PHYS4020

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1. Radiated power from a filament of infinite length

1.1. **Introduction.** We start in a very basic way. We just learned from Example 10.2 in Griffiths 4e (Gr4e) how to obtain the vector potential including retardation effects for a line current of infinite length (assumed to be along $\hat{z}$:

$$A_z(s,t) = \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty} \frac{I(t_r)}{\mathscr{r}} dz \ , \tag{1}$$

where t is the time at which the observation of the vector potential is made at location s, and the retarded time is defined as $t_r = t - \mathscr{r}/c$. Our goal here is not only to provide a precursor to the treatment of the radiation problem from a finite electric dipole (Section 11.1.2 in Gr4e), but actually to consider the fields generated for any distance s from the wire. This will allow us to understand better what retardation means.

By choosing the filament to be of infinite length we ignore the electric charges $\pm q(t)$ that appear at the endpoints of the dipole (at $z = \pm d/2$, where the separation d defines for initial charge q_0 the dipole moment $p_0 = q_0 d$). We can thus ignore the scalar potential, and just worry about the vector potential. The nice thing about it is that we understand the electric field to be purely derived as a curly field from

$$E_z(s,t) = -\frac{\partial A_z}{\partial t} \ . \tag{2}$$

The magnetic field follows in the usual way as $\vec{B} = \nabla \times \vec{A}$, which in the simple geometry (no z and no φ dependence due to symmetry) results in

$$B_\varphi(s,t) = -\frac{\partial A_z}{\partial s} \ . \tag{3}$$

The task is to calculate (1) and then to derive the electric and magnetic fields from (2) and (3) respectively. The phase relation between the two fields is of interest, and it will be shown that it becomes simple for distances s on the wavelength scale λ. Ultimately, our solution should be connected to the wave equation which $A_z(s,t)$ satisfies, since (1) is a formal solution to it. This will provide a connection to the more general treatment of antenna theory as provided in advanced electrical engineering textbooks.

1.2. **Calculation.** We begin for convenience with a step borrowed from Example 10.2 in Gr4e. We choose a point at $z = 0$ and a distance s away from the infinitely long, thin (infinitesimal) wire. The contributions to A_z from $z < 0$ and from $z > 0$ are equal and add, so we can choose to take twice the integral over positive z, and replace (1) by

$$A_z(s,t) = \frac{\mu_0}{2\pi} \int_0^\infty \frac{I(t_r)}{\mathscr{r}} dz \ , \tag{4}$$

As we integrate elements of the filament dz located at height z the distance between current element and the observation point is expressed as (cf. Fig. 10.4 in Gr4e)

$$\mathscr{r} = \sqrt{s^2 + z^2} \ . \tag{5}$$

Assuming a current of form

$$I(t) = I_0 \sin \omega t \tag{6}$$

we need to calculate

$$A_z(s,t) = \frac{\mu_0}{2\pi} I_0 \int_0^\infty \frac{\sin\left[\omega(t - \sqrt{s^2+z^2}/c)\right]}{\sqrt{s^2+z^2}} dz \ . \tag{7}$$

We can use the wavenumber $k = 2\pi/\lambda$ and the relation $\omega/c = k$ to write the above as

$$A_z(s,t) = \frac{\mu_0}{2\pi} I_0 \int_0^\infty \frac{\sin\left[\omega t - k\sqrt{s^2+z^2}\right]}{\sqrt{s^2+z^2}} dz \ . \tag{8}$$

The trigonometric relation $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ makes the calculation manageable and results in two integrals that can be reduced to standard form.

$$A_z(s,t) = \frac{\mu_0 I_0}{2\pi} \left(\sin(\omega t) \int_0^\infty \frac{\cos\left[k\sqrt{s^2+z^2}\right]}{\sqrt{s^2+z^2}} dz - \cos(\omega t) \int_0^\infty \frac{\sin\left[k\sqrt{s^2+z^2}\right]}{\sqrt{s^2+z^2}} dz \right) \ . \tag{9}$$

In an ideal world we would throw the two similar integrals at Mathematica and wait for the answer. Unfortunately, we need to play with a couple of substitutions before this approach nets a result. The first trick is to realize that the argument of the trig functions can be made simple by the change of variables:

$$y = \sqrt{s^2+z^2} \quad , \quad z = \sqrt{y^2 - s^2} \quad , \quad \frac{dz}{\sqrt{s^2+z^2}} = \frac{dy}{\sqrt{y^2-s^2}} \tag{10}$$

This is a smart move, since now we have

$$A_z(s,t) = \frac{\mu_0 I_0}{2\pi} \left(\sin(\omega t) \int_s^\infty \frac{\cos(ky)}{\sqrt{y^2-s^2}} dy - \cos(\omega t) \int_s^\infty \frac{\sin(ky)}{\sqrt{y^2-s^2}} dy \right) \ , \tag{11}$$

where we also implemented the appropriate lower integration boundary change. The integrals have an integrable singularity at that boundary. A further small step, namely the substitution $y = s\,x$ with $dy = s\,dx$ turns the integrals into standard forms which are known representations of two Bessel functions. Mathematica will simplify these integrals properly.

$$A_z(s,t) = \frac{\mu_0 I_0}{2\pi}\left(\sin(\omega t)\int_1^\infty \frac{\cos[(ks)x]}{\sqrt{x^2-1}}dx - \cos(\omega t)\int_1^\infty \frac{\sin[(ks)x]}{\sqrt{x^2-1}}dx\right) \ . \qquad (12)$$

Our main result for the subsection becomes

$$A_z(s,t) = -\frac{\mu_0 I_0}{4}\left(\sin(\omega t)\, Y_0(ks) + \cos(\omega t)\, J_0(ks)\right) \ . \qquad (13)$$

This is a useful result for various reasons. For us, the main point is that Mathematica knows properties of these functions (such as taking a derivative). Another reason is that we can let Mathematica verify that the wave equation for $A_z(s,t)$ is satisfied by (13) on the basis of the fact that the equation becomes closely related to Bessel's differential equation. Rather than going via the differential equation route (which is typically done in advanced textbooks) we helped Mathematica solve the integral, or alternatively, we could have found the integrals in a table, such as Gradshteyn and Ryzhik (they use J_0 for one integral, but call the other one N_0 instead of Y_0). The beauty of the approach is that you got exposure to small parts of the theoretical physics toolbox. The main purpose of our exercise, will be, however to connect to the idea of retardation! This is implied in the wave equation solution, but practical experience usually comes from the integral (1) from which we started.

1.3. **The fields E_z and B_φ and their interpretation.** The electric field is related very simply to A_z given in (13). Evaluation of (2) gives:

$$E_z(s,t) = \frac{\mu_0 I_0 \omega}{4}\left(\cos(\omega t)\, Y_0(ks) - \sin(\omega t)\, J_0(ks)\right) \ . \qquad (14)$$

We apply (3) to find the magnetic field (note how the chain rule yields a factor of k, and that the prime indicates the derivative maps):

$$B_\varphi(s,t) = \frac{\mu_0 I_0 k}{4}\left(\sin(\omega t)\, Y_0'(ks) + \cos(\omega t)\, J_0'(ks)\right) \ . \qquad (15)$$

The derivatives generate an index shift from 0 to 1, and a sign change. We also make the replacement $k = \omega/c$. Thus,

$$B_\varphi(s,t) = -\frac{\mu_0 I_0 \omega}{4\,c}\left(\sin(\omega t)\, Y_1(ks) + \cos(\omega t)\, J_1(ks)\right) \ . \qquad (16)$$

In this form, it is apparent that the magnetic field is suppressed compared to the electric field by a factor of c. The comparison of the electric field (14) and the magnetic field (16) makes it clear that the phase relation between these fields may be complicated. It simplifies only for large values of $k\,s$, when the Bessel functions relate to simple trigonometric functions.

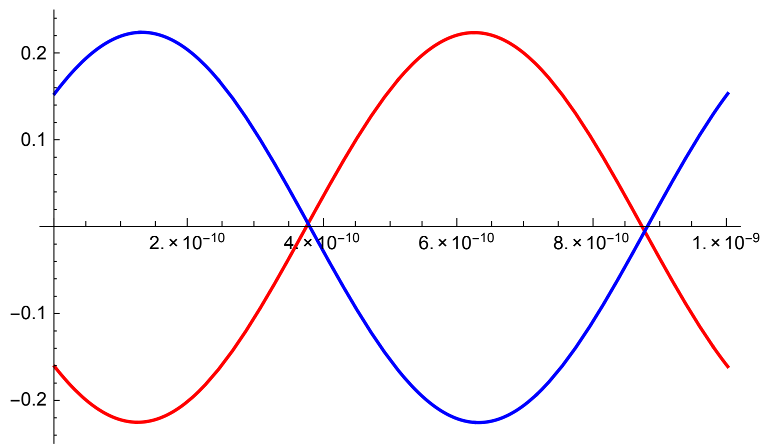
To graph $E_z(s,t)$ in comparison with the magnetic field we show $cB_\varphi(s,t)$ in order to have numbers that can use the same ordinate axis. We indicate in Fig. 1 the Mathematica code that defines the fields, chooses values for the frequency and the other parameters, and the actual plots over one period in time. In the first plot, for distance $s_0 = 2\,\lambda$ the fields appear to be out of phase. Note, however, that for a radially (in s) outgoing wave,

```
Ez = Cos[ω t] BesselY[0, k s] - Sin[ω t] BesselJ[0, k s];

cBφ = -Sin[ω t] BesselY[1, k s] - Cos[ω t] BesselJ[1, k s];

f = 1000 × 10^6; (* 1 Gigahertz *)
ω = 2 π f;
c = 3 × 10^8;
λ = c / f;
k = 2 π / λ;

PLe = Plot[Ez /. s -> 2 λ, {t, 0, 2 π / ω}, PlotStyle → Red];
PLb = Plot[cBφ /. s -> 2 λ, {t, 0, 2 π / ω}, PlotStyle → Blue];
Show[PLe, PLb]
```




```
PLe = Plot[Ez /. s -> λ / 20, {t, 0, 2 π / ω}, PlotStyle → Red];
PLb = Plot[cBφ /. s -> λ / 20, {t, 0, 2 π / ω}, PlotStyle → Blue];
Show[PLe, PLb, PlotRange → All]
```

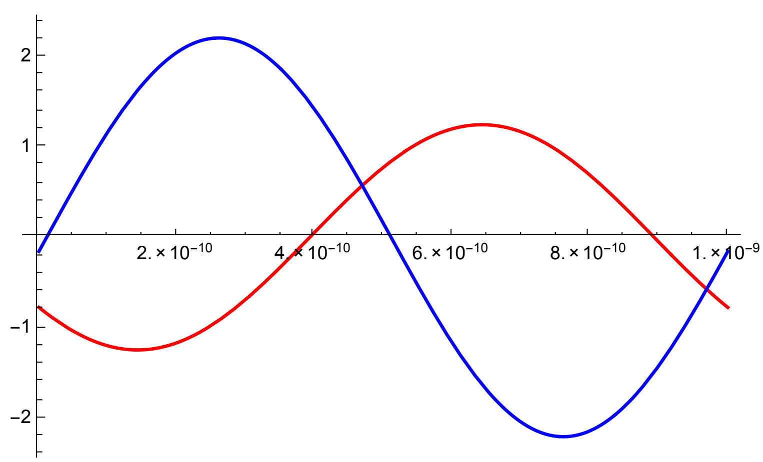


FIGURE 1. Traces of the functions $E_z(s_0, t)$ and $cB_\varphi(s_0, t)$ for one period of oscillation, and for two particular choices of distance s from the filament, namely $s_0 = 2\,\lambda$ in the first plot, and $s_0 = 0.05\,\lambda$ in the second. E_z is in red, while cB_φ is in blue. Note that the wavelength can be large for radio waves, i.e., a small fraction of λ may correspond to a sizeable distance, but for the present choice of $f = 1$ GHz it is short. We have taken out the pre-factors in (14) and (16) to go easy on the y-scale.

$\hat{z}\times\hat{\varphi} = -\hat{s}$, i.e., an electric field pointing along $-\hat{z}$ combined with a magnetic field pointing along $\hat{\varphi}$ results in an outward propagation direction $\vec{k} = k\hat{s}$. Thus, we consider the electric and magnetic fields to be in phase in this case.

Close to the wire (at fractions of the wavelength λ) things are complicated. We observe that, *(i)* the amplitude of cB_φ exceeds that of E_z; *(ii)* the phase relation is messy; *(iii)* the amplitude is larger for smaller values of s_0 than for larger distances from the wire. It is worth mentioning that the choice of frequency does not matter for the plots, since the observation points are chosen as multiples of the wavelength λ. One would just obtain a change on the time axis. The strength of the fields does depend on $\omega = 2\pi f$, however, the pre-factor that was dropped for plotting is given in (14) and (16) and shows proportionality to the AC frequency. In the next subsection it will be shown how the radiated power depends on ω.

One question worth asking is: 'what is the role of retardation?' Could one evaluate the vector potential in the quasi-static approximation? The answer is that the infinite-wire problem cannot be solved quasi-statically. The fact that as one goes further out in z that there are times at which the current is opposite in direction, and this leads to cancellations which make the integral converge. In the quasi-static approximation there is no dependence on k, and one obtains a logarithmically divergent result (as a function of wire length). In cases where the current is turned on at some time (e.g., $t = 0$), such as in Example 10.2 (Gr4e), the z-integral acquires a finite boundary, since at some far-away distance the retarded time becomes zero. In our present example, however, the AC current was chosen to exist for all times.

1.4. **The Poynting vector, its time average, and radiated power.** The flow of power is described by the time average of the quantity

$$\vec{S} = \vec{E}\times\vec{H} = \frac{1}{\mu_0}\vec{E}\times\vec{B}\ . \tag{17}$$

Based on our previous considerations we have only a radially outward component at large distances $s >> \lambda$, viz.,

$$S_s(s,t) = -E_z(s,t)B_\varphi(s,t)\ . \tag{18}$$

The Bessel functions have known asymptotic expansions at large argument $k\,s$. Combining the electric and magnetic fields in (18) results in two terms that are proportional to $\cos(\omega t)\sin(\omega t)$ which are irrelevant after the time average is introduced, and $\cos^2(\omega t)$ and $\sin^2(\omega t)$ terms which both average to one half. Thus,

$$\langle S_s\rangle = \frac{\mu_0 I_0^2}{16c}\omega^2\frac{1}{2}\left(Y_0(ks)J_1(ks) - Y_1(ks)J_0(ks)\right)\ . \tag{19}$$

In the asymptotic region the term in brackets simplifies, and

$$\langle S_s\rangle \ \rightarrow\ \frac{\mu_0 I_0^2}{16\pi c}\frac{\omega^2}{ks} \quad \text{for} \quad s >> \lambda\ . \tag{20}$$

To obtain the average power flow through a cylinder mantle at large radius $s >> \lambda$ we perform an area integral. Noting that the area vector for the cylinder mantle points in the same direction as $\vec{S}$, namely along $\hat{s}$ we obtain

$$\langle P \rangle = \langle S_s \rangle 2\pi s Z \ , \tag{21}$$

where Z is the chosen cylinder height to calculate the power. We also replace $k = 2\pi/\lambda$, and find

$$\langle P \rangle = \frac{\mu_0 I_0^2}{16\pi c} \omega^2 \lambda Z \ , \tag{22}$$

A dimensional analysis shows that this is indeed a power. Without the chosen height Z one obtains power/unit height.

This final result can be compared to the average power radiated by a finite electric dipole (ED) that has charge flowing between $z = +d/2$ and $z = -d/2$ with initial charge q_0 (in Gr4e this result is obtained as (11.22)):

$$\langle P_{ED} \rangle = \frac{\mu_0 q_0^2}{12\pi c} \omega^4 d^2 \ . \tag{23}$$

Here the fourth power of ω appears, since the current amplitude $I_0 = q_0 \omega$. For the charge sloshing hence and forth between $z = \pm d/2$ the geometry is different, the asymptotic surface through which the radiation passes is a sphere, but the radiation pattern is donut shaped and maximizes at polar angle $\theta = \pi/2$. The result is obtained in the $\lambda >> d$ limit.

Another point of comparison is the Larmor formula for an accelerated point charge (APC) (which moves slowly compared to the speed of light c). This result can be obtained from the ED radiation case by considering the electric dipole moment of an oscillating single point charge q displaced from the origin by position $\vec{d}(t)$. For details, see Example 11.2 in Gr4e.

$$P_{APC} = \frac{\mu_0 q^2 a^2}{6\pi c} \tag{24}$$

Observe how in this result the factors of length squared and inverse time to the fourth arise through the square of the acceleration of the point charge.

1.5. **Wave equation for A_z.** We close the circle by demonstrating that indeed the result (13) for $A_z(s,t)$ does satisfy the wave equation, to which (1) from which we started is a formal solution. The Laplacian reduces to a single terms (due to the absence of dependence on φ and z), and the equation we should verify reads

$$\frac{1}{c^2} \frac{\partial^2 A_z}{\partial t^2} = \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial A_z}{\partial s} \right) \ . \tag{25}$$

The result is demonstrated in Fig. 2. The pre-factor is omitted, since the partial differential equation is homogeneous, i.e., constants can be multiplied without destroying the solution.

In addition we show some solutions of (13) for a few times as a function of s on the scale of a few wavelengths. In Fig. 3 it is shown that the combination of two different types of Bessel functions (regular and irregular at the origin, i.e., J_0 and Y_0 respectively) allows to produce a vector potential A_z which can be (weakly) singular at the origin. The Bessel functions allow to have oscillating behaviour modulated by an amplitude that decreases with argument.

```
Az = Sin[ω t] BesselY[0, k s] + Cos[ω t] BesselJ[0, k s];

LHS = D[Az, {t, 2}] / c^2 /. ω → k c // FullSimplify

-k² (BesselJ[0, k s] Cos[c k t] + BesselY[0, k s] Sin[c k t])

RHS = D[s D[Az, s], s] / s /. ω → k c // FullSimplify

-k² (BesselJ[0, k s] Cos[c k t] + BesselY[0, k s] Sin[c k t])

LHS - RHS

0
```

FIGURE 2. Verification in Mathematica that the solution (13) for the Cartesian z-component of the vector potential, $A_z(s,t)$ does indeed satisfy the wave equation.

```
f = 1000 × 10^6; (* 1 Gigahertz *)
ω = 2 π f;
c = 3 × 10^8;
λ = c / f;
k = 2 π / λ;

Plot[{Az /. {t → π / 4}, Az /. {t → π / 3}, Az /. {t → 5 π / 6}},
 {s, 0, 3 λ}, PlotStyle → {Red, Blue, Green}]
```

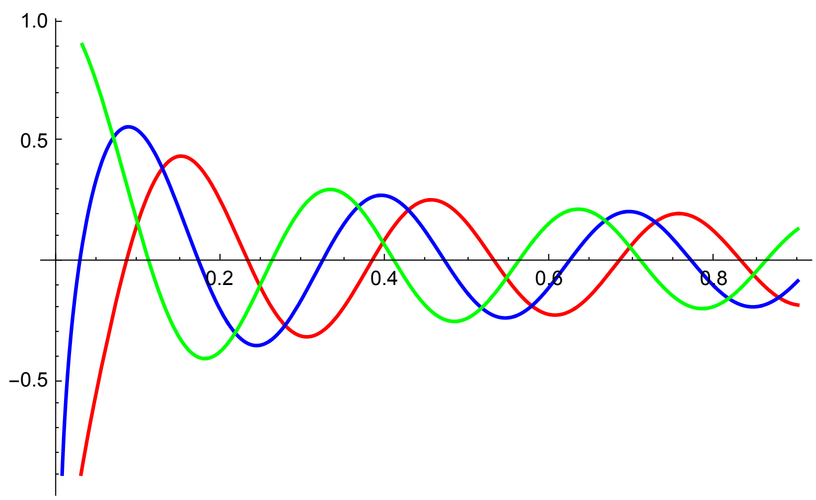


FIGURE 3. Plots of $A_z(s, t_j)$ for three times t_j.