-order approximation using an iterative procedure or through a non-linear least squares routine as in Anderson and Van Wincoop (2003). However, the closed form solution greatly simplifies the estimation procedure. Furthermore, Monte Carlo simulations suggest that the error arising from the first-order approximation used is small.

Intuitively, the general equilibrium transformed regressors capture the effect of the entire set of explanatory variables on any particular observed bilateral trade flow.[16] As a result, $\mu(\alpha, \beta)$ directly accounts for all (first-order) general equilibrium effects arising from the network structure of trade flows.[17]

We then find the gravity constants α and β which minimize the total squared error. From equation (29) (and the fact that a projection matrix is idempotent), the estimation of the gravity constants can be written as:

$$(\alpha^*, \beta^*) = \arg\min_{\alpha,\beta\in\mathbb{R}} \hat{\mathbf{y}}' \left(\mathbf{I} - \hat{\mathbf{T}} \left(\left(\mathbf{D}(\alpha,\beta)\hat{\mathbf{T}}\right)' \left(\mathbf{D}(\alpha,\beta)\hat{\mathbf{T}}\right)\right)^{-1} \left(\mathbf{D}(\alpha,\beta)\hat{\mathbf{T}}\right)'\right) \hat{\mathbf{y}}. \quad (30)$$

We pursue a simple grid search method. Rather than searching directly across all (α, β) space, we instead search across elasticities of substitution (σ) and labor shares (ζ), which has the advantage of being easier to compare to existing estimates.

5.3 Data

We now describe the data we use to calculate the welfare-maximizing trade friction reductions and estimate the gravity constants.

Our trade data comes from the CEPII gravity data set of Head, Mayer, and Ries (2010). This data set has several advantages: it covers bilateral trade flows between over two hundred countries, allowing us to construct the nearly complete world trade network; it includes both trade flow and GDP data, allowing us to measure own trade flows; and it is widely used, allowing comparability with other empirical studies. We clean the data in three steps. First, we construct own trade flows. To do so, we rely on the market clearing and balanced trade conditions, which implies that own trade is simply the difference between observed income and total exports or total imports, respectively.[18] Second, to avoid inferring infinitely high trade frictions between bilateral trade flows we replace any missing or zero bilateral trade

[16] One ought not be concerned that equation (19) provides elasticities for $\frac{\partial \ln X_{ij}}{\partial \ln K_{ij}}$ whereas the elasticities required for the general equilibrium estimator are the "hatted" elasticities $\frac{\partial \ln \hat{X}_{ij}}{\partial \ln \hat{K}_{kl}}$, as it is straightforward to show that $\frac{\partial \ln \hat{X}_{ij}}{\partial \ln \hat{K}_{kl}} = \frac{\partial \ln X_{ij}^{new}}{\partial \ln K_{kl}}$, i.e. the "hatted" elasticities are the same as the new elasticities. To see this, apply the comparative statics derivation in the proof of Proposition 3 to the global comparative static system of equations in equation (22).

[17] We show that our "general equilibrium" estimator of $\mu(\alpha, \beta)$ can provide both efficiency improvements and avoid certain problems of omitted variable bias when compared to the standard fixed effects estimator of μ using Monte Carlo simulations in Online Appendix B.7.

[18] If income exceeds total imports (exports), we define own trade flows as income less total exports (imports); if income exceeds both total imports and exports, we define own trade flows as income less the average of total imports and exports.

flows with a small positive value. Finally, we balance the trade flows; while this is not strictly necessary, it guarantees that the equilibrium is unique, and as a result, the elasticities we estimate are well-defined. To do so, we ignore the observed level of trade flows and instead treat the observed import *shares* $\lambda_{ij} \equiv \frac{X_{ij}}{\sum_i X_{ij}}$ as the true data. We then find the unique set of incomes that are consistent with those import shares and balanced trade by solving the following linear system of equations:

$$Y_i = \sum_j \lambda_{ij} Y_j.$$

By the Perron-Frobenius theorem, there exists a unique (to-scale) set of Y_i;[19] we pin down the scale with the normalization that $\sum_{i \in S} Y_i = 1$. Given these equilibrium Y_i, we then define the balanced trade flows $X_{ij}^b = \lambda_{ij} Y_j$.[20]

For the observables that change trade costs (i.e. $\hat{\mathbf{T}}$), we use WTO membership. The WTO was founded on January 1, 1995, replacing the General Agreement on Tariffs and Trade (GATT). Of the 201 countries in our trade data, 125 were original WTO members. Between 1995 and 2005, an additional twenty-one countries joined the WTO.[21] In what follows, we assume that, apart from a common time trend ν, the only change in bilateral frictions between 1995 and 2005 was a (common) reduction in trade costs (i.e. an increase in $\hat{K}_{ij}$) between new WTO members and all other WTO members:

$$\hat{K}_{ij} = \mu \hat{T}_{ij} + \nu,$$

where $\hat{T}_{ij}$ is an indicator variable equal to one if either i or j is a new WTO member and its trading partner is a new or existing WTO member. While this is admittedly a strong assumption, note that by focusing on the change in trade flows rather than their level, we allow for any effect of time-invariant variables (e.g. distance, common language, shared border, etc.) on trade frictions.

[19] The Perron-Frobenius theorem guarantees that there exists a unique (to-scale) strictly positive vector that solves $Y_i = \kappa \sum_j \lambda_{ij} Y_j$ for the largest value of $\kappa > 0$. Since import shares sum to one, it is straightforward to show that $\kappa = 1$ in this case: $\kappa = \frac{\sum_i Y_i}{\sum_i \sum_j \lambda_{ij} Y_j} = \frac{\sum_i Y_i}{\sum_j Y_j \sum_i \lambda_{ij}} = 1$.

[20] It is straightforward to see that these trade flows are balanced: $\sum_j X_{ji}^b = \sum_j \lambda_{ji} Y_i = Y_i \sum_j \frac{X_{ji}}{\sum_j X_{ji}} = Y_i = \sum_j \lambda_{ij} Y_j = \sum_j X_{ij}^b$.

[21] The new members were Albania, Armenia, Bulgaria, China, Ecuador, Estonia, Georgia, Croatia, Jordan, Kyrgyzstan, Cambodia, Lithuania, Moldova, Macedonia, Mongolia, Nepal, Oman, Panama, Saudi Arabia, and Taiwan.

5.4 Results

We now report the estimated gravity constants, the estimated effects of joining the WTO, and the welfare-maximizing trade policy implied by the observed trade flows and estimated gravity constants.

Gravity Constants

Figure 2 presents the R-squared of the general equilibrium estimator on the gains from the WTO across all values of the share of labor in the production function (ζ) and the elasticity of substitution (σ) (which maps one-to-one to the gravity constants). As is evident, the GE estimator maximizes the R-squared of the regression (or, equivalently, minimizes the squared error of equation (30)) when $\sigma = 14.775$ and $\zeta = 0.075$. Recalling that $\alpha = \frac{1}{1-\sigma\zeta}$ and $\beta = \frac{1-\zeta}{1-\zeta\sigma}$, this implies that the GE estimator most closely matches the data when α and β are both large and negative; at the maximum R-squared, $\alpha = -30.2$ and $\beta = -27.9$. From Proposition 3, it can be shown that as as α and β approach negative infinity, the difference between the direct effect of a trade cost shock and the indirect effect of a trade cost shock (i.e. $\frac{\partial \ln X_{ij}}{\partial \ln K_{ij}} - \frac{\partial \ln X_{kl}}{\partial \ln K_{ij}}$ for some $k \neq i$ and $j \neq l$) gets larger. Large negative gravity constants thus imply that the general equilibrium effects are small relative to the direct effects of joining the WTO.

How does imposing the general equilibrium conditions affect the fit of the model? A simple way of answering this question is to compare the fit of the GE estimator with a traditional fixed effects estimator where the estimated exporter and importer shifters of each country are not constrained to satisfy general equilibrium conditions. The R-squared of the traditional fixed effects estimator is 0.1978, which is substantially larger than the 0.0234 of the general equilibrium estimator; however, a better fit is to be expected given that the traditional fixed effects estimator includes 402 covariates compared to two covariates for the general equilibrium estimator.

The effect of the WTO

Figure 3 depicts the estimated effect of WTO membership on bilateral trade frictions. At the preferred estimates of the gravity constants, joining the WTO is associated with a 37 percent reduction in bilateral trade frictions (i.e. a 37 percent increase in K_{ij}). This result is similar given alternative parameter constellations: WTO membership is estimated to reduce bilateral trade frictions by 41 percent under either the EK parameter constellation or the AL parameter constellation.

Figure 4 illustrates the average estimated change in welfare for new WTO members, existing WTO members, and non-members for all combinations of gravity constants given the estimated coefficients reported in Figure 3 using the global comparative statics methodology of Section 4.2.2. As with the effect of the WTO on trade frictions, the three gravity constellations imply similar welfare effects: the maximum R-squared gravity constellation implies that the welfare in countries joining the WTO increased on average by 18 percent, compared to 13 percent and 12 percent increases with the EK and AL parameter constellations, respectively. Existing WTO members also benefit, albeit to a much smaller extent, with all three parameter constellations estimating roughly 1 percent increases in welfare. Non-WTO members, however, are hurt by the resulting trade diversion of other countries joining the WTO, although the welfare losses are less than half a percentage point under all three parameter constellations.

Welfare-maximizing trade friction changes

Given the estimated gravity constants, we proceed to determine the welfare-maximizing unilateral and bilateral trade friction reductions.[22] As an example, Figure 5 depicts the welfare-maximizing unilateral reduction in trade costs for the United States. The results are intuitive: to maximize welfare in the U.S., it should concentrate its import friction reductions on its major trading partners (e.g. Canada, Mexico, China, Brazil, and Western Europe), at the expense of reducing its trade frictions only a small amount with less important trading partners like African countries.

How much does the U.S. (or any other country) benefit from reducing its import costs unilaterally? Figure 6 depicts the Lagrange multiplier of the unilateral welfare maximization problem (25) for each country, which recall can be interpreted as the "potential unilateral gains for trade." The potential benefits of unilateral trade friction reductions are the smallest in countries with sizable domestic production relative to external trade such as the United States, India, and Russia. The potential gains for smaller countries which engage in substantial trade (e.g. Belgium) are larger. However, countries where political constraints results to restricted trade – for example, North Korea, Burma, Somalia, Cuba, and Iraq – face the largest potential gains from freer trade. Intuitively, these countries trade very little with a number of large trading partners, which implies they have a large marginal utility of relaxing those bilateral trade frictions.

Figure 7 depicts the "non-discriminatory" reduction in trade frictions that maximizes the

[22]The distribution of optimal trade friction changes are nearly identical when using either the AL or EK constellation of gravity constants; see Online Appendix B.7.

increase in world welfare.[23] Because the elasticity of world welfare to changes in bilateral trade frictions is proportional to the level of trade flows, the welfare-maximizing multilateral trade policy reduces trade frictions most for those countries with the largest trade flows, such as the United States, China, and countries in Western Europe and least for countries like those in Africa that trade less. Figure 8 illustrates the distribution of welfare effects of the optimal multilateral trade policy across countries. There is substantial heterogeneity in the welfare effects of the optimal policy: the countries that reduced their trade frictions most benefit from the policy, while the majority of countries are actually made slightly worse off from the resulting trade diversion. Of course, we emphasize that these quantitative results depend on the given constraint $G(\cdot)$ where all changes to bilateral trade frictions are assumed to be equally costly: the results would differ if there was heterogeneity across bilateral pairs in the cost of trade friction reductions, although the methodology developed above could still be applied.

6 Conclusion

In this paper, we first show that the general equilibrium forces in many gravity trade models depend crucially on the value of two "gravity constants." In particular, given observed trade flows, these gravity constants are sufficient to determine all comparative statics of the model. This result – along with a way of estimating the gravity constants themselves – allows us to determine how any change in bilateral trade frictions will affect the model equilibrium without needing to specify a particular underlying trade model. This paper hence contributes to a growing literature emphasizing that the micro-economic foundations are not particularly important for determining a trade model's macro-economic implications.

We use our results to analytically characterize the welfare-maximizing set of trade friction reductions (to a first order) subject to an arbitrary constraint. Surprisingly, certain special cases yield closed form solutions: for example, the welfare-maximizing non-discriminatory multilateral trade policy is the eigenvector corresponding to the largest eigenvalue of the observed trade flow matrix. In general, we view this as a necessary step toward determining optimal trade policy in a many location general equilibrium model.

By providing a universal framework for understanding the general equilibrium forces in gravity trade models, we hope that this paper provides a step toward unifying the quantitative general equilibrium approach with the gravity regression analysis common in the

[23]Since $\frac{\partial \ln W}{\partial \ln K_{ij}} \propto X_{ij} > 0$ from equation (16), the Perron-Frobenius theorem guarantees the vector of trade friction changes that solve equation (26) will be strictly positive, i.e. it will be optimal for all countries to reduce their trade frictions.

empirical trade literature. Toward this end, we have developed a toolkit that operationalizes all the theoretical results presented in this paper, including the calculation of the equilibrium, identification, calculation of local and global comparative statics, and estimation.[24] We also hope the tools developed here can be extended to understand other general equilibrium spatial systems, such as those governing the structure of cities.

[24]The toolkit is available for download on Allen's website.

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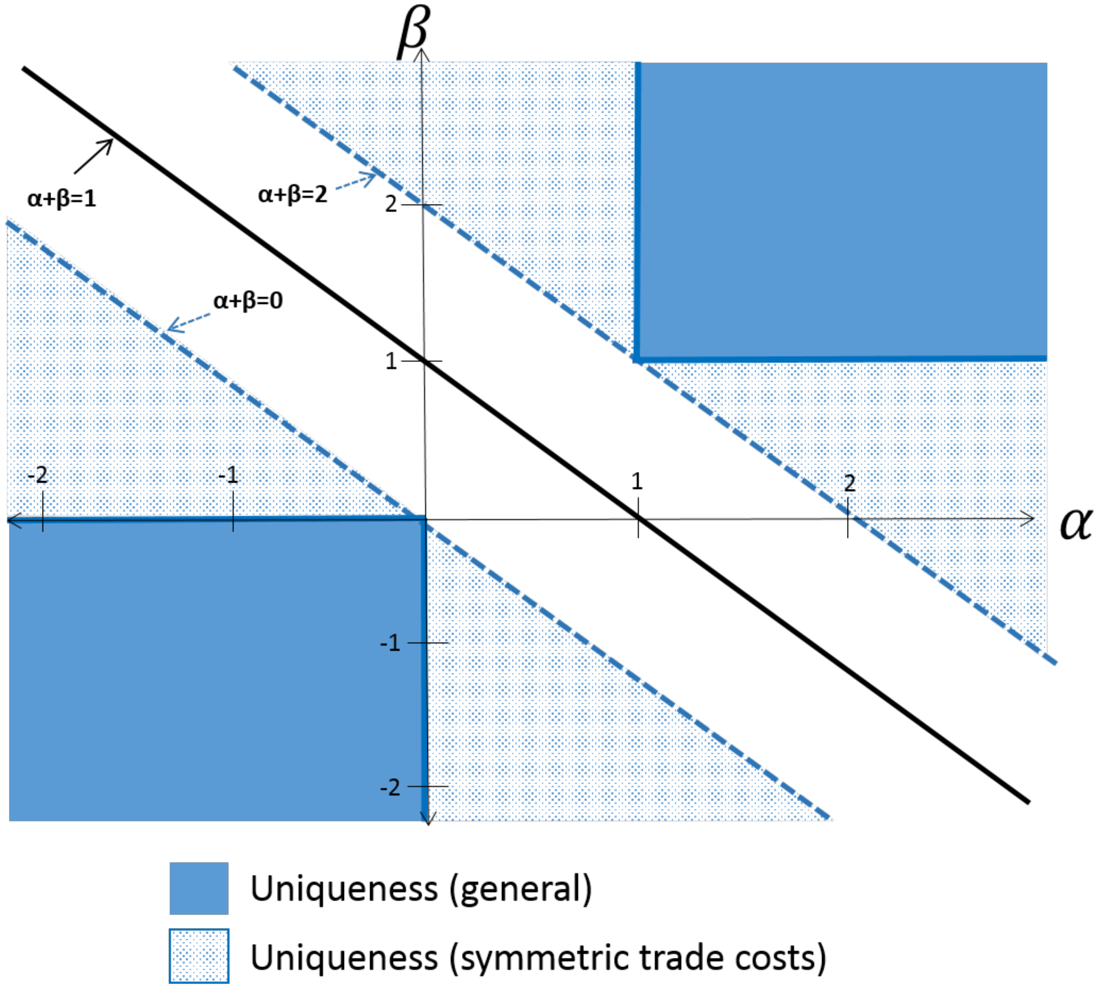
Table 1: Mapping of trade models to Universal Gravity framework

Model	Citation	Subs. Elast.	Hetero. Param.	Labor Share in Production	Additional Param.	Trade Elast.	Integrability Restrictions	Mapping to ρ	Mapping to α	Mapping to β	Condition for uniqueness (general)	Condition for uniqueness (quasi-symmetry)
Trade Models												
Armington, intermediate inputs	Armington (1969); Anderson (1979); Anderson and Van Wincoop (2003)	σ	N/A	ζ	N/A	$\sigma-1$	N/A	$(\sigma-1)(1-\zeta)$	$\frac{1}{1-\sigma}$	$\frac{1-\zeta}{1-\sigma\zeta}$	$\sigma\geq 1$	$\sigma\geq\frac{1}{2}$
Monopolistic competition, homogeneous firms, intermediate inputs	Krugman (1980) (with intermediates inputs)	σ	N/A	ζ	N/A	$\sigma-1$	N/A	$(\sigma-1)(1-\zeta)$	$\frac{1}{1-\sigma\zeta}$	$\frac{1-\zeta}{1-\sigma\zeta}$	$\sigma\geq 1$	$\sigma\geq\frac{1}{2}$
Perfect competition, intermediate inputs	Eaton and Kortum (2002); Dekle, Eaton, and Kortum (2008)	σ	θ (Frechet)	ζ	N/A	θ	$\theta>\sigma-1,\theta>0$	$\theta(1-\zeta)$	$\frac{1}{1-(\theta+1)\zeta}$	$\frac{1-\zeta}{1-(\theta+1)\zeta}$	$\theta>0$	$\sigma\geq\frac{1}{2}$
Monopolistic competition, heterogeneous firms, exporting fixed costs in destination	Melitz (2003); Eaton, Kortum, and Kramarz (2011); Arkolakis, Demidova, Klenow, and Rodríguez-Clare (2008); Chaney (2008)	σ	θ (Pareto)	1	N/A	θ	$\theta>\sigma-1,\theta>0$	θ	$-\frac{1}{\theta}$	0	$\theta>0$	$\theta>0$
Monopolistic competition, heterogeneous firms, exporting fixed costs in origin	Di Giovanni and Levchenko (2009)	σ	θ (Pareto)	1	N/A	θ	$\theta>\sigma-1,\theta>0$	θ	$-\frac{\sigma-1}{\theta\sigma-\sigma+1}$	0	$\sigma\geq 1$	$\sigma\geq 1$ or $\sigma\leq 1$ and $2\frac{1-\sigma}{\sigma}\geq\theta>\frac{1-\sigma}{\sigma}$
Monopolistic competition, heterogeneous firms, flexible exporting fixed costs, intermediate inputs	Arkolakis (2010) (with intermediate inputs)	σ	θ (Pareto)	ζ	η : share of exporting in foreign labor	θ	$\theta>\sigma-1,\theta>0$	$\theta(1-\zeta)$	$-\frac{(\sigma-1)}{(\theta+1)(\sigma-1)\zeta-(\sigma-1)+(1-\eta)(\theta-\sigma+1)}$	$-\frac{(\sigma-1)(1-\zeta)}{(\theta+1)(\sigma-1)\zeta-(\sigma-1)+(1-\eta)(\theta-\sigma+1)}$	$\sigma\geq 1$ or $\sigma<1$ and $(1-\sigma)(1-\zeta)>(1-\sigma)(1-\zeta)+\theta(\sigma-1)\zeta+(1-\eta)(\theta-\sigma+1)>0$	previous regions or $\sigma<1$ and $2(1-\sigma)(1-\zeta)>(1-\sigma)(1-\zeta)+\theta(\sigma-1)\zeta+(1-\eta)(\theta-\sigma+1)>0$
Extensions												
Multiple sectors	Chor (2010); Costinot, Donaldson, and Komunjer (2010)	σ	θ (Frechet)	1	η : sector spending share (assumed constant across sectors)	θ	N/A	θ	$-\frac{1}{\theta}$	0	$\theta>0$	$\theta>0$
Multiple sectors with intermediate goods	Caliendo and Parro (2010) (with symmetric IO structure and constant σ)	σ	θ (Frechet)	ζ	η : sector spending share (assumed constant across sectors)	θ	N/A	$\theta(1-\zeta)$	$\frac{1}{1-(\theta+1)\zeta}$	$\frac{1-\zeta}{1-(\theta+1)\zeta}$	$\theta>0$	$\theta>0$
Economic geography models	Allen and Arkolakis (2014); Helpman (1998); Redding (2014)	σ	N/A	1	a :productivity spillover; b: amenity spillover	$\sigma-1$	N/A	$\sigma-1$	$\frac{1-b}{1+b(\sigma-1)+\sigma a}$	$\frac{a+1}{1+b(\sigma-1)+\sigma a}$	(sufficient conditions) $\sigma\geq 1$, $a+b\leq 0$, $a>-1$, $b<1$	(sufficient conditions) $\sigma\geq\frac{1}{2}$, $a+b\leq 0$, $a>-1$, $b<1$

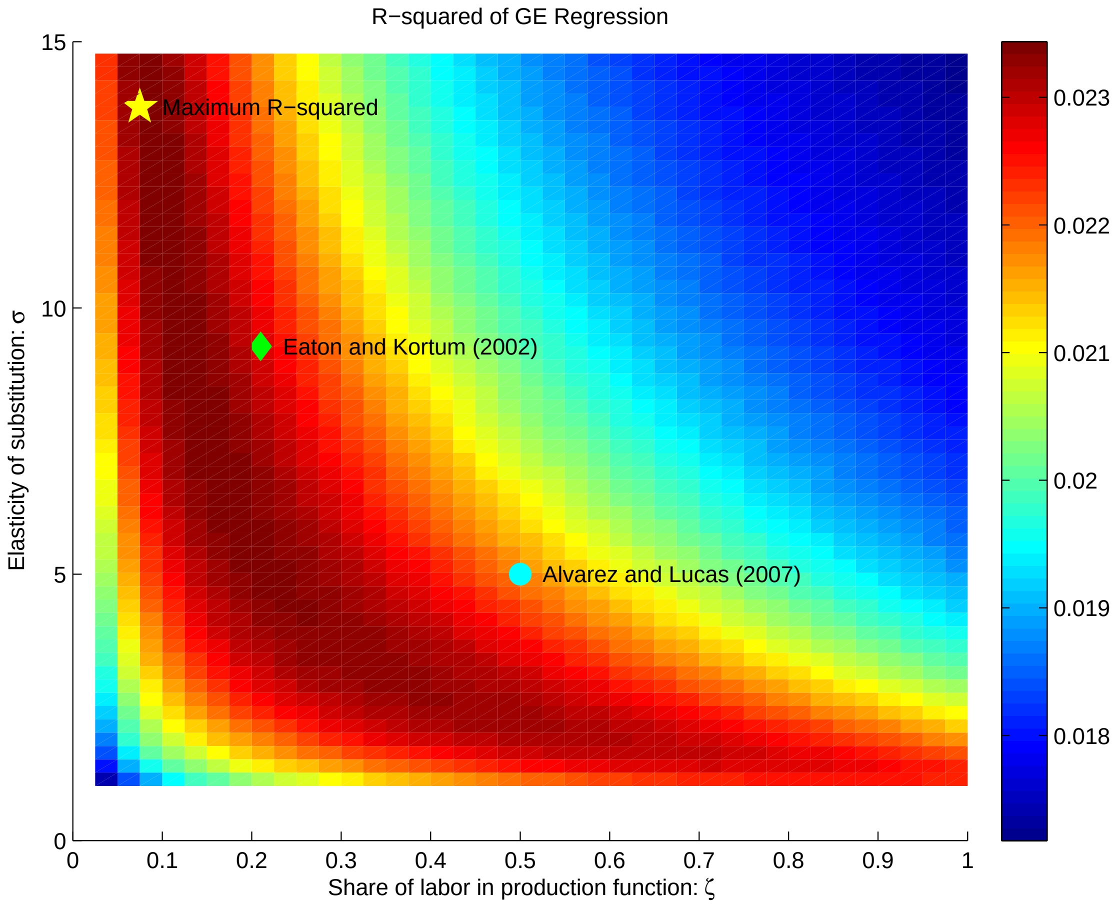
Notes: This table includes a (non-exhaustive) list of trade models that can be examined within the universal gravity framework.

Figure 1: Existence and Uniqueness



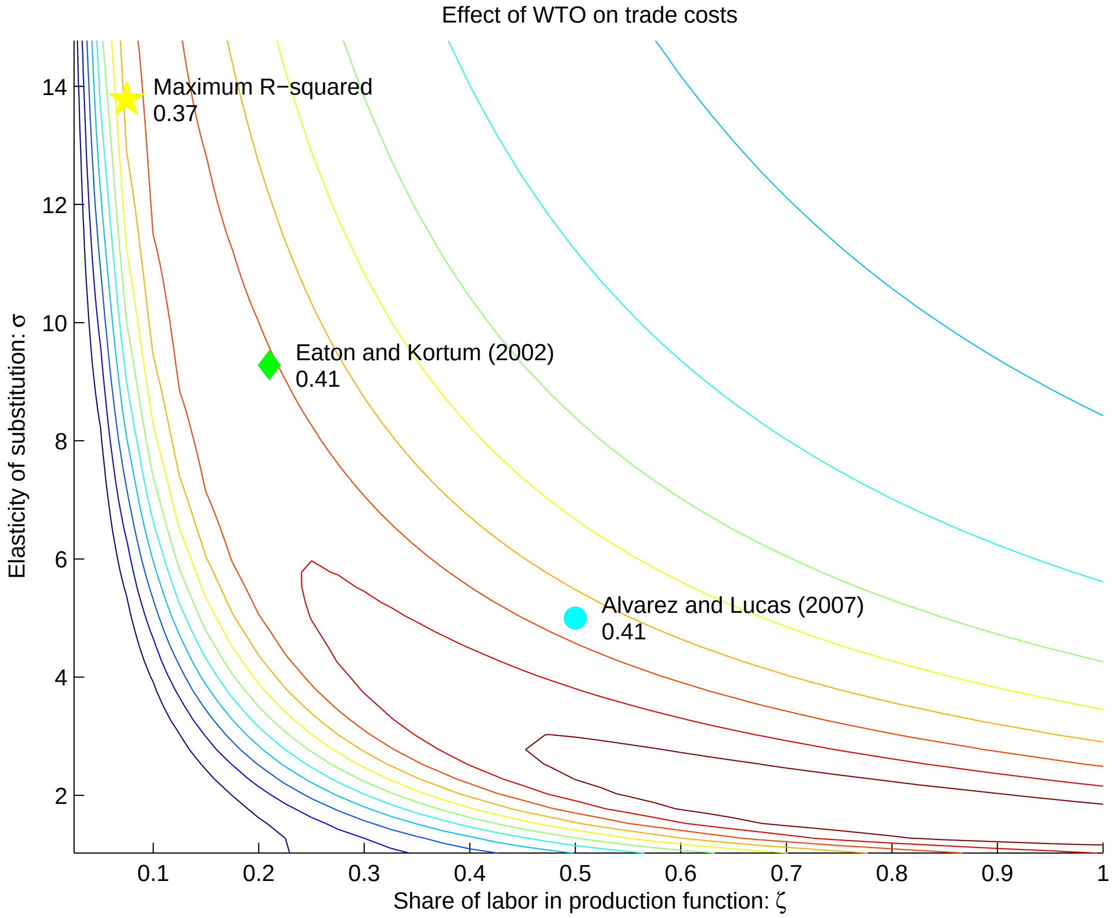
Notes: This figure shows the regions in (α, β) space for which the gravity equilibrium is unique generally and in the special case when trade frictions are quasi-symmetric. Existence can be guaranteed throughout the entire region with the exception of when $\alpha + \beta = 1$ or $\alpha = \beta = 1$.

Figure 2: Estimating the gravity constants



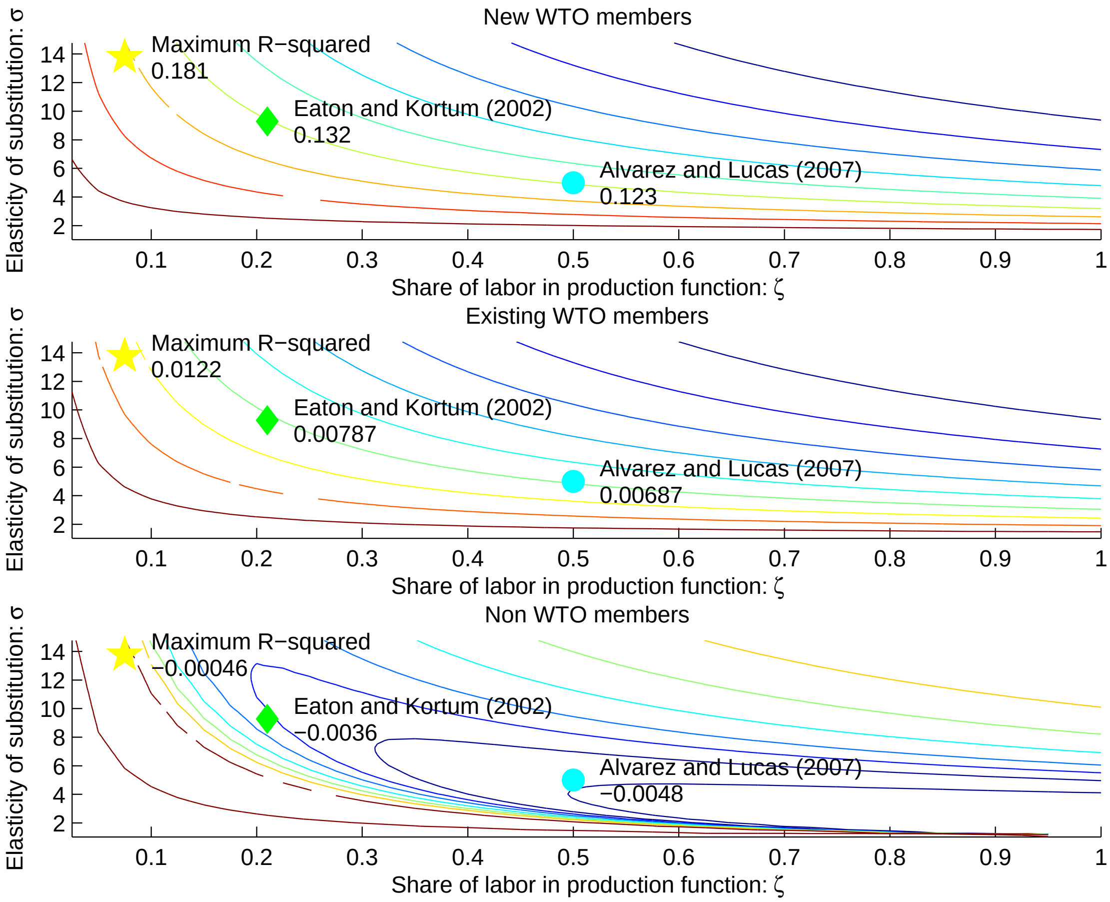
Notes: This figure shows the R-squared of the general equilibrium estimator for the effect of joining the WTO on bilateral trade flows for all combinations of the elasticities of substitution and the share of labor in the production function. The yellow star indicates where the R-squared is maximized (at $\sigma = 13.775$ and $\zeta = 0.075$), which corresponds to gravity constants of $\alpha = -30.2$ and $\beta = -27.9$. Also indicated are the parameter values used in Eaton and Kortum (2002) (diamond) and the Alvarez and Lucas (2007) (circle).

Figure 3: The effect of the WTO on bilateral trade frictions



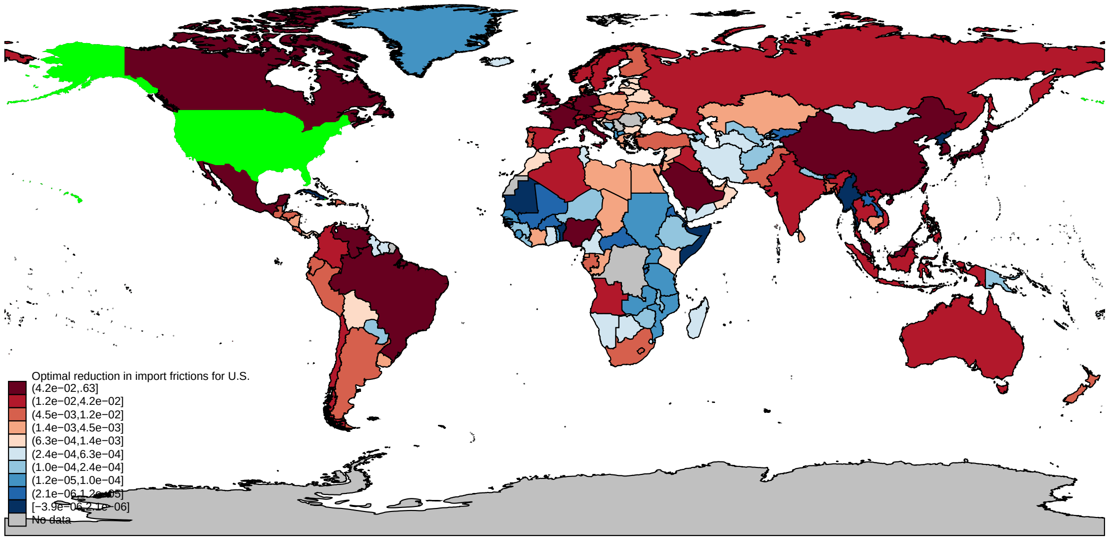
Notes: This figure shows the iso-contours of the estimated effect of the WTO on bilateral trade frictions for all combinations of the elasticities of substitution and the share of labor in the production function. The yellow star indicates where the R-squared is maximized; also indicated are the estimated gains from the WTO using parameter values from Eaton and Kortum (2002) (diamond) and Alvarez and Lucas (2007) (circle).

Figure 4: The effect of the WTO on welfare



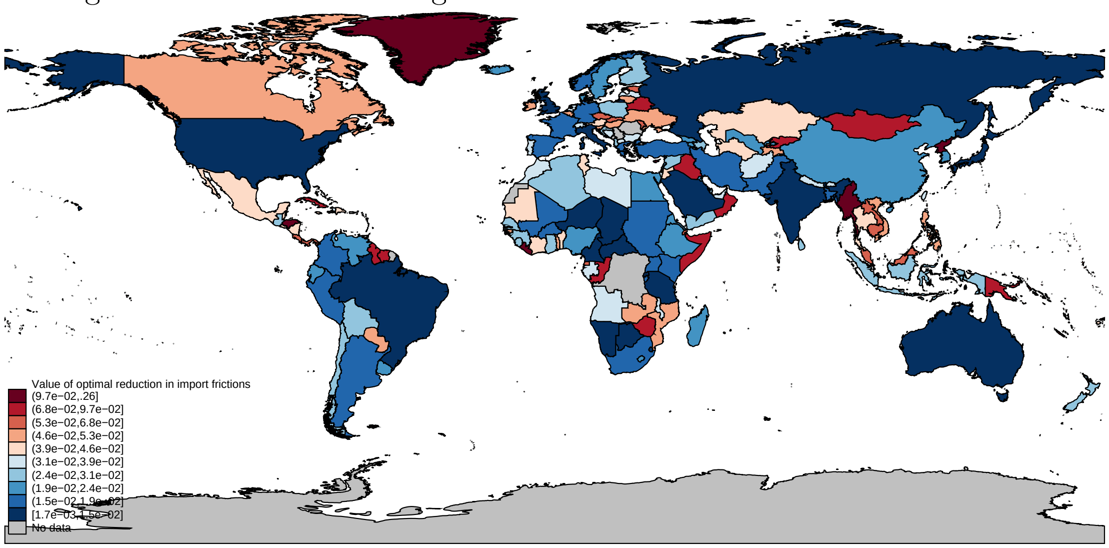
Notes: This figure shows the iso-contours of the estimated effect of the WTO on welfare for all combinations of the elasticities of substitution and the share of labor in the production function. The yellow star indicates where the R-squared is maximized; also indicated are the estimated gains from the WTO using parameter values from Eaton and Kortum (2002) (diamond) and Alvarez and Lucas (2007) (circle). The top panel reports the average change in welfare for all countries who joined the WTO between 1995 and 2005; the middle panel reports the average change in welfare for all countries already in the WTO in 1995; and the bottom panel reports the average change in welfare for all non-WTO members.

Figure 5: Optimal unilateral trade friction reduction for the U.S.



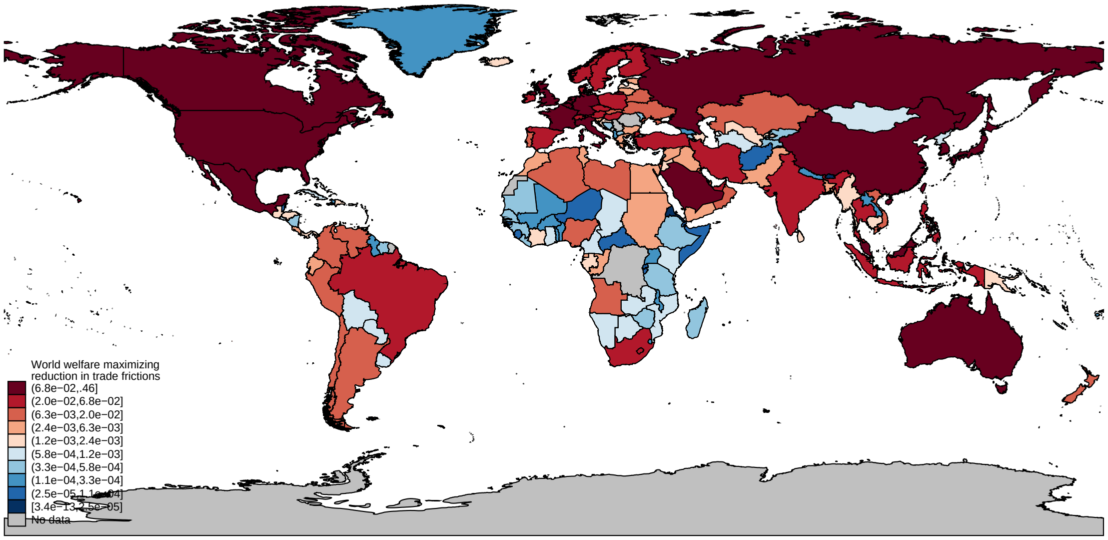
Notes: This figure shows the set of unilateral reductions in import trade frictions for the United States that maximize its change in welfare subject to the norm of the total reductions remaining constant. Countries are sorted by deciles; red indicates a greater reduction in trade frictions while blue indicates a smaller reduction in trade frictions.

Figure 6: Potential welfare gains from unilateral trade friction reductions



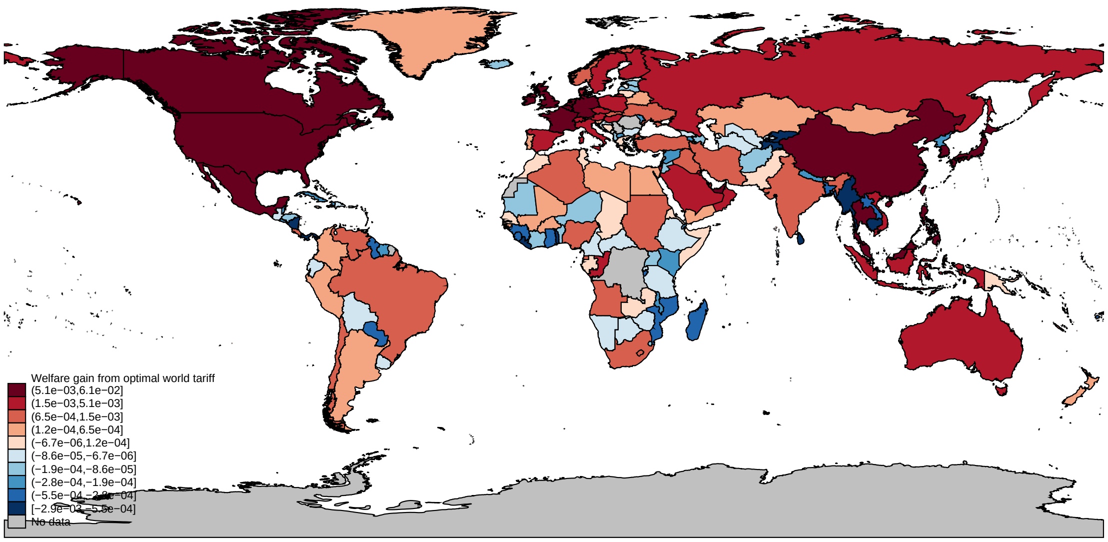
Notes: This figure shows the elasticity of each country's welfare to increasing the amount of unilateral trade friction reductions in the optimal way (i.e. the Lagrange multiplier of equation (25)). Countries are sorted by deciles; red indicates a greater potential welfare gain while blue indicates a smaller potential welfare gain.

Figure 7: World optimal multilateral trade friction reduction



Notes: This figure shows the set of non-discriminatory country reductions in trade frictions that maximizes the world welfare (where the country Pareto weights are those imposed by the competitive equilibrium). Countries are sorted by deciles; red indicates a greater reduction in trade frictions while blue indicates a smaller reduction (or even increase) in trade frictions.

Figure 8: Welfare gains from world optimal multilateral trade friction reduction



Notes: This figure shows distribution of welfare gains from an optimal non-discriminatory multilateral trade friction reduction. In particular, we report the welfare gain each country would achieve if all countries in the world were to alter their trade frictions in order to maximize world welfare (where the country Pareto weights are those imposed by the competitive equilibrium. Countries are sorted by deciles; red indicates a greater increase in welfare while blue indicates a smaller increase in welfare.

A Proofs

A.1 Proof of Theorem 1

We analyze a transformed system by defining $x_i \equiv B_i \gamma_i^{\alpha-1} \delta_i^{\beta}$ and $y_i \equiv B_i \gamma_i^{\alpha} \delta_i^{\beta-1}$. Then it can be shown that $\delta_i = x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{1-\alpha}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}}$ and $\gamma_i = x_i^{\frac{1-\beta}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}}$ so that for any set of $\{B_i\} \in \mathbb{R}_{++}^N$, $\{K_{ij}\} \in \mathbb{R}_{++}^{N\times N}$, $\{\alpha, \beta\} \in \{(\alpha, \beta) \in \mathbb{R}^2 | \alpha + \beta \neq 1\}$, the equilibrium of a general equilibrium gravity model can be written using

$$x_i = \sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}}, \tag{31}$$

and

$$y_i = \sum_j K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}, \tag{32}$$

and the world income is taken to be 1 as numeraire,

$$1 = \sum_i B_i^{\frac{1}{1-\alpha-\beta}} x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}}. \tag{33}$$

The proof of Theorem 1 proceeds in four parts. In the **first part**, we consider a general mathematical structure, for which the general equilibrium gravity model is a special case, and show existence for the general mathematical structure. Namely we show that for any positive F and H, and a, b, c, and d, there exits a solution to

$$x_i = \frac{\sum_j F_{i,j} x_j^a y_j^b}{\sum_{i,j} F_{i,j} x_j^a y_j^b} \tag{34}$$

$$y_i = \frac{\sum_j H_{ij} x_j^c y_j^d}{\sum_{i,j} H_{ij} x_j^c y_j^d}. \tag{35}$$

Note that their structure is different from the general equilibrium trade model in two ways. First, in the general equilibrium trade model, the denominators of the right-hand-side should be 1. Second, the general equilibrium trade model should also satisfy (33). We take care of these differences in the **second part**. We prove lemmas that will allow us to convert the existence result of the general mathematical result to the existence of the particular case of the gravity trade model. In the **third** and **fourth** parts, we use the system of the second part to prove existence and uniqueness for the general equilibrium trade models, respectively.

A.1.1 Part 1 : the general case

We start with the result for the general mathematical system, stated as the following lemma. For the proof, we use a version of Schauder's fixed point theorem (FPT for short). The original statement is found in Aliprantis and Border (2006).

Theorem 3. *(Schauder's FPT) Suppose that $D \subset R^N$, where D is a convex and compact set. If a continuous function $f : D \to D$ satisfies the condition that $f(D)$ is a compact subset of D, then there exists $x \in D$ such that $f(x) = x$.*

Lemma 1. *Consider the system the equation (34) and (35). Then the system has a positive solution $x, y \in \mathbb{R}^S_+$ and all its possible solutions are positive.*

Proof. To apply the Schauder's FPT, we set up a subset D of R^{2S} such that D satisfies the conditions in Schauder's FPT.

Now consider the system (34)-(35). We define the set Γ as

$$\Gamma \equiv \left\{(x, y) \in \Delta\left(R^S\right) \times \Delta\left(R^S\right); m_x \leq x_i \leq M_x,\ m_y \leq y_i \leq M_y \quad \text{for all } i\right\},$$

and the following constants

$$\begin{aligned} M_x &\equiv \max_{i,j} \frac{F_{i,j}}{\sum_i F_{i,j}} \\ m_x &\equiv \min_{i,j} \frac{F_{i,j}}{\sum_i F_{i,j}} \\ M_y &\equiv \max_{i,j} \frac{H_{i,j}}{\sum_i H_{i,j}} \\ m_y &\equiv \min_{i,j} \frac{H_{i,j}}{\sum_i H_{i,j}}. \end{aligned}$$

Γ is convex and compact subset of R^{2S}.

We define the following operator for $d = (x, y) \in \Gamma$.

$$\begin{aligned} Td &= T(x, y) \\ &= \left(\left(T^x(x, y)\right), \left(T^y(x, y)\right)\right), \end{aligned}$$

where

$$\begin{aligned} T^x_i(x, y) &= \frac{\sum_j F_{i,j} x_j^a y_j^b}{\sum_i \sum_j F_{i,j} x_j^a y_j^b} \\ T^y_i(x, y) &= \frac{\sum_j H_{i,j} x_j^c y_j^d}{\sum_i \sum_j H_{i,j} x_j^c y_j^d}. \end{aligned}$$

It is easy to show that

$$m_x \leq T_i^x(x,y) \leq M_x, m_y \leq T_i^x(x,y) \leq M_y$$

so that the operator T is from Γ to Γ.

To show that T is continuous, it suffices to show that T_i^x and T_i^y are continuous for all i. Since the range is compact, these functions are trivially continuous.

Since Schauder's FPT is applied for T, then there exists a solution to the system. Also by construction, any fixed points satisfy for all i,

$$\begin{aligned} 0 &< m_x \leq x_i \\ 0 &< m_y \leq y_i. \end{aligned}$$

$\square$

A.1.2 Part 2 : from the general mathematical system to GE trade model

Second, we prove a result that will allow us to map the general equilibrium gravity model to the general mathematical system. This subsection consists of several lemmas. In Lemma 2, in the general equilibrium trade models, the double-sum terms in (34) and (35) should coincide i.e.

$$\sum_i \sum_j F_{i,j} x_j^a y_j^b = \sum_i \sum_j H_{i,j} x_j^c y_j^d.$$

Next, in Lemma 3, we show that a simple transformation of (x, y) that solves equations (34) and (35) also solves (31) and (32). To show the existence of the general equilibrium trade model, we still need to show that the normalization equation (33) is satisfied. Lemma 4 takes care of this issue.

The following lemma shows that the double-sum terms should coincide with each other.

Lemma 2. *Suppose that (x, y) satisfies (34) and (35) with*

$$\begin{aligned} a &= \frac{\alpha}{1-\alpha-\beta}, & b &= \frac{1-\alpha}{\alpha+\beta-1} \\ c &= \frac{1-\beta}{\alpha+\beta-1}, & d &= \frac{\beta}{\alpha+\beta-1} \\ F_{i,j} &= K_{ij} B_j^{\frac{1}{1-\alpha-\beta}}, & H_{i,j} &= K_{ji} B_j^{\frac{1}{1-\alpha-\beta}}. \end{aligned}$$

Then we have

$$\sum_{i,j} K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} = \sum_{i,j} K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}.$$

Proof. Note that

$$x_i = \lambda_x \sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}},$$

where

$$\lambda_x = \sum_{i,j} K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}}.$$

Multiply both sides by $x_i^{\frac{1-\beta}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}}$, which yields:

$$x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} = \lambda_x \sum_j K_{ij} \left(B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} \right) \times \left(x_i^{\frac{1-\beta}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right).$$

Now sum over all i and rearrange to solve for λ_x:

$$\sum_i x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} = \lambda_x \sum_i \sum_j K_{ij} \left(B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} \right) \times \left(x_i^{\frac{1-\beta}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right) \iff$$

$$\begin{aligned}
\lambda_x &= \frac{\sum_i x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}}}{\sum_i \sum_j K_{ij} \left(B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} \right) \times \left(x_i^{\frac{1-\beta}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right)} \\
&= \frac{\sum_i x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}}}{\sum_i \left[\sum_j K_{ij} \left(B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} \right) \right] \times \left(x_i^{\frac{1-\beta}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right)}.
\end{aligned}$$

Now let us consider the second equilibrium condition:

$$y_i = \lambda_y \sum_j K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}$$

where

$$\lambda_y = \sum_{i,j} K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}.$$

Multiply both sides by $x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{1-\alpha}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}}$:

$$\begin{aligned} y_i \times \left(x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{1-\alpha}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right) &= \lambda_y \sum_j K_{ji} \left(B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}} \right) \times \left(x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{1-\alpha}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right) \iff \\ x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} &= \lambda_y \sum_j K_{ji} \left(B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}} \right) \times \left(x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{1-\alpha}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right) \end{aligned}$$

Now sum over all i and rearrange to solve for λ_y:

$$\begin{aligned} \sum_i x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} &= \lambda_y \sum_i \sum_j K_{ji} \left(B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}} \right) \times \left(x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{1-\alpha}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right) \iff \\ \lambda_y &= \frac{\sum_i x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}}}{\sum_i \sum_j K_{ji} \left(B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}} \right) \times \left(x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{1-\alpha}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right)} \\ &= \frac{\sum_i x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}}}{\sum_i \left[\sum_j K_{ij} \left(x_j^{\frac{\alpha}{\beta+\alpha-1}} y_j^{\frac{1-\alpha}{\beta+\alpha-1}} B_j^{\frac{1}{1-\alpha-\beta}} \right) \right] \times \left(B_i^{\frac{1}{1-\alpha-\beta}} x_i^{\frac{1-\beta}{\alpha+\beta-1}} y_i^{\frac{\beta}{\alpha+\beta-1}} \right)}. \end{aligned}$$

Comparing the expressions for λ_x and λ_y, we immediately have $\lambda_x = \lambda_y \equiv \lambda$. □

The previous lemma tells that there exists (x_i, y_i) satisfying the following set of the equations.

$$x_i = \lambda \sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} \tag{36}$$

$$y_i = \lambda \sum_j K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}. \tag{37}$$

In the general equilibrium trade models, (x_i, y_i) should solve the same system with $\lambda = 1$. The following lemma tells that a simple transformation of (x_i, y_i) solves (31) and (32).

Lemma 3. *There exits s such that (sx_i, y_i) satisfying (31) and (32), i.e. satisfying (36) and (37) with $\lambda = 1$.*

Proof. Now take s as

$$s = \left(\sum_{i,j} K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} \right)^{\frac{1}{1-\frac{\alpha}{\alpha+\beta-1}}}.$$

To complete the proof of the lemma, we have to show that if (x, y) is a solution to

$$\begin{aligned}
x_i &= \frac{\sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} (x_j)^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}}}{\sum_{i,j} K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} (x_j)^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}}} \\
y_i &= \frac{\sum_j K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} (x_j)^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}}{\sum_j K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} (x_j)^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}},
\end{aligned}$$

then (sx, y) is a solution to a general equilibrium trade model, and s is given by

$$s = \left(\sum_{i,j} K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} \right)^{\frac{1}{1-\frac{\alpha}{\alpha+\beta-1}}}.$$

Namely $(\widetilde{x}, \widetilde{y}) = (sx, y)$ solves

$$\begin{aligned}
(\widetilde{x}_i) &= \sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} (\widetilde{x}_j)^{\frac{\alpha}{\alpha+\beta-1}} \widetilde{y}_j^{\frac{1-\alpha}{\alpha+\beta-1}} \\
\widetilde{y}_i &= \sum_j K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} (\widetilde{x}_j)^{\frac{1-\beta}{\alpha+\beta-1}} \widetilde{y}_j^{\frac{\beta}{\alpha+\beta-1}}.
\end{aligned}$$

To prove this last point, note that

$$\begin{aligned}
sx_i &= \underbrace{\frac{s^{1-\frac{\alpha}{\alpha+\beta-1}}}{\sum_{i,j} K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} (x_j)^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}}}}_{=1} \sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} (sx_j)^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} \\
&= \sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} (sx_j)^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}}.
\end{aligned}$$

The equality holds by construction of s. Thus first equation that $\widetilde{x}_i = \sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} (\widetilde{x}_j)^{\frac{\alpha}{\alpha+\beta-1}} \widetilde{y}_j^{\frac{1-\alpha}{\alpha+\beta-1}}$ is satisfied. To show the second equation, it suffices to show

$$\sum_{i,j} K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} (\widetilde{x}_j)^{\frac{1-\beta}{\alpha+\beta-1}} \widetilde{y}_j^{\frac{\beta}{\alpha+\beta-1}} = 1.$$

This holds since[25]

$$\begin{aligned}\sum_{i,j} K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} \left(sx_j\right)^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}} &= t^{\frac{1-\beta}{\alpha+\beta-1}} \sum_{i,j} K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}} \\ &= \frac{\sum_{i,j} K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}}}{\sum_{i,j} K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}} = 1.\end{aligned}$$

The last equality holds from the previous lemma. □

Previous two lemmas imply that there exists (x_i, y_i) satisfying equations (31) and (32). The final touch is needed to ensure that there exists (x_i, y_i) satisfying (31), (32), and 33.

Lemma 4. *There exists (x_i, y_i) satisfying (31) (32), and (33).*

Proof. From Lemma 3, there exists (x_i, y_i) satisfying ((31)) and ((32)).

Now consider $(\widetilde{x}_i, \widetilde{y}_i) = \left(t^{\frac{\alpha-1}{1-\beta}} x_i, t y_i\right)$, where

$$t = \left[\sum_i B_i^{\frac{1}{1-\alpha-\beta}} \left(x_i\right)^{\frac{\alpha}{\beta+\alpha-1}} \left(y_i\right)^{\frac{\beta}{\beta+\alpha-1}}\right]^{-\frac{1-\beta}{\alpha-\beta}}.$$

It is easy to show

$$\begin{aligned}\widetilde{x}_i &= t^{-\frac{1-\alpha}{1-\beta}} x_i = t^{-\frac{1-\alpha}{1-\beta}} \sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} \left(x_j\right)^{\frac{\alpha}{\alpha+\beta-1}} \left(y_j\right)^{\frac{1-\alpha}{\alpha+\beta-1}} \\ &= \sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} \left(t^{-\frac{1-\alpha}{1-\beta}} x_j\right)^{\frac{\alpha}{\alpha+\beta-1}} \left(t y_j\right)^{\frac{1-\alpha}{\alpha+\beta-1}} \\ &= \sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} \left(\widetilde{x}_j\right)^{\frac{\alpha}{\alpha+\beta-1}} \left(\widetilde{y}_j\right)^{\frac{1-\alpha}{\alpha+\beta-1}} \\ \widetilde{y}_i &= t \sum_j K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} \left(x_j\right)^{\frac{1-\beta}{\alpha+\beta-1}} \left(y_j\right)^{\frac{\beta}{\alpha+\beta-1}} \\ &= \sum_j K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} \left(\widetilde{x}_j\right)^{\frac{1-\beta}{\alpha+\beta-1}} \left(\widetilde{y}_j\right)^{\frac{\beta}{\alpha+\beta-1}}.\end{aligned}$$

Thus $(\widetilde{x}_i, \widetilde{y}_i)$ still solves (31) and (32).

[25] If $\beta = 1$, then this last line is not true, since the equation for y is no longer dependent on x. In this case, however, existence and uniqueness follows immediately from Theorem 1 of Karlin and Nirenberg (1967), as the two integral equations can be treated as distinct from each other.

The world income induced by $(\widetilde{x}_i, \widetilde{y}_i) = \left(t^{\frac{\alpha-1}{1-\beta}} x_i, ty_i\right)$ is

$$\begin{aligned}
\sum_i B_i^{\frac{1}{1-\alpha-\beta}} \left(t^{\frac{\alpha-1}{1-\beta}} x_i\right)^{\frac{\alpha}{\beta+\alpha-1}} (ty_i)^{\frac{\beta}{\beta+\alpha-1}} &= \sum_i B_i^{\frac{1}{1-\alpha-\beta}} \left(t^{\frac{\alpha-1}{1-\beta}} x_i\right)^{\frac{\alpha}{\beta+\alpha-1}} (ty_i)^{\frac{\beta}{\beta+\alpha-1}} \\
&= t^{\frac{(\alpha-\beta)}{1-\beta}} \sum_i B_i^{\frac{1}{1-\alpha-\beta}} (x_i)^{\frac{\alpha}{\beta+\alpha-1}} (y_i)^{\frac{\beta}{\beta+\alpha-1}} = 1.
\end{aligned}$$

The last equality holds by the construction of t. □

A.1.3 Part 3: Existence for trade models

We next consider the existence of a strictly positive solution to the general equilibrium gravity model defined by equations (31), (32), and 33.

Proof. It directly follows from Proposition 4. □

A.1.4 Part 4: Uniqueness for trade models

We now consider the uniqueness of the general equilibrium gravity model. We prove uniqueness by contradiction.

Proof. Suppose that there are two solutions $(x, y), (\widetilde{x}, \widetilde{y})$ satisfying ((31)), ((32)), and (33). Then there are no constants t such that[26]

$$x = t\widetilde{x}. \tag{38}$$

Without loss of generality, we can assume that for all i,

$$\sum_j F_{i,j} = \sum_j H_{i,j} = 1.$$

Also we can take $(\widetilde{x}, \widetilde{y}) = (1, 1)$ since

$$\begin{aligned}
1 &= \sum_j F_{i,j} 1^a 1^b \\
1 &= \sum_j H_{i,j} 1^c 1^d.
\end{aligned}$$

[26]Such t exists when (x, y) and (x', y') gives the same real variables and the only difference comes from the price level.

Define

$$m_x \equiv \min_i x_i,\ M_x \equiv \max_i x_i,\ m_y \equiv \min_i y_i,\ M_y \equiv \max_i y_i.$$

From (38), $m_x\,(m_y)$ is strictly less than $M_x\,(M_y)$ respectively.

Given the relationship above between a, b, c and d and the gravity constants α and β, it is easy to show that the following inequalities hold:

$$\begin{aligned} c &< 0 < a \\ b &< 0 < d. \end{aligned}$$

Given that we have that

$$\begin{aligned} \max x_i &= M_x = \max \sum_j F_{i,j} x_j^a y_j^b \leq M_x^a m_y^b \\ \max y_i &= M_y = \max \sum_j H_{i,j} x_j^c y_j^d \leq m_x^c M_y^d \\ m_x = \min x_i &= \min \sum_j F_{i,j} x_j^a y_j^b \geq m_x^a M_y^b \\ m_y = \min y_i &= \min \sum_j H_{i,j} x_j^c y_j^d \geq M_x^c m_y^d. \end{aligned}$$

It is easy to show[27]

$$\left(\frac{M_x}{m_x}\right)^{1-a} \left(\frac{M_y}{m_y}\right)^{b} < 1,\ \left(\frac{M_x}{m_x}\right)^{c} \left(\frac{M_y}{m_y}\right)^{1-d} < 1.$$

[27] To obtain first equation, multiply first and third equation.

$$M_x \left(m_x^b M_y^b\right) \leq m_x \left(M_x^a m_y^b\right),$$

which is equivalent to

$$\left(\frac{M_x}{m_x}\right)^{1-a} \left(\frac{M_y}{m_y}\right)^{b} < 1.$$

For second equation, multiply second and fourth equation.

$$(M_y)\, M_x^c m_y^d \leq \left(m_x^c M_y^d\right) m_y,$$

which implies

$$\left(\frac{M_x}{m_x}\right)^{c} \left(\frac{M_y}{m_y}\right)^{1-d} \leq 1$$

Since $c = a - 1$, and $d = b + 1$,

$$\left(\frac{M_x}{m_x}\right)^{1-a}\left(\frac{M_y}{m_y}\right)^{b} < 1, \quad \left(\frac{M_x}{m_x}\right)^{a-1}\left(\frac{M_y}{m_y}\right)^{-b} < 1.$$

Therefore the following holds.

$$\left(\frac{M_x}{m_x}\right)^{1-a}\left(\frac{M_y}{m_y}\right)^{b} < 1 < \left(\frac{M_x}{m_x}\right)^{1-a}\left(\frac{M_y}{m_y}\right)^{b},$$

which is a contradiction. $\square$

A.2 Proof of Theorem 2

Proof. Part i) This relation comes from conditions C.2 and C.3 $\sum_i X_{i,j} = \sum_i X_{j,i}$, which, given C.1, is equivalent to:

$$\frac{K_i^A\gamma_i}{K_i^B\delta_i} = \frac{\sum_j \widetilde{K}_{i,j}K_j^A\gamma_j}{\sum_j \widetilde{K}_{i,j}K_j^B\delta_j} = \sum_j \frac{\widetilde{K}_{i,j}K_j^B\delta_j}{\sum_j\left(\widetilde{K}_{i,j}K_j^B\delta_j\right)} \times \frac{K_j^A\gamma_j}{K_j^B\delta_j}.$$

It is easy to show that

$$\frac{K_i^A\gamma_i}{K_i^B\delta_i} = 1,$$

is a solution to the problem. From the Perron-Frobenius theorem, this solution is unique up to scale. Therefore for some κ, we have

$$\gamma_i K_i^A = \kappa\delta_i K_i^B. \tag{39}$$

Part ii) The relation (39) implies

$$y_i = \frac{\gamma_i}{\delta_i}x_i = \kappa\frac{K_i^B}{K_i^A}x_i.$$

Substituting this expression into (31), we get

$$x_i = \kappa^{\frac{1-\alpha}{\alpha+\beta-1}}\sum_j \widetilde{K}_{i,j}K_i^A K_j^B B_j^{\frac{1}{1-\alpha-\beta}}\left(\frac{K_i^B}{K_i^A}\right)^{\frac{1-\alpha}{\alpha+\beta-1}} x_j^{\frac{1}{\alpha+\beta-1}}. \tag{40}$$

Also, if we substitute the same expression into (32), we get the exact same expression. Therefore one of the two equations is trivially satisfied. From Theorem 1 of Karlin and

Nirenberg (1967), the system has an unique solution if $\left|\frac{1}{\alpha+\beta-1}\right| \leq 1$, which is equivalent to the condition given in the statement of the theorem. □

A.3 Proof of Proposition 1

A.3.1 Part (i): The trade equilibrium solves the world income maximization problem.

Proof. To show that the trade equilibrium maximizes the world income, we show that the FONCs for the maximization problem coincide with the equilibrium conditions for the trade model. Mathematically we show that any solutions to the world income maximization satisfy the trade equilibrium conditions.

The associated Lagrangian of the maximization problem is:

$$\mathcal{L} : \sum_{i \in S} \sum_{j \in S} K_{ij} \gamma_i \delta_j - \sum_{i \in S} \kappa_i \left(\sum_j K_{ij} \gamma_i \delta_j - \sum_j K_{ji} \gamma_j \delta_i \right) - \lambda \left(\sum_{i \in S} \sum_{j \in S} K_{ij} \gamma_i \delta_j - \sum_{i \in S} B_i \gamma_i^{\alpha} \delta_i^{\beta} \right) \iff$$

$$\mathcal{L} : (1-\lambda) \sum_{i \in S} \sum_{j \in S} K_{ij} \gamma_i \delta_j - \sum_{i \in S} \kappa_i \left(\sum_j K_{ij} \gamma_i \delta_j - \sum_j K_{ji} \gamma_j \delta_i \right) + \lambda \sum_{i \in S} B_i \gamma_i^{\alpha} \delta_i^{\beta},$$

where $\{\kappa_i\}$ are the Lagrange multipliers on the balanced trade constraint and λ is the Lagrange multiplier on the aggregate factor market clearing.

First order conditions with respect to γ_i are:

$$(1-\lambda-\kappa_i) \sum_j K_{ij} \gamma_i \delta_j + \sum_j K_{ij} \gamma_i \delta_j \kappa_j + \alpha \lambda B_i \gamma_i^{\alpha} \delta_i^{\beta} = 0 \tag{41}$$

First order conditions with respect to δ_i are:

$$(1-\lambda+\kappa_i) \sum_j K_{ji} \gamma_j \delta_i - \sum_j K_{ji} \gamma_j \delta_i \kappa_j + \beta \lambda B_i \gamma_i^{\alpha} \delta_i^{\beta} = 0 \tag{42}$$

We first solve for the λ. Add the two FOC together and sum over all $i \in S$:

$$2(1-\lambda) \sum_i \sum_j K_{ij} \gamma_i \delta_j + \sum_i \sum_j (K_{ij} \gamma_i \delta_j - K_{ji} \gamma_j \delta_i) \kappa_j + (\alpha+\beta) \lambda \sum_i B_i \gamma_i^{\alpha} \delta_i^{\beta} = 0,$$

which implies

$$\lambda = \frac{2}{2-\alpha-\beta}. \tag{43}$$

The FONCs for γ_i and δ_i become:

$$B_i\gamma_i^\alpha\delta_i^\beta = \left(\frac{\alpha+\beta}{2\alpha}+\frac{2-\alpha-\beta}{2\alpha}\kappa_i\right)\sum_j K_{ij}\gamma_i\delta_j - \frac{2-\alpha-\beta}{2\alpha}\sum_j K_{ij}\gamma_i\delta_j\kappa_j \tag{44}$$

$$B_i\gamma_i^\alpha\delta_i^\beta = \left(\frac{\alpha+\beta}{2\beta}-\frac{2-\alpha-\beta}{2\beta}\kappa_i\right)\sum_j K_{ji}\gamma_j\delta_i + \frac{2-\alpha-\beta}{2\beta}\sum_j K_{ji}\gamma_j\delta_i\kappa_j \tag{45}$$

We now try to solve for the κ. Equating the two FOC yields:

$$\frac{\beta-\alpha}{2-\alpha-\beta}+\kappa_i = \frac{\sum_j\left(\frac{\alpha}{\alpha+\beta}K_{ji}\gamma_j\delta_i+\frac{\beta}{\alpha+\beta}K_{ij}\gamma_i\delta_j\right)\kappa_j}{\sum_j K_{ij}\gamma_i\delta_j}. \tag{46}$$

Substituting (46) back into the FOC for γ_i yields:

$$B_i\gamma_i^\alpha\delta_i^\beta = \sum_j K_{ij}\gamma_i\delta_j + \frac{2-\alpha-\beta}{\alpha+\beta}\sum_j\left(\frac{K_{ji}\gamma_j\delta_i - K_{ij}\gamma_i\delta_j}{2}\right)\kappa_j \tag{47}$$

Substituting (46) back into the FOC for δ_i yields:

$$B_i\gamma_i^\alpha\delta_i^\beta = \sum_j K_{ji}\gamma_j\delta_i + \frac{2-\alpha-\beta}{\alpha+\beta}\sum_j\left(\frac{K_{ij}\gamma_i\delta_j - K_{ji}\gamma_j\delta_i}{2}\right)\kappa_j \tag{48}$$

Note that equating the two FOC yields:

$$\sum_j K_{ji}\gamma_j\delta_i\kappa_j = \sum_j K_{ij}\gamma_i\delta_j\kappa_j,$$

where the second to last line imposed balanced trade. Hence the first order conditions become:

$$B_i\gamma_i^\alpha\delta_i^\beta = \sum_j K_{ij}\gamma_i\delta_j$$
$$B_i\gamma_i^\alpha\delta_i^\beta = \sum_j K_{ji}\gamma_j\delta_i.$$

Therefore the solution to the problem is unique and coincides with the allocation of the general equilibrium gravity model. $\square$

A.3.2 Part (ii) : The trade equilibrium solves the world welfare maximization problem.

Proof. With the assumption we made that the utility for country i is expressed in the following form

$$u_i = \left(B_i \left(\gamma_i\right)^{\alpha-1} \left(\delta_i\right)^{\beta-1}\right)^{1/\rho},$$

the welfare maximization problem is to maximize the weighted sum of $\{u_i\}_i$ subject to the same constraints. To show that the competitive allocation is Pareto efficient, we show that under a particular choice of (θ_i), the competitive allocation $\left(\gamma_i^{CE}, \delta_i^{CE}\right)_i$ solves the planning problem.

Set the Pareto weights $(\omega_i)_i$ as follows.

$$(\omega_i) = \sum_k \frac{\left(B_k\right)^{1/\rho} \left(\gamma_k^{CE}\right)^{(\alpha-1)/\rho} \left(\delta_k^{CE}\right)^{\rho(\beta-1)/\rho}}{\left(B_i\right)^{1/\rho} \left(\gamma_i^{CE}\right)^{(\alpha-1)/\rho} \left(\delta_i^{CE}\right)^{(\beta-1)/\rho} \frac{\sum_j (B_i)^{1/\rho} \left(\gamma_i^{CE}\right)^{\alpha} \left(\delta_i^{CE}\right)^{\beta}}{\sum_j K_{j,i} \gamma_j^{CE} \delta_i^{CE}}} \left(\omega_k\right).$$

From Karlin and Nirenberg (1967), we know there is a solution to the system.

The associated Lagrangian is

$$\mathcal{L} = \sum_i \omega_i B_i^{1/\rho} \gamma_i^{(\alpha-1)/\rho} \delta_i^{(\beta-1)/\rho} - \lambda \left(\sum_i \sum_j K_{ij} \gamma_i \delta_j - \sum_i B_i \gamma_i^{\alpha} \delta_i^{\beta}\right).$$

Taking the FONCs w.r.t. γ_i and δ_i, we get

$$\rho^{-1} (\alpha - 1) \omega_i B_i^{1/\rho} \gamma_i^{(\alpha-1)/\rho} \delta_i^{(\beta-1)/\rho} = \lambda \sum_j K_{ij} \gamma_i \delta_j - \alpha \lambda B_i \gamma_i^{\alpha} \delta_i^{\beta}$$
$$\rho^{-1} (\beta - 1) \omega_i B_i^{1/\rho} \gamma_i^{(\alpha-1)/\rho} \delta_i^{(\beta-1)/\rho} = \lambda \sum_j K_{ji} \gamma_j \delta_i - \lambda \beta B_i \gamma_i^{\alpha} \delta_i^{\beta}.$$

Adding the two equations, and solving for λ, we have

$$\lambda = \frac{1}{\rho} \frac{W}{Y}.$$

Substitute this expression into the FONCs.

$$\left(\frac{\alpha-1}{\alpha}\right) \left(\frac{\omega_i B_i^{1/\rho} \gamma_i^{(\alpha-1)/\rho} \delta_i^{(\beta-1)/\rho}}{W} Y^W - \sum_j K_{ij} \gamma_i \delta_j\right) + \sum_j K_{ij} \gamma_i \delta_j = B_i \gamma_i^{\alpha} \delta_i^{\beta}$$
$$\left(\frac{\beta-1}{\beta}\right) \left(\frac{\omega_i B_i^{1/\rho} \gamma_i^{(\alpha-1)/\rho} \delta_i^{(\beta-1)/\rho}}{W} Y^W - \sum_j K_{ji} \gamma_j \delta_i\right) + \sum_j K_{ji} \gamma_j \delta_i = B_i \gamma_i^{\alpha} \delta_i^{\beta}.$$

From the construction of ω_i, the bracket term is zero if we evaluate the system at $\left(\gamma_i^{CE}, \delta_i^{CE}\right)_i$.

$$\left(\omega_i B_i^{1/\rho}\left(\gamma_i^{CE}\right)^{(\alpha-1)/\rho}\left(\delta_i^{CE}\right)^{(\beta-1)/\rho}\frac{\sum_j B_j\left(\gamma_j^{CE}\right)^{\alpha}\left(\delta_j^{CE}\right)^{\beta}}{\sum_j \omega_i B_i^{1/\rho}\left(\gamma_i^{CE}\right)^{(\alpha-1)/\rho}\left(\delta_i^{CE}\right)^{(\beta-1)/\rho}}-\sum_j K_{ij}\gamma_i^{CE}\delta_j^{CE}\right)=0.$$

Then the second equation is solved at $\left(\gamma_i^{CE}, \delta_i^{CE}\right)_i$ since

$$\sum_j K_{ji}\gamma_j^{CE}\delta_i^{CE}=B_i\left(\gamma_i^{CE}\right)_i^{\alpha}\left(\delta_i^{CE}\right)^{\beta}.$$

□

A.4 Proof of Proposition 2

Proof. We need to show that there exist a unique set of $\{K_{ij}\}_{i\neq j}$ such that observed trade flows satisfy the model equilibrium conditions, i.e.:

$$\begin{aligned} K_{ij}\gamma_i\delta_j &= X_{ij} && (49)\\ B_i\gamma_i^{\alpha}\delta_i^{\beta} &= \sum_j X_{ij} && (50)\\ B_i\gamma_i^{\alpha}\delta_i^{\beta}+\bar{D}_i &= \sum_j X_{ji}, && (51)\end{aligned}$$

where recall that $\{X_{ij}\}$ are observed and $\{B_i\}$,$\{K_{ii}\}$, α, and β are chosen.

We proceed by construction. Choose $\bar{D}_i \equiv \sum_j X_{ji}-\sum_j X_{ij}$, $Y_i \equiv \sum_j X_{ij}$ $\gamma_i \equiv \left(\frac{X_{ii}}{K_{ii}}\right)^{-\frac{\beta}{\alpha-\beta}}\left(\frac{B_i}{Y_i}\right)^{-\frac{1}{\alpha-\beta}}$ $\delta_i \equiv \left(\frac{X_{ii}}{K_{ii}}\right)^{\frac{\alpha}{\alpha-\beta}}\left(\frac{B_i}{Y_i}\right)^{\frac{1}{\alpha-\beta}}$, and $K_{ij} \equiv \frac{X_{ij}}{\gamma_i\delta_j}$. We verify that these definitions satisfy equilibrium conditions given observed trade flows. First, note that (49) is satisfied by the construction of K_{ij}.

Second, note that:

$$\begin{aligned} B_i\gamma_i^{\alpha}\delta_i^{\beta} &= B_i\left(\left(\frac{X_{ii}}{K_{ii}}\right)^{-\frac{\beta}{\alpha-\beta}}\left(\frac{B_i}{Y_i}\right)^{-\frac{1}{\alpha-\beta}}\right)^{\alpha}\left(\left(\frac{X_{ii}}{K_{ii}}\right)^{\frac{\alpha}{\alpha-\beta}}\left(\frac{B_i}{Y_i}\right)^{\frac{1}{\alpha-\beta}}\right)^{\beta} \iff\\ B_i\gamma_i^{\alpha}\delta_i^{\beta} &= B_i\left(\frac{B_i}{Y_i}\right)^{\frac{\beta-\alpha}{\alpha-\beta}} \iff\\ B_i\gamma_i^{\alpha}\delta_i^{\beta} &= \sum_j X_{ij},\end{aligned}$$

so that equilibrium condition (50) is satisfied.

Finally, note that:

$$B_i\gamma_i^{\alpha}\delta_i^{\beta} + \bar{D}_i B_i\gamma_i^{\alpha}\delta_i^{\beta} + \bar{D}_i = \sum_j X_{ij} + \sum_j X_{ji} - \sum_j X_{ij} \iff = \sum_j X_{ji},$$

so that equilibrium condition (51) is satisfied. $\square$

A.5 Proof of Proposition 3

Part (i) We start by proving the first part with an application of the implicit function theorem.

Proof. Some notation is necessary. Define $y_i \equiv \ln \gamma_i$, $z_i \equiv \ln \delta_i$, $k_{ij} \equiv \ln K_{ij}$. Let $\vec{y} \equiv \{y_i\}$ and $\vec{z} \equiv \{z_i\}$ both be $N \times 1$ vectors and let $\vec{x} \equiv \{\vec{y}; \vec{z}\}$ be a $2N \times 1$ vector. Let $\vec{k} \equiv \{k_{ij}\}$ be a $N^2 \times 1$ vector. Now consider the function $f\left(\vec{x}; \vec{k}\right) : R^{2N} \times R^{N^2} \to R^{2N}$ given by:

$$f\left(\vec{x}; \vec{k}\right) = \begin{bmatrix} \left[B_i \left(\exp\{y_i\}\right)^{\alpha} \left(\exp\{z_i\}\right)^{\beta} - \sum_j \exp\{k_{i,j}\} \left(\exp\{y_i\}\right) \left(\exp\{z_j\}\right)\right]_i \\ \vdots \\ \left[B_i \left(\exp\{y_i\}\right)^{\alpha} \left(\exp\{z_i\}\right)^{\beta} - \sum_j \exp\{k_{j,i}\} \left(\exp\{y_j\}\right) \left(\exp\{z_i\}\right)\right]_i \end{bmatrix}.$$

In the general equilibrium trade model, we have:

$$f\left(\vec{x}; \vec{k}\right) = 0.$$

Full differentiation of the function hence yields:

$$f_{\vec{x}} D_{\vec{k}} \vec{x} + f_{\vec{k}} = 0, \tag{52}$$

where $f_{\vec{x}}$ is the $2N \times 2N$ matrix:

$$f_{\vec{x}}\left(\vec{x}; \vec{k}\right) = \begin{pmatrix} (\alpha - 1) Y & \beta Y - X \\ \alpha Y - X^T & (\beta - 1) Y \end{pmatrix},$$

where Y is a $N \times N$ diagonal matrix whose i^{th} diagonal is equal to Y_i and X is the $N \times N$ trade matrix.

Similarly, $f_{\vec{k}}$ is a $2N \times N^2$ matrix that depends only on trade flows:

$$f_{\vec{k}}\left(\vec{x},\vec{k}\right) = -\begin{pmatrix} X_{11} & \cdots & X_{1N} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & X_{21} & \cdots & X_{2N} & \cdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & X_{N1} & \cdots & X_{NN} \\ X_{11} & \cdots & 0 & X_{21} & \cdots & 0 & \cdots & X_{N1} & \cdots & 0 \\ 0 & \ddots & \vdots & 0 & \ddots & \vdots & \cdots & 0 & \ddots & \vdots \\ 0 & \cdots & X_{1N} & 0 & \cdots & X_{2N} & \cdots & 0 & \cdots & X_{NN} \end{pmatrix}$$

If $f_{\vec{x}}$ was of full rank, we could immediately invert equation (52) (i.e. apply the implicit function theorem) to immediately yield:

$$D_{\vec{k}}\vec{x} = -\left(f_{\vec{x}}\right)^{-1} f_{\vec{k}}.$$

However, because Walras Law holds and we can without loss of generality apply a normalization to $\{\gamma_i\}$ and $\{\delta_i\}$, we effectively have $N-1$ equations and $N-1$ unknowns. Hence, there exists an infinite number of solutions to equation (52), each corresponding to a different normalization. To find the solution that corresponds to our choice of world income as the numeraire, note that from equation (1):

$$\sum_l B_l \gamma_l^{\alpha} \delta_l^{\beta} = Y^W \implies$$
$$\sum_l Y_l \left(\alpha \frac{\partial \ln \gamma_l}{\partial \ln K_{ij}} + \beta \frac{\partial \ln \delta_l}{\partial \ln K_{ij}}\right) = 0. \tag{53}$$

We claim that if $\frac{\partial \ln \gamma_l}{\partial \ln K_{ij}} = X_{ij} \times \left(A_{l,i} + A_{N+l,j}\right) - c$ and $\frac{\partial \ln \delta_l}{\partial \ln K_{ij}} = X_{ij} \times \left(A_{N+l,i} + A_{l,j}\right) - c$, where $c \equiv \frac{1}{Y^W(\alpha+\beta)} X_{ij} \sum_l Y_l \left(\alpha \left(A_{l,i} + A_{N+l,j}\right) + \beta \left(A_{N+l,i} + A_{l,j}\right)\right)$, then $\frac{\partial \ln \gamma_l}{\partial \ln K_{ij}}$ and $\frac{\partial \ln \delta_l}{\partial \ln K_{ij}}$ solve equations (52) and (53). It is straightforward to see that our assumed solution ensures equation (52) holds, as the generalized inverse is a means of choosing from one of the infinitely many solutions; see James (1978). It remains to scale the set of elasticities appropriately to ensure that our normalization holds as well. Given our definition of the scalar c, it is straightforward to verify that equation (53) holds:

$$
\begin{aligned}
\sum_l Y_l\left(\alpha\frac{\partial\ln\gamma_l}{\partial\ln K_{ij}}+\beta\frac{\partial\ln\delta_l}{\partial\ln K_{ij}}\right) &= \begin{array}{l}\sum_l Y_l(\alpha\left(X_{ij}\times\left(A_{l,i}+A_{N+l,j}\right)-c\right)+\\ \quad+\beta\left(X_{ij}\times\left(A_{N+l,i}+A_{l,j}\right)-c\right))\end{array}\\
&= \begin{array}{r}X_{ij}\sum_l Y_l\left(\alpha\left(X_{ij}\times\left(A_{l,i}+A_{N+l,j}\right)\right)+\beta\left(X_{ij}\times\left(A_{N+l,i}+A_{l,j}\right)\right)\right)\\ -c\left(\alpha+\beta\right)\sum_l Y_l\end{array}\\
&= \begin{array}{r}X_{ij}\sum_l Y_l\left(\alpha\left(X_{ij}\times\left(A_{l,i}+A_{N+l,j}\right)\right)+\beta\left(X_{ij}\times\left(A_{N+l,i}+A_{l,j}\right)\right)\right)-\\ \left(\frac{1}{Y^W(\alpha+\beta)}X_{ij}\sum_l Y_l\left(\alpha\left(A_{l,i}+A_{N+l,j}\right)+\beta\left(A_{N+l,i}+A_{l,j}\right)\right)\right)\left(\alpha+\beta\right)Y^W\end{array}\\
&= 0,
\end{aligned}
$$

i.e. equation (53) also holds. More generally, different choices of c correspond to different normalizations. A particularly simple example is if we choose the normalization $\gamma_1 = 1$. Since this implies that $\frac{\partial\ln\gamma_1}{\partial\ln K_{ij}} = 0$, $c = X_{ij}\times\left(A_{1,i}+A_{N+1,j}\right)$. In this case, however, an alternative procedure is even simpler: the elasticities for all $i > 1$ can be calculated directly by inverting the $(2N-1)\times(2N-1)$ matrix generated by removing the first row and first column of $f_{\vec{x}}$. □

Part (ii). This part has several cases. Before formal proving the theorem, we introduce two simple lemmas that help to simplify the main proof. The proof of the first lemma is in Online Appendix B.6 and the proof of the second lemma can be found in Allen, Arkolakis, and Li (2014).

Lemma 5. *A and B are two matrices whose summation of the elements of every row, $\sum_j a_{ij}$, and $\sum b_{ij}$, are two constants $\bar{a}$ and $\bar{b}$. Then $\bar{a}\bar{b}$ is AB's eigenvalue and it is also the summation of each row in AB.*

and

Lemma 6. *Suppose A is a $N\times N$ positive matrix and λ_0 is its positive eigenvalue. Then the rank of $\lambda_0 I - A$ is $N-1$.*

Now we can proceed with the proof of the second part of the Proposition.

Proof. **Part ii-1)** Denote $\mathbf{B} = \begin{pmatrix}\mathbf{Y}^{-1} & 0\\ 0 & \mathbf{Y}^{-1}\end{pmatrix}$. Then

$$
\mathbf{BA} = \begin{pmatrix}\mathbf{Y}^{-1} & 0\\ 0 & \mathbf{Y}^{-1}\end{pmatrix}\begin{pmatrix}(\alpha-1)\,\mathbf{Y} & \beta\mathbf{Y}-\mathbf{X}\\ \alpha\mathbf{Y}-\mathbf{X}^T & (\beta-1)\,\mathbf{Y}\end{pmatrix} = \begin{pmatrix}(\alpha-1)\,\mathbf{I} & \beta\mathbf{I}-\mathbf{C}\\ \alpha\mathbf{I}-\mathbf{D} & (\beta-1)\,\mathbf{I}\end{pmatrix},
$$

where we define $\mathbf{C} \equiv \mathbf{Y}^{-1}\mathbf{X}$, $\mathbf{D} \equiv \mathbf{Y}^{-1}\mathbf{X}^T$, and $\mathbf{I}$ to be the $N \times N$ identity matrix. It is easy to verify that the row-summation of $\mathbf{C}$ and $\mathbf{D}$ are both constant and equal to 1 –i.e. $\mathbf{D}\overrightarrow{e} = \overrightarrow{e}$ where $\overrightarrow{e}$ is the identity vector– when we have trade balance. Since $\mathbf{Y}$ is a diagonal matrix, $\mathbf{B}$ is of full rank and the rank of $\mathbf{BA}$ is equal to the minimum of the ranks of $\mathbf{B}$ and $\mathbf{A}$ i.e. the rank of $\mathbf{A}$. Suppose $\alpha - 1 \neq 0$, implement the Gauss elimination on $\mathbf{BA}$, to get

$$\begin{pmatrix} (\alpha-1)\,\mathbf{I} & \beta\mathbf{I}-\mathbf{C} \\ \alpha\mathbf{I}-\mathbf{D} & (\beta-1)\,\mathbf{I} \end{pmatrix} = \begin{pmatrix} (\alpha-1)\,\mathbf{I} & \beta\mathbf{I}-\mathbf{C} \\ 0 & \frac{(\alpha-\mathbf{1})(\beta-\mathbf{1})\mathbf{I}-(\alpha\mathbf{I}-\mathbf{D})(\beta\mathbf{I}-\mathbf{C})}{(\alpha-1)} \end{pmatrix}. \tag{54}$$

Denote $E = (\alpha-\mathbf{1})\,(\beta-\mathbf{1})\,\mathbf{I} - (\alpha\mathbf{I}-D)\,(\beta\mathbf{I}-\mathbf{C})$. We now prove the four individual mutually excluding cases.

Let $\alpha, \beta \leq 0$: We have

$$E = (\mathbf{1}-\alpha)\,(\mathbf{1}-\beta)\,\mathbf{I} - (-\alpha\mathbf{I}+\mathbf{D})\,(-\beta\mathbf{I}+\mathbf{C})$$

The row summation $-\alpha\mathbf{I}+\mathbf{D}$ and $-\beta\mathbf{I}+\mathbf{C}$ are both positive since $1-\alpha > 0$ and $1-\beta > 0$. According to Lemma 5, $(1-\alpha)\,(1-\beta)$ is the eigenvalue of the positive matrix $(-\alpha\mathbf{I}+\mathbf{D})\,(-\beta\mathbf{I}+\mathbf{C})$. According to Lemma 6, the rank of E is $N-1$. Thus, the rank of $\mathbf{A}$ is $2N-1$.

Let $\alpha, \beta > 1$: The row summation of $\mathbf{D}$ is $1 < \alpha$. According to Theorem 3 in Allen, Arkolakis, and Li (2014) $\alpha\mathbf{I}-\mathbf{D}$ is invertible and the inverse $(\alpha\mathbf{I}-\mathbf{D})^{-1}$ is positive. As $(\alpha\mathbf{I}-\mathbf{D})\,\overrightarrow{e} = (\alpha-1)\,\overrightarrow{e}$, since the $(\alpha\mathbf{I}-\mathbf{D})^{-1}\,\overrightarrow{e} = \frac{1}{(\alpha-1)}\overrightarrow{e}$,. Thus, $(\alpha-1)^{-1}$ is the unique positive eigenvalue of positive matrix $(\alpha\mathbf{I}-D)^{-1}$. Similarly, $(\beta-1)^{-1}$ is the unique positive eigenvalue of positive matrix $(\beta\mathbf{I}-C)^{-1}$. Notice that

$$E = (\alpha\mathbf{I}-\mathbf{D})\,[\mathbf{F}-\mathbf{I}]\,(\beta\mathbf{I}-C)\,,$$

where $\mathbf{F} = (\alpha-\mathbf{1})\,(\alpha\mathbf{I}-\mathbf{D})^{-\mathbf{1}}\,(\beta-\mathbf{1})\,(\beta\mathbf{I}-\mathbf{C})^{-\mathbf{1}}$. $\mathbf{F}$ is a positive matrix and $\mathbf{F}\overrightarrow{e} = \overrightarrow{e}$ which means 1 is the eigenvalue of $\mathbf{F}$. Thus, according to above Lemma 6, the rank of $\mathbf{F}-\mathbf{I}$ is $N-1$. So again the rank of $\mathbf{E}$ is $N-1$, so that the $\mathbf{A}$ is $2N-1$.

Let $|\alpha| > 1$, $\beta = 1$ or $\alpha = 1$, $|\beta| > 1$: Without loss of generality, we only consider the case $\beta = 1$. Then implementing the Gaussian elimination on $\mathbf{BA}$, equation 54, we get

$$\begin{pmatrix} (\alpha-1)\,\mathbf{I} & \mathbf{I}-C \\ 0 & \frac{-(\alpha\mathbf{I}-\mathbf{D})(\mathbf{I}-C)}{(\alpha-1)} \end{pmatrix},$$

where from lemma 6, the rank of $\mathbf{I}-\mathbf{C}$ is $N-1$. As the for any eigenvalue λ of $\mathbf{D}$, it must be that $|\lambda| < \alpha$. $\alpha\mathbf{I}-\mathbf{D}$ is invertible which means $(\alpha\mathbf{I}-\mathbf{D})\,(\mathbf{I}-\mathbf{C})$ is of rank $n-1$. Thus, the rank of $\mathbf{A}$ is $2N-1$.

Part ii-2) In this case there is a linear restriction as from the results of Theorem 2 we have $\delta_i = \kappa\gamma_i$ (we neglect the constants without loss of generality). Then the system becomes

$$B_i\gamma_i^{\alpha+\beta} = \kappa^{1-\beta}\sum K_{i,j}\gamma_i\gamma_j.$$

With normalization C.5 it can be shown that $\kappa^{1-\beta} = 1$.

Following the same notation as before full differentiation yields 52 where $f_{\vec{x}}$ is now $N \times N$ matrix:

$$f_{\vec{x}}\left(\vec{x};\vec{k}\right) = (\alpha+\beta-1)\,Y - X,$$

and where Y is a $N \times N$ diagonal matrix whose i^{th} diagonal is equal to Y_i and X is the $N \times N$ trade matrix.

As before, if $f_{\vec{x}}$ was of full rank, we could immediately invert equation (52) (i.e. apply the implicit function theorem) to obtain:

$$D_{\vec{k}}\vec{x} = -\left(f_{\vec{x}}\right)^{-1} f_{\vec{k}}.$$

Therefore the derivative of γ_i w.r.t. $K_{k,l}$ is

$$\frac{\partial\gamma_i}{\partial K_{k,l}} = \begin{cases} -\left\{\left[\left(f_{\vec{x}}\right)^{-1}\right]_{i,k} X_{k,l} + \left[\left(f_{\vec{x}}\right)^{-1}\right]_{i,l} X_{l,k}\right\} & k \neq l \\ -\left[\left(f_{\vec{x}}\right)^{-1}\right]_{i,kN-k-1} X_{k,k} & k = l \end{cases}.$$

It is straightforward to show that the matrix $f_{\vec{x}}$ is of full rank whenever $\alpha+\beta \leq 0$ or $\alpha+\beta > 2$ and also that there are a finite number of α,β combinations that is not of full rank. To see this note that the matrix is not of full rank if there exists a set of $\{z_i\}$ such that:

$$\begin{aligned} \left((\alpha+\beta-1)\,Y - X\right)\mathbf{z} = 0 &\iff \\ z_iY_i\,(\alpha+\beta-1) = \sum_j X_{ij}z_j &\iff \\ z_i\,(\alpha+\beta-1) = \sum_j \frac{X_{ij}}{Y_i}z_j & \end{aligned}$$

Consider the related mathematical problem:

$$z_i\lambda = \sum_j \frac{X_{ij}}{Y_i}z_j \text{ s.t. } \|\mathbf{z}\| = 1. \tag{55}$$

Since $\frac{X_{ij}}{Y_i} > 0$, by the Perron-Frobenius theorem, there exists a unique set of strictly positive $\{z_i\}$ that solve this equation corresponding to the largest eigenvalue λ. It is immediate to show that $\lambda = 1$ by summing up (55) and the corresponding $z_i = 1$ for all i. Notice that, the absolute of any eigenvalue has to be strictly smaller, i.e. $|\alpha + \beta - 1| < 1$, or equivalently, $\alpha + \beta \in (0, 2)$. But the original equation that we assumed had a solution has an eigenvalue of $\alpha + \beta - 1$. Thus, no solution is in the region $|\alpha + \beta - 1| > 1 \iff \alpha + \beta \leq 0$ or $\alpha + \beta > 2$ and there exists a finite number of combinations of α, β so that there is no solution of the equation all in the range $\alpha + \beta \in (0, 2)$ (when $\alpha + \beta = 2$, then Perron-Frobenius guarantees the existence of a solution so that we know the matrix in non-invertible, and $f_{\vec{x}}$ is of full rank). $\square$

A.6 Proof of Proposition 4

Proof. We want to rewrite the equilibrium conditions in changes by defining $(\hat{x}_i) = x_i'/x_i$. Starting from (4) we have

$$
\begin{aligned}
\hat{\gamma}_i^{\alpha} \hat{\delta}_i^{\beta} &= \sum_j \frac{K_{ij}' \gamma_i' \delta_j'}{\sum_j K_{ij} \gamma_i \delta_j} \implies \\
\hat{\gamma}_i^{\alpha} \hat{\delta}_i^{\beta} &= \sum_j \pi_{ij} \hat{K}_{ij} \hat{\gamma}_i \hat{\delta}_j \implies \\
\hat{\gamma}_i^{\alpha-1} \hat{\delta}_i^{\beta} &= \sum_j \pi_{ij} \hat{K}_{ij} \hat{\delta}_j
\end{aligned}
$$

where $\pi_{ij} = X_{ij} / \sum_j X_{ij}$ represents the exporting shares. Similarly we can rewrite the second equilibrium condition, Equation (5), in changes as

$$
\begin{aligned}
\hat{\gamma}_i^{\alpha} \hat{\delta}_i^{\beta} &= \frac{\sum_j K_{ji}' \gamma_j' \delta_i'}{\sum_j K_{ji} \gamma_j \delta_i} \implies \\
\hat{\gamma}_i^{\alpha} \hat{\delta}_i^{\beta} &= \sum_j \lambda_{ij} \hat{K}_{ji} \hat{\gamma}_j \hat{\delta}_i \implies \\
\hat{\gamma}_i^{\alpha} \hat{\delta}_i^{\beta-1} &= \sum_j \lambda_{ji} \hat{K}_{ji} \hat{\gamma}_j
\end{aligned}
$$

where $\lambda_{ij} = X_{ij} / \sum_i X_{ij}$ represents the import shares. This system of equations in changes is the same as the system of equations in levels. As long as λ_{ij}, π_{ij} are the same and α,β are the same all the gravity models give the same changes in γ_i, δ_j for a given change in K_{ij}.

$\square$