

PHYS4020

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1. FARADAY'S LAW: THE GENERATION OF EMF

1.1. Introduction. Faraday's law is the basis for the generation of electricity: mechanical energy is converted into electrical energy by somehow producing time-varying magnetic fields, and picking up induced EMF. In an alternator, such as used in cars, and in some countries on bicycles a strong multi-polar permanent magnet is rotated in front of an assembly of coils, resulting in an AC voltage being produced. Under load an AC current flows (in cars, the current is rectified to charge the battery, and to power lights, to support the ignition system, etc.).

There are two fundamentally different processes that lead to the generation of EMF. On the one hand one can use the Lorentz force

$$\vec{F}_m = q \vec{v} \times \vec{B} \quad (1)$$

to produce charge separation by moving charges q through a constant magnetic field. The first-year physics example of dragging a metal bar of length $\gtrsim L$ across open-ended thin metal rails (separated by distance L) immersed in a homogeneous magnetic field at right angles is such an application. The Lorentz force produces charge separation, and one can look for force equilibrium by demanding that the produced electrostatic force $\vec{F}_e = q\vec{E}$ balances the charge separation.

$$\vec{F}_m + \vec{F}_e = \vec{0} \quad \text{or for magnitudes:} \quad \left| \frac{dx}{dt} \right| B = \frac{\Delta V}{L} \quad (2)$$

This results in a formula for the voltage drop ΔV produced on the rails. The circuit is open (no current flows), and the voltage is generated 'for free'. The bar is thicker than the rails, and is considered to be the source of the EMF with small internal resistance. This motional EMF $\mathcal{E} = \Delta V$ is proportional to the speed with which the bar is dragged, the length of the bar L , and to the strength of the magnetic field. This motional EMF is associated with a potential difference, i.e., there is a well-defined region with positive vs negative electric potential, i.e., one of the bars is at a positive potential with respect to the other.

The next step is to introduce a loop, where a current flows, e.g., by terminating the rails with a resistor, which acts as a load. Now the bar cannot be dragged for free, since there is a current passing through it, and a Lorentz force between this current and the magnetic field opposes the motion. It is customary to introduce the time-varying magnetic flux, and to explain the induced EMF and current orientation by the flux rule formulated by Lenz.

The induced current is oriented in such a way that the induced magnetic field opposes the change in magnetic flux produced by the change associated with the motion.

This leads to the same formula for the EMF as (2), and includes other possibilities beyond motional EMF. For the above example one can reduce the bar to a simple rail (similar resistivity), use a rail as the terminator, and one has a system where the EMF can be generated not by moving the bar, but by varying the strength of the magnetic field as a function of time. The EMF is now quite different, however, instead of one rail being positive against the other, the EMF has no beginning and no end. For reasons becoming apparent later, this EMF will be called curly EMF, since it will have electric field lines associated with it that go ‘in circles’.

Philosophically, one can take different points of view about this. One may take the (historical?) attitude that the flux rule of Lenz generalizes the simple Faraday result of motional EMF. One may also view this as a source of confusion, since the flux rule supersedes the motional EMF rule in simple geometrical settings, but may not be too helpful in giving the orientation of the current when the geometry is more complicated, such as discussed in Example 7.4 in Gr4e. The magnitude of the EMF can be obtained from the flux rule, by considering the area change as one disk area per revolution period T . D.J. Griffiths goes as far as stating that our first example of dragging a bar across rails in a perpendicular magnetic field is EMF generation by the magnetic (Lorentz) force, and should not be called an application of Faraday’s law, which he reserves for the generation of EMF without beginning or end by electric forces. The flux rule is a result from the integral formulation, which includes EMF generation by boundary conditions. This can be written as

$$\mathcal{E} = \oint_C \vec{E} \cdot d\vec{l} = -\frac{d\Phi_m}{dt} = -\iint_A \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}, \quad (3)$$

where the contour C bounds the area A . The motional EMF problem (where the Lorentz force separates charges) is associated with boundary conditions, i.e., a time-varying contour with area $A(t)$.

Stokes’ theorem can be used to arrive at a local statement from (3), for which we emphasize the case of stationary boundaries:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}. \quad (4)$$

This is Faraday’s law in differential form. It states that a time-varying magnetic field induces an electric field that cannot be derived from a scalar potential, since it is not curl-free. This is an EMF without beginning or end! We better learn about it using a concrete example.

1.2. Example: using the Ampèrian loop trick. Consider a region of space with a nearly homogeneous magnetic field $B_z(t)$ that is ramped up linearly in time. By nearly homogeneous we mean that there is an axis singled out which corresponds to the centre of the solenoids that produce the magnetic field. Consider a loop of radius R perpendicular to the field and centred on the axis. If you wish to think of the loop as a resistor wire,

fine. Perhaps, at some point we wish to insert a high-impedance voltmeter in the loop to check on our findings. The integral form of the law (3) appears to be tied to the concept of a loop, but the differential form gives us a partial differential equation that in principle can be solved for any locations, not just along the loop.

We have some experience with analyzing solutions to equations like (4), since Ampère's law is very similar. Thus, we realize that the solutions to the electric field (4) for our example of $\vec{B} = B_z(t)\hat{z}$ is obtained in the form of $E_\varphi(s)$. It is evident that we have singled out the central axis of the magnetic field about which the loop is taken (Gr4e does not mention this; why not?; without the special axis the analogy to Ampère's law fails).

The calculation (Example 7.7 in Gr4e) is now straightforward. The left-hand side in (3) for a loop of radius s becomes $E_\varphi(s)(2\pi s)$, and thus

$$E_\varphi(s)(2\pi s) = -\pi s^2 \frac{dB_z(t)}{dt} . \quad (5)$$

Apparently, for an increasing magnetic field (positive time derivative) the electric field runs clockwise, as viewed from above:

$$E_\varphi(s) = -\frac{s}{2} \frac{dB_z(t)}{dt} , \quad (6)$$

with the EMF growing with radius s linearly.

Let us make this example a bit more realistic, and also interesting. Suppose the magnetic field region is limited to $s \leq R$. The cylindrical symmetry is now imposed, and not left arbitrary. We just calculated the induced EMF as a function of cylindrical radius inside the magnetic field region. What is the induced EMF outside this region? A naïve guess on the basis of the differential form of Faraday's law (4) would state: no magnetic field, no induced electric field, after all the partial differential equation is local. It turns out that this is wrong, and the Ampèrian loop analogy helps us with this. Choosing such loops with $s > R$ implies that no additional dB_z/dt is enclosed, i.e., the right-hand side of (5) saturates. The loop length still grows with s on the left-hand side. Thus, the induced EMF falls off:

$$E_\varphi(s) = -\frac{R^2}{2s} \frac{dB_z(t)}{dt} \quad \text{for } s > R . \quad (7)$$

Thus, combining (6) for $s < R$ with (7) for $s > R$ we obtain a reasonable answer (unlike Example 7.7, which is really ill-posed: the result is ambiguous, it depends on the choice of central axis!). We have a finite, cylindrical-shaped magnetic field region, and by symmetry, we obtain an induced circular electric field about the central axis. It grows linearly towards the edge of the field region, and then drops off inversely proportional to the cylindrical radius. For $s = R$ the results from (6) and (7) match continuously with a derivative discontinuity, which is caused by the abrupt drop of the magnetic field from a constant to zero as s is moved from $s < R$ to $s > R$.

1.3. to be continued.