Uniformly accelerated disk (spinning up or down)

$\vartheta(t) = \vartheta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$, also abbreviated as $\vartheta \equiv \vartheta(t)$

$\vec{r}(t) = (R\cos\vartheta)\,\hat{\imath} + (R\sin\vartheta)\,\hat{\jmath}$

$\vec{v}(t) \equiv \frac{d}{dt}\vec{r}(t) = (-R\vartheta'\sin\vartheta)\,\hat{\imath} + (R\vartheta'\cos\vartheta)\,\hat{\jmath}$

where $\vartheta' = \omega_0 + \alpha t \equiv \omega(t)$, abbreviated as ω

Note: $\vec{v}(t)\cdot\vec{r}(t) = 0$, i.e., $\vec{r}(t) \perp \vec{v}(t)$ (perpendicular)

why? $\vec{r}\cdot\vec{v} = \underbrace{(R\cos\vartheta)}_{r_x}\underbrace{(-R\vartheta'\sin\vartheta)}_{v_x} + \underbrace{(R\sin\vartheta)}_{r_y}\underbrace{(R\vartheta'\cos\vartheta)}_{v_y}$

$= R^2\vartheta'(-\cos\vartheta\sin\vartheta + \cos\vartheta\sin\vartheta) = 0.$

We have shown: $\vec{v}(t)$ is tangential to the circle

Now calculate the acceleration vector:

$\vec{a}(t) \equiv \frac{d}{dt}\vec{v}(t) =$

$\frac{d}{dt}\left((-R\vartheta'\sin\vartheta)\,\hat{\imath} + (R\vartheta'\cos\vartheta)\,\hat{\jmath}\right)$

$a_x = \frac{d}{dt}(-R\vartheta'\sin\vartheta) = -R\left(\vartheta''\sin\vartheta + \vartheta'\cdot\vartheta'\cos\vartheta\right)$

product rule — from chain rule

$= -R\vartheta''\sin\vartheta - R(\vartheta')^2\cos\vartheta$ where $\vartheta = \vartheta(t)$!

$a_y = \frac{d}{dt}(R\vartheta'\cos\vartheta) = R\left(\vartheta''\cos\vartheta - (\vartheta')^2\sin\vartheta\right)$

$\vec{a} = \left(-R\vartheta''\sin\vartheta - R(\vartheta')^2\cos\vartheta\right)\hat{\imath} + \left(R\vartheta''\cos\vartheta - R(\vartheta')^2\sin\vartheta\right)\hat{\jmath}$

Now break this up into the sum of two vectors,

namely $\vec{a} = \vec{a}_{cp} + \vec{a}_{tang.}$ centripetal + tangential parts

How do we recognize which is which?

remember : $\vec{r} = R\cos\vartheta\,\hat{\imath} + R\sin\vartheta\,\hat{\jmath}$

$$\therefore \quad \vec{a}_{cp} = -R(\vartheta')^2\cos\vartheta\,\hat{\imath} - R(\theta')^2\sin\vartheta\,\hat{\jmath}$$

where $\vartheta' = \omega_0 + \alpha t = \omega = \omega(t)$ [when $\alpha \neq 0$]

$$\therefore \quad \vec{a}_{cp} = -\omega^2\vec{r}(t)$$

as in uniform circular motion, except $\omega = \omega(t)$ when $\alpha \neq 0$

$$\vec{a}_{tang} = -R\vartheta''\sin\vartheta\,\hat{\imath} + R\vartheta''\cos\vartheta\,\hat{\jmath}$$

$$= -R\alpha\sin\vartheta\,\hat{\imath} + R\alpha\cos\vartheta\,\hat{\jmath}$$

This vanishes when $\alpha = 0$ (uniform circular motion)

Why is it tangential? $\rightarrow$ has $\sin\vartheta / \cos\vartheta$ factors with $\hat{\imath}, \hat{\jmath}$ analogous to $\vec{v}(t)$

Compare $\vec{v}(t) = (-R\vartheta'\sin\vartheta)\,\hat{\imath} + (R\vartheta'\cos\vartheta)\,\hat{\jmath}$

$\omega = \omega_0 + \alpha t$ $\qquad = -R\omega\sin\vartheta\,\hat{\imath} + R\omega\cos\vartheta\,\hat{\jmath}$

$\alpha = \text{const.}$ $\qquad \vec{a}_{tang} = -R\alpha\sin\vartheta\,\hat{\imath} + R\alpha\cos\vartheta\,\hat{\jmath}$

Note: $\omega > 0$ means CCW rotation, $\omega < 0$ is CW rotation.

For $\alpha > 0$ (CCW acceleration) $\vec{a}_{tang}$ points in the same direction as CCW rotating $\vec{v}(t)$

Means: CCW spin-up – $\vec{a}_{tang}(t)$ makes $\vec{v}(t)$ longer.