Chapter 1

PREFERENCES REPRESENTATION AND RISK AVERSION

The main focus of this course is on individuals' consumption and investment decisions under uncertainty and their implications for asset valuation.

The main factor that influences the investor's decision is the risk attitude or risk preference. How to represent a person's risk preference?

1.1 Motivation

1.1.1 ***Historical Motivation: Resolution of Bernoulli Paradox (Machina, 1987)***

During the development of modern probability theory in the 17th century, mathematicians such as Blaise Pascal and Pierre de Fermat assumed that the attractiveness of a gamble offering the payoffs $(x_1, x_2, \ldots, x_n)$ with probability $(p_1, p_2, \ldots, p_n)$ was given by its expected value

$$\overline{x} = \sum_{i=1}^{n} p_i x_i.$$

The fact that individuals consider more than just expected value, however, was dramatically illustrated by an example posed by Nicholas Bernoulli in 1728 and now known as *the St.Petersburg* Paradox:

- *Suppose someone offers to toss a fair coin repeatedly until it comes up heads and to pay you \$1 if this happens on the first toss, \$2 if it takes two tosses to land a head, \$4 if it takes three tosses, \$8 if it takes four tosses, etc.. What is the largest sure gain you would be willing to forgo in order to undertake a single play of this game?*

Since this gamble offers a 1/2 chance of winning \$1, a 1/4 chance of winning \$2, etc., its expected value is

$$\frac{1}{2} \times \$1 + \frac{1}{4} \times \$2 + \frac{1}{8} \times \$4 + ... = \frac{1}{2} + \frac{1}{2} + ... = \infty.$$

So it should be preferred to any finite sure gain. However, it is clear that few people would forgo more than a moderate amount for a one-shot play.

The resolution of this paradox was proposed by Gabriel Cramer and Nicholas's cousin Dankiel Bernoulli. They argued that a gain of \$200 was not necessarily "worth" twice as much as a gain of \$100. This suggested that people don't use expected value to evaluate gamble, rather they use some sort of "expected utility function" to do so.

The expected utility function (or termed von-Neumann-Morgenstern utility) is defined as

$$V = \sum_{i=1}^{n} p_i U(x_i).$$

Thus, the sure gain ξ which would yield the same utility as the Petersburg gamble, i.e., the **certainty equivalent** of this gamble, is determined as

$$U(W+\xi) = E[U(W+z)] = \frac{1}{2} \times U(W+1) + \frac{1}{4} \times U(W+2) + \frac{1}{8} \times U(W+4) + ...,$$

where W is the wealth in hand. If the utility function is logarithmic form

$$U(W) = \ln W$$

and the initial wealth $W = \$50,000$, then, the individual's certainty equivalent amount ξ would only be about \$9, even though the gamble has an infinite expected value.

This historical approach assumes that individuals possess utility functions over wealth and then they maximize the expected utilities.

1.1.2 ***Modern motivation:***

The expected utility function is a useful form to characterize individuals' preferences. It is easy to work with and allows testable predictions. Also, it is itself testable. When we compute the expected utility, we need to determine the probability belief. There are two schools of thoughts concerning the probability belief:

1. Objective Probability Approach: take the choices over lotteries as primitive (i.e., they are what they are);

2. Subjective Probability Approach: this was first introduced by Savage (1972) who believes that the probability assessments are an integral part of an investor's preferences and thus purely subjective. It starts with preferences over prospects, not lotteries.

However, since this course is an introductory course, we use the objective probability assumption throughout the course, unless otherwise specified. After finishing this course, students are encouraged to investigate the implications of subjective beliefs on issues discussed in this course.

1.2 Preference Relation

Primitives:

1. A fixed set of the states of the world;

2. A set of lotteries: one outcome for one state of the world;

- Preference relation $\succeq$: two lotteries (x, y) are said to be in a (preference) binary relation $\succeq$ with $x \succeq y$ if x is preferred to y. If $(x, y\ )$ is not in this binary relation, then we write $x \not\succeq y$ and say x is not preferred to y.

- Preference relation $\sim$: two lotteries are said to be indifferent to each other if $x \succeq y$ and $y \succeq x$. We denote by $x \sim y$.

- Preference relation $\succ$: x is said to be strictly preferred to y if $x \succeq y$ and $y \not\succeq x$. We denote by $x \succ y$.

- A preference relation is a binary relation that is transitive and complete.

  1. $\succeq$ is transitive if $x \succeq y$ and $y \succeq z$ imply $x \succeq z$. That is, if x is preferred to y and y is preferred to z, then x is preferred to z.
  2. $\succeq$ is complete if for any pair of lottery (x, y), we either have $x \succeq y$ or $y \succeq x$. That is, any pair of lottery (x, y) can always to compared.
  3. preferences are linear in probabilities.

1.3 Expected Utility:

Let P be a probability defined on the state space Ω. A lottery outcome is a random variable whose probabilistic characteristics are specified by P. We define the distribution function for the lottery outcome as

$$F_x(z) \equiv P\{w \in \Omega : x_w \leq z\}.$$

If a preference relation has an expected utility representation with a utility function U on sure things, then the expected utility derived from x is

$$E[U(x)] = \int U(x)dF.$$

Axiom 1 $\succeq$ *is a preference relation on* P*.*

Axiom 2 *For all* $p, q, r \in P$ *and* $a \in (0, 1]$, $p \succ q$ *implies* $ap + (1-a)r \succ aq + (1-a)r$*.*

This Axiom is commonly called the *substitution axiom* or the *independence axiom*. Think $ap + (1-a)r$ and $aq + (1-a)r$ as compound lotteries, lottery r should not affect the preference relation between p and q.

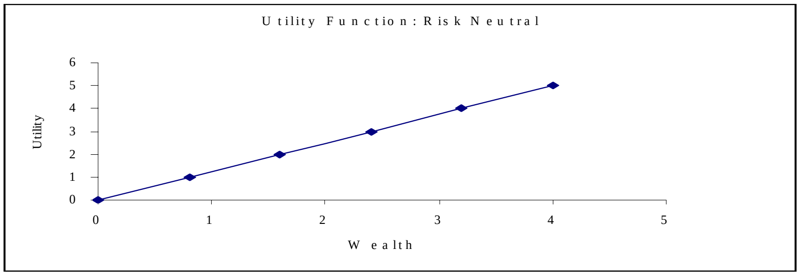
Axiom 3 *For all* $p, q, r \in P$*, if* $p \succ q \succ r$ *then there exists* $a, b \in (0, 1)$ *such that* p *implies* $ap + (1-a)r \succ q \succ bp + (1-b)r$*.*

This Axiom is commonly called the *archimedean axiom*. It implies that for $q \succ r$, for a small probability b, the compound lottery $bp + (1-b)r$ is never worse than q.Or for $p \succ q$, for a large probability a, the compound lottery $ap + (1-a)r$ is preferred to q.

1.4 Different Shapes of Utility Functions and their Implications

1.4.1 Linear Utility function

It is the same as ordering by expected value. In this case, the individual is said to be risk-neutral.

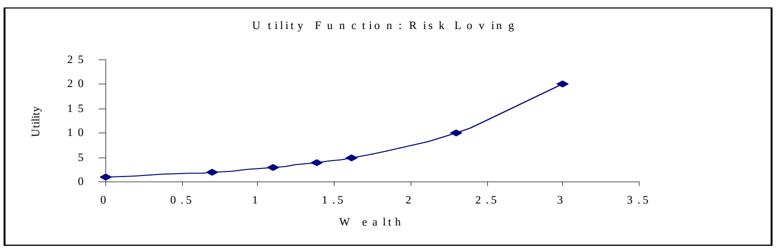


1.4.2

$$U' > 0 \qquad and \qquad U'' = 0$$

Convex utility function

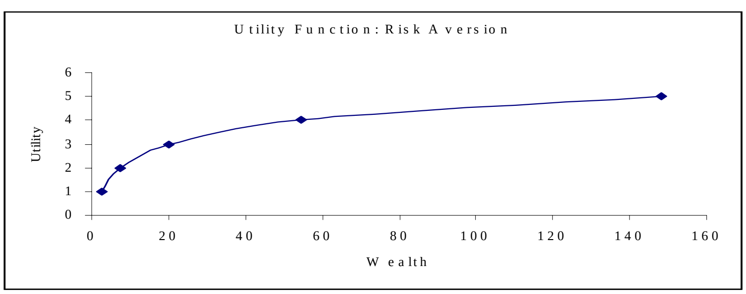
In this case, the individual is said to be risk-loving.



$$U' > 0 \qquad and \qquad U'' > 0$$

1.4.3 Concave utility function

In this case, the individual is said to be risk averse.



$$U' > 0 \qquad and \qquad U'' < 0$$

- Jenzen's inequality: U concave $\implies$ $U[E(x)] > E[U(x)]$

1.4.4 Implication of Risk Aversion

A risk-averse individual facing a loss and offered a fair insurance will always choose to fully insure.

- Notation

W :	initial wealth
L :	loss
p :	probability of loss
I :	insurance coverage
pI :	fair premium (risk-neutral insurance company's break even point)

The agent's problem is

$$\max_I EU = pU(W - pI - L + I) + (1-p)U(W - pI)$$

The first order condition with respect to I is

$$\begin{gathered} p(1-p)U'(W - pI - L + I) - (1-p)pU'(W - pI) = 0 \\ \Longrightarrow \\ U'(W - pI - L + I) = U'(W - pI) \end{gathered}$$

Therefore,

$$\begin{aligned} W - pI - L + I &= W - pI \\ &\Longrightarrow \\ L &= I \end{aligned}$$

This is a fair transfer of wealth across different states.

However, an expected utility maximizing individual facing an *unfair* insurance premium will not fully insure.

Exercise 1.4.1 *If an individual currently is in a risky position, show that he will always accept some part of a new lottery if (1) the lottery has a positive expected value; (2) the lottery is distributed independently of the current random wealth.*

1.5 Measure of Risk Premium

Compare with the risk-neutral agent, a risk aversion agent needs to be compensated by a risk premium. How to measure this risk premium?

We facilitate the discussion by using the concept of *Certainty Equivalent (CE)* which was first introduced by Markowitz.

$$U(CE) \equiv E[U(x)].$$

We have

$$CE = U^{-1}\{E[U(x)]\}.$$

The risk premium is defined as the difference between the expected value and CE. That is,

$$\pi = E(x) - CE = E(x) - U^{-1}\{E[U(x)]\}.$$

$\Longrightarrow$

$$U^{-1}\{E[U(x)]\} = E(x) - \pi.$$

$$E[U(x)] = U[E(x) - \pi].$$

π is the risk premium.

The individual starts with $E(x)$, π is the amount he would give up to avoid risk, $x - E(x)$, the equivalent variation of the risk.

1.6 Measure of the Degree of Risk Aversion

For two individuals, how do we determine which one is more risk averse? We need a measure to determine the degree of risk aversion based on the utility function, rather than on a specific lottery.

1.6.1 ***Absolute Risk Aversion***

Since the risk-averse utility is concave, an immediate candidate for measuring risk aversion is $-U''$, the second order derivatives. However, as shown below, this alone does not work. For example, person A has a function $U_A = U$ while person B has $U_B = a + bU$. In this case, these two people have the same behavior with respect to risk, but different U''.

One proper measure is $\frac{-U''}{U'}$, also termed as **Absolute Risk Aversion** (please see later discussion), because it makes economic sense. To see why, let us consider a lottery z with $E(z) = 0$. The risk premium for this lottery can be determined as

$$E[U(W + z)] = U[W + E(z) - \pi] = U(W - \pi).$$

Recall Taylor expansion:

$$f(x) = f(a) + (x - a)f'(a) + \frac{1}{2}(x - a)^2 f''(a) + o(x - a)^3$$

with

$$\lim_{x \to a} \frac{o(x - a)^3}{(x - a)^3} = 0.$$

We have

$$\begin{aligned}
& E[U(W+z)] \\
= \; & E\{U(W) + (W+z-W)U'(W) + \frac{1}{2}(W+z-W)^2U''(W) + o(W+z-W)^3\} \\
\approx \; & E[U(W)] + E(z)U'(W) + \frac{1}{2}E(z^2)U''(W). \\
= \; & U(W) + \frac{1}{2}Var(z)U''(W)
\end{aligned}$$

and

$$\begin{aligned}
& U(W-\pi) \\
= \; & U(W) + (W-\pi-W)U'(W) + o(W-\pi-W)^2 \\
\approx \; & U(W) - \pi U'(W).
\end{aligned}$$

Therefore,

$$U(W) + \frac{1}{2}Var(z)U''(W) \approx U(W) - \pi U'(W)$$

$\Longrightarrow$

$$\pi = \frac{1}{2}var(z)\left(-\frac{U''(W)}{U'(W)}\right) +'' local'' \text{ risk premium.}$$

The measure $-\frac{U''(W)}{U'(W)}$ only depends on the utility function, not a particular risk (e.g., lottery). It is also called **Absolute Risk Aversion** (ARA) because it measures an agent's risk attitude toward the absolute wealth level. In the next subsection, we introduce the concept of **Relative Risk Aversion** (RRA), which measures the risk attitude toward relative wealth level, i.e., the percentage of wealth level.

1.6.2 ***Relative Risk Aversion***

Another proper measure is the so-called **Relative Risk Aversion** (RRA). A person is said to be RRA because he cares about his relative wealth level, not the absolute level of wealth. The precise measure is $-W\frac{U''(W)}{U'(W)}$, which can be shown by completing the following exercise.

Exercise 1.6.1 *start with $W + Wz = W(1+z)$ (a propositional lottery), and derive*

$$\pi = \frac{1}{2}var(z)\left(-W\frac{U''(W)}{U'(W)}\right) +'' local'' \text{ risk premium.}$$

1.7 Implications of the ARA or RRA Risk Preference for Simple Portfolio Problem

We start with two assets: one risky asset with a return r and another riskless asset with a return r_f. The initial wealth is W and the Absolute Risk Aversion (ARA) $R_A(W) \equiv -\frac{U''(W)}{U'(W)}$. Denote x as the wealth invested in the risky asset.

- Claim: if $R'_A(W) < 0$, then $\frac{dx}{dW} > 0$.

Put it in words, the above claim indicates that an agent will invest more wealth into the risky asset if his ARA is decreasing with respect to his wealth level.

Proof:

The individual's problem is

$$\max_x E[U(\widetilde{W})] = E\{U[(x(1+r) + (W-x)(1+r_f)]\}$$

The first order condition is

$$E[U'(\widetilde{W})(r-r_f)] = 0$$

Then, we have

$$E\{(r-r_f)[(1+r_f)U''(\widetilde{W})dW + (r-r_f)U''(\widetilde{W})dx]\} = 0$$

So

$$\frac{dx}{dW} = \frac{(1+r_f)E[(r-r_f)U''(\widetilde{W})]}{-E[(r-r_f)^2U''(\widetilde{W})]}.$$

Since $-E[(r-r_f)^2U''(\widetilde{W})] > 0$, therefore

$$\text{sign}\left(\frac{dx}{dW}\right) = \text{sign}\left(E[(r-r_f)U''(\widetilde{W})]\right).$$

Let us consider two possible scenarios:

1. If $r > r_f$, then $\widetilde{W} > W(1+r_f)$.

$$\text{Since } R'_A(W) < 0, \text{ then } R(\widetilde{W}) < R[W(1+r_f)].$$

$$-\frac{U''(\widetilde{W})}{U'(\widetilde{W})} < -\frac{U''[W(1+r_f)]}{U'[W)(1+r_f)]}$$

$$U''(\widetilde{W}) > U'(\widetilde{W})\frac{U''[W(1+r_f)]}{U'[W)(1+r_f)]}$$

$\Longrightarrow$

$$(r-r_f)U''(\widetilde{W}) > (r-r_f)U'(\widetilde{W})\frac{U''[W(1+r_f)]}{U'[W)(1+r_f)]}.$$

$\Longrightarrow$

$$\begin{aligned} E[(r-r_f)U''(\widetilde{W})] &> E\left[(r-r_f)U'(\widetilde{W})\frac{U''[W(1+r_f)]}{U'[W)(1+r_f)]}\right] \\ &= \frac{U''[W(1+r_f)]}{U'[W)(1+r_f)]}E\left[(r-r_f)U'(\widetilde{W})\right] = 0. \end{aligned}$$

Therefore,

$$\text{sign}\left(\frac{dx}{dW}\right) = \text{sign}\left(E[(r-r_f)U''(\widetilde{W})]\right) > 0.$$

2. If $r < r_f$, then $\widetilde{W} < W(1+r_f)$.

$$\text{Since } R'_A(W) < 0, \text{ then } R(\widetilde{W}) > R[W(1+r_f)].$$

$$-\frac{U''(\widetilde{W})}{U'(\widetilde{W})} > -\frac{U''[W(1+r_f)]}{U'[W)(1+r_f)]}$$

$$U''(\widetilde{W}) < U'(\widetilde{W})\frac{U''[W(1+r_f)]}{U'[W)(1+r_f)]}$$

$\Longrightarrow$

$$(r-r_f)U''(\widetilde{W}) > (r-r_f)U'(\widetilde{W})\frac{U''[W(1+r_f)]}{U'[W)(1+r_f)]}.$$

$\Longrightarrow$

$$\begin{aligned} E[(r-r_f)U''(\widetilde{W})] &> E\left[(r-r_f)U'(\widetilde{W})\frac{U''[W(1+r_f)]}{U'[W)(1+r_f)]}\right] \\ &= \frac{U''[W(1+r_f)]}{U'[W)(r-r_f)]}E\left[(r-r_f)U'(\widetilde{W})\right] = 0. \end{aligned}$$

Therefore,

$$\text{sign}\left(\frac{dx}{dW}\right) = \text{sign}\left(E[(r-r_f)U''(\widetilde{W})]\right) > 0.$$

So we can conclude that

$$\frac{dx}{dW} > 0 \qquad if \qquad R'_A(W) < 0 \qquad\qquad Q.E.D.$$

The following exercise explore the implication of RRA on the risky investments. It implies that if an agent has a constant RRA, then he will invest a fixed portion of his wealth in the risk asset.

Exercise 1.7.1 *If $R_R(W) = -W\frac{U''(W)}{U'(W)}$ = constant, then $\lambda = \frac{x}{W}$ = constant.*

1.8 General issue of wealth-elasticity of the demand for the risky asset in a two-asset model

The detailed discussion can be found on page 24 in Huang and Litzenberger. The wealth-elasticity of the demand for the risky asset is defined as

$$\eta \equiv \frac{dx}{dW}\frac{W}{x} = 1 + \frac{E[(r-r_f)\widetilde{W}U''(\widetilde{W})]}{-xE[(r-r_f)^2U''(\widetilde{W})]}.$$

One can easily show the following:

Decreasing RRA (DRRA) $\quad \frac{dR_R(W)}{dW} < 0 \quad \rightarrow \quad \eta > 1$

Increasing RRA (IRRA) $\quad \frac{dR_R(W)}{dW} > 0 \quad \rightarrow \quad \eta < 1$

Constant RRA (CRRA) $\quad \frac{dR_R(W)}{dW} = 0 \quad \rightarrow \quad \eta = 1$

Moreover, we can denote the percentage of wealth invested in the risky asset as λ. That is,

$$\lambda = \frac{x}{W}.$$

Then, we have

$$\frac{d\lambda}{dW} = \frac{1}{W}\left(\frac{dx}{dW} - \frac{x}{W}\right) = \frac{x}{W^2}(\eta - 1).$$

Decreasing RRA (DRRA)	$\eta > 1$	$\rightarrow$	$\frac{d\lambda}{dW} > 0$
Increasing RRA (IRRA)	$\eta < 1$	$\rightarrow$	$\frac{d\lambda}{dW} < 0$
Constant RRA (CRRA)	$\eta = 1$	$\rightarrow$	$\frac{d\lambda}{dW} = 0$

- Possible Utility Functions:

  (1) extended power utility: CRRA

  $$U(z) = \frac{(A + Bz)^{B(c+1)} - 1}{(c+1)B}, \qquad \forall \qquad c \neq -1$$

  (2) log utility: CRRA

  $$U(z) = \ln(A + Bz).$$

  (3) exponential utility: CARA

  $$U(z) = \frac{A}{B}\exp(Bz).$$

Exercise 1.8.1 *Show that when $c \rightarrow -1$, the CRRA utility function*

$$U(z) = \frac{(A + Bz)^{B(c+1)} - 1}{(c+1)B}$$

approaches the logarithmic function

$$U(z) = \ln(A + Bz).$$

1.9 Comparison among Investors with Different Utility

U_1 and U_2 represent investors 1 and 2. If investor 2 is more risk averse than 1, then the following statements are equivalent:

1. Given only w_o, the set of lotteries acceptable to investor 2 is a subset of the set acceptable to investor 1;

2. $U_2 \equiv G(U_1)$ with some $G(\cdot)$ being increasing and concave;

3. $ARA(U_2, w) \geq ARA(U_1, w)$, for some w;

4. Markowitz risk premium: $\pi(U_2, w) \geq \pi(U_1, w)$

Exercise 1.9.1 *prove items 3 and 4.*

Note that items 2, 3 and 4 are useful results for comparative statics study.

Summary 1 *The main topics covered in this chapter are*
(i) the preference axioms
(ii) the expected utility, ARA, RRA
(iii) certainty equivalence, risk premium

Chapter 2

STOCHASTIC DOMINANCE

Suppose that there are two risky assets, under what conditions can we unambiguously say that an individual will prefer one risky asset to another under some minimal knowledge of utility? To put it differently, we are looking for "consensus" among preferences over distributions of random wealth: risky asset x has a distribution F and risky asset y has a distribution G, when can we conclude that the expected utility derived from having x is more than that derived from having y, i.e., $E[U(x)] \geq E[U(y)]$?

2.1 First Order Stochastic Dominance

To judge two risky assets in the sense of the First Order Stochastic Dominance (FOSD), we only assume that individuals always prefer more wealth to less. That is, the utility function is increasing ($U' > 0$), we do not put any restrictions on the second order derivative of the utility function, U''.

We say that risky asset x dominates risky asset y in the sense of FOSD, denoted by $x \underset{FOSD}{\geq} y$, if all individuals having increasing-continuous utility either prefer x to y or are indifferent between x and y. Intuition suggests that if the probability of asset x's rate of return in any state is not smaller than that of asset y's rate of return, then any nonsatiable individual will prefer x to y. This condition is not only sufficient but also necessary.

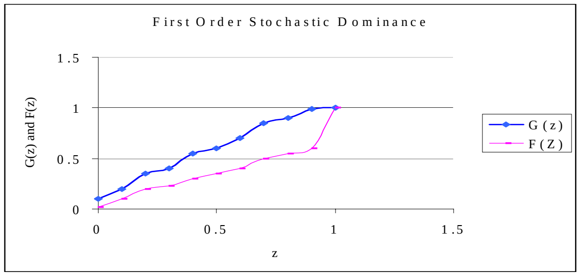
Mathematically, we have the three equivalent statements about FOSD:

1. $x \underset{FOSD}{\geq} y$ means $E_F[U(x)] \geq E_G[U(y)], \qquad \forall \qquad U \in \{U \mid U' > 0\}$;

2. $F(w) \leq G(w) \qquad \forall \qquad w$;

3. $x \underset{d}{=} y + \varepsilon \qquad and \qquad \varepsilon \geq 0$ (statement 2 implies 3, vice verse).

Below is the graphic illustration of FOSD:



2.1.1 *Proof of the Sufficient Condition: Statement (2) $\Longrightarrow$ Statement (1)*

Condition in (2) completely characterizes the consensus in (1). To prove it, we compute

$$\begin{aligned} E_F[U(x)] - E_G[U(y)] &= \int_{\underline{w}}^{\overline{w}} U(w)(dF(w) - dG(w)) \\ &= (F(w) - G(w))U(w) \mid_{\underline{w}}^{\overline{w}} - \int_{\underline{w}}^{\overline{w}} (F(w) - G(w))U'(w)dw. \end{aligned}$$

Since

$$\begin{aligned} F(\overline{w}) &= G(\overline{w}) = 1 \\ F(\underline{w}) &= G(\underline{w}) = 0 \end{aligned}$$

Then

$$(F(w) - G(w))U(w) \mid_{\underline{w}}^{\overline{w}} = 0.$$

Statement (2) indicates

$$F(w) < G(w).$$

With $U'(w) > 0$, we have

$$-\int_{\underline{w}}^{\overline{w}} (F(w) - G(w))U'(w)dw > 0$$

Therefore,

$$E_F[U(x)] - E_G[U(y)] > 0,$$

which proves statement (1).

2.1.2 ***Proof of the Necessary Condition: Statement (1) $\Longrightarrow$ Statement (2)***

This is a powerful result. Under normal circumstance, we can use (1) to show (2) holds. However, for this proof, we cannot directly show (2) from (1). Therefore, we will prove it by contradiction: assume (1), then we want to show (2) follows. Proof by contradiction requires us to show “not (2)” leads to “not (1)” [then contradicting to the true (1)].

“not (2)” implies that there exists some w^* so that

$$F(w^*) > G(w^*).$$

Consider an increasing utility function

$$\begin{aligned} U(w) &= 0, \text{ if } w \le w_0, \\ U(w) &= 1, \text{ if } w > w_0. \end{aligned}$$

Then,

$$E_F[U(x)] = \int_{\underline{w}}^{\overline{w}} U(w)dF(w) = \int_{\underline{w}}^{w_0} 0 \cdot dF + \int_{w_0}^{\overline{w}} 1 \cdot dF = 1 - F(w_0).$$

$$E_G[U(y)] = \int_{\underline{w}}^{\overline{w}} U(w)dG(w) = \int_{\underline{w}}^{w_0} 0 \cdot dG + \int_{w_0}^{\overline{w}} 1 \cdot dG = 1 - G(w_0).$$

Since for some w^*, we have

$$F(w^*) > G(w^*),$$

then it is possible that

$$\int_{\underline{w}}^{\overline{w}} U(w)dF < \int_{\underline{w}}^{\overline{w}} U(w)dG,$$

contradicting to (1), which implies “not (2)” is wrong. Hence, we have proven that “ (1)” leads to “(2)”.

2.1.3 ***Brief Discussion of Statement (3)***

Statement (3) requires $x \underset{d}{=} y + \varepsilon \quad \textit{and} \quad \varepsilon \ge 0.$ Then, we have

$$E[U(1 + r_x)] = E[U(1 + r_y + \varepsilon)] \ge E[U(1 + r_y)].$$

Hence, for an individual with increasing utility, he will always prefer x to y.

2.2 Second Order Stochastic Dominance (SOSD)

Now let us consider a risk-aversion agent whose utility has the following properties:

$$U \in \{U \mid U' > 0, U'' < 0\}.$$

What is the consensus among the risk averse investors regarding x and y?

We say that risky asset x dominates risky asset y in the sense of SOSD, denoted by $x \underset{SOSD}{\geq} y$, if all individuals having increasing-continuous concave utility either prefer x to y or are indifferent between x and y.

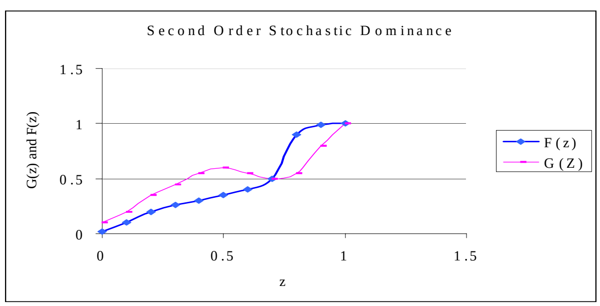
Mathematically, we have the three equivalent statements about SOSD:

1. $x \underset{SOSD}{\geq} y$ means $E_F[U(x)] \geq E_G[U(y)], \qquad \forall \qquad U \in \{U \mid U' > 0, U'' < 0\};$

2. $\begin{matrix} \int_{\underline{w}}^{w_0} [G(w) - F(w)]dw \geq 0 & \forall & w_0; \\ \int_{\underline{w}}^{w_0} [G(w) - F(w)]dw > 0 & \forall & \text{some } w_0. \end{matrix} \qquad \text{and} \quad E_G(w) = E_F(w).$

3. $y \underset{d}{=} x + \varepsilon \qquad \textit{and} \qquad E(\varepsilon \mid x) = 0.$

The following is the graphic illustration of SOSD:



2.2.1 Proof of the Sufficient Condition: Statement (2) $\Longrightarrow$ Statement (1)

Below, we present a brief proof:

$$\begin{aligned} E_F[U(x)] - E_G[U(y)] &= \int_{\underline{w}}^{\overline{w}} U(w)(dF(w) - dG(w)) \\ &= (F(w) - G(w))U(w) \mid_{\underline{w}}^{\overline{w}} - \int_{\underline{w}}^{\overline{w}} (F(w) - G(w))U'(w)dw. \end{aligned}$$

From the previous discussion, we know

$$(F(w) - G(w))U(w) \mid_{\underline{w}}^{\overline{w}} = 0.$$

Also, we integrate by parts to obtain:

$$
\begin{aligned}
& -\int_{\underline{w}}^{\overline{w}} (F(w)-G(w))U'(w)dw \\
= \quad & -U'(w)\int_{\underline{w}}^{w}[F(w)-G(w)]dw \mid_{\underline{w}}^{\overline{w}} + \int_{\underline{w}}^{\overline{w}} \left(\int_{\underline{w}}^{w}[F(z)-G(z)]dz\right) U''(w)dw.
\end{aligned}
$$

Given Statement (1), we have

$$
\begin{aligned}
-U'(w)\textstyle\int_{\underline{w}}^{w}[F(w)-G(w)]dw \mid_{\underline{w}}^{\overline{w}} &= -U'(\overline{w})\textstyle\int_{\underline{w}}^{\overline{w}}[F(w)-G(w)]dw \\
&= -U'(\overline{w})\left(w[F(w)-G(w)]\mid_{\underline{w}}^{\overline{w}} - \textstyle\int_{\underline{w}}^{\overline{w}} w[dF-dG]\right) \\
&= -U'(\overline{w})\left(\overline{w}[1-1]-[E_F(w)-E_G(w)]\right) = 0
\end{aligned}
$$

and

$$
\int_{\underline{w}}^{\overline{w}} \left(\int_{\underline{w}}^{w}[F(z)-G(z)]dz\right) U''(w)dw > 0.
$$

Therefore,

$$
-\int_{\underline{w}}^{\overline{w}} (F(w)-G(w))U'(w)dw > 0,
$$

which implies

$$
E_F[U(x)] - E_G[U(y)] > 0.
$$

This completes the proof.

2.2.2 ***Proof of the Necessary Condition: Statement (1) $\Longrightarrow$ Statement (2)***

Again, we prove it by contradiction. As stated earlier, proof by contradiction requires us to show "not (2)" leads to "not (1)" [then contradicting to the true (1)].

"not (2)" implies that there exists some w_0 so that

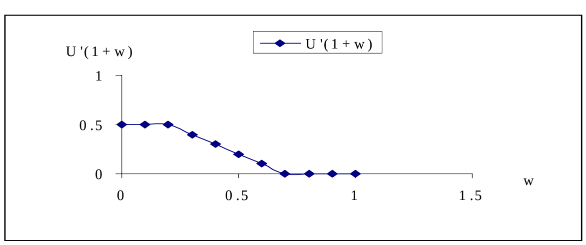
$$
\int_{\underline{w}}^{w_0} [G(w)-F(w)]dw \le 0 \qquad \forall \qquad \text{some } w_0.
$$

Consider the following increasing and concave utility function

$$
U(1+w) = \int_0^w \int_y^1 1_{[1+a,1+b]}(1+t)dtdy, \text{ for } w \in [0,1],
$$

which is continuous and differentiable and concave with

$$
U'(1+w) = \int_w^1 1_{[1+a,1+b]}(1+t)dt
$$



Then,

$$
\begin{aligned}
E_F[U(x)] - E_G[U(y)] &= \int_{\underline{w}}^{\overline{w}} U(w)[dF(w) - dG(w)] \\
&= \int_0^1 \left(\int_{\underline{w}}^{w} [F(z) - G(z)]dz \right) dU'(1+w) \\
&= -\int_a^b \int_{\underline{w}}^{w} [F(z) - G(z)]dzdw < 0.
\end{aligned}
$$

This contradicts to $x \underset{SOSD}{\geq} y$. Hence, Statement (1) implies Statement (2).

2.2.3 ***Brief Discussion of Statement (3):***

Given statement (3): $y \underset{d}{=} x + \varepsilon \qquad \textit{and} \qquad E(\varepsilon \mid x) = 0$, we have

$$
\begin{aligned}
E[U(y)] &= E[U(x+\varepsilon)] \\
&= E[E[U(x+\varepsilon) \mid x]] \\
&\leq E[U[E(x+\varepsilon \mid x)]] \\
&= E[U(x)].
\end{aligned}
$$

Therefore,

$$E[U(y)] \leq E[U(x)].$$

The inequality follows from the conditional Jensen's inequality. Therefore, for an individual with increasing and concave utility, he will always prefer x to y.

Summary 2 *The topics covered in this chapter are*

(i) First Order Stochastic Dominance;

(ii) Second Order Stochastic Dominance.

Chapter 3

PORTFOLIO FRONTIER AND TWO-FUND SEPARATION

In Chapter 2, we have discussed the FOSD and SOSD between two assets. The SOSD is very useful since economic agents are risk-averse. The next question we want to consider is how we can form a portfolio with a desired expected return while keeping the variance at the lowest (i.e., this portfolio dominates other portfolio with the same return in the SOSD sense).

The mean-variance model of asset choices has been used extensively in finance for its simplicity and tractability. The mean-variance analysis is supported by two conditions: (1) economic agents have quadratic utility functions, or (2) the assets' returns are normally distributed. To see this, consider the first case

$$\begin{aligned} EU(W) &= E(aW - bW^2) \\ &= aE(W) - bE(W^2) \\ &= aE(W) - b(E(W))^2 - bVar(W). \end{aligned}$$

In this case, the agents only care about the mean and variance of the returns. Consider the second situation where returns are normal variables whose distribution is completely characterized by mean and variance. In this case, even if a agent has a very general utility preference, his expected utility will be a function of mean and variance of the return.

3.1 Portfolios Frontier

Denote the expected returns for N assets by a vector $e = [E(r_1)\ E(r_2)\ ...\ E(r_N)]'$ and its covariance matrix as V. We can form a portfolio $w = [w_1\ w_2...w_N]'$ with a desired expected return $E(r_p)$. Given this primitives, we can find a portfolio with $E(r_p)$ whose variance is the lowest. Formally,

$$\min_w \frac{1}{2} w' V w$$

s.t.

$$\begin{aligned} w' \cdot e &= E(r_p), \\ w' \cdot 1 &= 1. \end{aligned}$$

Forming the Lagrangian, we have

$$L = \frac{1}{2} w' V w + \lambda_1 (E(r_p) - w' \cdot e) + \lambda_2 (1 - w' \cdot 1).$$

The first order conditions are

$$\begin{aligned}\frac{\partial L}{\partial w} &= Vw - \lambda_1 e - \lambda_2 1 = 0,\\ \frac{\partial L}{\partial \lambda_1} &= E(r_p) - w'e = 0,\\ \frac{\partial L}{\partial \lambda_2} &= 1 - w'1 = 0.\end{aligned}$$

Denote

$$A = 1'V^{-1}e, \qquad B = e'V^{-1}e, \qquad C = 1'V^{-1}1, \qquad D = BC - A^2 > 0.$$

Then the optimal portfolio weights are

$$w_p = \frac{BV^{-1}1 - AV^{-1}e}{D} + \frac{CV^{-1}e - AV^{-1}1}{D}E(r_p). \tag{3.1}$$

The variance of the portfolio is

$$\sigma_p^2 = w_p'Vw_p = \frac{1}{D}\left[CE(r_p)^2 - 2AE(r_p) + B\right]. \tag{3.2}$$

3.2 Minimum Variance Portfolio and its Features

We have shown that, given a desired expected return, we can find out a optimal portfolio with weights stated in equation (3.1) and the portfolio variance in (3.2). What is the expected return which corresponds to the global minimum variance? The answer to this question is the solution to the following problem

$$\min_{E(r_p)} \sigma_p^2 = \min_{E(r_p)} \frac{1}{D}\left[CE(r_p)^2 - 2AE(r_p) + B\right].$$

The return which yields the minimum global variance is

$$r_{\min\_var} = \frac{A}{C}$$

and the minimum global variance is

$$\sigma_{\min}^2 = \frac{1}{C}.$$

Its portfolio weights are

$$w_{\min\_var} = \frac{V^{-1}1}{C}.$$

We can examine the covariance between the global minimum variance portfolio with any frontier portfolio

$$cov(r_p, r_{\min\_var}) = w_p'Vw_{\min\_var} = \frac{1}{C} = \sigma_{\min}^2.$$

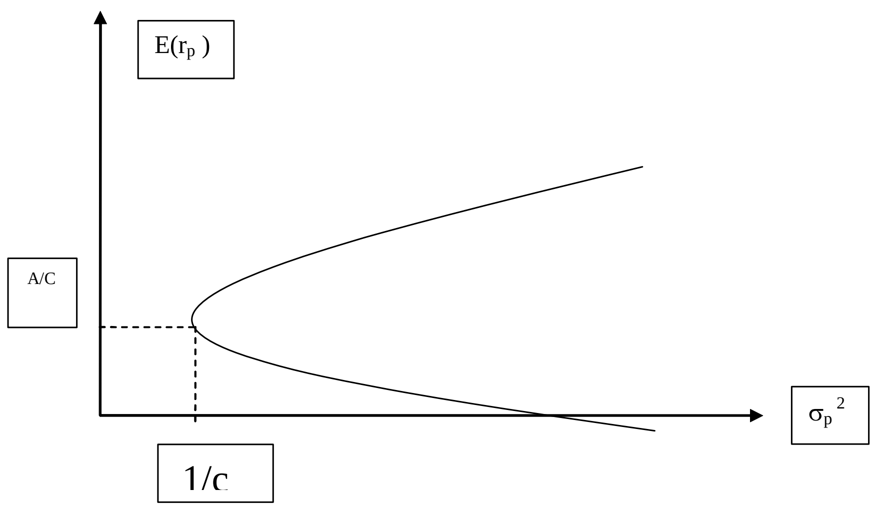
- The covariance between the global minimum variance portfolio with any frontier portfolio is equal to the global minimum variance.

3.3 Key Features of the Frontier Portfolio

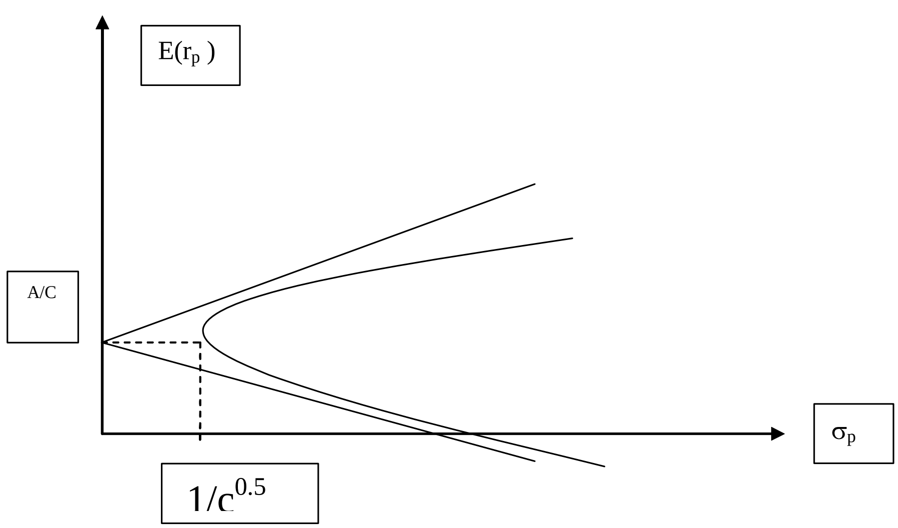
We first examine the relation between the desired expected return and its optimized variance. From equation (3.2), we have

$$\sigma_p^2 = w_p' V w_p = \frac{1}{D}\left[CE(r_p)^2 - 2AE(r_p) + B\right],$$

which depicts the following relationship



Similarly, we can depict the relationship between the desired expected return and the standard deviation as follows



- Portfolio frontier is divided into two sections. Any return higher than the return of the global minimum variance portfolio is efficient while any return less than the return of the global minimum variance portfolio is inefficient;

- We will only consider the efficient portfolio frontier.

Now we want show that any two frontier portfolios can generate the entire frontier. Take portfolio w_p and w_q as examples. Since both of them are on the frontier, then

$$\begin{aligned} w_p &= \frac{BV^{-1}1 - AV^{-1}e}{D} + \frac{CV^{-1}e - AV^{-1}1}{D} E(r_p), \\ w_q &= \frac{BV^{-1}1 - AV^{-1}e}{D} + \frac{CV^{-1}e - AV^{-1}1}{D} E(r_q). \end{aligned}$$

To form a portfolio z with p and q with weights $(a, 1-a)$, the portfolio z's expected return is

$$E(r_z) = aE(r_p) + (1-a)E(r_q).$$

Its portfolio weights are

$$\begin{aligned} w_z &= aw_p + (1-a)w_q \\ &= a\left[\frac{BV^{-1}1 - AV^{-1}e}{D} + \frac{CV^{-1}e - AV^{-1}1}{D} E(r_p)\right] \\ &\quad +(1-a)\left[\frac{BV^{-1}1 - AV^{-1}e}{D} + \frac{CV^{-1}e - AV^{-1}1}{D} E(r_q)\right] \\ &= \frac{BV^{-1}1 - AV^{-1}e}{D} + \frac{CV^{-1}e - AV^{-1}1}{D} E(r_z), \end{aligned}$$

which indicates that portfolio z is also on the frontier.

- The portfolio frontier can be generated by any two frontier portfolio.

3.4 Zero-covariance Portfolio for Any Frontier Portfolio

Consider two frontier portfolio w_p and w_q. Their optimal weights are

$$\begin{aligned} w_p &= \frac{BV^{-1}1 - AV^{-1}e}{D} + \frac{CV^{-1}e - AV^{-1}1}{D} E(r_p), \\ w_q &= \frac{BV^{-1}1 - AV^{-1}e}{D} + \frac{CV^{-1}e - AV^{-1}1}{D} E(r_q). \end{aligned}$$

We can compute the covariance between them as

$$cov(r_p, r_q) = w_p' V w_q = \frac{C}{D}\left[E(r_p) - \frac{A}{C}\right]\left[E(r_q) - \frac{A}{C}\right] + \frac{1}{C}.$$

The condition that portfolio q has a zero covariance with portfolio p (denoted as r_{zero-p}) is

$$E(r_{zero-p}) = \frac{A}{C} - \frac{D}{C^2\left[E(r_p) - \frac{A}{C}\right]}.$$

This zero-covariance portfolio is unique.

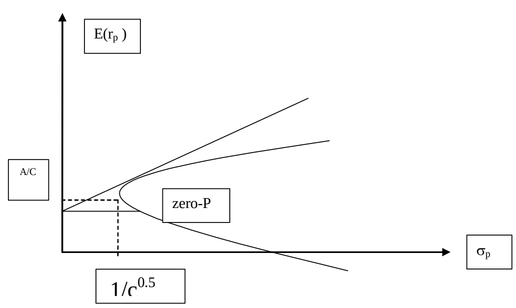
- If portfolio p is efficient, then its zero-covariance portfolio is an inefficient frontier portfolio, vice versa. This is true because

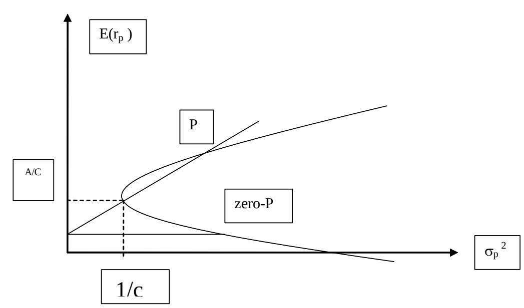
$$E(r_{zero-p}) = \frac{A}{C} - \frac{D}{C^2\left[E(r_p) - \frac{A}{C}\right]} < \frac{A}{C} \qquad when \qquad E(r_p) > \frac{A}{C},$$

vice versa.

- The location of the zero-covariance portfolio is shown in the following graph:

<u>*Location in the $E(r_p)\ -\ \sigma_p$ graph*</u> <u>*Location in the $E(r_p)\ -\ \sigma_p^2$ graph*</u>





Exercise 3.4.1 *For any portfolio q (not necessarily on the frontier), its return can be written as*

$$E(r_q) = E(r_{zero-p}) + \beta_{pq}[E(r_p) - E(r_{zero-p})],$$

where

$$\beta_{pq} = \frac{cov(r_p, r_q)}{var(r_p)}.$$

- The result in the above exercise establishes the zero-beta CAPM.

3.5 Two-Fund Separation

We have discussed the formation of efficient portfolio with N risky assets. Now we introduce the riskfree asset with a return r into the portfolio choice set and see whether the addition of the riskfree asset can improve agents' portfolio formation. The optimization problem becomes

Denote the expected returns for N assets by a vector $e = [E(r_1)\ E(r_2)\ ...\ E(r_N)]'$ and its covariance matrix as V. We can form a portfolio $w = [w_1\ w_2 ... w_N]'$ with a desired expected return $E(r_p)$. Given this primitives, we can find a portfolio with $E(r_p)$ whose variance is the lowest. Formally,

$$\min_w \frac{1}{2} w' V w$$

s.t.

$$w' \cdot e + (1 - w'1)r = E(r_p).$$

Forming the Lagrangian, we have

$$L = \frac{1}{2} w'Vw + \lambda[E(r_p) - w' \cdot e - (1 - w'1)r].$$

The first order conditions are

$$\begin{aligned} \frac{\partial L}{\partial w} &= Vw - \lambda(e - 1 \cdot r) = 0, \\ \frac{\partial L}{\partial \lambda} &= E(r_p) - w'e - (1 - w'1)r = 0. \end{aligned}$$

Denote

$$A = 1'V^{-1}e, \qquad B = e'V^{-1}e, \qquad C = 1'V^{-1}1, \qquad D = BC - A^2 > 0.$$

Then the optimal portfolio weights are

$$w_p = \frac{V^{-1}e - rV^{-1}1}{B - 2Ar + Cr^2}[E(r_p) - r].$$

The variance of the portfolio is

$$\sigma_p^2 = w_p'Vw_p = \frac{[E(r_p) - r]^2}{B - 2Ar + Cr^2}.$$

Or

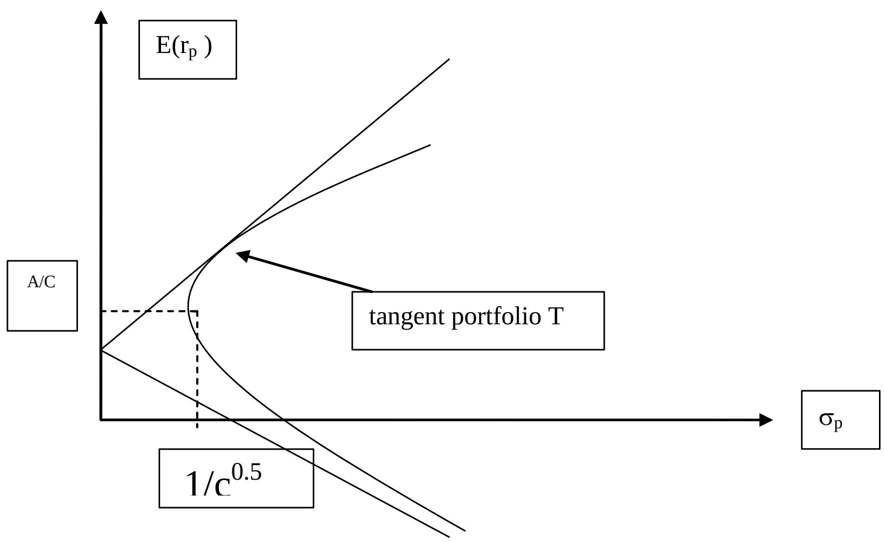
$$\sigma_p = \frac{E(r_p)-r}{\sqrt{B-2Ar+Cr^2}} \qquad when \quad E(r_p) \geq r$$

$$\sigma_p = \frac{E(r_p)-r}{\sqrt{B-2Ar+Cr^2}} \qquad when \quad E(r_p) < r.$$

The optimal result indicates that there is a linear relation between $E(r_p)$ and σ_p. In order to depict the precise relation, we need to know whether $r \geq A/C$ or $r < A/C$. Since we know that A/C is the return corresponding to the global minimum variance portfolio. As long as the global minimum variance is not zero, an economic meaningful condition on the riskfree rate is $r < A/C$, otherwise there exists arbitrage. Therefore, we only consider the case where $r < A/C$.*

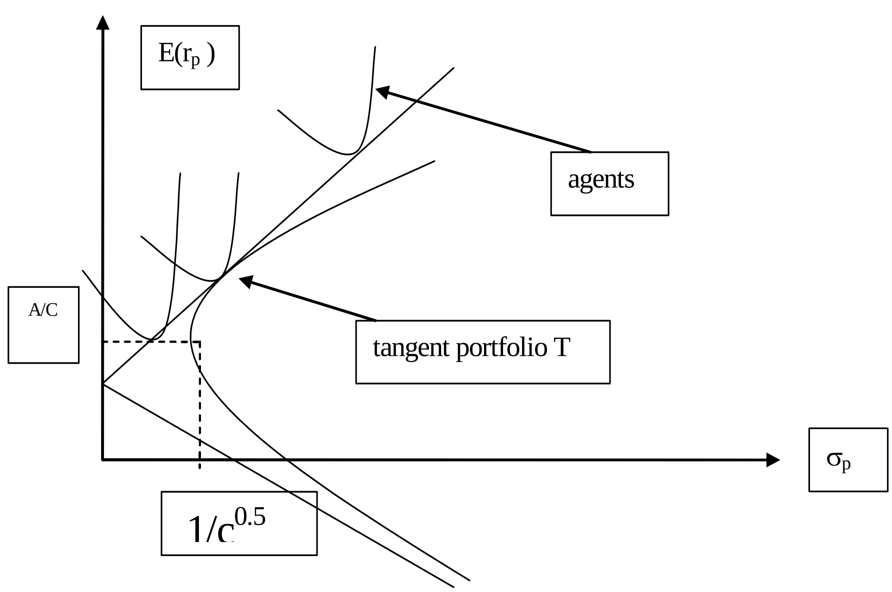
*Students can consult Huang and Litzenberger (1988) for other cases on page 79.

The Portfolio Frontier when $r < A/C$
in plane $E(r_p) - \sigma_p$



- The above diagram illustrate the two-fund separation result: any efficient portfolio can be generated by the riskfree asset and the tangent portfolio which is a basket of all risky asset with the highest Sharpe ratio.

Different Agents' Optimal Portfolio



Exercise 3.5.1 *Derive the optimal portfolio weights for the tangent portfolio and show that the return of any efficient portfolio can be written as*

$$E(r_p) = r + \beta_{Tp} E(r_T - r),$$

where

$$\beta_{Tp} = \frac{cov(r_T, r_p)}{var(r_T)}.$$

- If the market portfolio (r_m) is the tangent portfolio, then we obtain the CAPM

$$E(r_p) = r + \beta_{mp} E(r_m - r).$$

Summary 3 *The topics covered in this chapter are*
(i) Portfolio frontier with N risky assets;
(ii) Portfolio frontier with N risky assets and a riskfree asset;
(iii) Two-fund separation.

Chapter 4

ASSET VALUATION BY ARBITRAGE PRINCIPLE

There are three different approaches to value assets:

1. Arbitrage Principle;
2. Individual Agent Utility Optimization and Equilibrium Valuation (multi-agent): existence of optimality and equilibrium;
3. Equivalent Martingale Pricing Principal: derived from the equilibrium valuation.

Under regularity conditions, they are equivalent. We will start with the first approach. The second and third approaches are discussed in Chapters 5 and onward. To begin with, we first lay out the basic concepts and definitions.

4.1 Asset Valuation by No-Arbitrage: a Two-Period Set-up

The following discussion is based on Duffie (1996) Chapter 1.

Assumption 1 *There are I states of the economy $\{1, 2, ..., I\}$ to be realized at the end of the period.*

Assumption 2 *There are N assets in the economy and their dividend payments are described by the following matrix for the I possible states:*

$$D_{N\times I} = (d_{n\times i}) = \begin{pmatrix} d_{11} & d_{12} & ... & d_{1I} \\ d_{21} & d_{22} & ... & d_{2I} \\ : & : & : & : \\ d_{n1} & d_{n2} & & d_{NI} \end{pmatrix}.$$

(some of the d_{ni} can be negative). These assets are traded in the market with the corresponding prices $S_{N\times 1} = [S_1\ S_2\ ...\ S_N]' \in \Re^N$.

Definition 1 *A portfolio is described by a vector $\theta_{N\times 1} \in \Re^N$ where θ_n indicates the number of shares invested in asset n. The payoff of the portfolio is $D'\theta$.*

Definition 2 *For every portfolio θ, the no-arbitrage condition implies that*

$$\text{if} \qquad D'\theta \geq 0 \;(non-\text{negative } payoff), \qquad \text{then} \qquad S\prime\theta > 0 \;(\text{positive } price),$$

and

$$\text{if} \qquad D'\theta = 0 \;(zero\ payoff), \qquad \text{then} \qquad S\prime\theta \geq 0 \;(non\text{-}negative\ price).$$

(Please see Ingersoll (1987) for variation.)

Example 3 *a trivial example is $D = \dot{0}$. Then any portfolio $\theta \neq 0$ provides arbitrage if $S < 0$.*

Definition 4 *The state price vector $\pi \in \Re^I_{++}$ for the pair (D, S) is implied by $S = D\pi$.*

π_i is the price of a claim to one unit of the payoff in state i. This is the Arrow-Debreu price which provides the link between the dividends D and the asset prices S.

Example 5 *The asset n's price is determined as*

$$S_n = \sum_{i=1}^{I} \pi_i D_{ni}.$$

Theorem 6

$$\textit{No arbitrage} \qquad \Longleftrightarrow \qquad \textit{There exists a state price vector } \pi \textit{ for } (D, S)$$

4.1.1 ***Proof of the Sufficient Condition:***

We need to show the following:

$$\text{No arbitrage} \qquad \Longleftarrow \qquad \text{There exists a state price vector } \pi \text{ for } (D, S)$$

The proof is trivial. Given $S = D\pi$, we need to show that if $D'\theta \geq 0$, *then* $S\prime\theta \geq 0$. Since

$$S'\theta = (D\pi)'\,\theta = \pi' D'\theta,$$

and $\pi > 0$ (because $\pi \in \Re^I_{++}$), therefore,

$$S'\theta \geq 0.$$

4.1.2 ***Proof of the Necessary Condition:***

We need to show that

$$\text{No arbitrage} \quad \Longrightarrow \quad \text{There exists a state price vector } \pi \text{ for } (D, S)$$

The proof is non-trivial. Duffie (1996) uses the Hyperplane Separation Theorem to prove the necessary condition. However, some of us may not be familiar with the theorem. Therefore, we use "Dual Linear Programming" to prove it. Recall the following:

Primal problem:		Dual problem:
$\min_{y} \; y \cdot b$		$\max_{x} \; c \cdot x$
s.t.	$\Longleftrightarrow$	s.t.
$yA \geq c$		$Ax \leq b$
$y \geq 0$		$x \geq 0$

- The solutions to one problem are the shadow prices for the other problem;
- Dual of Dual is primal;
- If one has the feasible solution, then so is the other.

Our problem can be restated as

Primal problem:		Dual problem:
$\min_{\theta} \; \theta \cdot S$		$\max_{\lambda} \; 0 \cdot \lambda$
s.t.	$\Longleftrightarrow$	s.t.
$D'\theta \geq 0$		$D\lambda \leq S$
where θ is not constrainted.		$\lambda \geq 0$

Note that the dual problem corresponds to $b = S$, $A = D$ and $c = 0$ from the primal problem. We form the Lagrangian

$$L = \theta \cdot S + \lambda' \left(0 - D'\theta\right).$$

First order conditions:

$$\begin{aligned} \frac{\partial L}{\partial \theta} &= S - (\lambda' D')' = 0 \qquad \longrightarrow \qquad S = D\lambda \\ \frac{\partial L}{\partial \lambda} &= 0 - D'\theta = 0 \qquad \longrightarrow \qquad D'\theta = 0. \end{aligned}$$

Then

$$\theta \cdot S = S'\theta = (D\lambda)'\theta = \lambda' D'\theta = 0.$$

We know that asset prices are non-negative, in order for $S'\theta = 0$, we must have $\theta = 0$. Otherwise, no-arbitrage condition will be violated.

Now the primal has a solution $\theta = 0$. From the Dual Linear Programming Principle, the dual must have a solution

$$S = D\lambda \qquad with \qquad \lambda > 0,$$

where λ is the state price vector $\pi = \lambda$.

4.1.3 ***Prices of "Other Securities"***

Given (D, S) exhibits no arbitrage and given the state price vector π, let $(X_1, X_2, ..., X_J) \in \Re^I$ represents J new securities with a payoff matrix $\mathrm{D}^X_{J\times I}$, can we say anything about the price of the J securities?

The new security prices should be such so that there is no arbitrage. That is,

$$X = D^X_{J\times I}\pi_{I\times 1}.$$

Note that state prices need not to be unique. However, if the state price vector for (D, S) is unique, there is a unique price vector for these new securities that preserves no arbitrage.

Example 7 *If D is a full-rank square matrix, then state price vector is unique. That is, for each contingent state, there exists a portfolio to deliver a sure payoff. This is the case of complete market.*

4.2 Asset Pricing under Equivalent Probability Measure with a Risk-free Asset

Now we introduce the risk-free asset into the economy whose payoff is 1 regardless of the state. The dividend vector for the risk-free asset is

$$D^0_{1\times I} = \begin{pmatrix} 1 & 1 & ... & 1 \end{pmatrix}$$

The modify the dividend matrix is as follows:

Assumption 3 *There are $N+1$ assets in the economy with asset 0 being the risk-free asset. The corresponding dividend matrix is:*

$$\delta_{(N+1)\times I} = \begin{pmatrix} D^0_{1\times I} \\ \\ D_{N\times I} \end{pmatrix} = \begin{pmatrix} 1 & 1 & ... & 1 \\ \\ d_{11} & d_{12} & ... & d_{1I} \\ \\ d_{21} & d_{22} & ... & d_{2I} \\ \vdots & \vdots & \vdots & \vdots \\ d_{n1} & d_{n2} & & d_{NI} \end{pmatrix}.$$

Given the state price vector π for the pair (D, S), the price of the risk-free asset should also satisfy

$$\begin{aligned} B &= D^0_{1\times I}\pi = \begin{pmatrix} 1 & 1 & ... & 1 \end{pmatrix} \cdot \begin{pmatrix} \pi_1 \\ \pi_2 \\ ... \\ \pi_I \end{pmatrix} \\ &= \sum_{i=1}^{I} \pi_i. \end{aligned}$$

The rate of return on the bond (or the interest rate) is

$$1 + R = \frac{1}{B} = \frac{1}{\sum_{i=1}^{I} \pi_i}.$$

We redefine a new vector based on the state price vector

$$\widehat{\pi} = \begin{pmatrix} \widehat{\pi}_1 \\ \widehat{\pi}_2 \\ ... \\ \widehat{\pi}_I \end{pmatrix} = (1+R) \begin{pmatrix} \pi_1 \\ \pi_2 \\ ... \\ \pi_I \end{pmatrix} = \begin{pmatrix} \frac{\pi_1}{\sum_{i=1}^{I} \pi_i} \\ \frac{\pi_2}{\sum_{i=1}^{I} \pi_i} \\ ... \\ \frac{\pi_I}{\sum_{i=1}^{I} \pi_i} \end{pmatrix}.$$

Since $\pi >> 0$, so is $\widehat{\pi} >> 0$. Also, notice that

$$\sum_{i=1}^{I} \widehat{\pi}_i = \sum_{i=1}^{I} \frac{\pi_i}{\sum_{i=1}^{I} \pi_i} = 1.$$

In this case, each element of $\widehat{\pi}$ can be viewed as a probability because it satisfies the non-negativity condition and the sum of them adds to 1. Now we call

$$\widehat{\pi} = \begin{pmatrix} \widehat{\pi}_1 \\ \widehat{\pi}_2 \\ ... \\ \widehat{\pi}_I \end{pmatrix}$$

the Q probability measure.

This concept turns out to be a very useful and powerful tool for valuing assets. To see why, let us consider asset $n's$ price. From the previous discussion, we have

$$S_n = \sum_{i=1}^{I} \pi_i D_{ni}.$$

Restate the above equation in terms of $\widehat{\pi}_i = \frac{\pi_i}{\sum_{i=1}^{I} \pi_i}$, the price of asset n is

$$\mathbf{S}_n = \sum_{i=1}^{I} \widehat{\pi}_i \left(\sum_{i=1}^{I} \pi_i \right) \mathbf{D}_{ni} = \frac{1}{1+R} \sum_{i=1}^{I} \widehat{\pi}_i \mathbf{D}_{ni} = \frac{1}{1+R} \mathbf{E}^Q \left(D_n \right). \tag{4.1}$$

Equation (4.1) states that the price of asset n is the expected future dividends discounted at the risk-free rate where the expectation is taken under the Q probability measure. This equation illustrates the essence of asset pricing under the Equivalent probability measure. Such results can be extended to multi-period model (we will come back to this concept later).

4.3 Arbitrage Pricing Theory (APT)

The APT was proposed by Ross (1976) in a seminar paper published in the Journal of Economic Theory. He examines an economy with N assets whose returns are affected by $K << N$ independent factors. Precisely, the return of asset n can be characterized as

$$r_n = E(r_n) + b_{n1}F_1 + b_{n2}F_2 + ... + b_{nK}F_K + \varepsilon_n \qquad \forall \qquad n = 1, 2, ..., N$$

with

$$\varepsilon_n \perp \varepsilon_j \qquad \forall \qquad n = 1, 2, ..., N \qquad , \qquad j = 1, 2, ..., N \qquad and \qquad n \neq j$$

$$\varepsilon_n \perp F_k \qquad \forall \qquad n = 1, 2, ..., N \qquad and \qquad k = 1, 2, ..., K.$$

Let us form a portfolio p_k representing factor k whose weight =[0 0 0 1 0 ... 0]. Then, as $N \to \infty$, the APT theory states that asset n's expected return can be approximated as

$$E(r_n) = b_{n1}(E(p_1) - r_f) + b_{n2}(E(p_2) - r_f) + ... + b_{nK}(E(p_K) - r_f) \qquad \forall \qquad n = 1, 2, ..., N$$

where r_f is the riskfree interest rate.

To establish the APT result, the only economic assumption we need is the no-arbitrage assumption.

Proof:

Consider a zero-cost portfolio $w = (w_1, w_2, ..., w_n)$ with

$$\sum_{n=1}^{N} w_n = 0.$$

Its return should be

$$r_w = \sum_{n=1}^{N} w_n r_n = \sum_{n=1}^{N} w_n \left[E(r_n) + b_{n1}F_1 + b_{n2}F_2 + ... + b_{nK}F_K + \varepsilon_n \right].$$

Since the cost of the portfolio is zero, so the the return. Therefore,

$$r_w = \sum_{n=1}^{N} w_n \left[E(r_n) + b_{n1}F_1 + b_{n2}F_2 + ... + b_{nK}F_K\right] + \sum_{n=1}^{N} w_n \varepsilon_n = 0.$$

Now we choose the portfolio weights w to satisfy the following conditions:

$$\mid w_n \mid < \frac{1}{n}, \qquad \sum_{n=1}^{N} w_n = 0, \qquad \sum_{n=1}^{N} w_n b_{nk} = 0 \qquad \forall \qquad k = 1, 2, ..., K.$$

As $N \to \infty$, by law of large number,

$$\sum_{n=1}^{N} w_n \varepsilon_n = 0.$$

Therefore,

$$r_w = \sum_{n=1}^{N} w_n \left[E(r_n) + b_{n1}F_1 + b_{n2}F_2 + ... + b_{nK}F_K + \varepsilon_n\right] = \sum_{n=1}^{N} w_n E(r_n) = 0.$$

Now we have

$$\begin{aligned} w \cdot 1 &= 0, \\ w \cdot b_k &= 0 \qquad \forall \qquad k = 1, 2, ..., K, \\ w \cdot E(r) &= 0, \end{aligned}$$

which implies that $E(r) = [E(r_1) \quad E(r_2) ... \quad E(r_n)]'$ is a linear combination of $1_{N\times 1}$, and $b_1 = [b_{11} \quad b_{21} \quad ... \quad b_{N1}]'$, $b_2 = [b_{12} \quad b_{22} \quad ... \quad b_{N2}]', ..., b_K = [b_{1K} \quad b_{2K} \quad ... \quad b_{NK}]'$. Therefore, there must exist $\lambda_0, \lambda_1, ..., \lambda_K$ such that

$$E(r_n) = \lambda_0 + \lambda_1 b_{n1} + ... + \lambda_K b_{nK},$$

where b_{nk} $(k = 1, 2, ..., K)$ is the "sensitivity" of return on the *nth* asset to the *kth* factor. For the riskfree asset, the sensitivity parameters are zero, i.e.,

$$b_{01} = b_{02} = ... = b_{0K} = 0.$$

Therefore,

$$r_f = \lambda_0.$$

So we can rewrite the expected return for asset n as

$$E(r_n) - r_f = \lambda_1 b_{n1} + ... + \lambda_K b_{nK}.$$

Now we choose a portfolio p_k whose weights v_k (for $k = 1, 2, ..., K$) satisfy the following

$$v_k' b_k = 1 \qquad and \qquad v_k' b_l = 0 \qquad \forall \qquad k = 1, 2, ..., K \qquad l = 1, 2, ..., K \qquad k \neq l.$$

The return on this portfolio is

$$E(r_{pk}) - r_f = \lambda_k.$$

Hence, we can express the expected return on asset n as

$$E(r_n) - r_f = E(r_{p1} - r_f)b_{n1} + ... + E(r_{pK} - r_f)b_{nK}.$$

This proves the APT.

APT model describes a factor return generating mechanism. It is interesting to note the following:

- There is no assumption made about investors' risk preference, except that investors prefer more to less;
- There is no assumption made about the return distributions;
- The K factors are not identifiable: they are latent variables;
- Is it testable?

Summary 4 *The topics covered in this chapter are*
(i) No arbitrage, state prices vector and asset valuation;
(ii) equivalent probability vector and asset valuation;
(iii) Arbitrage Pricing Theory.

Chapter 5
AN ELEMENTARY INTERTEMPORAL ECONOMY

To examine individual agent's consumption and portfolio decision, we need to characterize the primitives of an economy. To begin with, we start with an economy in this section which is a simplified version of Lucas's (1978, Econometrica) tree economy. The purpose of describing this economy is to introduce students to some basic elements of intertemporal maximization over consumption and portfolio selection, which include

- how to set up an intertemporal utility maximization model;
- how to solve the maximization problem;
- how to define a competitive equilibrium.

5.1 Agents, Goods and Assets

Imagine an economy with a large number of agents, the size of which is normalized to one. All these agents are identical and so we can choose one of them as the "representative" agent.* Agents live for two periods and then die. Let t denote time. The first period is $t = 0$ and the second period is $t = 1$. Each agent is endowed with a "tree", which should be interpreted more generally as a tangible asset (e.g., the market portfolio). Since the total number of agents is normalized to one (million, for example), the total number of trees is also one (million, for example). A tree yields output (fruits) D_t^M to its owner at the beginning of period t. The output can be understood as aggregate output. Fruits are perishable. If they are not consumed in the period they are produced, they cannot be stored and carried over to the next period. In contrast, trees are perfectly durable.

The ownership of trees can be sold and bought in the asset market. An agent can issue shares on his tree and sell these shares to other agents. Normalize the number of shares on each tree to one. There is no currency in the economy and all

*There are studies which examine economies with heterogeneous agents. For examples and references, please consult Duffie (1997).

transactions are traded in terms of goods, i.e., the shares are sold for fruits. *In this context output can also be interpreted as dividends to the shares.* For example, if agent A bought the entire share of agent B's tree, he is entitled to the entire output generated by that tree, D_t^M, which is the dividend to the ownership of the tree. Similarly, if agent A bought only 50% of agent B's tree, he is entitled to only 50% of the tree's dividend. In general, let θ_t^M be the share of trees which the representative agent has at the beginning of period t. He is then entitled to receiving $D_t^M \theta_t^M$ units of fruit as dividend in period t. With this notation, $\theta_0^M = 1$ (the agent begins the life with one tree). In addition, there are N other contingent claim which yield dividends $D_{t,N+1}^A$. The number of shares held by the representative agent is denoted as $\theta_{t,N\times 1}^A$. These assets have a zero net supply. To simplify notations, we denote $\theta = [\theta^M \ \theta_{N\times 1}^A \ ']$ and $D = [D^M \ D_{N\times 1}^A \ ']$.

5.2 Budget Constraints for Consumption and Portfolio Decisions

We find the budget constraint in each period for the representative agent. Examine period $t = 0$ first. The agent's wealth is derived from the ownership of this tree and other assets. He receives dividend, $D_0'\theta_0$. After receiving the dividend he can sell (if he wants to) the share to other agents. Let $S_0 = [S_0^M \ S_{0,N\times 1}^A \ ']$ the price vector per share (in terms of goods) after dividend payments. The assets' values after dividend payments are $S_0'\theta_0$. Adding up the two parts, we have:

$$\begin{aligned} & \text{the agent's wealth at the beginning of period 0} \\ = \ & S_0'\theta_0 + D_0'\theta_0 = (S_0' + D_0')\theta_0. \end{aligned}$$

The agent uses this wealth to finance his expenditure in period 0. There are two items in the expenditure. The first is consumption of fruits in period 0, which is denoted c_0. The second is the purchase of shares. Let θ_1 be the number of shares the agent purchases in period 0, where the subscript 1 indicates that this will be the number of shares the agent has at the beginning of period 1. The expenditure on this purchase is $S_0'\theta_1$. We have:

$$\text{the agent's expenditure in period } 0 = c_0 + S_0'\theta_1.$$

Since the agent cannot spend more than his wealth, the agent faces the following budget constraint in period 0:

$$c_0 + S_0'\theta_1 \leq (S_0' + D_0')\theta_0. \tag{5.1}$$

The budget constraint in period 1, the end of life, is similar. With θ_1 shares of trees acquired in period 0, the agent gets $D_1'\theta_1$ units of goods in period 1. In addition, he can sell the shares after receiving the dividend. Let S_1 be the after-dividend price of a share in period 1 and θ_2 the number of shares the agent acquires in period 1.[†] Let c_1 be the consumption level in period 1. The budget constraint in period 1 is

$$c_1 + S_1'\theta_2 \leq (S_1' + D_1')\theta_1. \tag{5.2}$$

5.3 Intertemporal Utility Function

We postulate that the agent tries to maximize his happiness represented by a utility function, as discussed in Chapter 1. The agent can be made happier only by consuming more goods. That is, the utility function depends only on consumption in the two periods. If the agent consumes c_t units of goods in period t, he gets an instantaneous utility $u(c_t)$. Since the agent is likely to consume in both periods, the agent's life-time utility, or "intertemporal utility", depends on the entire "consumption program" $C = (c_0, c_1)$ rather than on consumption in one period alone. The intertemporal utility function, denoted $U(C)$, describes how the agent ranks different consumption programs.

One plausible measure of intertemporal utility is to treat today's and tomorrow's consumption equally. The corresponding intertemporal utility is simply the sum of instantaneous utilities in the two periods, $u(c_0) + u(c_1)$. A slightly more general form of the intertemporal utility is

$$U(C) = u(c_0) + \beta u(c_1). \tag{5.3}$$

In this specification, future utility has a weight β relative to current consumption. We need some reasonable restrictions on the intertemporal utility function in order to proceed with maximization:

(U1) The intertemporal utility is increasing in consumption (c_0, c_1). That is, the more the better. This is satisfied if and only if the function $u(\cdot)$ is an increasing function, i.e., $u' > 0$.

(U2) The utility function $U(C)$ is concave in (c_0, c_1) jointly. That is,

$$\frac{\partial^2 U}{\partial c_0^2} < 0, \; \frac{\partial^2 U}{\partial c_1^2} < 0, \; \frac{\partial^2 U}{\partial c_0^2} \cdot \frac{\partial^2 U}{\partial c_1^2} - \left(\frac{\partial^2 U}{\partial c_0 \partial c_1}\right)^2 > 0.$$

All these conditions are satisfied if and only if the instantaneous utility function $u(\cdot)$ is concave, i.e. $u'' < 0$.

[†] Of course, there is a question whether agents would want to buy the shares sold by a particular agent in period 1. We will see the implication of the model on this later.

(U3) Discounting: $0 < \beta < 1$. The agent discounts tomorrow's utility. The agent likes today's consumption better than tomorrow's consumption, other things being equal. The constant β is the subjective discount factor. The subjective discount rate is $\frac{1}{\beta} - 1$. The restriction $\beta \in (0,1)$ implies a positive discount rate.

Exercise 5.3.1 *With (5.3) show that if $U(C)$ is concave in c_0 and c_1 separately then it is also concave in (c_0, c_1) jointly. Show that the same result does not necessarily hold if $U(C) = c_0^{1/2} c_1^{1/2}$.*

Exercise 5.3.2 *For the maximization problem below we only need $U(C)$ to be quasi-concave in (c_0, c_1). Put differently, we only need the indifference curve, $U(c_0, c_1) = \bar{U}$ (a constant), to be convex towards the origin. Show that the function in (5.3) with $u'' < 0$ and the function $U(C) = c_0^{1/2} c_1^{1/2}$ are both quasi-concave.*

A useful example of the utility function is given by (5.3) with the instantaneous function $u(\cdot)$ being specialized to the following form:

$$u(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}, \qquad \gamma > 0.$$

This is a special form of the CRRA utility function presented in Chapter 1. The agent's risk attitude is completely summarized by the parameter γ (the risk aversion parameter). By varying the value of the parameter γ the function form includes interesting special cases such as the logarithmic function. For this particular utility function, the Relative Risk Aversion, defined as $-cu''(c)/u'(c)$ in Chapter 1, is equal to γ.

Exercise 5.3.3 *Show that when $u(c)$ is in the CRRA class, the intertemporal utility function (5.3) satisfies the conditions (U1) and (U2) as long as $\gamma > 0$.*

5.4 Intertemporal Maximization

The agent faces the following problem:

- The objective: to maximize the intertemporal utility $U(C)$.
- The choices:

  a consumption program $C = (c_0, c_1)$;

  a portfolio holding strategy (θ_1, θ_2).

- The given data (not up to the agent's choice):

  initial share of trees: $\theta_0^M = 1$ and $\theta_0^A = 0_{N\times 1}$;

  output (or dividends): D_0, and D_1;

  share prices: S_0 and S_1– since there are many other agents in the economy, each agent takes the prices as given (see more discussions later).

- The constraints:

  budget constraints: (5.1) and (5.2);

  non-negativity constraints: $c_0 \geq 0$, $c_1 \geq 0$.

The agent's consumption and portfolio decision can be expressed more formally as follows:

$$\begin{aligned} \max_{(c_0,c_1,\theta_1,\theta_2)} U(C) &= u(c_0) + \beta u(c_1) \\ s.t.\ c_0 &\geq 0,\ c_1 \geq 0,\ (5.1),\ (5.2). \end{aligned}$$

There are two important features of this problem.

- The consumption choices and the portfolio strategy (θ_1, θ_2) are not independent. They are connected by the budget constraints. For example, one cannot choose θ_1 arbitrarily without affecting the amount of consumption in period 0. Acquiring shares requires sacrificing current consumption.

- There is a trade-off between current and future consumption. For example, if the agent can sacrifice some of the consumption in period 0, he could increase the asset holdings θ_1 without violating the budget constraints. The increased asset holding will yield a higher income for consumption in period 1. This trade-off between consumption across time is called the intertemporal trade-off. Solving the intertemporal maximization problem is largely an exercise to find the best intertemporal trade-off without violating the specified constraints.

We now argue intuitively for the conditions that an optimal program (c_0, c_1) must satisfy, leaving the task of verifying them mathematically to the next section. To simplify the argument, we ignore the non-negativity constraints $c_0 \geq 0$ and $c_1 \geq 0$ since they are satisfied under usual utility functions (e.g., if the marginal utility of consumption approaches infinity when c_0 or c_1 approaches zero). Suppose $C = (c_0, c_1)$ is an optimal program for the agent. In period 1, there is no point spending resources to acquire assets, since the agent will die after consumption. The value of assets held in period 1 must be zero: $S_1'\theta_2 = 0$.[‡] It is best to consume all goods available in that period, which is

$$c_1 = (S_1' + D_1')\theta_1. \tag{5.4}$$

Now consider period 0. If c_0 is the best consumption choice in period 0, it must satisfy the following requirements:

- The budget constraint (5.1) holds with equality. That is,

$$c_0 + S_0'\theta_1 = (S_0' + D_0')\theta_0. \tag{5.5}$$

  If the budget constraint were in strict inequality, it would mean that some of the income is not spent. By increasing consumption in period 0 to make the budget constraint an equality the agent would be able to increase utility.

- The marginal utility (benefit) of consumption in period 0 equals the marginal opportunity cost of consumption.

We explore the second requirement further. Suppose that the agent increases consumption in period 0 by Δ_1 from the level c_0. The benefit from this increment is

$$u(c_0 + \Delta_1) - u(c_0).$$

The cost is that the agent must reduce the amount of assets acquired in period 0 and hence reduce consumption in period 1. With the increase in consumption, the agent can maintain the budget constraint (5.1) only if he reduces the amount of assets acquired in period 0 by Δ_2 such that

$$(c_0 + \Delta_1) + S_0'(\theta_1 - \Delta_2) \le (S_0' + D_0')\theta_0.$$

Assume that the agent will only reduce the share purchase for the tree (the market portfolio). Since (c_0, θ_1) satisfy (5.5), we can rewrite the above restriction as

$$S_0^M \Delta_2 \ge \Delta_1 + (c_0 + S_0'\theta_1) - (S_0' + D_0')\theta_0 = \Delta_1.$$

That is, the amount of the market portfolio acquired in period 0 must be curtailed by at least Δ_1/S_0^M in order to increase consumption in period 0 by Δ_1 and yet satisfy the budget constraint (5.1). The best the agent can do is to reduce θ_1^M by exactly $\Delta_2 = \Delta_1/S_0^M$.

[‡]If no one wants to buy the asset, the price of the asset must also be zero in equilibrium ($S_1 = 0$) in order to induce agents to hold any asset, as demonstrated later.

When the agent acquires fewer shares of the asset in period 0, his income at the beginning of period 1 will fall. Precisely, when θ_1^M falls by Δ_2, the income in period 1 falls by $(S_1^M + D_1^M)\Delta_2$. Since the agent eats all his income in period 1 (see (5.4)), his consumption in period 1 will fall by $(S_1^M + D_1^M)\Delta_2$ as well. With $\Delta_2 = \Delta_1/S_0^M$, the reduction in period-1 utility is

$$u(c_1) - u\left(c_1 - (S_1^M + D_1^M)\frac{\Delta_1}{S_0^M}\right).$$

Since period-1 utility is discounted, the opportunity cost of increasing period-0 consumption is

$$\beta\left[u(c_1) - u\left(c_1 - (S_1^M + D_1^M)\frac{\Delta_1}{S_0^M}\right)\right].$$

For the consumption choice c_0 to be optimal, the benefit and cost of a "sufficiently small" increase in period-0 consumption must be equal to each other. That is, for sufficiently small Δ_1,

$$u(c_0 + \Delta_1) - u(c_0) = \beta\left[u(c_1) - u\left(c_1 - (S_1^M + D_1^M)\frac{\Delta_1}{S_0^M}\right)\right].$$

Since this condition holds only for sufficiently small Δ_1, we divide the above equation by Δ_1 and take the limit $\Delta_1 \to 0$. Note that

$$\lim_{\Delta_1 \to 0} \beta \frac{u(c_1) - u\left(c_1 - (S_1^M + D_1^M)\frac{\Delta_1}{S_0}\right)}{\Delta_1} = \beta u'(c_1)\frac{(S_1^M + D_1^M)}{S_0^M}.$$

Then

$$u'(c_0) = \beta u'(c_1)\frac{(S_1^M + D_1^M)}{S_0^M}. \tag{5.6}$$

The left-hand side is the marginal benefit of increasing period-0 consumption from c_0 and the right-hand side is the marginal cost. The condition characterizes the optimal intertemporal trade-off.

Exercise 5.4.1 *If the agent increases period-*0 *consumption* c_0 *by* Δ_1 *and* Δ_1 *is made up by reducing the holding in asset* i*. Use the above argument and show that*

$$u'(c_0) = \beta u'(c_1)\frac{(S_1^i + D_1^i)}{S_0^i}.$$

Remark 1 *For an alternative (but familiar) interpretation of (5.6), try to regard goods in periods* 0 *and* 1 *as different goods.*[§] *Call them good-*0 *and good-*1*, respectively.*

[§]Goods can be different because either they have different physical features, or they are consumed in different dates (states), or both. Here goods in the two periods differ in dates, although they are physically identical.

*The agent can substitute consumption of one good for the other. For example, the agent can increase consumption of good-*0 *by reducing the consumption level of good-*1*. On the preference side, the "marginal rate of substitution" of good-*0 *consumption for good-*1 *consumption is*

$$MRS = \frac{u'(c_0)}{\beta u'(c_1)}.$$

*On the "technology" side, saving one unit of good-*0 *can increase the amount of asset by* $1/S_0^M$*, which can be used to increase consumption of good-*1 *by* $(S_1^M + D_1^M)/S_0^M$*. The "marginal rate of transformation" between the two goods is*

$$MRT = \frac{S_1^M + D_1^M}{S_0^M}.$$

If the consumption levels (c_0, c_1) *are optimal, the marginal rate of substitution and the marginal rate of transformation between the two goods must be equal. That is*

$$\frac{u'(c_0)}{\beta u'(c_1)} = \frac{(S_1^M + D_1^M)}{S_0^M}. \tag{5.7}$$

This condition is equivalent to (5.6) after re-arranging the terms.

We briefly summarize the results of the intuitive argument. For the consumption program (c_0, c_1) and the asset holding strategy (θ_1, θ_2) to be optimal, we must have:

- The value of assets acquired in period 1 is zero: $S_1'\theta_2 = 0$;
- The budget constraints in the two periods hold with equality;
- The choice of c_0 gives the optimal intertemporal trade-off. That is, the marginal benefit of consumption in period 0 equals its marginal (opportunity) cost; or, equivalently, the marginal rate of substitution equals the marginal rate of transformation between the goods in the two periods.

5.5 The Lagrangian Method

In this section we formally derive the results in the last section by using the Lagrangian method. Let λ_0 be the shadow price of the constraint (5.1), measured by period-0 utility, and λ_1 be the shadow price of the constraint (5.2), measured by period-1 utility. These shadow prices are called the current-value shadow prices because they are measured in terms of utility in the period corresponding to the constraint. In contrast, the present-value shadow prices are measured in terms of period-0 utility. Thus, the present-value shadow price of (5.1) is λ_0 and the present-value shadow price of (5.2) is $\beta\lambda_1$. The shadow price of a constraint is also called the Lagrangian multiplier of the constraint. With them we can combine the constraints with the objective function of the maximization to form the Lagrangian:

$$\begin{aligned} L &= u(c_0) + \beta u(c_1) + \lambda_0[(S'_0 + D'_0)\theta_0 - c_0 - S'_0\theta_1] \\ &\quad + \beta\lambda_1[(S'_1 + D'_1)\theta_1 - c_1 - S'_1\theta_2]. \end{aligned}$$

Note the following details of the Lagrangian:

- Move all terms of a constraint to one side and rewrite the constraint as "*terms* $\geq$ 0" – the right-hand side of the rewritten constraint must be zero and the inequality sign must be "$\geq$";
- Take only the left-hand side of the rewritten constraint into the Lagrangian;
- Use the present-value shadow prices of the constraints to combine the constraint with the objective function.

The construction of the Lagrangian also provides an economic meaning for the shadow price λ_0. It is the marginal value of wealth (or income) at the beginning of period 0. Similarly, λ_1 is the marginal value of wealth (or income) at the beginning of period 1, measured in terms of period-1 utility. Consider λ_0 for example. Being the shadow price of the budget constraint in period 0, it is the (implicit) marginal gain in utility from relaxing the constraint marginally. Namely, if the constraint is relaxed by a small amount Δ, the contribution to utility is $\lambda_0\Delta$. Since the constraint can be relaxed by Δ if the wealth level at the beginning of period 0 is increased by Δ, $\lambda_0\Delta$ measures the marginal value of this increase in wealth.

After setting up the Lagrangian, one can then find the first-order conditions for the choice variables $(c_0, c_1, \theta_1, \theta_2)$:

$$\text{for } c_0: \qquad \frac{\partial L}{\partial c_0} = u'(c_0) - \lambda_0 \leq 0, \qquad \text{“ = ” holds if } c_0 > 0;$$

$$\text{for } c_1: \qquad \frac{\partial L}{\partial c_1} = \beta u'(c_1) - \beta\lambda_1 \leq 0, \qquad \text{“ = ” holds if } c_1 > 0;$$

$$\text{for } \theta_1: \qquad \frac{\partial L}{\partial \theta_1} = -S_0\lambda_0 + \beta\lambda_1(S_1 + D_1) \leq 0, \qquad \text{“ = ” holds if } \theta_1 > 0;$$

$$\text{for } \theta_2: \qquad \frac{\partial L}{\partial \theta_2} = -\beta\lambda_1 S_1 \leq 0, \qquad \text{“ = ” holds if } \theta_2 > 0.$$

We interpret the condition for c_0; the other conditions are similar. It states that, if the choice of c_0 is optimal, the marginal utility of consumption c_0 cannot exceed the marginal cost of such consumption. The marginal cost of consumption is now measured by the marginal value of wealth at the beginning of period 0, λ_0. This is because consumption is financed by wealth.[¶] If the marginal utility of consumption exceeds the marginal value of wealth, agents will continue to increase consumption. On the other hand, if the marginal utility of consumption falls below the marginal value of wealth, agents will continue to reduce consumption. Since the lowest level of consumption is zero, the marginal utility consumption may still be lower than the marginal value of wealth at $c_0 = 0$. The inequality sign in the first-order condition allows for such a "corner" solution.

With usual utility functions, the solutions for (c_0, c_1) will be interior solutions, $c_0 > 0$ and $c_1 > 0$. Also, the solution for θ_1 will be positive. Thus the first-order conditions for (c_0, c_1, θ_1) will be simplified as follows:

$$\text{for } c_0: \qquad u'(c_0) = \lambda_0; \tag{5.8}$$

$$\text{for } c_1: \qquad u'(c_1) = \lambda_1; \tag{5.9}$$

$$\text{for } \theta_1: \qquad S_0\lambda_0 = \beta\lambda_1(S_1 + D_1). \tag{5.10}$$

Substituting λ_1 from (5.9) into (5.10), we get:

$$\lambda_0 = \beta\,((S_1 + D_1)./S_0)\,u'(c_1).$$

This coincides with the formula of the marginal value of wealth λ_0 obtained in the last section. Further substituting λ_0 from (5.8) yields the condition (5.6) that we obtained intuitively before.

The optimality condition for θ_2 is tricky. Since $\lambda_1 = u'(c_1) > 0$, we can rewrite the condition for θ_2 as

$$S_1 \geq 0, \qquad \text{“} = \text{” if } \theta_2 > 0.$$

This can be expressed further in a closed form as

$$S_1'\theta_2 = 0. \tag{5.11}$$

This is also intuitively obtained in the last section. Its interpretation is as follows. Since agents will die after period 1, they hold an asset (to their graves) only if the asset is costless to obtain. That is, $\theta_2 > 0$ only if $S_1 = 0$. If the asset is costly, the optimal amount of holdings is zero.

[¶]This measure is more general than the measure, $\beta u'(c_1)(S_1^M + D_1^M)/S_0^M$, which we used in the last section and the two coincide when (c_0, c_1, θ_1) are all positive (see later discusssion).

In addition to the choices for $(c_0, c_1, \theta_1, \theta_2)$, the Lagrangian enables us to recover the constraints of the maximization problem, given below:

$$\frac{\partial L}{\partial \lambda_0} = (S_0' + D_0')\theta_0 - c_0 - S_0'\theta_1 \geq 0, \qquad \text{“ = ” if } \lambda_0 > 0;$$

$$\frac{\partial L}{\partial \lambda_1} = (S_1' + D_1')\theta_1 - c_1 - S_1'\theta_2 \geq 0, \qquad \text{“ = ” if } \lambda_1 > 0.$$

The first condition recovers the budget constraint in period 0 and the second condition recovers the budget constraint in period 1. Note that the inequality signs in these conditions are "$\geq$" rather than "$\leq$". The interpretations for these conditions are straightforward. Take the first condition for example. It states that if the constraint is costly (i.e., $\lambda_0 > 0$), in which case we say that the constraint binds, it is optimal to choose (c_0, θ_1) to make the budget constraint in period 0 an equality. This is because wealth is valuable when $\lambda_0 > 0$, in which case it is optimal to spend all wealth either to consume or to acquire assets. Of course, when wealth is not valuable ($\lambda_0 = 0$), it does not matter whether or not the wealth is fully spent, and so the budget constraint can hold as strict inequality.

The optimality conditions (5.8) and (5.9) indicate that both λ_0 and λ_1 are positive. Thus, the budget constraints hold with equality:

$$c_0 + S_0'\theta_1 = (S_0' + D_0')\theta_0; \tag{5.12}$$

$$c_1 + S_1'\theta_2 = (S_1' + D_1')\theta_1. \tag{5.13}$$

We now have obtained all the requirements on optimality that we intuitively argued for in the last section. What is still to be determined is the asset prices (S_0, S_1). These prices are taken as given by each agent in the economy but must be consistent with the agent's decisions in equilibrium. The next section deals with this issue.

Exercise 5.5.1 *Envelope theorem. The first order conditions implicitly specify the choices $(c_0, c_1, \theta_1, \theta_2)$ as functions of the initial endowment θ_0. Denote these functions by $c_0(\theta_0)$, $c_1(\theta_0)$, $\theta_1(\theta_0)$, $\theta_2(\theta_0)$. The maximized Lagrangian is*

$$L^* = L(c_0(\theta_0), c_1(\theta_0), \theta_1(\theta_0), \theta_2(\theta_0); \theta_0).$$

Show that $dL^*/d\theta_0 = \partial L/\partial\theta_0 = \lambda_0(S_0 + D_0)$.

Exercise 5.5.2 *Let the intertemporal utility function be $U(C) = c_0^{1/2} c_1^{1/2}$ and assume that there is no contingent claims in the economy. Set up the Lagrangian and derive the first order conditions. Take θ_0, D_0, S_0 and D_1 as given, solve for c_0, c_1 and θ_1. Show that the agent's optimal choices yield $c_0 > 0$ and $c_1 > 0$.*

Exercise 5.5.3 *Repeat the above exercise with the following intertemporal utility function*

$$U(C) = u(c_0) + \beta u(c_1)$$

where

$$u(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}, \qquad \gamma > 0.$$

5.6 Competitive Equilibrium

The asset prices cannot be arbitrary in the economy. Although each agent is too small in the economy to affect the prices, their choices together will generate a particular demand pattern for the assets. The prices must be such that demand and supply equal each other, in which case the corresponding market is *cleared*. The same argument applies to the goods market (good prices are one in each period because everything here is measured in terms of goods).

We have the following market-clearing conditions:

- Asset markets clear:

$$\theta_1^M = 1; \qquad \theta_1^A = 0_{N\times 1}; \qquad \theta_2^M = 1; \qquad \theta_2^A = 0_{N\times 1}. \tag{5.14}$$

- Goods markets clear:

$$c_0 = D_0^M; \qquad c_1 = D_1^M. \tag{5.15}$$

Let us explain (5.14) first. In each period t $(= 0, 1)$, the total demand for shares is the demand by each agent, θ_t, multiplied by the total number of agents which we normalized to one (million, for example), i.e., $1 \cdot \theta_t = \theta_t$. To calculate the total supply of the market portfolio in each period, note that each agent is endowed with one tree at the beginning of period 0 ($\theta_0^M = 1$). The total supply in period 0 is $1 \cdot \theta_0^M = 1$. Since trees do not depreciate nor accumulate in this economy, the total supply of assets in period 1 is also 1. (5.14) sets the total demand for other contingent claims in each period t to the total supply of assets.

Eq. (5.15) can be explained similarly. Take period 0 for example. The total demand for goods (fruits) in period 0 is $1 \cdot c_0 = c_0$. The total supply of goods is the total output (or dividends) $1 \cdot D_0^M = D_0^M$. The market clearing condition requires that the demand for and the supply of goods are equal to each other.

Some of these conditions are redundant. Consider period 0 for example. Recall that the agent's maximization problem implies that the budget constraint in period 0 holds as equality. Given the equality form of the budget constraint, if the asset market clears in period 0 then the goods market automatically clears in period 0. Similarly, if the goods market clears in period 0, the asset market automatically clears in period 0. Thus, two of the four conditions in (5.14) and (5.15) are redundant. This principle is called the Walras Law. In the general form, it states that if $(n-1)$ of all n markets in a period are cleared, then the remaining market is automatically cleared.

The following exercise verifies the Walras law in the current economy.

Exercise 5.6.1 *Let $\theta_0^M = 1$. Show that if $\theta_1^M = 1$ and the budget constraint (5.1) holds with equality then $c_0 = D_0^M$.*

Now the time has come to define a competitive equilibrium.

Definition 1 *A competitive equilibrium in the described two-period economy consists of a consumption program (c_0, c_1), a portfolio holding strategy (θ_1, θ_2) and prices (S_0, S_1) such that the following conditions hold:*
(i) Given (S_0, S_1) (and (D_0, θ_0)), the consumption program and the asset holding strategy solve each agent's utility maximization problem;
(ii) The prices clear the asset and goods markets in each period.

The equilibrium is called a competitive equilibrium because each agent takes prices as given when choosing the consumption program and the portfolio holding strategy. The definition indicates the following procedure to solve for the equilibrium:

Step 1. Taking (S_0, S_1) as given, one solves the agent's utility maximization problem, which generates conditions (5.8) — (5.13). These conditions implicitly give the variables $(c_0, c_1, \theta_1, \theta_2, \lambda_0, \lambda_1)$ as functions of the prices, (S_0, S_1). An explicit characterization of these functions is usually difficult.

Step 2. In principle one substitutes the solutions for $(c_0, c_1, \theta_1, \theta_2, \lambda_0, \lambda_1)$ from Step 1 into the market clearing conditions to solve for (S_0, S_1). Since the explicit solutions for $(c_0, c_1, \theta_1, \theta_2, \lambda_0, \lambda_1)$ are difficult to obtain, this exercise is usually done in an indirect way. We first solve $(c_0, c_1, \theta_1, \theta_2)$ from the market clearing conditions and then substitute into the first-order conditions (5.8) — (5.13) to solve for $(S_1, S_2, \lambda_0, \lambda_1)$.

Let us go through these steps. The last section has already completed Step 1. In the second step, we set $c_0 = D_0^M$, $c_1 = D_1^M$, $\theta_1^M = 1$, and $\theta_2^M = 1$ to satisfy the market clearing conditions. Since we have imposed all four market clearing conditions, the two budget constraints (5.12) and (5.13) are automatically satisfied as a result of the Walras Law. The remaining four conditions, (5.8) — (5.11), help us to determine the remaining four variables $(S_1, S_2, \lambda_0, \lambda_1)$.

Substituting $c_0 = D_0^M$ and $c_1 = D_1^M$ into (5.8) and (5.9) one obtains the solutions for λ_0 and λ_1:

$$\lambda_0 = u'(D_0^M), \qquad \lambda_1 = u'(D_1^M).$$

Eq. (5.11), the optimal condition for θ_2, gives S_1. Since $\theta_2 = [1\ 0\ 0\ ...\ 0] > 0$, the condition implies $S_1 = 0$. That is, the asset must be costless to obtain in period 1 in order for agents to hold the amount supplied (one unit). Finally, for S_0, one substitutes the solutions for (λ_0, λ_1) and S_1 into (5.10) to obtain:

$$S_0 = \frac{\beta u'(D_1^M)}{u'(D_0^M)} D_1 = \frac{D_1}{\left(\frac{u'(D_0^M)}{\beta u'(D_1^M)}\right)}.$$

The after-dividend share price is equal to the discounted value of future dividend, where the gross discount rate is the marginal rate of substitution between periods 0 and 1, $\frac{u'(D_0^M)}{\beta u'(D_1^M)}$. Given the marginal rate of substitution, higher future dividend yields a higher share price. However, for the market portfolio, the equilibrium marginal rate of substitution also depends on future output and the overall dependence of the share price on D_1^M is analytically ambiguous. The following exercise explores this feature.

Exercise 5.6.2 *Let $u(c)$ be the CRRA function with relative risk aversion $\gamma > 0$.*
(i) Show that the share prices S_0 always increase with their future dividends D_1. Explain.

(ii) Show that the share prices S_0 always increase with the current aggregate dividend D_0^M. Explain.

(iii) Show that the share price S_0^M is an increasing function of future dividends D_1^M if and only if $\gamma < 1$. Explain why in the case $\gamma > 1$ the share price S_0^M falls when D_1^M increases.

In addition to pricing the share of assets, the model also implies a price of period-1 goods relative to period-0 goods. The following exercise asks you to find this relative price and explore its features.

Exercise 5.6.3 *Find the relative price of period-1 good to period-0 good that is consistent with the competitive equilibrium. Show that (i) the relative price is a decreasing function of period-1 output and an increasing function of period-0 output, and (ii) If $D_1^M = D_0^M$, the relative price equals β.*

Summary 1 *The main topics covered in this chapter are*

(i) Setting up the agent's consumption and portfolio selection problem;

(ii) Using the Lagrangian method to derive the agent's optimality conditions;

(iii) Using the marginal cost-benefit analysis to interpret the agent's intertemporal trade-offs;

(iv) Solving the agent's optimal consumption and portfolio selection strategy;

(v) Defining the competitive equilibrium and solving the equilibrium prices.

Chapter 6
AN ECONOMY WITH GOVERNMENT FINANCE AND INTEREST RATE

In previous chapter, we discussed individual's consumption and portfolio selection in a simplified Lucas's economy, where lending and borrowing are not possible. This section introduces government finance to fulfill this possibility. With the issue of government bonds, we can also discuss interest rates in the economy.

6.1 Government and Government Bonds

Before introducing a government into the simple, two-period economy, let us note the following feature of the economy discussed in previous chapter. All agents are the same so that the amount of borrowing among agents is zero in equilibrium. If prices are such that make one agent willing to borrow, then all other agents want to borrow as well; if prices are such that make one agent willing to lend, then all other agents want to lend as well. In the end, prices maintain an equilibrium in the market, and so the net amount of borrowing or lending is zero – Everyone consumes exactly the amount of endowment (output from the tree).

This particular feature of the representative-agent economy may not be appealing. One way to introduce borrowing and lending is, of course, to make these agents heterogeneous. Another way, adopted in this chapter, is to introduce a government and preserve the assumption of homogeneous private agents. Although private agents are homogeneous and cannot borrow from each other, they can borrow from or lend to the government. Since private agents are kept homogeneous, the competitive equilibrium is a simple modification of the one in the previous chapter.

We model the government in the simplest way. It is passive, taking the prices as given just as private agents do. Its actions are described as follows:

- Spending. The government spends g_0 in period 0 and g_1 in period 1. These amounts are exogenously set and are expressed in per-capita terms.

- Taxes. The government levies a lump-sum tax τ_0 in period 0 and τ_1 in period 1. These amounts can be chosen by the government.

- Bond issuing. In addition to taxes, the government can also issue one-period bonds in period 0 to finance its spending. The bonds are sold competitively to private agents. The government promises to pay one unit of goods in period 1 for each unit of such bonds. The bond is a one-period bond because it matures one period after it is issued. Note that the bond is a real bond – it is sold for and redeemed with goods.

Government spending is fixed here. The purpose of making this restriction is to focus on the government's financing methods rather than on its spending strategies. Namely, we are interested in whether different financing methods for the same spending matter in this economy. It is possible that the government bonds are negative, in which case the private sector owes to the government. Without loss of generality, we examine only the case of positive government borrowing.

We find the government budget constraints in the two periods. Consider period 0 first. The government starts the life with an outstanding debt θ_{B0} per capita (inherited historically). There are two types of expenditure in period 0 given below:

$$\begin{aligned} &\text{total government expenditure} \\ = \;&\text{spending on goods} \;+\; \text{debt payments} = g_0 + \theta_{B0}. \end{aligned}$$

Let $\overline{\theta}_{B1}$ be the quantity (per capita) of government bonds sold in period 0 and B_0 be the unit price of such bonds (in terms of goods in period 0). Then,

$$\begin{aligned} &\text{total government revenue in period 0} \\ = \;&\text{tax revenue} \;+\; \text{value of bonds} = \tau_0 + B_0\overline{\theta}_{B1}. \end{aligned}$$

The government budget constraint in period 0 is

$$g_0 + \theta_{B0} \leq \tau_0 + B_0\overline{\theta}_{B1}. \tag{6.1}$$

In period 1 the government is unable to raise revenue through bonds because agents do not want to purchase such bonds knowing that the bonds will yield return only after agents die. The tax revenue τ_1 is the government's sole source of revenue in period 1. The total expenditure in period 1 is

$$\text{spending on goods} \;+\; \text{debt repayment} = g_1 + \overline{\theta}_{B1}.$$

The government budget constraint in period 1 is

$$g_1 + \overline{\theta}_{B1} \leq \tau_1. \tag{6.2}$$

We assume that the government does not waste any revenue, and so the two budget constraints (6.1) and (6.2) hold with equality.

6.2 Competitive Equilibrium

With the introduction of taxes and bonds, each agent's budget constraints must be modified. In period 0, the agent's wealth now includes the government's payment on the inherited debt θ_{B0}, as well as the value and dividends of the tree. The agent's total initial wealth is $(S_0'+D_0')\theta_0+\theta_{B0}$. On the expenditure side there are two new items. One is the tax τ_0. The other is the value of bonds purchased in period 0. **Denote the quantity of bonds purchased by the agent by θ_{B1}, which is the demand for bonds by the private sector, to be distinguished from the supply of bonds by the government ($\bar{\theta}_{B1}$).*** Since each unit of bond is sold for B_0 units of goods, the agent's expenditure on bonds is $B_0\theta_{B1}$. The agent's budget constraint in period 0 is

$$c_0+\tau_0+S_0'\theta_1+B_0\theta_{B1}\le(S_0'+D_0')\theta_0+\theta_{B0}. \tag{6.3}$$

In period 1, the agent receives payment (redemption) on the bonds purchased in period 0. Since each unit of bond is paid one unit of goods, the value of redemption is θ_{B1}. The wealth at the beginning of period 1 is $(S_1'+D_1')\theta_1+\theta_{B1}$. The new element on the expenditure side is the tax τ_1. The agent's budget constraint in period 1 is

$$c_1+\tau_1+S_1'\theta_2\le(S_1'+D_1')\theta_1+\theta_{B1}. \tag{6.4}$$

The agent's maximization problem becomes

$$\max_{(c_0,c_1,\theta_1,\theta_2,\theta_{B1})} u(c_0)+\beta u(c_1) \quad s.t. \quad c_0,\ c_1\ge 0,\ (6.3),\ (6.4).$$

The additional choice variable is the amount of bonds purchased in period 0, θ_{B1}. Note that θ_{B0} is the amount of inherited bonds, which is not a choice variable. The agent also takes government spending (g_0,g_1), taxes (τ_0,τ_1) and share prices (S_0,S_1) as given.

Let λ_0 be the current-value shadow price of (6.3) and λ_1 the current-value shadow price of (6.4). The Lagrangian can be set up in the same way as in the last chapter and the results of optimization are summarized below:

- The two budget constraints hold as equality:

$$c_0+\tau_0+S_0'\theta_1+B_0\theta_{B1}=(S_0'+D_0')\theta_0+\theta_{B0}, \tag{6.5}$$

$$c_1+\tau_1+S_1'\theta_2=(S_1'+D_1')\theta_1+\theta_{B1}. \tag{6.6}$$

- The optimal choices of $(c_0,c_1,\theta_1,\theta_2)$ satisfy

$$u'(c_0)=\lambda_0, \tag{6.7}$$

$$u'(c_1)=\lambda_1, \tag{6.8}$$

$$S_0\lambda_0=\beta\lambda_1(S_1+D_1), \tag{6.9}$$

$$S_1\ge 0, \qquad \text{“} = \text{” if } \theta_2>0. \tag{6.10}$$

*We do not make such a distinction on b_0 because it is a given quantity to both the government and the private sector.

- The optimal choice of θ_{B1} satisfies

$$B_0\lambda_0 = \beta\lambda_1. \tag{6.11}$$

The optimality conditions for $(c_0, c_1, \theta_1, \theta_2)$ are exactly the same as in the last chapter. This is because the taxes are lump sum, which do not distort the agents' choices at the margin. (We do not mean that the four variables have the same equilibrium values as before.)

The optimality condition for θ_{B1} requires an explanation. Recall that λ_t is the marginal value of wealth at the beginning of period t, where $t = 0, 1$. Since each bond costs B_0 units of period-0 goods and each unit of period-0 good has a marginal value λ_0, the term $B_0\lambda_0$ is the marginal cost of bonds in terms of period-0 utility. Similarly, since each bond pays one unit of goods in period 1 and each unit of period-1 good has a marginal value λ_1 in terms of period-1 utility, the marginal benefit of a unit of bond in terms of period-1 utility is λ_1. Discounted back to period 0, the marginal benefit of a bond in terms of period-0 utility is $\beta\lambda_1$. Therefore, the condition (6.11) requires that the marginal cost and marginal benefit of bonds be equal to each other.

The introduction of the government also changes the goods market clearing conditions. Since the government spends a part of output, total spending on goods in period t is the sum $c_t + g_t$. This should be equal to total output (or aggregate dividend) D_t^M. The market clearing condition for the shares market is the same as before, because we assume that the government is not endowed with shares of the trees. In addition to the shares market and the goods market, now there is a new market - the bond market. We summarize the market-clearing conditions as follows:

- Shares markets clear:

$$\theta_1^M = 1, \qquad \theta_2^M = 1, \qquad \theta_1^A = 0_{N\times 1}, \qquad \theta_2^A = 0_{N\times 1}. \tag{6.12}$$

- Goods markets clear:

$$c_0 + g_0 = D_0^M, \qquad c_1 + g_1 = D_1^M. \tag{6.13}$$

- Bonds market clears:

$$\theta_{B1} = \overline{\theta}_{B1}. \tag{6.14}$$

A competitive equilibrium now consists of a consumption program (c_0, c_1), a share holding strategy (θ_1, θ_2), the amount of bond acquisition θ_{B1}, asset prices (S_0, S_1, B_0), and government policies $(\overline{\theta}_{B1}, \tau_0, \tau_1)$ such that

(i) Given asset prices and government policies, the agent's choices $(c_0, c_1, \theta_1, \theta_2, \theta_{B1})$ solve the maximization problem for consumption and portfolio holdings;

(ii) The shares, bond and goods markets all clear;

(iii) The government budget constraints (6.1) and (6.2) hold with equality.

As before, the equilibrium can be obtained by first solving the agent's maximization problem, which generates the conditions (6.5) — (6.11). Then we use the market clearing conditions to solve for $(c_0, c_1, \theta_1, \theta_2, \theta_{B1})$. Substituting these solutions back into (6.5) — (6.11) to recover $(\lambda_0, \lambda_1, S_0, S_1, B_0)$. We summarize the results below:

$$\begin{aligned} &c_0 = D_0^M - g_0, \qquad c_1 = D_1^M - g_1, \qquad \theta_1^M = \theta_2^M = 1, \\ &\theta_{B1} = \bar{\theta}_{B1}, \qquad \theta_1^A = 0_{N\times 1}, \qquad \theta_2^A = 0_{N\times 1}; \end{aligned} \tag{6.15}$$

$$S_1 = 0; \tag{6.16}$$

$$\lambda_0 = u'(c_0) = u'(D_0^M - g_0), \;\; \lambda_1 = u'(c_1) = u'(D_1^M - g_1); \tag{6.17}$$

$$S_0 = \frac{\beta u'(D_1^M - g_1)}{u'(D_0^M - g_0)} D_1; \tag{6.18}$$

$$B_0 = \frac{\beta u'(D_1^M - g_1)}{u'(D_0^M - g_0)}. \tag{6.19}$$

In particular, (6.18) is obtained by substituting the solutions for $(\lambda_0, \lambda_1, c_0, c_1)$ into (6.9), and (6.19) by substituting $(\lambda_0, \lambda_1, c_0, c_1)$ into (6.11).

Exercise 6.2.1 *Explain why an increase in government spending g_0 increases the marginal value of private wealth.*

In addition to (6.15) — (6.19), the competitive equilibrium also requires the government to honor its budget constraints. This requirement imposes restrictions on the policies $(\bar{\theta}_{B1}, \tau_0, \tau_1)$:

$$\tau_0 = g_0 + \theta_{B0} - B_0 \bar{\theta}_{B1}, \qquad \tau_1 = g_1 + \bar{\theta}_{B1}. \tag{6.20}$$

There are only two equations here and we have three policy variables. There are not enough equations to determine all the three policy variables. We look into this issue in the next section.

6.3 Government Finance and Intertemporal Constraints

6.3.1 Ricardian Equivalence

Since we do not have enough equations to determine the equilibrium level of each of the three policy variables, the government can use different combinations of tax τ_0 and bonds $\bar{\theta}_{B1}$ to finance spending in period 0. It is interesting to ask

Do different levels of bonds $\bar{\theta}_{B1}$ matter to the competitive equilibrium?

To phrase the question differently, imagine that the government wants to cut tax in period 0 but does not want to change the spending profile (g_0, g_1). To keep the same level of spending, the government can issue more bonds in period 0 to finance the tax cut. Does this bond financing method change the equilibrium value of any of the real variables such as consumption, asset holdings, and asset prices? The answer to this question is No.

Ricardian Equivalence. For any given government spending profile (g_0, g_1) and initial public debt θ_{B0}, issuing more government bonds $\bar{\theta}_{B1}$ to finance a tax cut in τ_0 does not affect any of the real variables $(c_0, c_1, \theta_1, \theta_2, S_0, S_1, B_0, \lambda_0, \lambda_1)$, provided that the government maintains the budget constraints in the two periods.

It is easy to see that the Ricardian Equivalence holds in the current model. The equilibrium solutions for $(c_0, c_1, \theta_1, \theta_2, S_0, S_1, B_0, \lambda_0, \lambda_1)$ are given by (6.15) — (6.19), which depend on the government's spending profile (g_0, g_1) but not on the financing methods $(\tau_0, \tau_1, \bar{\theta}_{B1})$. Once the spending profile is fixed, changes in the financing methods do not change any of the real variables. Issuing more bonds in period 0 merely changes the amount of bonds held by the private sector, as $\theta_{B1} = \bar{\theta}_{B1}$ in equilibrium, and the tax paid by the private sector. Since the additional bonds are issued to finance a tax cut, the changes in the value of bonds held by the private sector, $B_0\theta_{B1}$, and the tax paid cancel out, leaving the agent's consumption decision and asset acquisition unaffected.

We can make the above argument more precisely by examining the budget constraints of the private agent and the government. Consider period 0 first. The government's budget constraint in period 0 requires

$$\tau_0 + B_0\bar{\theta}_{B1} = g_0 + \theta_{B0}.$$

When both spending g_0 and the initial debt θ_{B0} are fixed, as we have maintained, total government expenditure in period 0, $(g_0+\theta_{B0})$, is fixed. To maintain the budget constraint, the total government revenue must also be fixed – every unit of cut in the tax τ_0 must be met by a unit increase in the value of bonds, $B_0\bar{\theta}_{B1}$. Since in equilibrium the amount of bonds held by the private sector equals the amount supplied, the equilibrium value of the sum $\tau_0 + B_0\theta_{B1}$ is also unchanged by the increase in $\bar{\theta}_{B1}$. If we re-arrange the agent's budget constraint in period 0 as

$$c_0 + S_0'\theta_1 = (S_0' + D_0')\theta_0 + \theta_{B0} - (\tau_0 + B_0\theta_{B1}),$$

it is immediately clear that the decisions on (c_0, θ_1) depend only on the sum $(\tau_0 + B_0\theta_{B1})$, not on τ_0 or $B_0\theta_{B1}$ individually. Since the equilibrium value of the sum $(\tau_0 + B_0\theta_{B1})$ is unchanged by the increase in bond issuing, the decisions on (c_0, θ_1) should not change either.

A similar argument applies to period 1. The government's budget constraint in period 1 requires

$$\tau_1 - \bar{\theta}_{B1} = g_1.$$

Since spending g_1 is fixed, the difference $\tau_1 - \overline{\theta}_{B1}$ is also fixed: every unit of increase in the bonds issued in period 0, $\overline{\theta}_{B1}$, must be paid by increasing period-1 tax τ_1. Since $\theta_{B1} = \overline{\theta}_{B1}$ in equilibrium, the difference $\tau_1 - \overline{\theta}_{B1}$ is also unchanged by the increase in government bond issuing. From the agent's budget constraint in period 1:

$$c_1 + S_1'\theta_2 = (S_1' + D_1')\theta_1 - (\tau_1 - \theta_{B1}),$$

it is clear that the agent's decisions on (c_1, θ_2) will be unaffected in equilibrium by the change in $\overline{\theta}_{B1}$.

The share prices (S_0, S_1) do not change with $\overline{\theta}_{B1}$. This is because the share prices clear the shares markets in the two periods, where neither the demand (θ_1, θ_2) nor the supply of the shares have changed. Why does the bond price B_0 remain unchanged when its supply increases? A quick answer to this question is that, with the shift of the supply curve of bonds, the demand curve for bonds also shifts so that the same price B_0 clears the bond market. It is more difficult to answer why the demand curve for bonds shifts. We return to this question two subsections later.

6.3.2 Intertemporal Budget Constraints

The analysis in the last subsection reveals an important restriction on the government's financing technique: For every unit of increase in bond issuing in period 0, the government must increase the tax by exactly one unit in period 1. There is no free-lunch for the government. The tax cut in period 0 financed by bond issuing is only temporary – the bond financing merely delays the tax from period 0 to period 1. Since such bond financing does not affect the equilibrium, it must be the present value of the taxes, not the distribution of taxes across periods, that matters in equilibrium. This argument can be captured by the government's intertemporal budget constraint, which states that

$$\begin{aligned} &\text{the present value of the government's spending plus the initial debt} \\ \leq\ &\text{the present value of the government's tax revenue.} \end{aligned}$$

Let us obtain the intertemporal budget constraint by intuitive arguments first. The government spends g_0 in period 0 and g_1 in period 1. To calculate the present value of spending, we need to convert spending in period 1 into a period-0 value. To do so, let the *gross real interest rate* be R_0. That is, lending one unit of goods in period 0 yields R_0 units of goods in period 1. The spending g_1 is worth g_1/R_0 in period 0 and then

$$\text{the present value of the government's spending} = g_0 + \frac{g_1}{R_0}.$$

Similarly,

$$\text{the present value of tax revenue} = \tau_0 + \frac{\tau_1}{R_0}.$$

With an initial debt θ_{B0} to pay, the government's intertemporal budget constraint is

$$g_0 + \frac{g_1}{R_0} + \theta_{B0} \leq \tau_0 + \frac{\tau_1}{R_0}. \tag{6.21}$$

This constraint holds as equality since we assume that the government does not waste any tax revenue. This intertemporal budget constraint shows that the present value of tax revenue must be the same to finance the fixed present value of spending and debt, regardless of the amount of bonds issued in period 0.

Similarly, what matters for the agent's consumption choices and asset holding strategy is the agent's intertemporal budget constraint, which requires

$$\begin{array}{rl} & \text{the agent's present value of consumption} \\ \leq & \text{the agent's present value of income (wealth)} \\ & \quad -\text{the agent's present value of tax liability.} \end{array}$$

The agent's present value of consumption is $c_0 + \frac{c_1}{R_0}$ and the present value of tax liability is $\tau_0 + \frac{\tau_1}{R_0}$. His wealth is the sum of the holding on initial government bonds, θ_{B0}, and the value of the share, $(S_0' + D_0')\theta_0$. The agent's intertemporal budget constraint is

$$c_0 + \frac{c_1}{R_0} \leq (S_0' + D_0')\theta_0 + \theta_{B0} - \left(\tau_0 + \frac{\tau_1}{R_0}\right). \tag{6.22}$$

Now we derive the intertemporal budget constraints from the constraints in the two periods. By comparing the results with (6.21) and (6.22), we can see what the real interest rate R_0 should be. Start with the government's budget constraints in the two periods which we re-produce below

$$g_0 + \theta_{B0} \leq \tau_0 + B_0\bar{\theta}_{B1}, \qquad g_1 \leq \tau_1 - \bar{\theta}_{B1}.$$

Multiply the second constraint by B_0 and add onto the first constraint. (To preserve the inequality, perform the addition on the two sides of the constraints separately.) We have an alternative form of the government's intertemporal budget constraint:

$$g_0 + B_0 g_1 + \theta_{B0} \leq \tau_0 + B_0\tau_1. \tag{6.23}$$

We can carry out a similar exercise on the agent's budget constraints in the two periods to obtain the following intertemporal budget constraint:

$$\begin{aligned} c_0 + B_0 c_1 \quad \leq \quad & (S_0' + D_0')\theta_0 + \theta_{B0} - (\tau_0 + B_0\tau_1) \\ & +[B_0(S_1' + D_1') - S_0']\theta_1 - B_0 S_1'\theta_2. \end{aligned} \tag{6.24}$$

The last two terms are the profit from buying and selling shares during the lifetime. In a competitive equilibrium, such profit must be zero. Indeed, $S_1 = 0$ implies that the last term is zero. Also, dividing the pricing equation (6.18) by (6.19) yields $S_0/B_0 = D_1$. Thus the agent's intertemporal budget constraint in equilibrium is

$$c_0 + B_0 c_1 \leq (S_0' + D_0')\theta_0 + \theta_{B0} - (\tau_0 + B_0\tau_1). \tag{6.25}$$

Compare (6.23) with (6.21) and (6.25) with (6.22). The only way to make the two types of intertemporal budget constraints consistent with each other is to have:

$$R_0 = \frac{1}{B_0}. \tag{6.26}$$

This is *not* a coincidence! To see why it should hold in this economy, note that by issuing bonds the government is borrowing from the private sector and by purchasing the bonds the private sector is lending to the government. By selling one unit of bond in the bond market at a price B_0, the government borrows B_0 units of goods in period 0. The repayment is one unit of goods in period 1. The gross real interest rate the government faces is the amount of repayment on each unit of bond divided by the price of the bond, i.e., $1/B_0$.

To make a convincing argument that only the intertemporal budget constraints matter for agents' choices, it is necessary to show that the agent's choices are the same under the intertemporal budget constraints as under the two separate constraints (one for each period). To do so, formulate the agent's maximization problem using the intertemporal budget constraint. We use (6.24) instead of (6.25) as the agent's intertemporal constraint, because (6.25) has already used some of the optimality conditions that we want to derive.

$$\max_{(c_0,c_1,\theta_1,\theta_2)} \; u(c_0) + \beta u(c_1) \quad s.t.\ c_0 \geq 0,\, c_1 \geq 0,\, (6.24).$$

The Lagrangian associated with this maximization problem is

$$\begin{aligned} L \;=\; & u(c_0) + \beta u'(c_1) + \lambda_0\{(S_0' + D_0')\theta_0 + \theta_{B0} - (\tau_0 + B_0\tau_1) \\ & -B_0S_1'\theta_2 + [B_0(S_1' + D_1') - S_0']\theta_1 - (c_0 + B_0c_1)\}. \end{aligned}$$

Note that the Lagrangian multiplier for the intertemporal constraint is the same multiplier λ_0 that we used for the agent's period-0 constraint, as all incomes and expenditures are now converted into period-0 values.

The optimality conditions for c_0 and θ_2 are the same as before, given by (6.7) and (6.10). The optimality condition for c_1 is:

$$\beta u'(c_1) = B_0\lambda_0.$$

Substituting $\lambda_0 = u'(c_0)$ yields the same pricing equation for B_0 as (6.19).

The optimality condition for θ_1 is

$$B_0(S_1 + D_1) - S_0 \leq 0, \quad \text{“} = \text{”} \text{ if } \theta_1 > 0.$$

We may rewrite this condition using the relation $R_0 = 1/B_0$:

$$(S_1 + D_1)\,./S_0 \leq R_0, \quad \text{“} = \text{”} \text{ if } \theta_1 > 0.$$

The left-hand side of the condition is the gross rate of return to a share for all assets, which is defined as the return on the share $(S_1 + D_1)$ divided by the current price of the share. For future use, let us denote the gross rate of return to the share by

$$R_{s0} = (S_1 + D_1)\,./S_0. \tag{6.27}$$

The optimality condition for θ_1 can then be expressed as $R_{s0} \leq R_0$ and “ = ” if $\theta_1 > 0$. This condition is a result of the arbitrage between the bond market and the share market: The rate of return to holding the government bonds, R_0, must be at least equal to the rate of return to holding a share of trees or any contingent claim (any of the element in R_{s0}).

Of course, both bonds and shares are held by the private sector in the competitive equilibrium. In this case the optimality condition for θ_1 becomes $R_{s0} = R_0$, or equivalently, $S_0 = B_0(S_1 + D_1)$. Substituting the pricing equation for B_0 we obtained above, the condition becomes identical to the pricing equation (6.18). Therefore the maximization problem using the intertemporal budget constraint indeed yields the same solutions as the original maximization problem does.

Exercise 6.3.1 *Spending policies matter. Compare two economies which are the same except that the government spending profile is* (g_0, g_1) *in economy* $E1$ *and* $(g_0 + \Delta, g_1 - \delta)$ *in economy* $E2$*. Let* B_0 *be the equilibrium price for bond in economy A. The two quantities* (Δ, δ) *satisfy* $\Delta = B_0\delta$*. That is, the government spending profiles in the two economies have the same present value* if *both profiles are discounted by the same bond price as in economy* $E1$*. Compute the equilibrium bond price in economy* $E2$*. Why is it different from* B_0*?*

The alternative formulation of the maximization problem also illustrates a strong result: Equilibrium rates of return to bonds and to shares must be equal to each other. We will return to this result later. For now, we want to examine the real interest rate further.

6.3.3 Bond Price and Real Interest Rate

Formulating the agent's maximization problem using the intertemporal budget constraint also reveals that the bond price B_0 is the relative price of future consumption goods in terms of current consumption goods. By multiplying consumption in period 1 by B_0 we can add with consumption in period 0. Similarly, incomes in the two periods are added up with period-1 income being multiplied by B_0.

The alternative interpretation of B_0 gives a clue to the earlier question of why the demand curve for bonds shifts up when the supply of bonds increases in period 0. As a relative price of goods, its equilibrium value must clear the goods market. If the demand curve for bonds did not shift up, the bond price would fall, which in turn would imply a decrease in the relative price of period-1 goods to period-0 goods. Agents would curtail consumption in period 0 and increase consumption in period 1. This could not be an equilibrium outcome in the goods market because the supplies in the goods market in the two periods are fixed at D_0 and D_1, respectively. In fact, to clear the goods markets, the relative price of future consumption goods to current consumption goods must remain the same as before after an increase in the supply of bonds. The only way to achieve this outcome is an upward shift in the demand curve for bonds.

As the inverse of the bond price, the real interest rate R_0 can be interpreted as the relative price of consumption goods in period 0 relative to consumption goods in period 1. The real interest rate depends on four aspects of the economy: The output profile (D_0^M, D_1^M), the government spending profile (g_0, g_1), the subjective discount factor β and the form of the marginal utility function $u'(\cdot)$. The dependence of the real interest rate on the first three factors can be readily identified.

- The steeper the output profile (D_0^M, D_1^M), the higher the real interest rate. That is, the higher the relative output D_1^M/D_0^M, the higher the interest rate. This is because the relative abundance of goods in period 1 increases the relative value of consumption in period 0 to consumption in period 1. The interest rate must increase in order to induce agents to lend.

- The steeper the government spending profile (g_0, g_1), the lower the real interest rate. The explanation is similar to the above. The higher government spending in period 1 implies that goods available for private consumption is reduced, which increases the relative value of consumption in period 1 to consumption in period 0.

- The higher the discount factor β (i.e., the more patient agents are), the lower the real interest rate. Equivalently, the lower the discount rate $\frac{1}{\beta} - 1$, the lower the real interest rate. As agents value future more, the more likely they will save and lend. The demand for bonds will be higher, the bond price will be higher and so the real interest rate will be lower.

It is more difficult to identify how the real interest rate depends on the form of the marginal utility function. The following example tries to find the dependence with an explicit form of the utility function.

Example 1 *Let $u(c)$ be the CRRA function with relative risk aversion $\gamma > 0$. The marginal utility is $u'(c) = c^{-\gamma}$ and its feature is summarized by γ. The real interest rate is*

$$R_0 = \frac{1}{B_0} = \frac{1}{\beta} \cdot \left(\frac{D_1^M - g_1}{D_0^M - g_0} \right)^{\gamma}.$$

We can compute the derivative of the interest rate with respect to γ:

$$\frac{dR_0}{d\gamma} = \frac{1}{\beta} \cdot \left(\frac{D_1^M - g_1}{D_0^M - g_0} \right)^{\gamma} \cdot \ln \left(\frac{D_1^M - g_1}{D_0^M - g_0} \right)$$

Thus, an increase in the relative risk aversion γ increases the real interest rate if and only if $D_1^M - g_1 > D_0^M - g_0$.

Exercise 6.3.2 *With R_0 as the relative price of period-0 goods to period-1 goods, the elasticity of substitution between the two goods can be defined*

$$\textit{elasticity of intertemporal substitution} = -\frac{d \ln(c_0/c_1)}{d \ln R_0}.$$

Show that with the CRRA function the elasticity of intertemporal substitution is $1/\gamma$.

Exercise 6.3.3 *Replace the intertemporal utility function by* $U(C) = c_0^{1/2} c_1^{1/2}$. *Find the equilibrium real interest rate and examine how it depends on the output profile and government spending profile. Show that the elasticity of intertemporal substitution is one.*

We have carried out the discussion so far using the gross real interest rate. For the net real interest rate r_0, it is worthwhile noting that it is not necessarily positive. Only when the bond is sold at a discount, i.e., $B_0 < 1$, does the real interest rate exceed zero. In light of the pricing equation for bonds, $r_0 > 0$ only if the subjective discount rate $\frac{1}{\beta} - 1$ is sufficiently high, or the output for private consumption grows sufficiently fast over time, or both.

6.4 Asset Pricing Formula

From the pricing equations for bonds and shares we have found that rates of return to the two assets must be the same in equilibrium. Such results can be generalized. In this section, we intend to conduct the following analyses:

- Unify the asset pricing equations for all assets;
- Find the relationship between rates of return to different assets.

Let us begin the analysis by rewriting the asset pricing equations (6.18) and (6.19). Dividing through (6.18) by S_0 and using the definition (6.27), we have

$$R_{s0} \frac{\beta u'(D_1^M - g_1)}{u'(D_0^M - g_0)} = 1. \tag{6.28}$$

Conducting a similar exercise on (6.19) yields

$$R_0 \frac{\beta u'(D_1^M - g_1)}{u'(D_0^M - g_0)} = 1. \tag{6.29}$$

The two pricing equations (6.28) and (6.29) have exactly the same format! Therefore, we unify the pricing equation as

$$R^i \frac{\beta u'(D_1^M - g_1)}{u'(D_0^M - g_0)} = 1 \qquad \forall \qquad i = M, 1, 2, ..., N, B. \tag{6.30}$$

The assets considered above all have certain payoffs in period 1. The bond is paid one unit of good and the share and other contingent claims are paid D_1 dividends plus its future value S_1. However, an asset may have uncertain future payoffs. We will extend the above two-period certainty model in the next section to include uncertainties.

We can rewrite (6.30) by dividing $\beta u'(c_1)/u'(c_0)$ on both sides. The procedure generates the following strong result:

An alternative form of the asset pricing formula: *All* assets in the two-period economy must have the same rate of return that is given by

$$R^i = \frac{u'(D_0^M - g_0)}{\beta u'(D_1^M - g_1)}. \tag{6.31}$$

Although this result is equivalent to (6.30) in the current economy, it does not permit extensions to more complicated economies as easily as (6.30) does. Nevertheless, rewriting the asset pricing equation in the above form gives an alternative explanation to the pricing equation. Recall that the marginal rate of substitution between period-0 goods and period-1 goods is $u'(c_0)/[\beta u'(c_1)]$, the right-hand side of (6.31). The rate of return to an asset can be interpreted as the marginal rate of transformation between the two types of goods, since by holding the asset an agent can transform consumption from period 0 to period 1. The pricing equation (6.31) requires that the marginal rate of transformation and the marginal rate of substitution be equal to each other. Since the marginal rate of substitution depends on the consumption levels in the two periods, which are the same regardless of the asset purchased, all assets must offer the same expected rate of return.

We supply two cautionary notes to the strong result in (6.30).

(i) First, and perhaps apparently, saying that different assets have the same expected rate of return does not imply that they have the same price. This is because different assets can have different payoffs. In the simple two-period economy, the higher the payoff, the higher the price of the asset. Since the *rate* of return is the ratio between the payoff and the price of the asset, an asset that has both a high payoff and a high price can have the same rate of return as another asset that has both a low payoff and a low price.

(ii) Second, the result of equal rates of return to different assets is derived here from an economy where there is no randomness in output and consumption. When such variables are random and correlated differently with returns to different assets, there can be differences between rates of returns to those assets. For example, the rate of return to stocks is typically higher than that to bonds in reality and the difference might be due to the higher risk involved in stocks.

However, if an asset has a random payoff which is not correlated with the aggregate output, the pricing formula (6.30) can be generalized to

$$E(R^i) = \frac{u'(D_0^M - g_0)}{\beta u'(D_1^M - g_1)}. \tag{6.32}$$

Let us take the following lottery as an example and analyze its price: the lottery pays 1 unit of goods in period 1 with a probability $\alpha \in (0, 1)$. That is, the holder of the lottery gets 1 unit of good in period 1 with probability α and 0 unit of goods with probability $1 - \alpha$. The expected return to the lottery is

$$\alpha \cdot 1 + (1 - \alpha) \cdot 0 = \alpha.$$

Suppose the lottery is sold in period 0 at price B_L. The rate of return to the lottery, denoted $E(R_L)$, is

$$E(R_L) = \frac{\alpha}{B_L}.$$

We show that $E(R_L)$ obeys the general pricing equation (6.32).

Consider the marginal cost and marginal benefit analysis of purchasing the lottery. The marginal cost of the lottery is B_L units of goods in period 0, which can be translated into $B_L\lambda_0$ of period-0 utility. With probability α, the lottery pays 1 unit of goods which yields a marginal utility λ_1 in period 1; with probability $1-\alpha$ the lottery pays 0 unit of goods which, of course, yields zero utility. The expected marginal benefit of the lottery in terms of period-1 utility is

$$\alpha \cdot \lambda_1 + (1-\alpha)\cdot 0 = \alpha\lambda_1.$$

Discounted into period 0, the marginal benefit of the lottery in terms of period-0 utility is $\beta\alpha\lambda_1$. If the lottery is purchased by agents in the competitive economy, the marginal cost and marginal benefit associated with the purchase must be equal to each other. Therefore, $B_L\lambda_0 = \beta\alpha\lambda_1$. Substituting the solutions for (λ_0, λ_1) obtained before, we have the price of the lottery:

$$B_L = \beta\alpha\frac{u'(D_1^M - g_1)}{u'(D_0^M - g_0)}. \tag{6.33}$$

Dividing through the equation by B_L and using the relation $E(R_L) = \alpha/B_L$, we can then easily show that the expected rate of return to the lottery, $E(R_L)$, obeys the general pricing formula (6.32).

Exercise 6.4.1 *This exercise derives (6.33) formally from the maximization problem. Assume that each agent is endowed with zero unit of the lottery but can issue or purchase any quantity of such lottery in the competitive market.*
(i) Modify the agent's budget constraints in the two periods to include the possibility of lottery purchase and payoffs.
(ii) From the budget constraints in (i) derive the intertemporal budget constraint.
(iii) Derive the optimality conditions.
(iv) Define the competitive equilibrium and establish (6.33).

The following exercise further illustrates the generality of the pricing formula by pricing a secondary security whose payoffs depend on the payoffs to the lottery.

Exercise 6.4.2 *Consider the following security whose payoff in period* 1 *depends on the payoff to the lottery examined above. The security pays A (>0) units of goods if the payoff on each unit of the lottery exceeds* $1/2$ *units of goods and* $-A$ *units of goods if the payoff on the lottery does not exceed* $1/2$ *units of goods.*
(i) Calculate the expected rate of return to this security, denoted ER_d;
(ii) Use the marginal cost-benefit analysis to find the price of the security, denoted B_d;

(iii) Show that ER_d obeys (6.30);
(iv) Show that the security has a negative price when $\alpha < 1/2$.
(v) Suppose $\alpha > 1/2$. If the lottery gets repaid with a higher probability than before, how would the price ratio between the security and the lottery, B_d/B_L, change?

For assets whose payoffs are correlated with the aggregate output, the pricing equation (6.30) cannot be easily extended. We need to formally set up the model and derive the pricing equation. Such exercise is presented in the next section.

6.5 Asset Pricing with Uncertainty

Assume that there are I possible states at $t = 1$. Each state i occurs with probability p_i. The payoffs from the tree (or the market portfolio) and other contingent claims in each state i is denoted by a vector D_1^i. The agent's utility function now is

$$U(C) = u(c_0) + \beta E\left(u(c_1)\right) = u(c_0) + \beta \sum_{i=1}^{I} p_i u(c_1^i).$$

The agent's maximization problem becomes

$$\max_{(c_0, c_1, \theta_1, \theta_2, \theta_{B1})} u(c_0) + \beta \sum_{i=1}^{I} p_i u(c_1^i)$$

s.t.

$$c_0 + \tau_0 + S_0'\theta_1 + B_0\theta_{B1} \leq (S_0' + D_0')\theta_0 + \theta_{B0}.$$
$$c_1^i + \tau_1 + S_1^{i'}\theta_2^i \leq (S_1^{i'} + D_1^{i'})\theta_1 + \theta_{B1}, \qquad \forall \qquad i = 1, 2, ..., I.$$

Based on the analysis in the previous section, we know that the agent will not hold any asset by the end of period 1 and $\theta_2 = 0$. Also, the above two budget constraints will hold in equality. Therefore, we can form the following Lagrangian:

$$\begin{aligned} L &= u(\tau_0 + S_0'\theta_1 + B_0\theta_{B1} - (S_0' + D_0')\theta_0 - \theta_{B0}) \\ &\quad + \beta \sum_{i=1}^{I} p_i u(\tau_1' - (S_1^{i'} + D_1^{i'})\theta_1 - \theta_{B1}). \end{aligned}$$

The first order conditions are:

$$\frac{\partial L}{\partial \theta_1} = u'(c_0)S_0 - \beta \sum_{i=1}^{I} p_i u'(c_1^i)(S_1^i + D_1^i) = 0.$$

$$\frac{\partial L}{\partial \theta_{B1}} = u'(c_0)B_0 - \beta \sum_{i=1}^{I} p_i u'(c_1^i) = 0.$$

Rearrange the above two first order conditions, we have

$$S_0 = \sum_{i=1}^{I} \frac{\beta p_i u'(c_1^i)}{u'(c_0)}(S_1^i + D_1^i) = E\left(\frac{\beta u'(c_1)}{u'(c_0)}(S_1 + D_1)\right), \tag{6.34}$$

and

$$B_0 = \sum_{i=1}^{I} \frac{\beta p_i u'(c_1^i)}{u'(c_0)} = E\left(\frac{\beta u'(c_1)}{u'(c_0)}\right). \tag{6.35}$$

Exercise 6.5.1 *Recall the no-arbitrage valuation discussed in Chapter 3. Show* $\pi = [\frac{\beta p_1 u'(c_1^1)}{u'(c_0)} \quad \frac{\beta p_2 u'(c_1^2)}{u'(c_0)} \quad \cdots \quad \frac{\beta p_I u'(c_1^I)}{u'(c_0)}]$ *is the state price vector which does not permit arbitrage.*

The above exercise indicates that utility maximization is equivalent to no-arbitrage condition. **We can use the marginal utility benefit and cost analysis to prove the equivalence. To state differently, in a competitive equilibrium, the asset prices are such that there exists no arbitrage.**

Now we turn to the derivation of the famous CAPM based on the two pricing equations (6.34) and (6.35). Define the interest rate and the asset returns as

$$\begin{aligned} 1+r &= \frac{1}{B_0}, \\ 1+r_M &= \frac{S_1^M + D_1^M}{S_0^M}, \\ 1+r_n &= \frac{S_1^n + D_1^n}{S_0^n} \qquad \forall \qquad n = 1, 2, ..., N. \end{aligned}$$

Since the price of the market portfolio

$$\begin{aligned} S_0^M &= E\left(\frac{\beta u'(c_1)}{u'(c_0)}(S_1^M + D_1^M)\right) \\ 1 &= E\left(\frac{\beta u'(c_1)}{u'(c_0)}(1+r_M)\right) = E\left(\frac{\beta u'(c_1)}{u'(c_0)}\right)E(1+r_M) + cov\left(\frac{\beta u'(c_1)}{u'(c_0)}, r_M\right) \\ &= \frac{E(1+r_M)}{1+r} + cov\left(\frac{\beta u'(c_1)}{u'(c_0)}, r_M\right). \end{aligned}$$

or

$$E(r_M) = r - (1+r)cov\left(\frac{\beta u'(c_1)}{u'(c_0)}, r_M\right).$$

Similarly,

$$E(r_n) = r - (1+r)cov\left(\frac{\beta u'(c_1)}{u'(c_0)}, r_n\right).$$

Therefore,

$$E(r_n) - r = \frac{cov\left(\frac{\beta u'(c_1)}{u'(c_0)}, r_n\right)}{cov\left(\frac{\beta u'(c_1)}{u'(c_0)}, r_M\right)}\left(E(r_M) - r\right). \tag{6.36}$$

This equation (6.36) is the consumption-based CAPM. In equilibrium, if there is no government, the consumption becomes $c_0 = D_0^M$ and $c_1 = D_1^M$. Then the above pricing equation can be further written as

$$E(r_n) - r = \frac{cov\left(\frac{\beta u'(D_1^M)}{u'(D_0^M)}, r_n\right)}{cov\left(\frac{\beta u'(D_1^M)}{u'(D_0^M)}, r_M\right)}\left(E(r_M) - r\right) = \frac{cov\left(u'(D_1^M), r_n\right)}{cov\left(u'(D_1^M), r_M\right)}\left(E(r_M) - r\right).$$

However, our usual CAPM is expressed as

$$E(r_n) - r = \frac{cov\left(r_M, r_n\right)}{var\left(r_M\right)}\left(E(r_M) - r\right).$$

Exercise 6.5.2 *Under what conditions the consumption-based CAPM is identical to the usual CAPM? (hint: look for specific utility function or specific distribution for the dividends and returns.)*

Summary 1 *In this chapter we have covered the following main issues:*
(i) Bond financing and Ricardian equivalence;
(ii) Intertemporal budget constraints;
(iii) The real interest rate and its determinants;
(iv) A general asset pricing formula.

Chapter 7

ECONOMIES WITH LONGER HORIZONS

The two-period model has generated interesting results such as the Ricardian equivalence and asset pricing formula. The two-period horizon, however, is also quite restrictive. One restriction is that all bonds are one-period bonds – a government must pay all its bonds one period after it issues them. The short maturity makes it difficult to examine the term structure of interest rates. Allowing for a longer horizon also permits us to analyze the relationship between long-term interest rates and short-term interest rates. In addition, we can also examine the Ricardian equivalence. Would it continue to hold when the government can issue long-term bonds to finance tax cuts? An intuitive answer is Yes, but it is useful to confirm the answer with explicit modelling of a long horizon. Below we will first extend the two-period economy to a three-period economy and then into an infinite horizon economy.

7.1 A Three-Period Economy

7.1.1 Description of the Environment

Let t denote time, $t = 0, 1, 2$. As in the two-period economy, all private agents are identical and the size of the population is normalized to one. Each agent is endowed with one tree, which yields output D_t^M in period t. Goods are perishable across periods but trees are perfectly durable. The representative agent's ownership of trees at the beginning of period t is denoted θ_t, with $\theta_0 = 1$.

To simplify notation, we omit the N contingent claims in this chapter. Adding them does not change the nature of the analysis, only makes the notation complicated since we have to deal with dividend vector, share holding vector and price vector.

The agent's consumption in period t is denoted c_t and the consumption program during his lifetime is $C = (c_0, c_1, c_2)$. The agent's intertemporal utility is represented by the following function:

$$U(C) = u(c_0) + \beta u(c_1) + \beta^2 u(c_2).$$

As before, the intertemporal utility function is strictly increasing and concave in the consumption program. The agent maximizes the intertemporal utility subject to the intertemporal budget constraint.

Before specifying the agent's budget constraint, let us describe the government's actions as below:

- Spending: (g_0, g_1, g_2). In each period t, the government spends g_t on goods. The amount is exogenously set.
- Taxes: (τ_1, τ_2, τ_3). In each period t, the government collects a lump-sum tax τ_t. The government can decide how much to collect in conjunction with bond financing.
- One-period bonds: $(\bar{\theta}_{B(0,1)}, \bar{\theta}_{B(1,2)})$. In period t $(= 0, 1)$, the government can issue $\bar{\theta}_{B(t,t+1)}$ amount of bonds that are redeemed in period $t+1$ with one unit of goods. Bonds are sold competitively in the market for a price $B(t, t+1)$ in period t $(= 0, 1)$. The government is unable to raise revenue by issuing bonds in the third period $(t = 2)$ because agents will not buy them unless the price is zero.*
- Two-period bonds: $\bar{\theta}_{B(0,2)}$. In period 0 the government can issue a two-period bond which pays one unit of goods in period 2. The price of the two-period bond is $B(0, 2)$ in period 0. The government is unable to issue two-period bonds in period 1 since the maturity is beyond agents' life span.

Let the agent's purchase of one-period bonds be $\theta_{B(0,1)}$ and the purchase of two-period bonds be $\theta_{B(0,2)}$. As before, let the price of shares be S_0^M, S_1^M and S_2^M in the three periods, respectively. In period 0, the agent's wealth is $(S_0^M + D_0^M)\theta_0$ (note that we assume that the government's inherited debt is zero). The agent's possible expenditure includes consumption c_0, tax payment τ_0, acquisition of shares θ_1, purchase of one-period bond $\theta_{B(0,1)}$ and purchase of two-period bonds $\theta_{B(0,2)}$. Thus,

$$\begin{aligned} & \text{the agent's total expenditure in period 0} \\ = \; & c_0 + \tau_0 + S_0^M\theta_1 + B(0,1)\theta_{B(0,1)} + B(0,2)\theta_{B(0,2)}. \end{aligned}$$

The agent's budget constraint in period 0 is

$$c_0 + \tau_0 + S_0^M\theta_1 + B(0,1)\theta_{B(0,1)} + B(0,2)\theta_{B(0,2)} \le (S_0^M + D_0^M)\theta_0. \tag{7.1}$$

In period 1, the agent collects dividends from the shares he purchased in period 0 and gets payment on the one-period bonds. Since the two-period bonds purchased in period 0 have not matured, the agent does not receive any payment from such bonds. His total wealth at the beginning of period 1 is $(S_1^M + D_1^M)\theta_1 + \theta_{B(0,1)}$ and his budget constraint in period 1 is

$$c_1 + \tau_1 + S_1^M\theta_2 + B(1,2)\theta_{B(1,2)} \le (S_1^M + D_1^M)\theta_1 + \theta_{B(0,1)}. \tag{7.2}$$

In period 2, the two-period bonds purchased in period 0 are matured, which pay one unit of goods per bond. The agent's wealth at the beginning of period 2 is $(S_2^M + D_2^M)\theta_2 + \theta_{B(1,2)} + \theta_{B(0,2)}$. The agent's budget constraint is

$$c_2 + \tau_2 + S_2^M\theta_3 \le (S_2^M + D_2^M)\theta_2 + \theta_{B(1,2)} + \theta_{B(0,2)}. \tag{7.3}$$

*The government's inherited debt at the beginning of period 0 is assumed to be zero.

To obtain the intertemporal budget constraint from the budget constraints in the three periods, recall that the bond price $B(0,1)$ is the relative price of period-1 goods to period-0 goods. Similarly, the bond price $B(1,2)$ is the relative price of period-2 goods to period-1 goods. Thus, the relative price of period-2 goods to period-0 goods is $B(0,1)B(1,2)$. We can use these relative prices to convert the budget constraints in periods 1 and 2 into period-0 values. That is, multiply the budget constraint in period 1 by $B(0,1)$ and the budget constraint in period 2 by $B(0,1)B(1,2)$. We have:

$$B(0,1)(c_1 + \tau_1 + S_1^M\theta_2 + B(1,2)\theta_{B(1,2)}) \leq B(0,1)(S_1^M + D_1^M)\theta_1 + B(0,1)\theta_{B(0,1)},$$

$$B(0,1)B(1,2)(c_2 + \tau_2 + S_2^M\theta_3) \leq B(0,1)B(1,2)[(S_2^M + D_2^M)\theta_2 + \theta_{B(1,2)} + \theta_{B(0,2)}].$$

Add these two constraints with the budget constraint in period 0. Cancelling out the terms $B(0,1)\theta_{B(0,1)}$ and $B(0,1)B(1,2)\theta_{B(1,2)}$ from the two sides of the constraint and re-arranging terms, we have

$$\begin{aligned}
&c_0 + B(0,1)c_1 + B(0,1)B(1,2)c_2 \\
&\leq (S_0^M + D_0^M)\theta_0 - (\tau_0 + B(0,1)\tau_1 + B(0,1)B(1,2)\tau_2) \\
&-B(0,1)B(1,2)S_2^M\theta_3 + [B(0,1)(S_1^M + D_1^M) - S_0^M]\theta_1 \\
&+B(0,1)[B(1,2)(S_2^M + D_2^M) - S_1^M]\theta_2 \\
&+[B(0,1)B(1,2) - B(0,2)]\theta_{B(0,2)}.
\end{aligned} \tag{7.4}$$

This constraint looks complicated but the last four terms on the right-hand side are all zero as a result of maximization, as we will see immediately.

The agent's maximization problem is to choose a consumption program (c_0, c_1, c_2), a share holding strategy $(\theta_1, \theta_2, \theta_3)$ and purchases of bonds $(\theta_{B(0,1)}, \theta_{B(1,2)}, \theta_{B(0,2)})$ to solve

$$\max \quad U(C) = u(c_0) + \beta u(c_1) + \beta^2 u(c_2) \qquad \text{s.t.} \quad (7.4).$$

Note that we have suppressed the non-negativity constraints on consumption. Let the Lagrangian multiplier of the intertemporal budget constraint be λ_0 (the marginal value of wealth at the beginning of period 0). The optimality conditions are displayed below:

$$\text{for } c_0\text{: } u'(c_0) = \lambda_0; \tag{7.5}$$

$$\text{for } c_1\text{: } \beta u'(c_1) = B(0,1)\lambda_0; \tag{7.6}$$

$$\text{for } c_2\text{: } \beta^2 u'(c_2) = B(0,1)B(1,2)\lambda_0; \tag{7.7}$$

$$\text{for } \theta_1\text{: } B(0,1)(S_1^M + D_1^M) \leq S_0^M, \quad \text{“}=\text{” if } \theta_1 > 0;$$

$$\text{for } \theta_2\text{: } B(0,1)B(1,2)(S_2^M + D_2^M) \leq B(0,1)S_1^M, \quad \text{“}=\text{” if } \theta_2 > 0;$$

$$\text{for } \theta_3\text{: } B(0,1)B(1,2)S_2^M \geq 0, \quad \text{“}=\text{” if } \theta_3 > 0;$$

for $\theta_{B(0,2)}$: $B(0,1)B(1,2) \le B(0,2)$, "$=$" if $\theta_{B(0,2)} > 0$.

The conditions for consumption are easy to explain. The marginal value of wealth at the beginning of periods 0, 1 and 2 is λ_0, $B(0,1)\lambda_0$ and $B(0,1)B(1,2)\lambda_0$, respectively. For consumption in each period t, the marginal utility of consumption must be equal to the marginal cost of consumption which is represented by the marginal value of wealth at the beginning of period t.

The conditions for share holdings are now familiar conditions. The condition for θ_1 states that the rate of return to the one-period bond purchased in period 0 must be at least as high as the rate to shares. The condition for θ_2 states a similar requirement for period 1. The condition for θ_3 states that the agent will purchase shares in the last period of life only if the share price S_2^M is zero. In equilibrium, as the market clearing conditions require, all quantities $(\theta_1, \theta_2, \theta_3)$ are positive and so the three conditions hold with equality:

$$B(0,1)(S_1^M + D_1^M) = S_0^M; \tag{7.8}$$

$$B(1,2)(S_2^M + D_2^M) = S_1^M; \tag{7.9}$$

$$S_2^M = 0. \tag{7.10}$$

The condition for $\theta_{B(0,2)}$ needs some explanation which we will supply two subsections later. If the two-period bonds are held by the private sector, as the competitive equilibrium requires, $\theta_{B(0,2)} = \overline{\theta}_{B(0,2)}$ and so the condition for $\theta_{B(0,2)}$ holds with equality:

$$B(0,2) = B(0,1)B(1,2). \tag{7.11}$$

With the conditions (7.8) — (7.11), the agent's intertemporal budget constraint (7.4) can be simplified as

$$\begin{aligned} &c_0 + B(0,1)c_1 + B(0,1)B(1,2)c_2 \\ &\le (S_0^M + D_0^M)\theta_0 - (\tau_0 + B(0,1)\tau_1 + B(0,1)B(1,2)\tau_2). \end{aligned} \tag{7.12}$$

Namely, the present value of consumption and tax liabilities cannot exceed the agent's wealth.

7.1.2 An Extension of the Ricardian Equivalence

In the last chapter we have shown that issuing one-period bonds to finance a tax cut does not change equilibrium values of consumption, share holdings and asset prices. The introduction of two-periods bonds allows us to extend the Ricardian equivalence to richer government financing methods. Let us start with the government's budget constraints in the three periods:

$$g_0 \le \tau_0 + B(0,1)\overline{\theta}_{B(0,1)} + B(0,2)\overline{\theta}_{B(0,2)}; \tag{7.13}$$

$$g_1 \le \tau_1 - \overline{\theta}_{B(0,1)} + B(1,2)\overline{\theta}_{B(1,2)}; \tag{7.14}$$

$$g_2 \le \tau_2 - \overline{\theta}_{B(1,2)} - \overline{\theta}_{B(0,2)}. \tag{7.15}$$

As before we assume that these constraints hold as equality.

Suppose that the government wants to cut tax in period 0 by Δ. The government does not have to finance the tax by issuing one-period bonds which must be repaid in period 1. Rather, the government can defer the payments to period 2 by issuing more two-period bonds, as described below:

- Using long-term bonds to finance tax cuts: To finance a tax cut Δ in period 0, the government increases the amount of two-period bonds issued in period 0 by Δ_1. The additional bonds are redeemed in period 2 by collecting Δ_2 more taxes. The amount of one-period bonds issued in periods 0 and 1 and the tax in period 1 are maintained the same as before, i.e., $(\overline{\theta}_{B(0,1)}, \overline{\theta}_{B(1,2)}, \tau_1)$ are unchanged.

We first solve for the amounts (Δ_1, Δ_2) and then show that the financing method does not change the government's present value of revenue, provided that the government budget constraints hold with equality. In particular, the present value of government revenue is equal to the present value of taxes. Since only the present value of taxes enters the agent's intertemporal budget constraint (7.12), the financing method will not affect the agent's optimal choices.

To find (Δ_1, Δ_2), start with period 0. To maintain the government budget constraint in period 0 with equality, the increase in two-period bonds must satisfy

$$g_0 = (\tau_0 - \Delta) + B(0,1)\overline{\theta}_{B(0,1)} + B(0,2)[\overline{\theta}_{B(0,2)} + \Delta_1]. \tag{7.16}$$

Note that the amount of one-period bonds issued in period 0 is assumed to be the same as before (but see later discussion and exercise). Re-arrange the above constraint as

$$B(0,2)\Delta_1 - \Delta = g_0 - [\tau_0 + B(0,1)\overline{\theta}_{B(0,1)} + B(0,2)\overline{\theta}_{B(0,2)}].$$

The right-hand side of this equation is zero, since the original spending and policies satisfy the government budget constraint in period 0. Thus, to finance the tax cut Δ, the government must increase the amount of two-period bonds by $\Delta/B(0,2)$.

The government budget constraint in period 1 is unchanged because the tax τ_1, the redemption on previous bonds $\overline{\theta}_{B(0,1)}$and new bonds $\overline{\theta}_{B(1,2)}$ are unchanged under our assumption. That is, (7.14) still holds with equality.

In period 2, the additional tax revenue Δ_2 must be enough to maintain the budget constraint:

$$g_2 = (\tau_2 + \Delta_2) - \overline{\theta}_{B(1,2)} - [\overline{\theta}_{B(0,2)} + \Delta_1]. \tag{7.17}$$

Re-arranging terms, we have

$$\Delta_2 - \Delta_1 = g_2 - [\tau_2 - \overline{\theta}_{B(1,2)} - \overline{\theta}_{B(0,2)}].$$

The right-hand side is zero, because the original policies satisfy the government budget constraint in period 2. Thus, to redeem the additional two-period bonds issued in period 0, the government must increase the tax in period 2 by $\Delta/B(0,2)$.

The government budget constraints in the three periods are (7.16), the equality form of (7.14) and (7.17). Transform these constraints by multiplying the budget constraint in period 1 by $B(0,1)$ and the constraint in period 2 by $B(0,1)B(1,2)$. Adding up these transformed constraints with (7.16) yields

$$\begin{aligned}
& g_0 + B(0,1)g_1 + B(0,1)B(1,2)g_2 \\
= \quad & (\tau_0 + B(0,1)\tau_1 + B(0,1)B(1,2)\tau_2) + (B(0,2)\Delta_1 - \Delta) \\
& + B(0,1)B(1,2)(\Delta_2 - \Delta_1) + [B(0,2) - B(0,1)B(1,2)]\bar{\theta}_{B(0,2)}.
\end{aligned}$$

As shown above, $B(0,2)\Delta_1 = \Delta$ and $\Delta_2 = \Delta_1$. The second and third terms of the right-hand side of the above equation are zero. Since bond prices satisfy (7.11), the last term on the right-hand side is zero. The government's intertemporal budget constraint becomes

$$g_0 + B(0,1)g_1 + B(0,1)B(1,2)g_2 = \tau_0 + B(0,1)\tau_1 + B(0,1)B(1,2)\tau_2. \tag{7.18}$$

That is, the present value of spending equals the present value of tax revenue.

The present value of government revenue consists solely of the tax revenue. Financing the tax cut in period 0 by issuing two-period bonds delays the tax to period 2. Each unit of tax cut in period 0 requires more than one unit of tax increase in period 2. Precisely, for each unit of tax cut in period 0, the government must issue $1/B(0,2)$ units of two-period bonds which require an increase in period-2 tax by the same amount. The present value of the tax increase in period 2 is $B(0,1)B(1,2) \cdot 1/B(0,2) = 1$, which exactly equals the value of tax cut in period 0. Given bond prices, the present value of taxes are unchanged by the government's financing method and so the agent's choices of $(c_t, \theta_{t+1})_{t=0}^2$ are unchanged.

To establish the Ricardian equivalence, we need to show that asset prices are unchanged by the government's financing method. This part is easy. Substituting λ_0 from (7.5) into (7.6) and using goods markets clearing conditions $c_0 = D_0^M - g_0$ and $c_1 = D_1^M - g_1$, we have

$$B(0,1) = \beta \frac{u'(D_1^M - g_1)}{u'(D_0^M - g_0)}. \tag{7.19}$$

Since (D_0^M, D_1^M, g_0, g_1) do not change with the financing method, the price $B(0,1)$ is unchanged. Similarly, from (7.6) — (7.7) and goods markets clearing conditions we can show that the bond price $B(1,2)$ is unchanged by the government's financing method:

$$B(1,2) = \beta \frac{u'(D_2^M - g_2)}{u'(D_1^M - g_1)}. \tag{7.20}$$

The price of the two-period bond is

$$B(0,2) = B(0,1)B(1,2) = \beta^2 \frac{u'(D_2^M - g_2)}{u'(D_0^M - g_0)}, \tag{7.21}$$

which is also independent of the government's financing method.

Similarly, from the optimality conditions (7.8) – (7.10) we can show that share prices are unaffected by the government's financing method. In particular, $S_2^M = 0$ as before and

$$S_t^M = \beta(S_{t+1}^M + D_{t+1}^M)\frac{u'(D_{t+1}^M - g_{t+1})}{u'(D_t^M - g_t)}, \qquad \text{for } t = 0, 1. \tag{7.22}$$

The above procedure establishes the Ricardian equivalence with respect to two-period bond financing method. To be transparent, we have imposed the restriction that the government does not change the tax in period 1 or the amount of one-period bonds issued in periods 0 and 1. The restriction can be relaxed, as shown below in the exercises.

Exercise 7.1.1 *Consider the following policy changes. The government issues δ_1 units of additional one-period bonds in period 0, δ_2 units of additional one-period bonds in period 1, and A units of additional two-period bonds in period 0. (All these changes can be negative in principle). Suppose that the government wants to keep the taxes in periods 1 and 2 unchanged.*
(i) Show that $A = -\delta_2$ and $\delta_1 = B(1,2)\delta_2$.
(ii) Show that the tax in period 0 should be unchanged in order to maintain the government budget constraint as equality.

Exercise 7.1.2 *Consider a more complicated policy change. The government cuts the tax in period 0 by Δ. To finance the cut, the government increases one-period bonds by δ_1 and δ_2 in period 0 and period 1 respectively, raises taxes by Δ_1 and Δ_2 in period 1 and period 2 respectively, and issues A units of additional two-period bonds in period 0. (Again all these changes can be negative in principle.)*
(i) Show that $\Delta_1 = \delta_1 - B(1,2)\delta_2$ and $\Delta_2 = \delta_2 + A$.
(ii) Show that the present value of taxes must be unchanged (for given bond prices), i.e., $-\Delta + B(0,1)\Delta_1 + B(0,1)B(1,2)\Delta_2 = 0$.

In light of these exercises, we can extend the Ricardian equivalence to the following:

Ricardian Equivalence: Equilibrium values of consumption, share holdings and asset prices are independent of the government's tax-bond financing method and the maturities of the bonds, provided that the government's budget constraints hold as equality with a given spending profile (g_0, g_1, g_2).

7.1.3 Bond Maturity and the Term Structure of Interest

The illustration of the Ricardian equivalence in the previous subsection has frequently used the equilibrium relation between the price of two-period bonds and prices of one-period bonds. This relation, given by (7.11), is a result of no-arbitrage between different bonds. Consider the following two strategies for the agent:

- Strategy #1: use one unit of good to buy one-period bonds in period 0; when the bonds mature in period 1, use the payment to buy one-period bonds in period 1, hold them to maturity (period 2) and redeem them.

- Strategy #2: use one unit of good to buy two-period bonds in period 0; hold the bonds to maturity (period 2) and redeem them.

The cost of the two strategies is the same – the investment is one unit of good in period 0, which has a marginal utility $u'(c_0)$. If the agent holds both one-period and two-period bonds, the marginal utility of the return in period 2 must be the same for both strategies. If strategy #1 yielded less than strategy #2, the agent would invest only in strategy #2. If everyone did the same (as all agents are indeed the same), there would no demand for one-period bonds. This could not be an equilibrium situation, because the zero demand for one-period bonds would press their prices down and hence push their return up until the return equals that to strategy #2. The equality between the return to the two strategies can be called an arbitrage-free condition between one-period and two-period bonds – it ensures that an arbitrage between the two yields no profit.

We now show that the arbitrage-free condition is (7.11). Consider strategy #1. Since the price of one-period bonds in period 0 is $B(0,1)$, the investment of one unit of goods can buy $1/B(0,1)$ units of such bonds. When the bonds mature in period 1, each bond pays one unit of goods and so the investment yields $1/B(0,1)$ units of goods in period 1. Since these payments are re-invested to purchase one-period bonds in period 1 at a price $B(1,2)$, the number of one-period bonds purchased is $\frac{1}{B(0,1)}/B(1,2) = \frac{1}{B(0,1)B(1,2)}$. When these bonds mature in period 2, the total return is $\frac{1}{B(0,1)B(1,2)}$. Now examine strategy #2. Since the price of two-period bonds in period 0 is $B(0,2)$, the investment of one unit of goods can purchase $1/B(0,2)$ units of two-period bonds. These bonds are held to maturity which yield $1/B(0,2)$ units of goods in period 2. The two investment strategies must yield the same return and so $1/B(0,2) = 1/(B(0,1)B(1,2))$, i.e., $B(0,2) = B(0,1)B(1,2)$.

The above explanation of the arbitrage-free condition assumes that the holder of two-period bonds does not sell the two-period bonds before it matures. When agents are allowed to sell the long-term bonds before it matures, the agent might have another investment strategy:

- Strategy #3: use one unit of good to buy two-period bonds in period 0; hold the bonds to period 1, re-sell them in the market and use the receipt to buy one-period bonds.

Intuition suggests that this strategy should not make any profit over the other two strategies, as shown in the following exercise:

Exercise 7.1.3 *Show that the re-sale price of two-period bonds in period* 1 *is* $B(0,2)/B(0,1)$ *and so strategy #3 generates a return* $1/(B(0,1)B(1,2))$.

We can also express the arbitrage-free condition (7.11) in terms of rates of return. Using the definition in the previous chapter, the rate of return from period t to period $t+1$ ($t = 0,1$) is

$$R(t,t+1) = \frac{1}{B(t,t+1)}. \tag{7.23}$$

These are short-term interest rates. The long-term interest rate in the current case is the rate of return to the long-term investment strategy #2. Denote such gross rate of return by $R(0, 2)$. Then

$$R(0,2) = \frac{1}{B(0,2)}. \tag{7.24}$$

The arbitrage-free condition (7.11) then implies

$$R(0,2) = R(0,1)R(1,2). \tag{7.25}$$

The two-period rate of return is equal to the product of one-period rates of return. This is called the *term structure of interest rates*: it connects rates of return to bonds that differ in maturity terms.

7.1.4 Asset Pricing Equation

The asset pricing equation is a direct extension of the equation in the last chapter. From the optimality results (7.19) and (7.20), it is apparent that the rate of return to one-period bond between periods t and $t+1$ obeys the following pricing equation:

$$\beta R(t,t+1)\frac{u'(D_{t+1}^M - g_{t+1})}{u'(D_t^M - g_t)} = 1, \text{ for } t = 0, 1.$$

With (7.24) and (7.21), the gross rate of return to two-period bonds obeys

$$\beta^2 R(0,2)\frac{u'(D_2^M - g_2)}{u'(D_0^M - g_0)} = 1.$$

Both pricing equations require that the marginal rate of substitution and the marginal rate of transformation between selected dates be equal to each other. In the formula for one-period bonds, marginal rates of substitution and transformation are between dates t and $t+1$. In the formula for two-period bonds, the selected dates are dates 0 and 2.

Formally, let us define the marginal rate of substitution between dates t and $t+k$ as

$$MRS_{t,t+k} = \frac{u'(c_t)}{\beta^k u'(c_{t+k})}.$$

Suppose an asset issued at t matures at $t+k$ and yields a gross expected rate of return $ER(t, t+k)$ between periods t and $t+k$, where $t = 0, 1$, $k = 1, 2$ and $t+k \leq 2$. Then $ER(t, t+k) = MRS_{t,t+k}$. Note that

$$MRS_{t,t+k} = MRS_{t,t+1} \cdot MRS_{t+1,t+2} \cdot \cdots \cdot MRS_{t+k-1,t+k}.$$

In particular, $MRS_{0,2} = MRS_{0,1} \cdot MRS_{1,2}$, which implies $ER(t, t+k) = R(t, t+1)R(t+1, t+2) \cdots R(t+k-1, t+k)$.

Exercise 7.1.4 *Suppose that there is an asset sold in period* 0 *that promises to pay one unit of one-period bond in period* 1 *(which in turn will pay one unit of goods in period* 2*). Let the price of such asset be* q_{b0}*. Use the marginal cost-benefit analysis to show that the rate of return to the asset between periods* 0 *and* 1*, denoted* R_b*, satisfies*

$$\beta R_b \frac{u'(D_1^M - g_1)}{u'(D_0^M - g_0)} = 1.$$

7.2 An Infinite-Horizon Economy

The main results of the three-period economy can be easily extended into an economy where each agent lives for an infinite number of periods. Since the procedure is similar to the extension from a two-period economy to a three-period economy, we keep the exercise brief and spend a large amount of effort to derive the intertemporal budget constraint.

7.2.1 Agent's Maximization Problem

Now the representative agent lives for an infinite number of periods. The agent is endowed with a tree which yields a stream of dividends, denoted $\{D_t^M\}_{t=0}^{\infty}$. The agent's intertemporal utility is

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad \beta \in (0,1).$$

The agent tries to choose a consumption program, $\{c_t\}_{t=0}^{\infty}$, share holding strategies, $\{\theta_{t+1}\}_{t=0}^{\infty}$, and purchases of one-period bonds, $\{\theta_{B(t,t+1)}\}_{t=0}^{\infty}$, to maximize the intertemporal utility. For the moment we assume that the only type of bonds available is one-period bonds. Note that the agent's choices are sequences of variables. When choosing these sequences, the agent takes sequences of taxes, $\{\tau_t\}_{t=0}^{\infty}$, share prices, $\{S_t^M\}_{t=0}^{\infty}$, and bond prices, $\{B(t,t+1)\}_{t=0}^{\infty}$, as given.

The agent's budget constraint in each period t is

$$c_t + \tau_t + S_t^M \theta_{t+1} + B(t,t+1)\theta_{B(t,t+1)} \leq (S_t^M + D_t^M)\theta_t + \theta_{B(t-1,t)}. \tag{7.26}$$

The historically inherited government debt is assumed to be zero. To obtain the intertemporal budget constraint, we need to convert period-t values into period-0 value. This is done with the intertemporal price of period-t goods, i.e., the relative price of period-t goods to period-0 goods. Recall that in the three-period economy that the intertemporal price of period-1 goods is $B(0,1)$ and the intertemporal price of period-2 goods is $B(0,1)B(1,2)$. Extending this notion into period t, we have the intertemporal price of period-t goods as

$$B(0,t) = B(0,1)B(1,2)\cdots B(t-1,t), \text{ for } t \geq 1. \tag{7.27}$$

This is also the price of a t-period bonds issued in period 0 that matures in period t. We normalize $B(0,0) = 1$, since the relative price of period-0 goods to themselves is unity. The definition of the intertemporal price implies

$$B(0,t+1) = B(0,t)B(t,t+1). \tag{7.28}$$

Re-arrange the budget constraint at t and multiply it by the intertemporal price $B(0,t)$:

$$B(0,t)(c_t+\tau_t) \leq B(0,t)[(S_t^M+D_t^M)\theta_t - S_t^M\theta_{t+1}] + B(0,t)[\theta_{B(t-1,t)} - B(t,t+1)\theta_{B(t,t+1)}].$$

Add up the constraints from $t = 0$ to ∞:

$$\begin{aligned}\sum_{t=0}^{\infty} B(0,t)[c_t + \tau_t] \quad &\leq \quad \sum_{t=0}^{\infty} B(0,t)[(S_t^M + D_t^M)\theta_t - S_t^M \theta_{t+1}] \\ &\quad + \sum_{t=0}^{\infty} B(0,t)[\theta_{B(t-1,t)} - B(t,t+1)\theta_{B(t,t+1)}].\end{aligned}$$

The left-hand side is the present value of consumption and tax liability. The right-hand side is messy so we need to simplify it. Start with the second summation on the right-hand side. With the relation (7.28), we have

$$\begin{aligned}&\textstyle\sum_{t=0}^{\infty} B(0,t)[\theta_{B(t-1,t)} - B(t,t+1)\theta_{B(t,t+1)}] \\ &= \textstyle\sum_{t=0}^{\infty} B(0,t)\theta_{B(t-1,t)} - \sum_{t=0}^{\infty} B(0,t+1)\theta_{B(t,t+1)} \\ &= \textstyle\sum_{t=0}^{\infty} B(0,t)\theta_{B(t-1,t)} - \sum_{t=1}^{\infty} B(0,t)\theta_{B(t-1,t)} \\ &= B(0,0)\theta_{B(-1,0)} = 0.\end{aligned}$$

This result says that, since the agent starts the life without any debt or credit to the government ($\theta_{B(-1,0)} = 0$), the present value of his purchase of government bonds in the lifetime must be zero. All bonds are redeemed.

The same intuition can be applied to the present value of the agent's purchase of shares, the first term on the right-hand side of the agent's intertemporal budget constraint. Since the agent starts life with one share of the tree, the present value of purchases of shares during the lifetime should not generate additional values. To confirm the intuition requires the use of equilibrium asset pricing equations which do not come into the picture until we solve the competitive equilibrium. Nevertheless, we can rewrite the term to separate the value of initial wealth, $(S_0^M + D_0^M)\theta_0$, from the possible profit from buying and selling shares during the lifetime:

$$\begin{aligned}&\sum_{t=0}^{\infty} B(0,t)[(S_t^M + D_t^M)\theta_t - S_t^M \theta_{t+1}] \\ =\quad & \sum_{t=0}^{\infty} B(0,t)(S_t^M + D_t^M)\theta_t - \sum_{t=0}^{\infty} B(0,t) S_t^M \theta_{t+1} \\ =\quad & (S_0^M + D_0^M)\theta_0 + \sum_{t=1}^{\infty} B(0,t)(S_t^M + D_t^M)\theta_t - \sum_{t=0}^{\infty} B(0,t) S_t^M \theta_{t+1} \\ =\quad & (S_0^M + D_0^M)\theta_0 + \sum_{t=0}^{\infty} B(0,t+1)(S_{t+1}^M + D_{t+1}^M)\theta_{t+1} - \sum_{t=0}^{\infty} B(0,t) S_t^M \theta_{t+1} \\ =\quad & (S_0^M + D_0^M)\theta_0 + \sum_{t=0}^{\infty} B(0,t)\left[B(t,t+1)(S_{t+1}^M + D_{t+1}^M) - S_t^M\right]\theta_{t+1}.\end{aligned}$$

Therefore, the agent's intertemporal budget constraint is

$$\begin{aligned}&\textstyle\sum_{t=0}^{\infty} B(0,t)[c_t+\tau_t]\\ &\le (S_0^M+D_0^M)\theta_0+\textstyle\sum_{t=0}^{\infty} B(0,t)\left[B(t,t+1)(S_{t+1}^M+D_{t+1}^M)-S_t^M\right]\theta_{t+1}.\end{aligned} \tag{7.29}$$

Exercise 7.2.1 *Suppose that the agent has lived for k periods and tries to plan his actions from period k onward. The proper intertemporal budget constraint would be one that discounts the future back to period k instead of period 0, since the first k periods have already passed. Define the relative price of period-t goods to period-k goods as*

$$B(k,t)=B(k,k+1)B(k+1,k+2)\cdots B(t-1,t),\ \textit{for}\ t\ge k+1$$

and normalize $B(k,k)=1$. Derive the intertemporal budget constraint from period k to period ∞.

The agent's intertemporal maximization problem is

$$\max\ \sum_{t=0}^{\infty}\beta^t u(c_t),\ \text{s.t. (7.29)}.$$

Let the Lagrangian multiplier for (7.29) be λ_0, the marginal value of wealth at the beginning of period 0. The optimality conditions are:

$$\text{for } c_t\ (t\ge 0):\quad \beta^t u'(c_t)=\lambda_0 B(0,t); \tag{7.30}$$

$$\text{for } \theta_{t+1}\ (t\ge 0):\quad B(t,t+1)(S_{t+1}^M+D_{t+1}^M)\le S_t^M,\quad \text{``$=$'' if } \theta_{t+1}>0.$$

The optimality condition for c_t equates the marginal utility of consumption at t, $\beta^t u'(c_t)$, and the marginal value of wealth at t, $\lambda_0 B(0,t)$, which represents the marginal cost of consumption at t. The optimality condition for θ_{t+1} states that the rate of return to holding shares from period t to period $t+1$ cannot exceed the rate of return to the one-period bond between periods t and $t+1$. If agents do hold shares, as in the equilibrium, rates of return to the two assets must be the same:

$$B(t,t+1)(S_{t+1}^M+D_{t+1}^M)=S_t^M. \tag{7.31}$$

This condition implies that the profit of buying and selling shares during the agent's lifetime is zero and so the present value of shares is the initial value, $(S_0^M+D_0^M)\theta_0$. The agent's intertemporal budget constraint reduces to

$$\sum_{t=0}^{\infty} B(0,t)[c_t+\tau_t]\le (S_0^M+D_0^M)\theta_0. \tag{7.32}$$

Since $\lambda_0>0$, this constraint must hold with equality as a result of agent's maximization.

7.2.2 Competitive Equilibrium

To define a competitive equilibrium, we first describe the government's actions. The government has an exogenously set spending profile, $\{g_t\}_{t=0}^{\infty}$. To finance spending, the government collects taxes, $\{\tau_t\}_{t=0}^{\infty}$, and issues one-period bonds, $\{\bar{\theta}_{B(t,t+1)}\}_{t=0}^{\infty}$. Similar to the private agent's budget constraint, the government intertemporal budget constraint requires that the present value of spending should not exceed the present value of taxes, as shown in the following exercise:

Exercise 7.2.2 *The government's budget constraint in each period t (≥ 0) is*

$$g_t = \tau_t + B(t,t+1)\bar{\theta}_{B(t,t+1)} - \bar{\theta}_{B(t-1,t)}.$$

With the assumption $\bar{\theta}_{B(-1,0)} = 0$, show that the government's intertemporal budget constraint is

$$\sum_{t=0}^{\infty} B(0,t)g_t = \sum_{t=0}^{\infty} B(0,t)\tau_t. \tag{7.33}$$

A competitive equilibrium in this economy consists of a consumption sequence, $\{c_t\}_{t=0}^{\infty}$, a sequence of share holdings, $\{\theta_{t+1}\}_{t=0}^{\infty}$, bond holdings, $\{\theta_{B(t,t+1)}\}_{t=0}^{\infty}$, taxes, $\{\tau_t\}_{t=0}^{\infty}$, and bond issuing, $\{\bar{\theta}_{B(t,t+1)}\}_{t=0}^{\infty}$, share prices, $\{S_t^M\}_{t=0}^{\infty}$, and bond prices, $\{B(t,t+1)\}_{t=0}^{\infty}$, such that
(i) given government policies and asset prices, the agent's consumption sequence, share holdings and bond holdings solve the agent's intertemporal maximization problem over the consumption and portfolio holding strategy;
(ii) the government's tax-bond financing policies satisfy the government intertemporal budget constraint for the exogenously given spending sequence;
(iii) goods and assets markets clear in each period, i.e., $c_t + g_t = D_t^M$, $\theta_{t+1} = 1$ and $\theta_{B(t,t+1)} = \bar{\theta}_{B(t,t+1)}$ for all $t = 0, 1, \cdots \infty$.

The competitive equilibrium can be solved in the familiar way. An immediate implication of the competitive equilibrium is that the Ricardian equivalence holds, since both the government's and the agent's intertemporal budget constraints are independent of the bonds issued – they only depend on the present value of taxes which equals the present value of the exogenously set spending profile.

7.2.3 Asset Prices and Bubbles

Equilibrium asset prices can be derived from the optimality conditions. First, with $B(0,0) = 1$, (7.30) implies $\lambda_0 = u'(c_0)$. Substituting this solution back into (7.30) yields

$$B(0,t) = \beta^t \frac{u'(c_t)}{u'(c_0)} = \beta^t \frac{u'(D_t^M - g_t)}{u'(D_0^M - g_0)} = \frac{1}{MRS_{0,t}}.$$

This is the pricing equation for a t-period bond that is sold in period 0 and matures in period t, although we have not explicitly modeled a t-period bond. The price of a one-period bond between any adjacent periods t and $t+1$ can be recovered from (7.28):

$$B(t,t+1) = \frac{B(0,t+1)}{B(0,t)} = \frac{MRS_{0,t}}{MRS_{0,t+1}} = \frac{1}{MRS_{t,t+1}} = \beta\frac{u'(D_{t+1}^M - g_{t+1})}{u'(D_t^M - g_t)}.$$

If $R(t,t+1) = 1/B(t,t+1)$ is the short-term interest rate between periods t and $t+1$ and $R(0,t) = 1/B(0,t)$ is the t-period interest rate between periods 0 and t (≥ 1), the term structure of interest rates is

$$R(0,t) = R(0,1)R(1,2)...R(t-1,t).$$

The term structure of interest rates is usually represented graphically by the yield curve. The yield on a one-period bond is simply the net interest rate associated with that bond. For example, the yield on a one-period bond that is issued at t and matures at $t+1$ is $r_{t,t+1} = R(t,t+1) - 1$. The yield on an t-period bond is the average (internal) rate of return to the bond in each period. For example, the yield on an n-period bond that is issued at 0 and matures at t is $r_{0,t} \equiv [R(0,t)]^{1/t} - 1$. The yield curve plots $r_{0,t}$ on the vertical axis of a graph against the maturity t on the horizontal axis. From the definition of $r_{0,t}$ and the formula for $R(0,t)$ we know that

$$\ln[1 + r_{0,t}] = \frac{1}{t}\ln R(0,t) = \frac{1}{t}\left[\ln(1 + r_{0,1}) + \ln(1 + r_{1,2}) + \cdots + \ln(1 + r_{t-1,t})\right].$$

If one-period yields are all sufficiently small, then $\ln(1 + r_t) \approx r_t$ and we can approximate the term structure as follows:

$$r_{0,t} \approx \frac{r_{0,1} + r_{1,2} + \cdots + r_{t-1,t}}{t}.$$

The yield on an t-period bond is roughly equal to the average of one-period yields during the maturity length of the t-period bond. The yield curve is sloping upward at a particular value t if and only if $r_{0,t+1} > r_{0,t}$ which, roughly speaking, requires one-period yields to be increasing with time. The following exercise makes a more precise statement.

Exercise 7.2.3 *Show that the yield curve is sloping upward at a particular value t if and only if $r_{t,t+1} > r_{0,t}$.*

Share prices can be found through (7.31). An important feature of the share price is that it is forward-looking: the price at any period t depends on future dividends and consumption. To solve for the share price, let us start at $t = 0$. The pricing equation for shares is $S_0^M = B(0,1)(S_1^M + D_1^M)$. Similarly, $S_1^M = B(1,2)(S_2^M + D_2^M)$. Then

$$\begin{aligned} S_0^M &= B(0,1)[B(1,2)(S_2^M + D_2^M) + D_1^M] \\ &= B(0,1)D_1^M + B(0,1)B(1,2)D_2^M + B(0,1)B(1,2)S_2^M \\ &= B(0,1)D_1^M + B(0,2)D_2^M + B(0,2)S_2^M. \end{aligned}$$

We can repeatedly substitute future share prices away on the right-hand side. After t times of substitution, the formula becomes

$$S_0^M = B(0,1)D_1^M + B(0,2)D_2^M + B(0,3)D_3^M + \cdots + B(0,t)D_t^M + B(0,t)S_t^M. \quad (7.34)$$

Presumably the substitution process can be repeated indefinitely. We need to know the limit of the last term when $t \to \infty$. In an economy with a finite number of periods, the end value of the share is always zero, because agents do not have any incentive to buy such shares. For an infinite-horizon economy, the last period does not exist and so the same restriction does not apply. How can we tie up the end of the share pricing equation?

One reasonable assumption is that bonds are sold at a discount, i.e., $B(0,t) < 1$ for all t. In this case the t-period bond has a declining price as t increases, since $B(0,t) = B(0,1)B(1,2)\cdots B(t-1,t)$. It may seem appealing to impose a restriction that long-term bond prices decline faster than share prices increase. In this case, as t increases, the product $B(0,t)S_t^M$ becomes smaller and smaller and eventually approaches zero. This assumption can be put as

$$\lim_{t\to\infty} B(0,t)S_t^M = 0. \quad (7.35)$$

If this assumption is satisfied, the pricing equation (7.34) can be repeated to the limit $t \to \infty$ to generate

$$S_0^M = B(0,1)D_1^M + B(0,2)D_2^M + B(0,3)D_3^M + \cdots = \sum_{t=1}^{\infty} B(0,t)D_t^M.$$

Namely, the share price is a discounted value of the future dividends, where the discount factor for period-t dividend is the t-period bond price $B(0,t)$. This solution is called the *fundamental solution* for the share price, because it depends only on the fundamentals of the share price – the dividends and discount factors. Since $B(0,t) = 1/MRS_{0,t}$, the fundamental solution for S_0^M can be written as

$$S_0^M = \sum_{t=1}^{\infty} D_t^M \frac{\beta^t u'(D_t^M - g_t)}{u'(D_0^M - g_0)} \quad (7.36)$$

The fundamental solution for the share price S_t^M for any $t = 0, 1, \cdots, \infty$ can be derived similarly.

Fundamental solution for the share price:

$$S_t^M = \sum_{\tau=t+1}^{\infty} B(t,\tau)D_\tau^M = \sum_{\tau=t+1}^{\infty} D_\tau^M \frac{\beta^{\tau-t} u'(D_\tau^M - g_\tau)}{u'(D_t^M - g_t)}, \text{ for } t = 0, 1, \cdots, \infty. \quad (7.37)$$

The following exercise illustrates the fundamental solution in a special case.

Exercise 7.2.4 *Let $u(c) = \ln c$, $D_t^M = \alpha D_{t-1}^M$, and $g_t = \alpha g_{t-1}$ for all t, where the constant α satisfies $\alpha > \beta$. Show that for all $t = 0, 1, \cdots, \infty$, the bond price and the fundamental solution for the share price in period i are*

$$B(t,t+1) = \beta/\alpha, \; S_t^M = \frac{\beta}{1-\beta} D_t^M.$$

The share price may not always satisfy the restriction (7.35). When a solution for S^M violates the restriction (7.35), we say that there are "bubbles" in share prices. The following proposition shows that bubbles can indeed be a part of the solution for the share price.

Proposition 1 *Bubbles in share prices: If S_t^M is the fundamental solution for the share price at time t, then for any constant A, $S_t^{M^{bub}}$ given below is also a solution for the share price at time t:*

$$S_t^{M^{bub}} = S_t^M + \frac{A}{B(0,t)}. \tag{7.38}$$

Exercise 7.2.5 *Prove the above proposition. (Hint: Substitute (7.38) into the pricing equation $S_t^M = B(t,t+1)(S_{t+1}^M + D_{t+1}^M)$ and show that it satisfies the equation.)*

The "bubble" term $A/B(0,t)$ captures two features of this non-fundamental solution. First, for any arbitrary constant A, the formula (7.38) gives one solution. There are infinitely many such solutions and the fundamental variables like dividends do not help the selection among these solutions. Second, the solutions "pop up" or "dive down". When one-period bonds are less than one, as they are in reality, the t-period bond price is a decreasing function of t. As $B(0,t)$ falls with t, the term $A/B(0,t)$ rises in absolute value. If $A > 0$, the stock price increases as t increases; if $A < 0$, the stock price eventually falls with t. Some analysts believe that bubbles are an important cause of abnormal behavior in stock prices like stock market crush (corresponding to $A < 0$).

Summary 1 *In this chapter we have carried out the following analyses:*
(i) Extended a two-period economy to a three-period economy to show that the Ricardian equivalence holds even when the government can issue long-term bonds;
(ii) Derived the term structure of interest rates;
(iii) Extended these conclusions to an infinite-horizon economy and showed the possibility of bubbles.

Chapter 8
UNCERTAINTY AND ASSET PRICES

The models in previous chapters are useful for illustrating equilibrium asset prices and restrictions on government financing methods. For simplicity we have abstracted from fundamental uncertainty in the economy. The abstraction yielded the unrealistic result that bonds and shares (stocks) must have the same rate of return. In this chapter we examine how uncertainty affects assets prices.

There can be two types of uncertainty. One is specific to each individual agent and is termed *idiosyncratic risks*. The other type of uncertainty affects all agents in the economy and is termed *aggregate uncertainty*. Examples of aggregate risks include shocks to aggregate output, the representative agent's tastes or the government spending profile. In this chapter, we will focus on shocks to output and ignore all other types of uncertainty. The basic discussion is based on Duffie (1996) Chapters 3 and 4.

8.1 Stochastic Output and Conditional Expectations

Let us now assume that each tree yields a random stream of dividends $\{D_t^M\}_{t=0}^{\infty}$, perhaps due to random changes of weather or technology. Output in period t is observed at the beginning of period t before anything else happens in period t. The particular observed value of D_t^M is the realization of the random variable at t. At time t the agent does not know precisely what output in the future will be. The agent may use observed output and other statistics to forecast future output.

To formulate expectations on random variables, let us briefly discuss some well-known random variables. A special sequence of random variables is identically and independently distributed variables (*iid*). Roughly speaking, *iid* random variables are independent of any information available at the present. For example, outcomes of fair coin flips are *iid* variables, since the outcome of each flip is independent of other flips. With a fair coin flip, heads and tails will show up with the same probability. This particular feature, however, is not necessary for an *iid* sequence.

Example 1 *Suppose that output at any time t has two possible realizations, high ($D_t^M = 100$) and low ($D_t^M = 80$). The realization "high" occurs with probability $3/4$ and the realization "low" occurs with probability $1/4$. These probabilities are independent of the realizations of D^M in any other period. The sequence $\{D_t^M\}$ is a sequence of iid random variables.*

We can use the above example to introduce a few operations on random variables.

- Unconditional expected value (unconditional mean). The unconditional expected value of D^M can be understood as the long-run mean of the variable and is often denoted $E(D^M)$. In the above example, we have

$$\begin{aligned} E(D_t^M) &= 100 \times prob(D_t^M = 100) + 80 \times prob(D_t^M = 80) \\ &= 100 \times \frac{3}{4} + 80 \times \frac{1}{4} = 95, \end{aligned}$$

  where the notation $prob(D_t^M = 100)$ stands for the probability of $D_t^M = 100$.

- Conditional expected value. The conditional expected value of D_t^M, say, conditional on D_{t-1}^M, is the prediction on D_t^M knowing that D_{t-1}^M has a particular realization. It is often written as $E(D_t^M \mid D_{t-1}^M)$. Presumably, observing D_{t-1}^M might improve the prediction on D_t^M if D_{t-1}^M and D_t^M has some relation. Thus, $E(D_t^M \mid D_{t-1}^M)$ and $E(D_t^M)$ are not equal to each other for most random variables. More generally, let us denote

$$I_t = \text{all information available in period } t.$$

  The notation $E(D_{t+1}^M \mid I_t)$ denotes the expectation of D_{t+1}^M conditional on the information I_t and is often shortened to $E_t(D_{t+1}^M)$. The two objects $E_t(D_{t+1}^M)$ and $E(D_{t+1}^M)$ are often different. In the above example, however, the two are the same because D_{t+1}^M is independent of any other variable in the example.

- Variance. The unconditional variance of a variable D_t^M is defined as $Var(D_t^M) = E[(D_t^M - ED_t^M)^2]$. In the current example,

$$\begin{aligned} Var(D_t^M) &= (100-95)^2 \times prob(D_t^M = 100) + (80-95)^2 \times prob(D_t^M = 80) \\ &= (100-95)^2 \times \frac{3}{4} + (80-95)^2 \times \frac{1}{4} = 75. \end{aligned}$$

- Covariance. The covariance between D_t^M and another variable, say D_{t-1}^M, is defined as

$$Cov(D_t^M, D_{t-1}^M) = E[(D_t^M - ED_t^M)(D_{t-1}^M - ED_{t-1}^M)].$$

  If the covariance between two variables are positive, the two variables are said to be positively correlated; if the covariance is negative, the two variables are negatively correlated; if the covariance is zero, the two variables are uncorrelated.

Note that the pair (D_t^M, D_{t-1}^M) in the above example can have four possible realizations: $(100, 100)$, $(100, 80)$, $(80, 100)$, and $(80, 80)$. To calculate the covariance between D_t^M and D_{t-1}^M, we need to know the joint probability of the two variables, i.e., the probability for each of the four pairs of realizations. In the current case, since D_t^M and D_{t-1}^M are independent of each other, we have

$$prob(D_t^M = 100, D_{t-1}^M = 100) = prob(D_t^M = 100) \times prob(D_{t-1}^M = 100) = \frac{3}{4} \times \frac{3}{4} = \frac{9}{16}.$$

Similarly, we can calculate the joint probabilities for other pairs: $(100, 80)$ occurs with probability $3/16$, $(80, 100)$ occurs with probability $3/16$ and $(80, 80)$ occurs with probability $1/16$. It is then easy to confirm that D_t^M and D_{t-1}^M are uncorrelated, as shown below:

$$\begin{aligned} Cov(D_t^M, D_{t-1}^M) &= (100-95)(100-95) \times \frac{9}{16} + (100-95)(80-95) \times \frac{3}{16} \\ &\quad +(80-95)(100-95) \times \frac{3}{16} + (80-95)(80-95) \times \frac{1}{16} \\ &= 0. \end{aligned}$$

From this exercise we may conclude that if two variables are independent, they are uncorrelated. (The reverse is not true: If two variables are uncorrelated, they may not necessarily be independent of each other.)

The expectation operator has the following properties (Try to verify them with the above example):

- For any constant k and a random variable x, $E(kx) = kE(x)$ and $E(x + k) = E(x) + k$. The conditional expectations operator has the same property.
- For any sequence of random variables $\{x_t\}$ where x_t is realized at t, the chain rule of conditional expectations applies:

$$E_t(x_{t+n}) = E_t(E_{t+1}(x_{t+n})) \text{ for any } n \geq 1. \tag{8.1}$$

  For example, to predict at t the value of a future variable x_{t+2}, one can simply predict what the prediction will be at $t + 1$.

- For any two random variables f and g, we have

$$E_t(f_{t+1} \cdot g_{t+1}) = E_t(f_{t+1}) \cdot E_t(g_{t+1}) + cov_t(f_{t+1}, g_{t+1}). \tag{8.2}$$

  In this formula, $cov_t(f_{t+1}, g_{t+1})$ is the conditional covariance between f_{t+1} and g_{t+1} which is given by

$$cov_t(f_{t+1}, g_{t+1}) = E_t \left\{ [f_{t+1} - E_t(f_{t+1})] \cdot [g_{t+1} - E_t(g_{t+1})] \right\}.$$

The random variable D^M in the above example can be decomposed into two parts:

$$D_t^M = 95 + \varepsilon_t,$$

where $\{\varepsilon_t\}$ is a sequence of *iid* variables that have zero mean and constant variance ($= 75$ in the example). This sequence of random variables is called white noise variables. In general, a sequence of white noise variables, denoted $\{\varepsilon_t\}$, has the following properties:

- Zero expected value: $E(\varepsilon_t) = 0$ for all t;
- Constant variance: $Var(\varepsilon_t) = \sigma^2$ for all t;
- For each t, ε_t is independent of any other variable ε_i $(i \neq t)$ in the sequence.

The requirement that one variable is independent of another variable may be too strong to be necessary. In most applications, what is relevant is whether the variable can be predicted by available observations. That is, whether $E_t(D^M_{t+1})$ is the same as $E(D^M_{t+1})$ for a variable D^M_{t+1}. The following example is one where available observations can help predicting future values of the variable.

Example 2 *Suppose that an economy can be in only two states: boom and bust. If the economy is in boom this year, the economy is likely to be in boom next year, say, with probability* 0.7*. But there is a chance, with probability* 0.3*, that the economy will turn into a bust. If the economy is in bust this year, the economy is likely to continue the bust next year, say, with probability* 0.8*. With probability* 0.2*, the economy will change into a booming situation. Let us denote the situation of the economy in period t by D^M_t, where $D^M = 1$ indicates boom and $D^M = 0$ indicates bust. The transition of the economy between period t and $t+1$ can be summarized by a* transition matrix*:*

		D^M_{t+1} 0	1
D^M_t	0	0.8	0.2
	1	0.3	0.7

If the economy is in a bust at t, the prediction on D^M_{t+1} is

$$
\begin{aligned}
E(D^M_{t+1} \quad &| \quad D^M_t = 0) \\
&= \quad 0 \times prob(D^M_{t+1} = 0 \mid D^M_t = 0) + 1 \times prob(D^M_{t+1} = 1 \mid D^M_t = 0) \\
&= \quad 0 \times 0.8 + 1 \times 0.2 = 0.2.
\end{aligned}
$$

If the economy is in a boom at t, the prediction on D^M_{t+1} is

$$
\begin{aligned}
E(D^M_{t+1} \quad &| \quad D^M_t = 1) \\
&= \quad 0 \times prob(D^M_{t+1} = 0 \mid D^M_t = 1) + 1 \times prob(D^M_{t+1} = 1 \mid D^M_t = 1) \\
&= \quad 0 \times 0.3 + 1 \times 0.7 = 0.7.
\end{aligned}
$$

The prediction on D^M_{t+1} depends on the state of the economy at t.

An important property of the process in the above example is that all information at t that is useful for predicting the future variable D^M_{t+1} is summarized by the current realization D^M_t. The prediction of D^M_{t+1} using all information available at t is the same as the prediction at t on D^M_{t+1} using only D^M_t. To illustrate this property, let us consider another example.

Example 3 *Suppose a sequence of random variables* $\{D_t^M\}_{t\geq 0}$ *obeys the following process*

$$D_{t+1}^M = \alpha D_t^M + \varepsilon_{t+1}, \qquad |\alpha| < 1, \tag{8.3}$$

where $\{\varepsilon_t\}_{t\geq 0}$ *is a sequence of white noise variables. We say that* D^M *follows a first-order auto-regressive process, AR1 for short. For an AR1 process, observing the value of* D_t^M *helps predicting the future variable* D_{t+1}^M*:*

$$E(D_{t+1}^M \mid D_t^M) = E(\alpha D_t^M + \varepsilon_{t+1} \mid D_t^M) = \alpha D_t^M + E(\varepsilon_{t+1} \mid D_t^M) = \alpha D_t^M.$$

If one uses all information available at t *to forecast* D_{t+1}^M*, the conditional expectation is the same as the one obtained with only the observation on* D_t^M*:*

$$\begin{aligned} E(D_{t+1}^M \quad &| \quad I_t) \\ &= \quad E(\alpha D_t^M + \varepsilon_{t+1} \mid I_t) = \alpha D_t^M + E(\varepsilon_{t+1} \mid I_t) \\ &= \quad \alpha D_t^M = E(D_{t+1}^M \mid D_t^M), \end{aligned}$$

In fact, in this example the prediction at t *of* all *future variables* $\{D_{t+i}^M\}_{i\geq 1}$ *using* D_t^M *is as good as the prediction of future variables using all information available at* t*.*

Exercise 8.1.1 *With the AR1 process show that* $E(D_{t+i}^M \mid I_t) = E(D_{t+i}^M \mid D_t^M)$ *for all* $i = 1, 2, \cdots, \infty$*.*

When a process of random variables satisfies the property $E(D_{t+i}^M \mid I_t) = E(D_{t+i}^M \mid D_t^M)$, the process is termed a *Markov process.* In all discussions in this chapter we assume that output follows a Markov process. A straightforward extension of the process (8.3) is

$$D_{t+1}^M = (1-\alpha)D^{M^*} + \alpha D_t^M + \varepsilon_{t+1}, \qquad |\alpha| < 1. \tag{8.4}$$

With this specification, the constant D^{M^*} is the "long-run" level of D^M, the unconditional mean of D^M, as shown below.

Exercise 8.1.2 *Let* D^M *follow the process in (8.4). Suppose that time starts at* $-\infty$ *rather than at* 0 *and that* $\lim_{t\to\infty} \alpha^t D_{-\infty}^M = 0$*.*
(i) Solve D_t^M *as a function of only* D^{M^*} *and the* ε*'s.*
(ii) Show that the unconditional expectation of D_{t+1}^M *is* D^{M^*} *for any* t*. Contrast the result with the conditional expectation* $E(D_{t+1}^M \mid D_t^M)$*.*

8.2 Utility Maximization under Uncertainty

We now turn to the agent's maximization problem. To simplify exposition, let us abstract bonds and the government from the model for the moment. When output is random, the representative agent faces income risks. At the beginning of the life, the agent tries to make a consumption plan and a share holding plan to maximize the intertemporal utility. Such plans have two characteristics:

- They depend on the information available at time of the planning (the beginning of the life), I_0;
- They are *contingent plans*: Plans for consumption and share holdings in a future period t are contingent on what output will be up to and including period t and on what actions the agents has taken up to t.

The agent can make the plan contingent on the *state* where the agent is. As it turns out, the Markov property of the output process simplifies the description of the agent's state at any time t to two elements. One is the exogenous state, which is the realization of output t, D_t^M. The other is the endogenous state, which is the amount of shares the agent bought in the previous period, θ_t. The agent's state at time t is (θ_t, D_t^M). Consumption c_t and asset holdings θ_{t+1} in period t can be written as functions of the state, $c_t(\theta_t, D_t^M)$ and $\theta_{t+1}(\theta_t, D_t^M)$. These symbols make it clear that future consumption levels and asset holdings are random variables at 0, because future output is random at 0. The share price in period t is also a random variable at 0. Since share price clears the shares market, it depends on aggregate variables but not on individuals' asset holdings. A reasonable conjecture is that S_t depends only on aggregate output D_t^M, $S_t = S_t(D_t^M)$ (we must verify this conjecture later). Since the price depends on the state, we call it state contingent price.

The intertemporal utility index, which depends on current and future consumption levels, is also uncertain. To give a well-defined objective function, we postulate that the agent maximizes the expected value of the intertemporal utility index, conditional on the information available at time 0. Using the notation $E_0(\cdot)$ to denote conditional expectations at time 0, the agent's objective function is

$$E_0\left(\sum_{t=0}^{\infty}\beta^t u(c_t)\right).$$

It is important to note that all expectations $E(\cdot)$ are calculated with the agent's probability assessment (or the actual probability measure), unless otherwise specified. This actual probability measure is called the $P-measure$. We make such a distinction here so that students will not be confused by the concept "equivalent probability $Q-measure$" later. The expectation under the Q-measure is denoted as $E^Q(\cdot)$.

As before, the agent faces a budget constraint in each period t:

$$c_t + S_t\theta_{t+1} \leq (S_t + D_t^M)\theta_t. \tag{8.5}$$

This constraint must hold for each every possible state at t, (θ_t, D_t^M). For different states, the choices (c_t, θ_{t+1}) must adjust (according to the contingent plan) to satisfy the constraint. For example, if there are ten different possible values of the state (θ_t, D_t^M), the agent are subject to ten different constraints at t when choosing consumption and asset holding plans at time 0. If the possible values of output occupy an interval, there will be infinitely many constraints the agent faces at time 0 just for period t! Since time t also runs from 0 to ∞, there would be too many constraints for the maximization problem to be formulated easily.

To reduce the complexity of the problem, let us imagine that period 0 has already passed and the agent is at the beginning of period 1. The agent's state in period 1 is (θ_1, D_1^M). Suppose that the agent can solve the maximization problem from period 1 onward subject to all constraints in periods 1 through ∞, given the state at time 1. The *maximized* expected intertemporal utility from period 1 onward must be a function of the state at 1. Let us denote this function by $V(\theta_1, D_1^M)$, which is called the value function. Now go back one period into period 0. Given that the future utility is lumped together by the value function, the agent's intertemporal utility from period 0 onward can be written as

$$u(c_0) + \beta E_0 V(\theta_1, D_1^M).$$

The conditional expectation sign E_0 must be put before $V(\theta_1, D_1^M)$ because the latter is a random variable at time 0. The agent's maximization problem at time 0 becomes a much simpler problem:

$$\max_{(c_0,\theta_1)} \; u(c_0) + \beta E_0 V(\theta_1, D_1^M) \text{ s.t. } c_0 + S_0\theta_1 \le (S_0 + D_0^M)\theta_0.$$

Only the budget constraint in period 0 appears in the above problem. This does not mean that the agent disregards the budget constraints in other periods – they are already taken care of in the process of obtaining the value function $V(\theta_1, D_1^M)$.

The principle involved in the above simplification is as follows. Suppose that a plan $\{c_t, \theta_{t+1}\}_{t=0}^{\infty}$ is optimal. Then after period 0 is passed, the remaining plan $\{c_t, \theta_{t+1}\}_{t=1}^{\infty}$ must be optimal for the remaining horizon under the state (θ_1, D_1^M). This principle is called the dynamic programming principle. In fact the principle applies to any period t. That is, if a plan $\{c_k, \theta_{k+1}\}_{k=t}^{\infty}$ is optimal for periods t through ∞, then after period t is passed the remaining plan $\{c_k, \theta_{k+1}\}_{k=t+1}^{\infty}$ must be optimal for the remaining periods under the state $(\theta_{t+1}, D_{t+1}^M)$. Therefore, for a given state (θ_t, D_t^M), the choice (c_t, θ_{t+1}) can be found by solving the following problem:

$$\max_{(c_t,\theta_{t+1})} \; u(c_t) + \beta E_t V(\theta_{t+1}, D_{t+1}^M) \text{ s.t. } c_t + S_t\theta_{t+1} \le (S_t + D_t^M)\theta_t. \tag{8.6}$$

Note that expectations at t are conditional on information available at t and so the symbol E_t appears before $V(\theta_{t+1}, D_{t+1}^M)$. The result of the maximization in (8.6) is the maximized intertemporal utility from period t to period ∞ which, according to our notation, is nothing but $V(\theta_t, D_t^M)$. Thus, the function V must satisfy the following equation:

$$V(\theta_t, D_t^M) = \max_{(c_t,\theta_{t+1})} \; u(c_t) + \beta E_t V(\theta_{t+1}, D_{t+1}^M) \text{ s.t. } c_t + S_t\theta_{t+1} \le (S_t + D_t^M)\theta_t. \tag{8.7}$$

This is an equation whose unknown is a function rather than a variable. For this reason, we call (8.7) the functional equation of dynamic programming. It is also called the Bellman equation.

It is a difficult task to show that there is such a function V that satisfies the functional equation. We do not intend to tackle the problem here. The comforting facts are:

- For most of the analysis, we do not need to know the solution for V, as long as we know it exists.
- The solution for V indeed exists under fairly general conditions (see later exercise for an example.)
- It can be also shown that the value function $V(\theta, D^M)$ is an increasing function of θ. This is not surprising, because if the agent brought into period t more shares (θ_t), he should be able to derive a higher future from period t onward.

To make calculus work, we will assume that the function $V(\theta, D^M)$ is differentiable and concave in θ. The maximization problem in (8.6) can be solved with the Lagrangian method. Let λ_t be the marginal value of wealth at the beginning of period t (which depends on the realized state (θ_t, D_t^M) that is known at the beginning of period t). λ_t is also the shadow price of the budget constraint in period t. We can then form the Lagrangian for the problem in (8.6) as

$$L_t = u(c_t) + \beta E_t V(\theta_{t+1}, D_{t+1}^M) + \lambda_t[(S_t + D_t^M)\theta_t - c_t - S_t\theta_{t+1}].$$

The optimality conditions for (c_t, θ_{t+1}) are

$$\text{for } c_t\text{:} \quad u'(c_t) = \lambda_t, \tag{8.8}$$

$$\text{for } \theta_{t+1}\text{:} \quad \beta E_t V_\theta(\theta_{t+1}, D_{t+1}^M) = \lambda_t S_t, \tag{8.9}$$

where V_θ is the partial derivative of $V(\theta, D^M)$ with respect to θ. These conditions can be explained using the marginal cost-benefit analysis but we omit such explanation. Also, since $\lambda_t > 0$ by the optimality condition for c_t, we have

$$\frac{\partial L_t}{\partial \lambda_t} = (S_t + D_t^M)\theta_t - c_t - S_t\theta_{t+1} = 0. \tag{8.10}$$

The optimality condition for θ_{t+1} depends on the value function V. Although we cannot know the form of the value function V without solving the functional equation, it is possible to find the derivative V_θ. The following exercise provides the answer.

Exercise 8.2.1 *Show that*

$$V_\theta(\theta_t, D_t^M) = \lambda_t(S_t + D_t^M) = u'(c_t) \cdot (S_t + D_t^M). \tag{8.11}$$

Hint: First show that the maximized value of the Lagrangian is equal to $V(\theta_t, D_t^M)$ and then use the envelope theorem of maximization (see chapter 4).

The equation (8.11) also holds for $t+1$ (why?). Using the result to substitute for $V_\theta(\theta_{t+1}, D^M_{t+1})$ in (8.9), we have

$$S_t u'(c_t) = \beta E_t (S_{t+1} + D^M_{t+1}) u'(c_{t+1}). \tag{8.12}$$

Note that this equation is similar to the counterpart in the deterministic economy, except the conditional expectation symbol $E_t(\cdot)$. The condition requires that the marginal cost of acquiring a unit of asset be equal to the expected marginal benefit from the asset. The marginal cost (in terms of period t utility) is the unit price of the share, S_t, times the marginal utility of goods in period t, $\lambda_t = u'(c_t)$. The marginal benefit of the share is the return to the share $(S_{t+1} + D^M_{t+1})$ times the marginal utility of goods in $t+1$. Since future output is uncertain, the marginal benefit of the share is random, depending on the realizations of future output, and hence it is the expected marginal benefit that is appropriate for evaluating the choice of shares θ_{t+1}.

For any given state (θ_t, D^M_t), (8.11) and (8.12) solve for the function $c_t(\theta_t, D^M_t)$ and $\theta_{t+1}(\theta_t, D^M_t)$ while the functional equation gives V. The random sequence $\{c_t(\theta_t, D^M_t), \theta_{t+1}(\theta_t, D^M_t)\}_{t=0}^\infty$ forms the agent's optimal contingent plan. The following exercise illustrates this plan and the solution for V in a special case.

Exercise 8.2.2 *Let $u(c) = \ln c$. Assume that output follows*

$$\ln D_{t+1} - \ln D_t = \alpha + \epsilon_{t+1}.$$

In this case the fundamental solution for the share price is $S_t = \frac{\beta}{1-\beta} D^M_t$ (shown later).
(i) Show that the optimal plan is to save β fraction of the wealth and consume $1-\beta$ fraction of the wealth in each period. That is, the optimal plan is

$$c_t = (1-\beta)(S_t + D^M_t)\theta_t \text{ and } S_t\theta_{t+1} = \beta(S_t + D^M_t)\theta_t.$$

(Hint: Show that this plan satisfies (8.12).)
(ii) Show that the value function is

$$V(\theta_t, D^M_t) = A_t + B_t \ln\left(D^M_t \theta_t\right),$$

and determine A_t and B_t. Confirm that the value function is indeed increasing and concave in θ.

8.3 Competitive Equilibrium and Asset Pricing

8.3.1 Competitive Equilibrium

A competitive equilibrium can be defined as a contingent plan $\{c_t(\theta_t, D^M_t), \theta_{t+1}(\theta_t, D^M_t)\}_{t=0}^\infty$, a value function $V(\theta, D^M)$ and a sequence of price functions $\{S_t(D^M_t)\}_{t=0}^\infty$ such that
(i) given the price sequence, the contingent plan solves the agent's expected utility maximization problem;
(ii) the value function satisfies the Bellman equation;

(iii) goods and share markets clear, i.e., for each state (θ_t, D_t^M) in period t and for each t, $c_t(\theta_t, D_t^M) = D_t^M$ and $\theta_{t+1}(\theta_t, D_t^M) = 1$.

Again, we want to emphasize that, in a competitive equilibrium, the asset prices are such that there exist no arbitrage in the economy. As before, we can use the marginal utility benefit and cost analysis prove this.

The share price is given implicitly by (8.12). Substituting goods market clearing conditions, we have

$$S_t = \beta E_t \left[(S_{t+1} + D_{t+1}^M) \frac{u'(D_{t+1}^M)}{u'(D_t^M)} \right]. \tag{8.13}$$

As before, the share price has the forward-looking feature. The price at time t depends on the expectation of the price and output in period $t+1$, which in turn depend on the price and output in period $t+2$, and so on. The end result is that the share price depends on expected discounted dividends in all future periods. In contrast to the share price in a deterministic economy, it is the expected discounted dividends, not the actual dividends, that affect the current asset price – agents do not have the perfect foresight on future dividends.

The share price can be solved by repeatedly substituting the formula (8.13) into itself. To do so, let us temporarily denote the marginal utility of one share at t by $x_t = S_t u'(D_t^M)$ and marginal utility of dividends at t by $d_t = D_t^M u'(D_t^M)$. (8.13) states that the marginal value (in terms of utility) of a share at t is equal to the expected future marginal utility of a share plus the marginal utility of dividends at $t+1$:

$$x_t = E_t[\beta x_{t+1} + \beta d_{t+1}]. \tag{8.14}$$

To repeatedly substitute the formula into itself is cumbersome when the conditional expectation sign appears. To get rid of the conditional expectation sign, observe that the sum, $\beta x_{t+1} + \beta d_{t+1}$, can be expressed as the sum of x_t and an unpredictable noise variable, say, ε_{t+1}. Precisely, the expectation on ε_{t+1} conditionally on the information at t is zero and

$$x_t = \beta x_{t+1} + \beta d_{t+1} - \varepsilon_{t+1}, \quad E_t(\varepsilon_{t+1}) = 0.$$

(To verify this equation, one can take the conditional expectation $E_t(\cdot)$ on both sides and note $E_t x_t = x_t$ (as x_t is known at time t). The exercise recovers (8.14).) Now the formula can be repeatedly substituted into itself. The fundamental solution for S_t is given below:

- Assume that the expected marginal utility of a share in $t+i$, discounted to period-t, approaches zero as i approaches infinity, i.e.,

  $$\lim_{i \to \infty} \beta^i E_t[S_{t+i} u'(D_{t+i}^M)] = 0.$$

  The fundamental solution for the share price is

  $$S_t = E_t \left(\sum_{i=1}^{\infty} \beta^i \frac{D_{t+i}^M u'(D_{t+i}^M)}{u'(D_t^M)} \right). \tag{8.15}$$

As in the deterministic economy, the discounted expected marginal utility of a future share may not approach zero. In this case bubbles can develop in the share price. Precisely, let $\{A_t\}$ be any sequence of random variables such that $A_t = \beta E_t A_{t+1}$. With S_t being given by (8.15), the bubbles are

$$S_t^{bub} = S_t + \frac{A_t}{u'(c_t)}. \tag{8.16}$$

A particular bubble is such that $A_t = A\beta^{-t}$, where A is an arbitrary constant. As before, there are infinitely many bubble solutions for the share price.

Exercise 8.3.1 *Let the utility function be $u(c) = \ln c$. Show that the fundamental solution for the share price is $S_t = \frac{\beta}{1-\beta} D_t^M$.*

Exercise 8.3.2 *Verify that (8.16) is a solution to (8.13).*

Before leaving this subsection, let us define the rate of return to a share between t and $t+1$ as $1 + r_{t,t+1}^m = (S_{t+1} + D_{t+1}^M)/S_t$. The pricing equation can be written as

$$\beta E_t \left(\left(1 + r_{t,t+1}^m\right) \frac{u'(D_{t+1}^M)}{u'(D_t^M)} \right) = 1. \tag{8.17}$$

This is a general pricing equation for assets: Any other asset which has a rate of return

$$1 + r_{t,t+1}^n = \frac{S_{t+1}^n + D_{t+1}^n}{S_t^n} \qquad \forall \qquad n = 1, 2, ..., N.$$

$r_{t,t+1}^n$ between t and $t+1$ must also satisfy the above equation, with $r_{t,t+1}^m$ being replaced by $r_{t,t+1}^n$.

8.3.2 Consumption-Based Dynamic CAPM and Risk Premium

we derive the dynamic consumption-based CAPM from the fundamental pricing equation 8.17. Let us first examine a one-period bond issued at t that pays one unit of goods at the maturity date $t+1$ regardless of the state of the economy at $t+1$. Since the bond's payoff is certain, this bond is risk-free. We want to see how the risk-free feature is represented by the pricing equation of the bond. Let the price of this bond be $B(t, t+1)$ at t and the rate of return be

$$1 + r_{t,t+1} = \frac{1}{B(t, t+1)}$$

between t and $t+1$. Then $B(t, t+1)$ must satisfy an equation similar to (8.17), i.e.,

$$\beta E_t \left((1 + r_{t,t+1}) \frac{u'(D_{t+1}^M)}{u'(D_t^M)} \right) = \beta E_t \left(\frac{1}{B(t, t+1)} \cdot \frac{u'(D_{t+1}^M)}{u'(D_t^M)} \right) = 1$$

Since the bond price is known at time t when the bond is issued, there is nothing uncertain about the rate of return $r_{t,t+1}$. In this case, $r_{t,t+1}$ can be taken out of the expectation sign and so the rate of return of the one-period risk-free bond is

$$E_t \left(\frac{\beta u'(D_{t+1}^M)}{u'(D_t^M)} \right) = \frac{1}{1 + r_{t,t+1}}. \tag{8.18}$$

Furthermore, for the market portfolio, we have

$$\begin{aligned} 1 &= E_t\left(\frac{\beta u'(D_{t+1}^M)}{u'(D_t^M)}(1+r_{t,t+1}^m)\right) \\ &= E_t\left(\frac{\beta u'(D_{t+1}^M)}{u'(D_t^M)}\right)E_t(1+r_{t,t+1}^m) + cov_t\left(\frac{\beta u'(D_{t+1}^M)}{u'(D_t^M)}, r_{t,t+1}^m\right) \\ &= \frac{E_t(1+r_{t,t+1}^m)}{1+r_{t,t+1}} + cov_t\left(\frac{\beta u'(D_{t+1}^M)}{u'(D_t^M)}, r_{t,t+1}^m\right). \end{aligned}$$

or

$$E_t(r_{t,t+1}^m) = r_{t,t+1} - (1+r_{t,t+1})cov_t\left(\frac{\beta u'(D_{t+1}^M)}{u'(D_t^M)}, r_{t,t+1}^m\right).$$

Similarly,

$$E_t(r_{t,t+1}^n) = r_{t,t+1} - (1+r_{t,t+1})cov_t\left(\frac{\beta u'(D_{t+1}^M)}{u'(D_t^M)}, r_{t,t+1}^n\right). \tag{8.19}$$

Therefore,

$$E_t(r_{t,t+1}^n) - r_{t,t+1} = \frac{cov_t\left(u'(D_{t+1}^M), r_{t,t+1}^n\right)}{cov_t\left(u'(D_{t+1}^M), r_{t,t+1}^m\right)}\left[E_t(r_{t,t+1}^m) - r_{t,t+1}\right] \tag{8.20}$$

The above equation (8.20) is the consumption-based dynamic CAPM.

The asset pricing equation (8.20) enables us to introduce a notion of riskiness of an asset. If such an asset is risky, or riskier than a one-period risk-free bond, the asset must provide a higher expected rate of return than the one-period risk-free bond. The higher expected rate of return is a premium to compensate for agents' loss from being exposed to the risk. Conversely, the asset is less risky than the risk-free bond if agents hold the asset even though the expected rate of return to the asset is lower than the rate of return to the risk-free bond. Thus the *risk premium* involved in holding an asset of a rate of return $r_{t,t+1}^n$ can be intuitively defined as

$$\text{risk premium } = E_t r_{t,t+1}^n - r_{t,t+1}.$$

An asset is risk free if its expected rate of return is the same as the rate of return to a risk-free bond. But, what feature of an asset induces a zero risk premium? We first provide the answer below and then support the answer:

- (zero-risk premium or)Risk-free assets: An asset is risk free if and only if its rate of return $r_{t,t+1}^n$ satisfies

$$E_t\left(r_{t,t+1}^n \cdot \beta\frac{u'(D_{t+1}^M)}{u'(D_t^M)}\right) = E_t\left(r_{t,t+1}^n\right) \cdot E_t\left(\beta\frac{u'(D_{t+1}^M)}{u'(D_t^M)}\right), \tag{8.21}$$

which implies

$$cov_t\left(\frac{\beta u'(D_{t+1}^M)}{u'(D_t^M)}, r_{t,t+1}^n\right) = 0.$$

Delaying the interpretation for this condition, we show that the feature (8.21) is indeed necessary and sufficient for $E_t r_{t,t+1}^n = r_{t,t+1}$. Start with the pricing equation for an asset. If the asset yields a rate of return $r_{t,t+1}^n$, then $r_{t,t+1}^n$ must satisfy the pricing equation (8.17), with $r_{t,t+1}^m$ be replaced by $r_{t,t+1}^n$. That is,

$$E_t\left(\left[1 + r_{t,t+1}^n\right] \cdot \beta\frac{u'(D_{t+1}^M)}{u'(D_t^M)}\right) = 1. \tag{8.22}$$

If (8.21) holds for such asset, then

$$1 + E_t\left[r_{t,t+1}^n\right] = \left[E_t\left(\beta\frac{u'(D_{t+1}^M)}{u'(D_t^M)}\right)\right]^{-1} = 1 + r_{t,t+1},$$

which confirms the risk-free nature of the asset.

The interpretation is that asset n has an uncertain rate of return, but **has no systematic risk since it is uncorrelated with the aggregate marginal rate of substitution**. Such an asset only has idiosyncratic risk, therefore, it can only earn a riskfree return (in the usual CAPM framework, such an asset is called the zero-beta asset).

The feature (8.21) is not always possessed by an asset, as shown below:

Exercise 8.3.3 *Let $S_t = \frac{\beta}{1-\beta}D_t^M$ and $u(c)$ be in the CRRA class with relative risk aversion $\gamma = 0.5$. Also, let $D_{t+1}^M = \alpha_{t+1}D_t^M$, where α_{t+1} is a random variable which has two possible realizations, "good" and "bad". For any t, The good realization, $\alpha_t = 4$, occurs with probability $1/3$ and the bad realization, $\alpha_t = 1/4$, occurs with probability $2/3$. Show that (8.17) holds but (8.21) does not. In particular, show that*

$$E_t r_{t,t+1}^m > \left(E_t\beta\frac{u'(D_{t+1}^M)}{u'(D_t^M)}\right)^{-1} = r_{t,t+1}.$$

In Exercise 8.3.3 the expected rate of return to the share exceeds the risk-free rate. A positive risk premium is required for agents to hold the shares. To find the reason for the positive risk premium, let us term the ratio $\beta\frac{u'(D_{t+1}^M)}{u'(D_t^M)}$ the growth rate of the marginal utility of consumption between t and $t+1$, which is the inverse of the marginal rate of substitution between consumption in the two periods. If the realization of output is good ($\alpha_{t+1} = 4$), the share price is $S_{t+1} = 4D_t^M\beta/(1-\beta)$, the return to the share is $(S_{t+1} + D_{t+1}^M) = 4D_t^M/(1-\beta)$ and the rate of return to the share is $\frac{4}{\beta} - 1$. With the same realization, the growth rate of output between t and $t+1$ is $D_{t+1}^M/D_t^M = 4$ and the growth rate of the marginal utility of consumption is

$\beta/4$. Similarly, we can calculate the rate of return to the share and the growth rate of the marginal utility when the realization is bad. The results are summarized below:

	net rate of return	output growth	growth rate of marginal utility
good state	$\frac{4}{\beta}-1$	4	$\frac{\beta}{4}$
bad state	$\frac{1}{4\beta}-1$	$\frac{1}{4}$	4β

The rate of return to the share is higher in the good state than in the bad state. At the same time, output growth is higher in the good state than in the bad state and the growth rate of the marginal utility is lower in the good state than in the bad state. Thus, the rate of return to the share in Exercise 8.3.3 is positively correlated with output growth and negatively correlated with the growth rate of the marginal utility of consumption. This is the feature which generates a positive risk premium for the share in the above example.

Why is a negative correlation between an asset's rate of return and the growth rate of the marginal utility of consumption intimately connected to the riskiness of the asset? This is because such an asset does not provide good hedging against future income risks. An asset that provides good hedging against income risks should deliver a high return when time is bad and a low return when time is good. This requires the asset's return to be positively correlated with the growth rate of the marginal utility of consumption. An asset like the one in Exercise 8.3.3 whose return is negatively correlated with the marginal utility of consumption does exactly the opposite. When the time is bad, it delivers a low return which cannot help increasing consumption much; when the time is good, it delivers a high return which exacerbates the low marginal utility of consumption. This asset exacerbates the income risk and a positive risk premium is required to induce agents to hold it.

The above argument can be made precise by deriving the expected rate of return to an arbitrary asset. As shown earlier,

$$\begin{aligned} & E_t\left((1+r^n_{t,t+1})\cdot\beta\frac{u'(D^M_{t+1})}{u'(D^M_t)}\right) \\ = \quad & \frac{(1+E_t r^n_{t,t+1})}{1+r_{t,t+1}} + cov_t\left(r^n_{t,t+1},\beta\frac{u'(D^M_{t+1})}{u'(D^M_t)}\right). \end{aligned}$$

Then the risk premium of asset n is

$$\begin{aligned} & \text{risk premium} = -(1+r_{t,t+1})\cdot cov_t\left(r^n_{t,t+1},\beta\frac{u'(D^M_{t+1})}{u'(D^M_t)}\right) \\ & = \frac{cov_t\left(u'(D^M_{t+1}),r^n_{t,t+1}\right)}{cov_t\left(u'(D^M_{t+1}),r^m_{t,t+1}\right)}\left[E_t(r^m_{t,t+1})-r_{t,t+1}\right]. \end{aligned} \tag{8.23}$$

This result confirms our earlier intuition, which can be summarized below:

- Risk premium. If an asset's rate of return is negatively correlated with the growth rate of the marginal utility of consumption (i.e., the covariance in (8.23) is negative), the asset is risky and so must pay a positive risk premium in order to induce agents to hold it. If the asset's rate of return is positively correlated with the growth rate of marginal utility, agents are willing to hold it even though it pays a negative risk premium.

When the asset in consideration is a share (an equity), the risk premium associated with holding the share is called the equity premium. With the existence of the equity premium, we no longer have the equality between expected rates of return to bond and to equality. The equity premium is typically positive as in Exercise 8.3.3.

Exercise 8.3.4 *Show that the equity premium is positive in Exercise 8.3.3 by calculating the conditional covariance between the rate of return to the share and the growth rate of the marginal utility of consumption.*

8.3.3 Term Structure of Interest Rates

The introduction of risks also leads to a revision of the term structure of interest. This is because short-term bonds and long-term bonds may not have the same risk. To illustrate the main point, let us compare two investment strategies that cover periods t through $t+2$:

- Strategy #1: Use one unit of good to purchase one-period risk-free bonds at t, hold them to $t+1$ and use the repayment on bonds to purchase one-period bonds in $t+1$ at whatever price the bond will have then. The price of the bond is $B(t,t+1)$ at t and $B(t+1,t+2)$ at $t+1$.

- Strategy #2: Use one unit of good to purchase two-period bonds at t and hold them to maturity. The two-period bond pays one unit of goods in $t+2$ regardless of the state of the economy at $t+2$. The price of the two period bond is $B(t,t+2)$ at t.

The two strategies have the same cost measured by the marginal utility of one unit of goods, which is $u'(D_t^M)$. The main difference between them is that, at time t, the return to strategy #2 in terms of goods is certain but the return to strategy #1 is uncertain. For strategy #2, each unit of two-period bond yields one unit of good regardless of the state of the economy. Since one unit of goods at t buys $1/B(t,t+2)$ units of two-period bonds, the return to strategy #2 is $1/B(t,t+2)$ units of goods at $t+2$, which is known at time t. The rate of return to strategy #2, $R(t,t+2)=1/B(t,t+2)$, is also known at t. For strategy #1, one unit of goods buys $1/B(t,t+1)$ units of one-period bonds at t, which yields $1/B(t,t+1)$ units of consumption goods at $t+1$. This receipt is re-invested at $t+1$ in one-period bonds. The return to strategy #1 is $1/(B(t,t+1)B(t+1,t+2))$, which is uncertain at t since the price of one-period bonds at $t+1$, $B(t+1,t+2)$, is unknown at t. The rate of return to strategy #1, $R(t,t+1)R(t+1,t+2)=1/(B(t,t+1)B(t+1,t+2))$, is also uncertain at t. Of course, returns in terms of utility are uncertain for both strategies, as the marginal utility of consumption at $t+2$ is uncertain. Strategy #1 generates a return in utility $u'(D^M_{t+2})/(B(t,t+1)B(t+1,t+2))$ and strategy #2 generates a return in utility $u'(D^M_{t+2})/B(t,t+2)$.

Since strategy #2 eliminates the uncertainty in the period-$(t+2)$ payoff but strategy #1 does not, there is a sense that strategy #1 is riskier than strategy #2 in "normal" cases and should generate a risk premium in order to induce agents to hold it. Because the two strategies have the same cost, the risk premium on strategy #1 should be exactly such that the expected return in terms of utility to the two strategies be the same. That is, in equilibrium we must have

$$E_t\left(\frac{u'(D^M_{t+2})}{B(t,t+1)B(t+1,t+2)}\right) = E_t\left(\frac{u'(D^M_{t+2})}{B(t,t+2)}\right).$$

The two-period bond price $B(t,t+2)$ and the one-period bond price $B(t,t+1)$ are both known at time t so we can take them out of the expectation sign to obtain

$$B(t,t+2) = B(t,t+1)\cdot E_t(u'(D^M_{t+2})) \Bigg/ E_t\left(\frac{u'(D^M_{t+2})}{B(t+1,t+2)}\right). \tag{8.24}$$

This is a restriction between the two-period bond price and one-period bond prices (which is equivalent to the **no arbitrage condition**). Since interest rates are the inverse of the bond prices, the above equation implicitly determines the term structure of interest rates.

We can also obtain a pricing formula for two-period bonds in a way similar to (8.22). Since a two-period bond generates a return two periods after its issuing, the proper growth rate of the marginal utility in the pricing equation should be the one from period t to period $t+2$. With this modification of (8.22), the rate of return to strategy #2 must satisfy

$$E_t\left(R(t,t+2)\cdot\beta^2\frac{u'(D^M_{t+2})}{u'(D^M_t)}\right) = 1.$$

Since $R(t,t+2)$ is known at time t, we can take it out of the conditional expectation sign to obtain

$$B(t,t+2) = \frac{1}{R(t,t+2)} = E_t\left(\beta^2\frac{u'(D^M_{t+2})}{u'(D^M_t)}\right). \tag{8.25}$$

The two pricing formulas (8.24) and (8.25) give the same price in equilibrium, as shown in the following exercise:

Exercise 8.3.5 *Assume that prices of one-period bonds obey (8.18).*
(i) Show that the following relations hold

$$\begin{aligned} E_t u'(D^M_{t+2}) &= \frac{1}{\beta}E_t[B(t+1,t+2)u'(D^M_{t+1})], \\ E_t\frac{u'(D^M_{t+2})}{B(t+1,t+2)} &= \frac{1}{\beta^2}B(t,t+1)u'(D^M_t). \end{aligned}$$

(ii) Use the results in (i) to show that (8.24) can be simplified to

$$B(t,t+2) = \beta E_t \left[B(t+1,t+2) \frac{u'(D_{t+1}^M)}{u'(D_t^M)} \right]. \tag{8.26}$$

(iii) Use result (ii) to show that $B(t,t+2)$ *obeys (8.25).*

The risk premium on strategy #1 is implicit in (8.24). Since (8.24) is equivalent to (8.26), we use the latter. Applying (8.2) to the right-had side of (8.26) and using (8.18) we have

$$B(t,t+2) = B(t,t+1) \cdot E_t B(t+1,t+2) + cov_t \left(B(t+1,t+2), \beta \frac{u'(D_{t+1}^M)}{u'(D_t^M)} \right). \tag{8.27}$$

To translate this result into the term structure of interest rates, we may use the definitions of one-period and two-period net interest rates, $r_{t,t+1} = R(t,t+1) - 1$, $r_{t,t+2} = R(t,t+2) - 1$, and the following approximations.

- Approximation: Suppose that r, r_a and r_b are all sufficiently close to zero. Then

$$\frac{1}{1+r} \approx 1 - r, \quad r_a r_b \approx 0.$$

Applying these approximations to (8.27) yields

$$r_{t,t+2} = r_{t,t+1} + E_t(r_{t+1,t+2}) - cov_t \left(B(t+1,t+2), \beta \frac{u'(D_{t+1}^M)}{u'(D_t^M)} \right). \tag{8.28}$$

This is the term structure between two-period and one-period interest rates. The term structure of interest differs from the one in a deterministic environment in two ways.

- It is the (conditionally) expected future interest rate rather than the future interest rate itself that enters the term structure. This is because the interest rate in the future is random at the time t when the agent considers the relative performance of strategy #1 to strategy #2.

- The term structure is modified by the risk premium associated with strategy #1, which can be defined as

$$\begin{aligned} \text{risk premium in strategy \#1} &= [r_{t,t+1} + E_t(r_{t+1,t+2})] - r_{t,t+2} \\ &= cov_t \left(B(t+1,t+2), \beta \frac{u'(D_{t+1}^M)}{u'(D_t^M)} \right). \end{aligned}$$

  When the sum of the current one-period rate and the expected future one-period rate exceeds the two-period rate, the excess rate is a positive risk premium that must be paid to induce agents to follow strategy #1 as well as strategy #2. When the sum of the current one-period rate and the expected future one-period rate falls short of the two-period rate, the low rate is a negative risk premium associated with strategy #1.

The risk premium associated with strategy #1 is equal to the covariance between the future one-period bond price $B(t+1, t+2)$ and the growth rate of the marginal utility of consumption between t and $t+1$. The risk premium on strategy #1 is positive if and only if the future bond price is positively correlated with the growth rate of the marginal utility of consumption between t and $t+1$. To explain why this is so, note that the relative risk of strategy #1 to strategy #2 is entirely generated by the uncertainty in the future one-period bond price at $t+1$: If an agent follows strategy #1, he cannot be certain at time t about the amount of one-period bonds he can buy with the payment he receives from the one-period bond he purchases at t. This relative uncertainty can be good or bad, depending on the income level at $t+1$. To illustrate, suppose that the income at $t+1$ is low, i.e., D^M_{t+1} is low. In this case the growth rate of the marginal utility of consumption between t and $t+1$ is high. The agent would like to consume more at $t+1$. If the future bond price is positively correlated with the growth rate of the marginal utility of consumption, $B(t+1, t+2)$ will be high as well and so purchasing bonds costs a large amount of goods. Since the agent's consumption is already low, the high bond price exacerbates the situation. In this sense the uncertainty in the future bond price is bad and a positive risk premium must be paid to compensate the agent for the risk. On the other hand, if the future bond price is negatively correlated with the growth rate of the marginal utility of consumption, $B(t+1, t+2)$ will be low and so purchasing bonds does not cost a large amount of goods. The low expense on bonds helps to keep consumption at $t+1$ from falling too much. In this sense the uncertainty in the future bond price is good and the risk premium is negative.

As in the deterministic economy, the agent may also consider the possibility of reselling the two-period bond before maturity. That is, he can use one unit of good to purchase two-period bonds at t, hold them to period $t+1$, resell the bonds at whatever price available at $t+1$ and use the receipts to purchase one-period bonds in $t+1$. Let the resale price of the two-period bond be x_{t+1} in period $t+1$, which is uncertain at t. The restriction on x_{t+1} is given in the following exercise.

Exercise 8.3.6 *Assume that the one-period bond price satisfies (8.18). Show that for the reselling strategy to generate the same expected marginal utility as strategy #2 does, the resale price of the two-period bond must satisfy*

$$E_t[x_{t+1}u'(D^M_{t+1}) - \beta u'(D^M_{t+2})] = 0.$$

The yield curve now depends not only on the (expected) future one-period rate, but also on the risk premium. If $r_{t,t+n}/n$ denotes the internal rate of return for each period to a n-period bond, then $r_{t,t+n}/n$ may fall with n even when the expected one-period rate in the future is increasing.

8.4 Martingale Price Process and the Equivalent Martingale Pricing Principle

First, we present the formal definition of a martingale. A process $Y = \{Y(t); t = 0, 1, ..., T\}$ is a martingale adapted to filtration $\mathfrak{F}$ under probability $P - measure$ if

$$E_t[Y(\tau)] = Y(t) \qquad \forall \qquad \tau \geq t.$$

We claim that the equilibrium price for any asset plus its accumulated dividends (normalized) is a martingale under an equivalent probability measure - the $Q - measure$.

To prove this, we need to find out such a probability measure Q. Recall the pricing equation (8.13) for the market portfolio

$$S_t^M = E_t\left[(S_{t+1}^M + D_{t+1}^M)\frac{\beta u'(D_{t+1}^M)}{u'(D_t^M)}\right].$$

This equation is also satisfied by any asset n and the riskfree asset. That is

$$S_t^n = E_t\left[(S_{t+1}^n + D_{t+1}^n)\frac{\beta u'(D_{t+1}^M)}{u'(D_t^M)}\right].$$

and

$$B(t, t+1) = E_t\left[\frac{\beta u'(D_{t+1}^M)}{u'(D_t^M)}\right].$$

If there are s states at time t+1, we call$\left\{\frac{\beta u'(D_{t+1}^{M^s})}{u'(D_t^M)}\right\}$the state price vector.

Before we use terms like **Radon-Nikodym derivative ,** let us present some intuitive argument: rewrite the above equation as

$$\sum\nolimits_{P-measure} \frac{1}{B(t,t+1)} \frac{\beta u'(D_{t+1}^M)}{u'(D_t^M)} \cdot P(\cdot \mid \mathfrak{F}_t) = 1.$$

Since $\sum\nolimits_{P-measure} \frac{1}{B(t,t+1)} \frac{\beta u'(D_{t+1}^{M^s})}{u'(D_t^M)} \cdot P^s(\cdot \mid \mathfrak{F}_t) = 1$ and $\frac{1}{B(t,t+1)} \frac{\beta u'(D_{t+1}^{M^s})}{u'(D_t^M)} \cdot P^s(\cdot \mid \mathfrak{F}_t) > 0$. We can treat each element $\frac{1}{B(t,t+1)} \frac{\beta u'(D_{t+1}^{M^s})}{u'(D_t^M)} \cdot P^s(\cdot \mid \mathfrak{F}_t)$ in state s as a conditional probability (the $Q - measure$) with respect to the filtration $\mathfrak{F}_t$. That is

$$Q^s(\cdot \mid \mathfrak{F}_t) = \frac{1}{B(t,t+1)} \frac{\beta u'(D_{t+1}^{M^s})}{u'(D_t^M)} \cdot P^s(\cdot \mid \mathfrak{F}_t)$$

and

$$\sum\nolimits_{Q-measure} Q^s(\cdot \mid \mathfrak{F}_t) = \sum\nolimits_{P-measure} \frac{1}{B(t,t+1)} \frac{\beta u'(D_{t+1}^{M^s})}{u'(D_t^M)} \cdot P^s(\cdot \mid \mathfrak{F}_t) = 1.$$

Note that the above $Q - measure$ can be more rigorously introduced by use **Radon-Nikodym derivative** as

$$dQ \mid \mathfrak{F}_t = \frac{1}{B(t,t+1)} \frac{\beta u'(D^M_{t+1})}{u'(D^M_t)} \cdot dP \mid \mathfrak{F}_t.$$

See Duffie (1996) for such an elegant treatment.

To show the martingale, we start with the equilibrium price

$$\begin{aligned} S^n_t &= E_t\left[(S^n_{t+1} + D^n_{t+1}) \frac{\beta u'(D^M_{t+1})}{u'(D^M_t)}\right] \\ &= \sum\nolimits_{P-measure} (S^{n^s}_{t+1} + D^{n^s}_{t+1}) \cdot \frac{\beta u'(D^{M^s}_{t+1})}{u'(D^M_t)} \cdot P^s(\cdot \mid \mathfrak{F}_t) \\ &= \sum\nolimits_{Q-measure} (S^{n^s}_{t+1} + D^{n^s}_{t+1}) \cdot B(t,t+1) \cdot Q^s(\cdot \mid \mathfrak{F}_t) \\ &= B(t,t+1) E^Q_t (S^n_{t+1} + D^n_{t+1}). \end{aligned}$$

$\Rightarrow$

$$\frac{S^n_t}{B(t,t+1)} = E^Q_t (S^n_{t+1} + D^n_{t+1}).$$

Define the normalized price for asset n with a normalization period h as

$$\widetilde{S}^n_{t,t+h} = \frac{S^n_t}{B(t,t+h)}$$

and the normalized accumulated dividends for asset n with a normalization period h as

$$\widetilde{d}^n_{t,t+h} = \sum_{\tau=0}^{t} \frac{D^n_\tau}{B(\tau,t+h)}, \qquad \forall \qquad t = 0,1,2,...,T-h.$$

Then

$$\begin{aligned} \widetilde{S}^n_{t,t+1} + \widetilde{d}^n_{t,t+1} &= \frac{S^n_t}{B(t,t+1)} + \sum_{\tau=0}^{t} \frac{D^n_\tau}{B(\tau,t+1)} \\ &= E^Q_t\left(S^n_{t+1} + D^n_{t+1} + \sum_{\tau=0}^{t} \frac{D^n_\tau}{B(\tau,t+1)}\right) \\ &= E^Q_t\left(\frac{S^n_{t+1}}{B(t+1,t+1)} + \sum_{\tau=0}^{t} \frac{D^n_\tau}{B(\tau,t+1)} + \frac{D^n_{t+1}}{B(t+1,t+1)}\right) \\ &= E^Q_t\left(\frac{S^n_{t+1}}{B(t+1,t+1)} + \sum_{\tau=0}^{t+1} \frac{D^n_\tau}{B(\tau,t+1)}\right) \\ &= E^Q_t(\widetilde{S}^n_{t+1,t+1} + \widetilde{d}^n_{t+1,t+1}). \end{aligned}$$

Therefore, we have shown that the normalized equilibrium price and the accumulated dividends is a Martingale under the $Q - measure$.

Note that the above discussion can also been found in Huang and Litzenberger from page 223 to page 256 in chapter 8.

Exercise 8.4.1 *Show*

$$\widetilde{S}^n_{t,\tau} + \widetilde{d}^n_{t,\tau} = E^Q_t(\widetilde{S}^n_{\tau,\tau} + \widetilde{d}^n_{\tau,\tau}) \qquad \forall \qquad \tau \geq t.$$

Now we can turn to the Equivalent Martingale Pricing Principle. In fact, we have just shown that, for any asset, its price can be expressed as

$$\begin{aligned} S^n_t &= E_t\left[(S^n_{t+1} + D^M_{t+1})\frac{\beta u'(D^M_{t+1})}{u'(D^M_t)}\right] = B(t,t+1)E^Q_t(S^n_{t+1} + D^n_{t+1}) \\ &= \frac{1}{1+r(t,t+1)}E^Q_t(S^n_{t+1} + D^n_t). \end{aligned}$$

In other words, asset n's price at time t is equal to the expected future price plus dividend discounted at the riskfree rate where the expectation is calculated under the equivalent probability $Q - measure$.

Recall in the two-period model, we have shown that

No arbitrage $\Longleftrightarrow$ existence of state prices that "value" securities.

By normalizing (i.e., choosing the $Q - measure$), the vector of state prices becomes a probability vector so that securities are valued in a risk-neutral way, with this probability vector. This is an equivalent characterization of no-arbitrage. This is the approach in the dynamic/arbitrage valuation. The vector of state prices (or probability) that value securities is called the **pricing kernel**.

This is a powerful result for asset valuation. It indicates that, if we can find an equivalent probability $Q - measure$, we don't need to know the agent's utility function to determine the price, we can discount the future payoffs at the riskfree risk if the expectation is calculated with the $Q - measure$.

Such a result has been extensively used for pricing derivative assets. However, one needs to exercise some caution when implementing this approach. For example, if a derivative security's value can be analytically expressed in terms of the parameters associated with the $Q{-}measur\dot{e}$, how can one obtain the estimates of such parameters from actual observed price history of the underlying asset?

Exercise 8.4.2 *Show*

$$\begin{aligned} S^n_t &= \frac{1}{1+r(t,\tau)}E^Q_t(S^n_\tau + d^n_\tau) \\ & \quad \textit{where} \\ d^n_\tau &= \sum_{i=1}^{\tau-t}(1+r(t+i,\tau))D^n_{t+i}. \end{aligned}$$

Chapter 9

OTHER TYPES OF ASSETS

9.1 Forward Contracts

When the future is uncertain, securities that deliver non-random returns may be demanded by rational agents even though they have a low rate of return. The risk-free bond is one example. Another example is a forward contract (or futures contract). A forward contract is a contract between two agents which specifies that one of the agents buys a good or security from the other agent at a future date at a predetermined price. To illustrate, consider two agents in the described economy, agent A and agent B, who meet at time t. Agent A signs a contract with agent B, which specifies:

- Agent A will buy from agent B one share at a future date $t+h$ at a price $F_t(S_t, h)$. Agent A is not obliged to pay agent B at t and agent B is not obliged to supply agent A one share at t. Only when time $t+h$ comes is the contract fulfilled, at which time the market price of the share can in general be different from the price specified by the contract.

What is the value of $F_t(S_t, h)$ that is consistent with equilibrium? Assume the signing of the contract itself is costless. Consider the marginal utility and cost of buying one share at a price $F_t(S_t, h)$. By buying one share, agent A gives up $F_t(S_t, h)$ units of goods in period $t+h$. Since each unit of good has a marginal utility $u'(D^M_{t+h})$, the marginal cost of the buying the share is $F_t(S_t, h)u'(D^M_{t+h})$. To calculate the marginal benefit of such a purchase, note that agent A can immediately sell the share in period $t+h$ at the market price S_{t+h}. The marginal benefit of the purchase is thus $S_{t+h}u'(D^M_{t+h})$. Since the market price of the share at time $t+h$ and the predetermined price are not necessarily the same, the marginal benefit and cost of purchasing the share are not necessarily equal to each other. In fact, the two prices will differ for some states of the economy as long as the market price is a random variable. However, such equality between the actual marginal benefit and cost is not necessary for agents to be rational. What is necessary is that the marginal benefit and cost of purchasing the share are *perceived* to be the same at the time of signing the contract. That is, the expected marginal benefit and cost, conditional on information available at t, are the same. If the expected marginal cost of buying the share exceeds the expected marginal utility of the purchase, agent A will not sign the contract. If the expected marginal cost of buying the share is less than the expected marginal utility of the purchase, agent B will not sign the contract. Thus, the only equilibrium value of $F_t(S_t, h)$ must satisfy

$$E_t\left[F_t(S_t, h)u'(D_{t+h}^M)\right] = E_t\left[S_{t+h}u'(D_{t+h}^M)\right].$$

Because the price $F_t(S_t, h)$ is determined at t, it is known at time t and so we can take it out of the expectation sign. Then

$$F_t(S_t, h) = \frac{E_t\left[S_{t+h}u'(D_{t+h}^M)\right]}{E_t\left[u'(D_{t+h}^M)\right]}. \tag{9.1}$$

Let us compare the forward share price $F_t(S_t, h)$ with the expected market price at $t+h$. Using the formula (8.2), we can rewrite (9.1) as

$$F_t(S_t, h) - E_t(S_{t+h}) = \frac{cov_t(S_{t+h}, u'(D_{t+h}^M))}{E_t\left[u'(D_{t+h}^M)\right]}. \tag{9.2}$$

The forward share price is less than the expected market price of the share in the future if and only if the share price in the future is positively correlated with future output, i.e., $cov_t(S_{t+h}, u'(D_{t+h}^M)) < 0$. To explain this result, suppose that the future price of the share and future output are positively correlated. We argue intuitively that the forward price $F_t(S_t, h)$ should be less than the expected market price. Consider agent A (analyzing agent B's behavior leads to the same conclusion). By selling the share purchased at the predetermined price $F_t(S_t, h)$, agent A's receipts of goods is $S_{t+h}/F_t(S_t, h)$ units of goods in period $t+h$. Since the market price at $t+h$ is positively correlated with output at $t+h$, such receipt will be low when output is low in period $t+h$ and hence does not provide good hedging against income fluctuations. Agent A's incentive to buy the share is weak and so the forward price must be low. Conversely, if the market price of the share in the future is negatively correlated with future output, buying the share at the predetermined price and reselling it at the market price provides good hedging against income fluctuations. Agent A has a strong incentive to buy the share at a predetermined price and so the forward price should be higher than the expected future price of the share.

In the derivation for $F_t(S_t, h)$, we implicitly assumed that agent A will use goods at time $t+h$ to pay for the share at the predetermined price. Suppose, to agent A's disappointment, that goods are perceived to be scarce in period $t+h$. As a prudent agent, agent A might want to purchase n-period bonds at t and use the repayment from bonds in period $t+h$ to finance the purchase of the share at the predetermined price $F_t(S_t, h)$. If he chooses such a strategy, he must purchase $F_t(S_t, h)$ units of h-period bonds at t in order to finance the purchase of the share at time $t+h$ (since each bond repays one unit of goods). The amount of goods used to purchase the n-period bonds at t is $F_t(S_t, h)B(t, t+h)$. Would this strategy make any difference? The following exercise provides an answer.

Exercise 9.1.1 *Suppose that agent A purchases $F_t(S_t, h)$ units of h-period bonds at t and uses the payment on these bonds to finance the purchase at $t+h$ of the share* *at the predetermined price $F_t(S_t, h)$. Show that, if the bond price $B(t, t+h)$ satisfies (8.25), this financing strategy does not make any gain or loss (in terms of expected marginal utility) relative to the strategy of directly using goods in $t+h$ to finance the purchase.*

The forward contract discussed above is a binding agreement on buying and selling future shares at a given price. Forward contracts can also be designed on buying and selling future bonds. For example, agent A can sign a contract with agent B at time t to purchase one unit of one-period bond in period $t+h$ at a pre-specified price $F_t(B(t+h, t+h+1), h)$. We can again use the expected marginal utility-cost analysis to find such a price that is consistent with equilibrium. The marginal cost of purchasing the bond at the price $F_t(B(t+h, t+h+1), h)$ in terms of period-$(t+h)$ utility is $F_t(B(t+h, t+h+1), h)u'(D^M_{t+h})$. Since the agent can immediately resell the bond in period $t+h$ at the market price for bonds, $B(t+h, t+h+1)$, the marginal benefit is $B(t+h, t+h+1)u'(D^M_{t+h})$. For agents to sign the contract at t, the pre-specified price must satisfy

$$E_t[F_t(B(t+h, t+h+1), h)u'(D^M_{t+h})] = E_t[B(t+h, t+h+1)u'(D^M_{t+h})]. \tag{9.3}$$

Exercise 9.1.2 *Show that (9.3) can be simplified to*

$$F_t(B(t+h, t+h+1), h) = \frac{B(t, t+h+1)}{B(t, t+h)}, \tag{9.4}$$

where $B(t, t+h)$ is the price of a h-period bond issued at t.

The introduction of forward bond contracts leads to another notion of risk premium. To ease exposition, let $h = 1$. If agent A signs a forward contract to buy one-period bonds in period $t+1$ at a fixed price, he can successfully eliminate the fluctuations in the bond price at $t+1$. If fluctuations in the bond price are "bad", the agent must pay a higher price than expected future bond price in order to eliminate this risks. Thus the difference between the bond forward price, $F_t(B(t+1, t+2), 1)$, and the expected market price for bonds, $E_t(B(t+1, t+2))$, is a measure of the risk premium for holding one-period bonds for two consecutive periods. Let us check what this measure looks like. Substituting (8.26) into (9.4) to eliminate $B(t, t+2)$ yields

$$\begin{aligned} & F_t(B(t+1, t+2), 1) - E_t(B(t+1, t+2)) \\ &= \tfrac{1}{B(t,t+1)} cov_t \left(B(t+1, t+2), \beta \tfrac{u'(D^M_{t+1})}{u'(D^M_t)} \right). \end{aligned} \tag{9.5}$$

Consistent with the earlier result, holding one-period bonds for two consecutive periods has a positive risk premium if the future bond price is positively correlated with the growth rate of the marginal utility of consumption between t and $t+1$.

9.2 Options and Derivative Securities

Bonds and stocks are primary securities in this economy. One can design secondary securities whose payoffs in the future depend on future prices of these primary securities. These securities are derivative securities. Options are examples of derivative securities. Options whose future payoffs are contingent on future bond prices are bond options and options whose payoffs are contingent on future stock prices are stock options. To illustrate bond options, consider the following economy:

Example 4 *Let the utility function be $u(c) = \ln c$. Output follows the process $D^M_{t+1} = \alpha_{t+1} D^M_t$, where α_{t+1} is a random variable which has three possible realizations, "good" ($\alpha = 4$), "moderate" ($\alpha = 1$) and "bad" ($\alpha = 1/4$). The transition matrix for α is*

		α_{t+1}		
		1/4	1	4
	1/4	1/2	1/4	1/4
α_t	1	1/3	1/3	1/3
	4	1/4	1/4	1/2

For example, if the state of the economy is bad at t ($\alpha_t = 1/4$), the economy at $t+1$ will be in a bad state with probability 1/2, in a moderate state with probability 1/4 and in a good state with probability 1/4. Since the economy is more likely to be in a bad (good) state at $t+1$ if the economy is in a bad (good) state at t, the states of the economy are positively correlated over time.

In this economy, the bond price at time t is

$$B(t, t+1) = \beta E_t \frac{u'(D^M_{t+1})}{u'(D^M_t)} = \beta E_t \frac{\frac{1}{D^M_{t+1}}}{\frac{1}{D^M_t}} = \beta E_t \frac{D^M_t}{D^M_{t+1}} = \beta E_t \frac{1}{\alpha_{t+1}}.$$

The bond price at time t depends on the state of the economy at t (i.e., on what α_t is). There are three possible values for the bond price, each corresponding to a state. If the economy is in a bad state at t, the bond price is denoted B^b_t. If the economy is in a moderate state at t, the bond price is denoted B^m_t. If the economy is in a good state at t, the bond price is denoted B^g_t. Given the transition matrix, we can calculate these prices as follows:

$$B^b_{t,t+1} = \beta\left(\frac{1}{2}\times\frac{1}{1/4} + \frac{1}{4}\times\frac{1}{1} + \frac{1}{4}\times\frac{1}{4}\right) = \frac{37\beta}{16};$$

$$B^m_{t,t+1} = \beta\left(\frac{1}{3}\times\frac{1}{1/4} + \frac{1}{3}\times\frac{1}{1} + \frac{1}{3}\times\frac{1}{4}\right) = \frac{7\beta}{4};$$

$$B^g_{t,t+1} = \beta\left(\frac{1}{4}\times\frac{1}{1/4} + \frac{1}{4}\times\frac{1}{1} + \frac{1}{2}\times\frac{1}{4}\right) = \frac{11\beta}{8}.$$

Now someone has invented the following security at t:

- Call option: The security gives the holder the right to buy a unit of one-period bond at the future date $t+1$ at a pre-specified price K. If the holder of the security does not buy the bond at $t+1$, the security is expired and the holder receives nothing.

This security is similar to a forward contract – the price of a future transaction is fixed by the contract in advance. The difference is that the above security gives the holder, say, agent A, the *option* not to buy at $t+1$. The pre-specified level K is termed the *strike price* or exercise price. The date $t+1$ is the expiration date of the call option. If the bond price exceeds the strike price, i.e., $B(t+1,t+2) > K$, the situation is called "in the money", where it is optimal to buy the bond at the strike price and make the gain $B(t+1,t+2)-K$. If agent A buys the bond, the call option is *exercised* and the strike price is also called the exercise price. If $B(t+1,t+2) < K$, the situation is called "out of the money", where it is optimal not to exercise the option since it would generate a loss $B(t+1,t+2)-K$. If $B(t+1,t+2) = K$, the situation is called "at the money", where exercising and not exercising the option generate the same payoff 0.

The call option can be sold in the market. How much would agent A be willing to pay for the call option? (When an agent buys a call option, he is said to "long a call".) Clearly, the price of the call option depends on two features:

- The strike price. The strike price determines not only the price for the possible future transaction but also how likely the option will be exercised.
- The state of the economy at t when the option is sold. The state of the economy at t determines agents' expectations about future bond prices and how likely the option will be exercised.

To find the price of the call option on bonds, we examine the cost and the expected benefit of buying the call option. Let us assume $K = 7\beta/4$ and that the state of the economy at t is $\alpha_t = 1/4$ ("bad"). Denote the corresponding price of the call option by CL_t^b to emphasize its dependence on the state of the economy. If agent A buys one unit of the option, the cost is CL_t^b units of goods at time t. In terms of utility, the cost is

$$\text{cost of a call option in terms of utility} = CL_t^b \cdot u'(D_t^M) = \frac{CL_t^b}{D_t^M}.$$

The benefit is the chance that agent A can make a gain by buying a bond at the strike price less than or equal to the market price. The actions of agent A at time $t+1$ and the gains are summarized as follows:

	$B(t+1,t+2)$		
$K = 7\beta/4$	$\frac{37\beta}{16}$ ($\alpha_{t+1} = \frac{1}{4}$)	$\frac{7\beta}{4}$ ($\alpha_{t+1} = 1$)	$\frac{11\beta}{8}$ ($\alpha_{t+1} = 4$)
actions	exercise	indifferent	not exercise
gains (goods)	$\frac{37\beta}{16} - \frac{7\beta}{4} = \frac{9\beta}{16}$	0	0
gains (utility)	$\frac{9\beta}{16} u'(D_{t+1}^M) = \frac{9\beta}{4D_t^M}$	0	0

Notice that the gains in terms of utility are different when the future state of the economy (α_{t+1}) varies.

Since the future benefit from purchasing the option depends on the future state α_{t+1}, agent A must assess the likelihood of each state at time t. The expectations are conditional on the current state α_t. Given $\alpha_t = 1/4$, the transition matrix for α states that $\alpha_{t+1} = 1/4$ with probability 1/2, $\alpha_{t+1} = 1$ with probability 1/4 and $\alpha_{t+1} = 4$ with probability 1/4. Thus the expected benefit in terms of utility from purchasing the option is

$$\frac{1}{2} \times \frac{9\beta}{4D_t^M} + \frac{1}{4} \times 0 + \frac{1}{4} \times 0 = \frac{9\beta}{8D_t^M}.$$

In equilibrium, the discounted value of this expected benefit must be equal to the cost of the option. That is, $\frac{9\beta^2}{8D_t^M} = \frac{CL_t^b}{D_t^M}$. The call option price sold in a bad state of the economy at t is

$$CL_t^b = 9\beta^2/8.$$

The call option price sold in other states of the economy can be solved similarly:

Exercise 9.2.1 *In the above economy show that a call option on bonds sold in a moderate state at t is $CL_t^m = 3\beta^2/4$ and the price in a good state of the economy is $CL_t^g = 9\beta^2/16$.*

Note that $CL_t^b > CL_t^m > CL_t^g$. This is because in this example the bond price is negatively correlated with output growth and because the state of the economy is positively correlated over time. When the state of economy is bad at t, as we assumed $\alpha_t = 1/4$, the economy at $t+1$ is likely to be in a bad state as well under the positive correlation of the states over time. At the same time, the bond price at $t+1$ is likely to be high, since bond prices are negatively correlated with output in this example. A call option purchased at a bad state has a high expected gain at $t+1$ and hence must be sold for a high price.

To illustrate the dependence of the call option price on the strike price, let us now lower the strike price to $K = 6\beta/4$ but continue to assume $\alpha_t = 1/4$ ("bad"). If agent A buys a call option at t, his gains at time $t+1$ are summarized as follows:

	$B(t+1,t+2)$		
	$\frac{37\beta}{16}$ ($\alpha_{t+1} = \frac{1}{4}$)	$\frac{7\beta}{4}$ ($\alpha_{t+1} = 1$)	$\frac{11\beta}{8}$ ($\alpha_{t+1} = 4$)
actions	exercise	exercise	not exercise
gains (goods)	$\frac{37\beta}{16} - \frac{6\beta}{4} = \frac{13\beta}{16}$	$\frac{7\beta}{4} - \frac{6\beta}{4} = \frac{\beta}{4}$	0
gains (utility)	$\frac{13\beta}{16}u'(D_{t+1}^M) = \frac{13\beta}{4D_t^M}$	$\frac{\beta}{4}u'(D_{t+1}^M) = \frac{\beta}{4D_t^M}$	0

The gains in utility expected at t is

$$\frac{1}{2} \times \frac{13\beta}{4D_t^M} + \frac{1}{4} \times \frac{\beta}{4D_t^M} + \frac{1}{4} \times 0 = \frac{27\beta}{16D_t^M}.$$

In this case the call option price at t is $CL_t^b = 27\beta^2/16$. This is higher than the call option price when $K = 7\beta/4$. This is true also when the state at time t is m or g.

Exercise 9.2.2 *When $K = 6\beta/4$, show that a call option on bonds sold in a moderate state at t is $CL_t^m = 7\beta^2/6$ and the price in a good state of the economy is $CL_t^g = 7\beta^2/8$.*

As it turns out, the above result on call option prices can be made consistent with the pricing equation for bonds and shares, (8.22). To use (8.22), we first calculate the return on the call option. A call option on bonds generates a return $B(t+1, t+2) - K$ if $B(t+1, t+2) \geq K$ and 0 otherwise. Thus the return is $\max(B(t+1, t+2) - K, 0)$. (A diagram is useful here.) If we denote the rate of return to the call option by $R_t^{CL}(K)$ then

$$R_t^{CL}(K) = \frac{\max(B(t+1, t+2) - K, 0)}{CL_t(K)}.$$

We claim that R_t^{CL} must satisfy the equation (8.22) with $R^i(t, t+1)$ being replaced by $R_t^{CL}(K)$. Thus,

$$CL_t(K) = \beta E_t\left(\max(B(t+1, t+2) - K, 0) \cdot \frac{u'(D_{t+1}^M)}{u'(D_t^M)}\right). \tag{9.6}$$

Exercise 9.2.3 *Consider the economy examined in this subsection. Let $K = 7\beta/4$ and assume that the state of the economy at t is $\alpha_t = 1/4$.*
(i) Calculate R_t^{CL} for $\alpha_{t+1} = 1/4$, 1 and 4 respectively.
(ii) Show that the price CL_t^b calculated from (9.6) is the same as the above, $9\beta^2/8$.

Having examined an option to buy bonds, we now examine an option to sell bonds in the future at a pre-specified price – a put option.

- A put option on (one-period) bonds at t is a security that gives the holder the option to sell one-period bonds in the future date $t+1$ at a pre-specified price K. If the option is not exercised at $t+1$, the option is expired.

If the option is in the money, $B(t+1, t+2) < K$, it is optimal to exercise the option. If the option is in out of the money, $B(t+1, t+2) > K$, it is optimal not to exercise the option. If the option is at the money $B(t+1, t+2) = K$, the agent is indifferent between exercising and not exercising. Let the price of a put option be PT_t^b at time t when the state is bad. Like the price CL_t^b, it can be calculated by examining the cost and expected benefit of buying the option. We leave this as an exercise (below). Rather, we use the short-cut, (8.22), to derive its equilibrium price. The gain to the put option is $\max(K - B(t+1, t+2), 0)$ and the rate of return to the put option, denoted $R_t^{PT}(K)$, is

$$R_t^{PT}(K) = \frac{\max(K - B(t+1, t+2), 0)}{PT_t(K)}.$$

(A diagram is useful here to illustrate the return to the put.) We claim that R_t^{PT} satisfies the pricing equation (8.22) and so the put price is

$$PT_t(K) = \beta E_t \left(\max(K - B(t+1, t+2), 0) \cdot \frac{u'(D_{t+1}^M)}{u'(D_t^M)} \right). \tag{9.7}$$

Exercise 9.2.4 *Consider the economy examined in this subsection. Let $K = 7\beta/4$ and assume that the state of the economy at t is $\alpha_t = 1/4$.*
(i) Use the cost-benefit analysis to show that the price of a put option on bond at t and in state "bad" is $PT_t^b = 3\beta^2/128$.
(ii) Use (9.7) to show that same result.

Put and call options are closely related. A call option generates a gain only if $B(t+1, t+2)$ exceeds the strike price and a put option generates a gain only if the strike price exceeds $B(t+1, t+2)$. For the same strike price K, if exercising a call option generates a gain, then exercising a put option does not; and vice versa. This connection between the put price and the call price is the put-call parity condition. More precisely, consider an investment strategy that buys a call option at t (long a call) and sells a put option simultaneously (short a put) with the same strike price K. The net cost of such investment is $CL_t(K) - PT_t(K)$. The return to the investment at $t+1$ is

$$\begin{aligned} & \max(B(t+1, t+2) - K, 0) - \max(K - B(t+1, t+2), 0) \\ = \; & \max(B(t+1, t+2) - K, 0) + \min(B(t+1, t+2) - K, 0) = B(t+1, t+2) - K. \end{aligned}$$

(A diagram is useful here to illustrate the equality.) Applying the pricing equation (8.22) to this investment strategy, we have

$$CL_t(K) - PT_t(K) = \beta E_t \left[(B(t+1, t+2) - K) \cdot \frac{u'(D_{t+1}^M)}{u'(D_t^M)} \right]. \tag{9.8}$$

In fact, subtracting (9.7) from (9.6) generates the same result.

There is an interesting link among the call price, the put price and the forward bond price, as illustrated below:

Exercise 9.2.5 *Show that $CL_t(K) = PT_t(K)$ if the strike price is equal to the one-period forward bond price, i.e., if $K = F_t(B(t+1, t+2), 1)$. Can you explain why such an equality emerges at this special strike price?*

Exercise 9.2.6 *Consider put and call options at t on shares rather than on bond. Derive the pricing formulas for such put and call options and the corresponding put-call parity condition.*

9.3 Derivative Security Valuation with Equivalent Martingale Pricing Principle

We use the above example to illustrate the equivalent Martingale approach. Recall that the $Q - measure$ is defined as

$$Q^s(\cdot \mid \mathfrak{F}_t) = \frac{1}{B(t,t+1)} \frac{\beta u'(D_{t+1}^{M^s})}{u'(D_t^M)} \cdot P^s(\cdot \mid \mathfrak{F}_t).$$

We know that

$$B_{t,t+1}^b = \beta\left(\frac{1}{2}\times 4 + \frac{1}{4}\times 1 + \frac{1}{4}\times\frac{1}{4}\right) = \frac{37\beta}{16};$$

$$B_{t,t+1}^m = \beta\left(\frac{1}{3}\times\frac{1}{4} + \frac{1}{3}\times 1 + \frac{1}{3}\times 4\right) = \frac{7\beta}{4};$$

$$B_{t,t+1}^g = \beta\left(\frac{1}{4}\times 4 + \frac{1}{4}\times 1 + \frac{1}{2}\times\frac{1}{4}\right) = \frac{11\beta}{8}.$$

Given $D_{t+1}^M = \alpha_{t+1} D_t^M$ and $u(c) = \ln c$. We have

$$\begin{aligned} Q^s(\cdot \quad | \quad \mathfrak{F}_t) &= \frac{1}{B(t,t+1)} \frac{\beta/(\alpha_{t+1}D_t^M)}{1/D_t^M} \cdot P^s(\cdot \mid \mathfrak{F}_t) \\ &= \frac{\beta}{\alpha_{t+1}B(t,t+1)} \cdot P^s(\cdot \mid \mathfrak{F}_t). \end{aligned}$$

The transition matrix under $Q - measure$ is

		$Q_{t+1} \mid \mathfrak{F}_t$		
		bad	moderate	good
Q_t	bad	$\frac{\beta}{\frac{1}{4}\cdot\frac{37\beta}{16}} \times \frac{1}{2} = \frac{32}{37}$	$\frac{\beta}{1\cdot\frac{37\beta}{16}} \times \frac{1}{4} = \frac{4}{37}$	$\frac{\beta}{4\cdot\frac{37\beta}{16}} \times \frac{1}{4} = \frac{1}{37}$
	moderate	$\frac{\beta}{\frac{1}{4}\cdot\frac{7\beta}{4}} \times \frac{1}{3} = \frac{16}{21}$	$\frac{\beta}{1\cdot\frac{7\beta}{4}} \times \frac{1}{3} = \frac{4}{21}$	$\frac{\beta}{4\cdot\frac{7\beta}{4}} \times \frac{1}{3} = \frac{1}{21}$
	good	$\frac{\beta}{\frac{1}{4}\cdot\frac{11\beta}{8}} \times \frac{1}{4} = \frac{8}{11}$	$\frac{\beta}{1\cdot\frac{11\beta}{8}} \times \frac{1}{4} = \frac{2}{11}$	$\frac{\beta}{4\cdot\frac{11\beta}{8}} \times \frac{1}{2} = \frac{1}{11}$

Given the strike price $K = 7\beta/4$

$K = 7\beta/4$	$B(t+1,t+2)$ $\frac{37\beta}{16}$ $(\alpha_{t+1} = \frac{1}{4})$	$\frac{7\beta}{4}$ $(\alpha_{t+1} = 1)$	$\frac{11\beta}{8}$ $(\alpha_{t+1} = 4)$
actions	exercise	indifferent	not exercise
gains (goods)	$\frac{37\beta}{16} - \frac{7\beta}{4} = \frac{9\beta}{16}$	0	0

Then the call price is

$$CL_t^b = B(t,t+1)E_t^Q[\max(B(t+1,t+2) - K, 0)] = \frac{37\beta}{16} \times \left(\frac{32}{37}\cdot\frac{9\beta}{16}\right) = \frac{9\beta^2}{8}$$

$$CL_t^m = B(t,t+1)E_t^Q[\max(B(t+1,t+2) - K, 0)] = \frac{7\beta}{4} \times \left(\frac{16}{21}\cdot\frac{9\beta}{16}\right) = \frac{3\beta^2}{4}$$

$$CL_t^g = B(t,t+1)E_t^Q[\max(B(t+1,t+2)-K,0)] = \frac{11\beta}{8} \times \left(\frac{8}{11} \cdot \frac{9\beta}{16}\right) = \frac{9\beta^2}{16}$$

which confirms the results obtained before.

We can also verify the results with $K = 6\beta/4$ but continue to assume $\alpha_t = 1/4$ ("bad"). If agent A buys a call option at t, his gains at time $t+1$ are summarized as follows:

	$B(t+1,t+2)$		
	$\frac{37\beta}{16}$ $(\alpha_{t+1} = \frac{1}{4})$	$\frac{7\beta}{4}$ $(\alpha_{t+1} = 1)$	$\frac{11\beta}{8}$ $(\alpha_{t+1} = 4)$
actions	exercise	exercise	not exercise
gains (goods)	$\frac{37\beta}{16} - \frac{6\beta}{4} = \frac{13\beta}{16}$	$\frac{7\beta}{4} - \frac{6\beta}{4} = \frac{\beta}{4}$	0

$$CL_t^b = B(t,t+1)E_t^Q[\max(B(t+1,t+2)-K,0)] = \frac{37\beta}{16} \times \left(\frac{32}{37} \cdot \frac{13\beta}{16} + \frac{4}{37} \cdot \frac{\beta}{4}\right) = \frac{27\beta^2}{16}$$

$$CL_t^m = B(t,t+1)E_t^Q[\max(B(t+1,t+2)-K,0)] = \frac{7\beta}{4} \times \left(\frac{16}{21} \cdot \frac{13\beta}{16} + \frac{4}{21} \cdot \frac{\beta}{4}\right) = \frac{7\beta^2}{6}$$

$$CL_t^g = B(t,t+1)E_t^Q[\max(B(t+1,t+2)-K,0)] = \frac{11\beta}{8} \times \left(\frac{8}{11} \cdot \frac{13\beta}{16} + \frac{2}{11} \cdot \frac{\beta}{4}\right) = \frac{7\beta^2}{8}.$$

Example 5 *Let us consider a call option written on the market portfolio with maturity $t+1$. The strike price is $K = \frac{\beta}{1-\beta}\ D_t^M$. Recall that the stock price under $u(c) = lnc$ is $S_t = \frac{\beta}{1-\beta} D_t^M$.*

	S_{t+1}		
	$\frac{\beta}{1-\beta}\ \frac{D_t^M}{4}$	$\frac{\beta}{1-\beta}\ D_t^M$	$\frac{\beta}{1-\beta}\ 4D_t^M$
	$(\alpha_{t+1} = \frac{1}{4})$	$(\alpha_{t+1} = 1)$	$(\alpha_{t+1} = 4)$
actions	*not exercise*	*not exercise*	*exercise*
gains (goods)	0	0	$\frac{\beta}{1-\beta}4D_t^M - \frac{\beta}{1-\beta}\ D_t^M = \frac{3\beta D_t^M}{1-\beta}$

$$CL_t^b = B(t,t+1)E_t^Q[\max(S_{t+1}-K,0)] = \frac{37\beta}{16} \times \left(\frac{1}{37} \cdot \frac{3\beta D_t^M}{1-\beta}\right) = \frac{3\beta^2 D_t^M}{16(1-\beta)}$$

$$CL_t^m = B(t,t+1)E_t^Q[\max(S_{t+1}-K,0)] = \frac{7\beta}{4} \times \left(\frac{1}{21} \cdot \frac{3\beta D_t^M}{1-\beta}\right) = \frac{\beta^2 D_t^M}{4(1-\beta)}$$

$$CL_t^g = B(t,t+1)E_t^Q[\max(S_{t+1}-K,0)] = \frac{11\beta}{8} \times \left(\frac{1}{11} \cdot \frac{3\beta D_t^M}{1-\beta}\right) = \frac{3\beta^2 D_t^M}{8(1-\beta)}.$$

Exercise 9.3.1 *(1) Use the equilibrium pricing equation to price the above call option on the market portfolio and verify the prices obtained under the equivalent Martingale approach.*

$$CL_t = E_t[\frac{u'(D_{t+1}^M)}{u'(D_t^M)} \max(S_{t+1}-K,0)].$$

(2) Use both the equilibrium pricing equation and the equivalent Martingale approach to price the corresponding put option on the market portfolio and verify the put-call parity condition.

Summary 1 *In this section we have covered the following main topics:*
(i) Intertemporal maximization when there is aggregate uncertainty;
(ii) Asset pricing equations in a stochastic economy;
(iii) Consumption-based dynamic CAPM;
(iv) Equivalent Martingale Pricing Principle;
(v) The notion of risk and the term structure of interest rates;
(vi) Pricing equation for forward contracts, call options and put options.

Chapter 10

CONTINUOUS-TIME FINANCE

The chapter casts the discussions in Chapters 8 and 9 in a continuous-time set-up. We will first discuss the individual consumption-portfolio problem. The lecture note is based on Merton's work (see references in the course outline). Once we understand the individual optimization problem, we then proceed with the definition of competitive equilibrium. The discussion is built on a few research papers and a few chapters from Duffie (1997). Further, we will use the equilibrium valuation to discuss asset valuation, including derivative security valuation. The no-arbitrage valuation method is presented at the end.

10.1 Individual Consumption and Portfolio Selection in a Continuous-Time Set Up

Consider an economy with identical agents. The representative agent is risk-averse and has a finite lifetime horizon T. The agent's risk preference is described by a smooth, time-additive expected utility function: $U(c) = E\left[\int_0^T u(c_t, t)dt\right]$. Similar to previous discussion, we adopt the constant relative-risk-aversion (CRRA) period utility function for the investor:

$$u(c_t, t) = e^{-\phi t}\frac{{c_t}^{1-\gamma}}{1-\gamma}, \tag{10.1}$$

where $\phi > 0$ is the rate of time preference (similar to the discount β in the discrete-time set-up) and $\gamma > 0$ is the coefficient of relative risk aversion. Since each agent is small, therefore he acts as a price-taker.

10.1.1 A Two-Asset Case

There is a financial market in the economy where agents can trade any asset instantaneously without any transaction and trading restriction. To begin with, we consider a simple economy where there is a risk-free asset B and two risky assets S_1 and S_2. The prices of these three assets are exogenous and evolve according to the following processes:

Assumption 1 *The risk-free rate is constant and the pure discount bond price evolves as*

$$\frac{dB}{B} = rdt.$$

Assumption 2 *The prices of the two risky assets are governed by the following set of geometric Brownian motions**

$$\frac{dS_1}{S_1} = \mu_1 dt + \sigma_1 dz_1,$$

$$\frac{dS_2}{S_2} = \mu_2 dt + \sigma_2 dz_2, \tag{10.2}$$

where the correlation between dz_1 and $d\,\mathbf{z}_2$ is ρ.

An agent is endowed with θ_{B_0} quantity of bond, θ_{1_0} number of share of the first asset and θ_{2_0} number of share of the second asset. Denote the security prices at time t by a vector $X_t = (B_t,\ S_{1t}, S_{2t})$, the corresponding dividends by $D_t = (q_b,$ $q_1,\ q_2)$ (which are zero by the assumption) and the portfolio holdings by $\theta_t = (\theta_{B_t},$ $\theta_{1_t},\ \theta_{2_t})$. The agent's objective is to chooses an optimal consumption and portfolio policy $\{c_t,\ \theta_t \in (0, T)\}$ to maximize his expected lifetime utility:

$$\max_{\{c_t,\theta_t\}} E\left[\int_0^T u(c_t, t)\mathrm{d}t\right]$$

s.t.

$$\int_0^t c_\tau \mathrm{d}\tau = \theta_0 \cdot X_0 - \theta_t \cdot X_t + \int_0^t \theta_\tau \cdot \mathrm{d}D_\tau + \int_0^t \theta_\tau \cdot \mathrm{d}X_\tau.$$

Note that the above budget constraint is equivalent to the counterpart in the discrete-time set-up. It intuitively states that the agent's cumulative consumption up to time t is financed by the proceeds from the net selling of marketable securities, plus cumulative dividends and capital gains.

However, to solve for the optimal consumption and portfolio holding, we need to use the same concept used for the discrete case. That is, the Bellman equation which involves the value function. To do so, we define the agent's portfolio wealth as

$$W_t = \theta_{B_t} B_t + \theta_{1_t} S_{1t} + \theta_{2_t} S_{2t}.$$

Define the percentages of his wealth invested in the two risky assets $\mathbf{x}_t = (x_{1t}, x_{2t})'$, then we have

$$x_{1t} = \tfrac{\theta_{1_t} S_{1t}}{W_t}, \qquad x_{2t} = \tfrac{\theta_{2_t} S_{2t}}{W_t}. \qquad x_{bt} = 1 - x_{1t} - x_{2t}.$$

Following Merton (1971) and Chapter 5 of Merton, we derive the budget constraint below:

$$\begin{aligned} \frac{dW_t}{W_t} &= \mu_w dt + \sigma_w dz_w \\ &= \left(r + x_{1t}(\mu_1 - r) + x_{2t}(\mu_2 - r) - \frac{c_t}{W_t}\right) dt + x_{1t}\sigma_1 dz_{1t} + x_{2t}\sigma_2 dz_{2t} \end{aligned}$$

*For simplicity, we assume zero dividends for the two risky assets.

with

$$\begin{aligned}\mu_w &= r + x_{1t}(\mu_1 - r) + x_{2t}(\mu_2 - r) - \frac{c_t}{W_t} \\ \sigma_w^2 &= x_{1t}^2\sigma_1^2 + 2\rho x_{1t}\sigma_1 x_{2t}\sigma_2 + x_{2t}^2\sigma_2^2.\end{aligned}$$

The correlation between $\mathrm{d}W/W$ and $\mathrm{d}S_1/S_1$ is

$$cor(\frac{dW}{W}, \frac{dS_1}{S_1}) = x_{1t}\rho\sigma_1\sigma_2 + x_{2t}\sigma_2^2$$

and the covariance between $\mathrm{d}W/W$ and $\mathrm{d}S_2/S_2$ is

$$cor(\frac{dW}{W}, \frac{dS_2}{S_2}) = x_{1t}\sigma_1^2 + \rho\sigma_1\sigma_2 x_{2t}.$$

Optimal Solution

Denote the value function as

$$J(W, S_1, S_2, t) = \max_{x,c} E_t\left[\int_t^T u(c(\tau), \tau)\mathrm{d}\tau\right].$$

Then we have the functional equation (just like in the discrete case) involving the value function:

$$J(W, S_1, S_2, t) = \max_{x,c} E_t\left(J(W, S_1, S_2, t + \Delta t) + \left[\int_t^{t+\Delta t} u(c(\tau), \tau)\mathrm{d}\tau\right]\right).$$

We apply the optimal control rule:

$$\Psi(x, c; W, S_1, S_2, t) = u(c(t), t) + \mathfrak{L}(\mathfrak{J}),$$

where $\mathfrak{L}(\mathfrak{J})$ is the differential generator of J associated with its control function:

$$\begin{aligned}\mathfrak{L}(\mathfrak{J}) = \frac{\partial J}{\partial t} + \frac{\partial J}{\partial W}W\mu_W + \tfrac{1}{2}\frac{\partial^2 J}{\partial W^2}W^2\sigma_W^2 + \frac{\partial J}{\partial S_1}S_1\mu_1 \\ + \frac{\partial^2 J}{\partial W\partial S_1}S_1 W * cor(W, S_1) + \tfrac{1}{2}\frac{\partial^2 J}{\partial S_1^2}S^2\sigma_1^2 + \frac{\partial J}{\partial S_2}S_2\mu_2 \\ + \frac{\partial^2 J}{\partial W\partial S_2}S_2 W * cor(W, S_2) + \tfrac{1}{2}\frac{\partial^2 J}{\partial S_2^2}S^2\sigma_2^2 + \frac{\partial^2 J}{\partial S_1\partial S_2}S_1 S_2\rho\sigma_1\sigma_2.\end{aligned}$$

We assume the bequest function for the agent at time T is $J(W, S_1, S_2, T) = 0$. The optimal consumption c_t^* and the optimal weights $(x_{1t}^*,\, x_{2t}^*)$ are solved by maximizing

$\Psi(x, c; W, S_1, S_2, t)$. The first order conditions are:

$$\frac{\partial \Psi}{\partial c} = \frac{\partial u}{\partial c} - \frac{\partial J}{\partial W} = 0, \tag{A.1}$$

$$\frac{\partial \Psi}{\partial x_1} = \frac{\partial J}{\partial W} W \frac{\partial \mu_W}{\partial x_1} + \frac{\partial^2 J}{\partial W^2} W^2 \sigma_W \frac{\partial \sigma_W}{\partial x_1}$$

$$+ \frac{\partial^2 J}{\partial W \partial S_1} S_1 W \frac{\partial cov(W, S_1)}{\partial x_1} + \frac{\partial^2 J}{\partial W \partial S_2} S_2 W \frac{\partial cov(W, S_2)}{\partial x_1} = 0, \tag{A.2}$$

$$\frac{\partial \Psi}{\partial x_2} = \frac{\partial J}{\partial W} W \frac{\partial \mu_W}{\partial x_2} + \frac{\partial^2 J}{\partial W^2} W^2 \sigma_W \frac{\partial \sigma_W}{\partial x_2}$$

$$+ \frac{\partial^2 J}{\partial W \partial S_2} S_2 W \frac{\partial cov(W, S_2)}{\partial x_2} + \frac{\partial^2 J}{\partial W \partial S_1} S_1 W \frac{\partial cov(W, S_1)}{\partial x_2} = 0, \tag{A.3}$$

together with

$$\Psi(x^*, c^*; W, S_1, S_2, t) = u(c^*(t), t) + \mathfrak{L}(\mathfrak{J}) = 0, \tag{A.4}$$

$$J(W, S_1, S_2, T) = 0. \tag{A.5}$$

As in the discrete case, we guess the following form for the value function

$$J(W, t, f) = e^{-\phi t} F(t, T)^{-\gamma} \frac{{W_t}^{1-\gamma}}{1-\gamma}.$$

From equation (A.1), we obtain

$$c_t^* = F(t, T) W_t.$$

Equations (A.2) and (A.3) are used to solve for the optimal portfolio holdings. The time-dependent function $F(t, T)$ is solved by using equations (A.4) and (A.5). The optimal solutions are summarized below:

$$x_{1t}^* = \frac{s_1}{\gamma \sigma_1} - \rho \frac{\sigma_2}{\sigma_1} x_{2t}^* = \frac{s_1 - \rho s_2}{\gamma \sigma_1 (1 - \rho^2)}, \qquad x_{2t}^* = \frac{s_2 - \rho s_1}{\gamma \sigma_2 (1 - \rho^2)}, \qquad \forall \qquad 0 < t < T,$$

with

$$\begin{aligned}
s_1 &= \frac{\mu_1 - r}{\sigma_1}, & s_2 &= \frac{\mu_2 - r}{\sigma_1}, \\
a &= \frac{\phi}{\gamma} - \frac{1-\gamma}{\gamma}\left(r + \frac{s_m^2}{2}\right), & F(t, T) &= f(a, t, T) = \frac{a}{1 - e^{-a(T-t)}}, \\
V &= \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}, & \boldsymbol{\mu} - \mathbf{r} &= \begin{bmatrix} \mu_1 - r \\ \mu_2 - r \end{bmatrix} \qquad s = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}. \\
s_m &= \frac{\mu_m - r}{\sigma_m} = \sqrt{(\boldsymbol{\mu} - r)' V^{-1} (\boldsymbol{\mu} - r)}.
\end{aligned}$$

Properties of the Optimal Portfolio

$$x^* = \begin{bmatrix} x_{1t}^* \\ \\ x_{2t}^* \end{bmatrix} = \frac{1}{\gamma} \begin{bmatrix} \frac{s_1 - \rho s_2}{\sigma_1(1-\rho^2)} \\ \\ \frac{s_2 - \rho s_1}{\sigma_2(1-\rho^2)} \end{bmatrix} = \frac{1}{\gamma} V^{-1} [\boldsymbol{\mu} - \mathbf{r}] \, .$$

Recall that, in the discrete-time framework, we have derived the market portfolio as follows:

Denote the portfolio weights as

$$w = \begin{pmatrix} w_1 \\ \\ w_2 \end{pmatrix}.$$

The market portfolio is obtained by solving the following maximization problem

$$\max_w \frac{\mu_m - r}{\sigma_m}$$

s.t.

$$\begin{aligned} w'e &= 1, \qquad e' = \begin{pmatrix} 1 & 1 \end{pmatrix}, \\ w'\boldsymbol{\mu} &= \mu_m, \qquad w'Vw = \sigma_m. \end{aligned}$$

The optimal solution is

$$\begin{aligned} w^* &= \frac{V^{-1}(\boldsymbol{\mu} - r)}{e'V^{-1}(\boldsymbol{\mu} - r)}, \\ \mu_m &= r + \frac{(\boldsymbol{\mu} - r)'V^{-1}(\boldsymbol{\mu} - r)}{e'V^{-1}(\boldsymbol{\mu} - r)}, \\ \sigma_m^2 &= \frac{(\boldsymbol{\mu} - r)'V^{-1}(\boldsymbol{\mu} - r)}{[e'V^{-1}(\boldsymbol{\mu} - r)]^2}, \\ s_m &= \frac{\mu_m - r}{\sigma_m} = \sqrt{(\boldsymbol{\mu} - r)'V^{-1}(\boldsymbol{\mu} - r)} \\ Cov(R_m, R) &= \frac{\boldsymbol{\mu} - r}{e'V^{-1}(\boldsymbol{\mu} - r)} \\ Corr(R_m, R) &= \frac{s}{\sigma_m e'V^{-1}(\boldsymbol{\mu} - r)}. \end{aligned}$$

Given this endogenized market portfolio, we can easily derive the following:

$$\boldsymbol{\mu} = re + \boldsymbol{\beta}(\mu_m - r)$$

where

$$\boldsymbol{\beta} = \frac{V^{-1}(\boldsymbol{\mu} - r)}{\sigma_m^2 e'V^{-1}(\boldsymbol{\mu} - r)}.$$

The optimal portfolio obtained by an investor is proportional to the market portfolio. That is

$$x^* = \frac{1}{\gamma} V^{-1} [\boldsymbol{\mu} - \mathbf{r}] = \frac{1}{\gamma} e' V^{-1} [\boldsymbol{\mu} - \mathbf{r}] \, w^*.$$

This result confirms the two-fund separation theory we obtained in the discrete-time framework. That is, every investor holds a combination of the market portfolio and the riskfree asset. The optimal holding in the market portfolio depends on the investor's risk-aversion. A more risk-averse investor will hold less in the market portfolio. The percentage holding is constant over.

10.1.2 A Two-asset Case Revisited

In addition to the economy described above, there is also a traded market portfolio which is formed in the way derived above so that $\beta_i = \rho_{mi} \frac{\sigma_i}{\sigma_m} = \frac{V^{-1}(\boldsymbol{\mu}_i - r)}{\sigma_m^2 e' V^{-1}(\boldsymbol{\mu} - r)}$,

$$\mu_i = r + \beta_i(\mu_m - r), \qquad for \qquad i = 1, 2.$$

For simplicity, we assume that the prices of the market portfolio follows

$$\frac{dS^M}{S^M} = \mu_m dt + \sigma_m dz_m,$$

where the correlation between $\mathrm{d}z_m$ and d $\mathbf{z}_i$ is ρ_{mi}. Denote he expected returns and volatilities of the two risky assets and the market portfolio as

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_m \\ \mu_1 = r + \beta_1(\mu_m - r) \\ \mu_2 = r + \beta_2(\mu_m - r) \end{pmatrix}, \qquad V = \boldsymbol{\sigma} \Sigma \boldsymbol{\sigma}$$

with

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_m \end{pmatrix} \qquad and \qquad \Sigma = \begin{pmatrix} 1 & \rho_{12} & \rho_{1m} \\ \rho_{12} & 1..... & \rho_{2m} \\ \rho_{1m} & \rho_{2m} & 1 \end{pmatrix}.$$

As shown above, the CAPM holds for any of the two risky stocks. Therefore, the non-systematic variance for a risky asset is

$$v_i^2 = (1 - \rho_{mi}^2)\sigma_i^2 \qquad for \qquad i = 1, 2.$$

We reformulate the portfolio and consumption problem for the investor where the portfolio choice set includes the riskfree asset, the two risky assets and the market portfolio. The investor's objective is still to chooses an optimal consumption and portfolio policy $\{c_t, \theta_t \in (0, T)\}$ to maximize his expected lifetime utility:

$$\max_{\{c_t, \theta_t\}} E\left[\int_0^T u(c_t, t)\mathrm{d}t\right]$$

s.t.

$$\int_0^t c_\tau \mathrm{d}\tau = \theta_0 \cdot X_0 - \theta_t \cdot X_t + \int_0^t \theta_\tau \cdot \mathrm{d}D_\tau + \int_0^t \theta_\tau \cdot \mathrm{d}X_\tau.$$

Now the agent's portfolio wealth is

$$W_t = \theta_{B_t} B_t + \theta_{M_t} S_t^M + \theta_{S_{1t}} S_{1t} + \theta_{S_{2t}} S_{2t}.$$

Define the percentages of his wealth invested in the two risky assets $\mathbf{x}_t = (x_{mt}, x_{st})'$, then we have

$$x_{mt} = \frac{\theta_{m_t} S_t^M}{W_t}, \qquad x_{it} = \frac{\theta_{s_{it}} S_{it}}{W_t}. \qquad x_{bt} = 1 - x_{mt} - x_{1t} - x_{2t}.$$

Following Merton (1971) and Chapter 5 of Merton, we derive the budget constraint below:

$$\begin{aligned}
\frac{dW_t}{W_t} &= \mu_w dt + \sigma_w dz_w \\
&= \left(r + x_{mt}(\mu_m - r) + \sum_{i=1}^{2} x_{it}(\mu_i - r) - \frac{c_t}{W_t} \right) dt + x_{mt}\sigma_m dz_{mt} + \sum_{i=1}^{2} x_{it}\sigma_i dz_{it}
\end{aligned}$$

with

$$\begin{aligned}
\mu_w &= r + x_{mt}(\mu_m - r) + \sum_{i=1}^{2} x_{it}(\mu_i - r) - \frac{c_t}{W_t} \\
\sigma_w^2 &= x_{mt}^2\sigma_m^2 + x_{1t}^2\sigma_1^2 + x_{2t}^2\sigma_2^2 + 2\rho_{m1}x_{mt}\sigma_m x_{1t}\sigma_1. \\
&\quad + 2\rho_{m2}x_{mt}\sigma_m x_{2t}\sigma_2 \ + 2\rho_{12}x_{1t}\sigma_1 x_{2t}\sigma_2.
\end{aligned}$$

The correlations are

$$\begin{aligned}
cor(\frac{dW}{W}, \frac{dS_1}{S_1}) &= x_{mt}\rho_{m1}\sigma_m\sigma_1 + x_{1t}\sigma_1^2 + x_{2t}\rho_{12}\sigma_1\sigma_2, \\
cor(\frac{dW}{W}, \frac{dS_2}{S_2}) &= x_{mt}\rho_{m2}\sigma_m\sigma_2 + x_{2t}\sigma_2^2 + x_{1t}\rho_{12}\sigma_1\sigma_2, \\
cor(\frac{dW}{W}, \frac{dS^M}{S^M}) &= x_{mt}\sigma_m^2 + \rho_{m1}\sigma_m\sigma_1 x_{1t} + \rho_{m2}\sigma_m\sigma_2 x_{2t}.
\end{aligned}$$

10.1.3 Optimal Solution

Denote the value function as

$$J(W, S^M, S_1, S_2, t) = \max_{x,c} E_t \left[\int_t^T u(c(\tau), \tau) \mathrm{d}\tau \right].$$

Then we have the functional equation (just like in the discrete case) involving the value function:

$$J(W, S^M, S_1, S_2, t) = \max_{x,c} E_t \left(J(W, S^M, S_1, S_2, t + \Delta t) + \left[\int_t^{t+\Delta t} u(c(\tau), \tau) \mathrm{d}\tau \right] \right).$$

We apply the optimal control rule:

$$\Psi(x, c; W, S^M, S_1, S_2, t) = u(c(t), t) + \mathfrak{L}(\mathfrak{J}),$$

where $\mathfrak{L}(\mathfrak{J})$ is the differential generator of J associated with its control function:

$$\mathfrak{L}(\mathfrak{J}) = \frac{\partial J}{\partial t} + \frac{\partial J}{\partial W} W\mu_W + \tfrac{1}{2}\frac{\partial^2 J}{\partial W^2} W^2\sigma_W^2 + \frac{\partial J}{\partial S^M} S^M \mu_m + \tfrac{1}{2}\frac{\partial^2 J}{\partial S^{M^2}} S^{M^2}\sigma_m^2$$

$$+\frac{\partial J}{\partial S_1} S_1\mu_1 + \tfrac{1}{2}\frac{\partial^2 J}{\partial {S_1}^2} {S_1}^2\sigma_1^2 + \frac{\partial J}{\partial S_2} S_2\mu_2 + \tfrac{1}{2}\frac{\partial^2 J}{\partial S_2^2} S_2^2\sigma_2^2$$

$$+\frac{\partial^2 J}{\partial W \partial S^M} S^M W * cov(W, S^M) + \frac{\partial^2 J}{\partial W \partial S_1} S_1 W * cov(W, S_1)$$

$$+\frac{\partial^2 J}{\partial W \partial S_2} W S_2 * cov(W, S_2) + \frac{\partial^2 J}{\partial S_1 \partial S_2} S_1 S_2 \rho_{12}\sigma_1\sigma_2$$

$$+\frac{\partial^2 J}{\partial S^M \partial S_1} S^M S_1 \rho_{m1}\sigma_m\sigma_1 + \frac{\partial^2 J}{\partial S^M \partial S_2} S^M S_2 \rho_{m2}\sigma_m\sigma_2.$$

We assume the bequest function for the agent at time T is $J(W, S^M, S_1, S_2, T) = 0$. The optimal consumption c_t^* and the optimal weights (x_{mt}^*, x_{1t}^*, x_{2t}^*) are solved by maximizing $\Psi(x, c; W, S^M, S_1, S_2, t)$. The first order conditions are:

$$\frac{\partial \Psi}{\partial c} = \frac{\partial u}{\partial c} - \frac{\partial J}{\partial W} = 0, \qquad (A.1)$$

$$\frac{\partial \Psi}{\partial x_m} = \frac{\partial J}{\partial W} W \frac{\partial \mu_W}{\partial x_m} + \frac{\partial^2 J}{\partial W^2} W^2 \sigma_W \frac{\partial \sigma_W}{\partial x_m} + \frac{\partial^2 J}{\partial W \partial S^M} S^M W \frac{\partial cov(W, S^M)}{\partial x_m}$$

$$+\frac{\partial^2 J}{\partial W \partial S_1} S_1 W \frac{\partial cov(W, S_1)}{\partial x_m} + \frac{\partial^2 J}{\partial W \partial S_2} S_2 W \frac{\partial cov(W, S_2)}{\partial x_m} = 0, \qquad (A.2)$$

$$\frac{\partial \Psi}{\partial x_i} = \frac{\partial J}{\partial W} W \frac{\partial \mu_W}{\partial x_i} + \frac{\partial^2 J}{\partial W^2} W^2 \sigma_W \frac{\partial \sigma_W}{\partial x_i} + \frac{\partial^2 J}{\partial W \partial S^M} S^M W \frac{\partial cov(W, S^M)}{\partial x_i}$$

$$+\frac{\partial^2 J}{\partial W \partial S_1} S_1 W \frac{\partial cov(W, S_1)}{\partial x_i} + \frac{\partial^2 J}{\partial W \partial S_2} S_2 W \frac{\partial cov(W, S_2)}{\partial x_i} = 0, \qquad (A.3)$$

together with

$$\Psi(x^*, c^*; W, S^M, S_1, S_2, t) = u(c^*(t), t) + \mathfrak{L}(\mathfrak{J}) = 0, \qquad (A.4)$$

$$J(W, S^M, S_1, S_2, T) = 0. \qquad (A.5)$$

As in the discrete case, we guess the following form for the value function

$$J(W, t, f) = e^{-\phi t} F(t, T)^{-\gamma} \frac{{W_t}^{1-\gamma}}{1-\gamma}.$$

From equation (A.1), we obtain

$$c_t^* = F(t, T) W_t.$$

Equations (A.2) and (A.3) are used to solve for the optimal portfolio holdings. The time-dependent function $F(t, T)$ is solved by using equations (A.4) and (A.5). The optimal solutions are summarized below:

$$x^*_{mt} = \frac{s_m}{\gamma \sigma_m}, \qquad x^*_{it} = 0, \qquad \forall \qquad 0 < t < T,$$

with

$$s_m = \frac{\mu_m - r}{\sigma_m}, \qquad F(t, T) = f(a, t, T) = \frac{a}{1 - e^{-a(T-t)}}, \qquad \textit{and} \qquad a = \frac{\phi}{\gamma} - \frac{1-\gamma}{\gamma}\left(r + \frac{s_m^2}{2}\right).$$

Important features of optimal consumption policy:

- The consumption in the period of $T - t$ is $c^* \times (T - t) = f(a, t, T)(T - t)W_t$. It is easy to show that $f(a, t, T)(T - t)$ is less than 1. That is, the optimal consumption policy indicates that the agent always consumes a portion of his wealth. By the end of the agent's life (i.e., $t \to T$), $f(a, t, T)(T - t) \to 1$. Therefore, he consumes all his wealth.

- Given the relation between the optimal consumption and wealth, we can use Ito's Lemma to derive the dynamics for the optimal consumption plan as follows:

$$\frac{dc}{c} = \frac{1}{\gamma}\left[r - \phi + \frac{1}{2}(1 + \frac{1}{\gamma})s_m^2\right] dt + \frac{s_m}{\gamma} dz_m \equiv \mu_c dt + \sigma_c dz.$$

- From the above process, we know that the agent's consumption uncertainty is completely induced by that of the market portfolio. The consumption growth rate is $\mu_c = \frac{1}{\gamma}\left[r - \phi + \frac{1}{2}(1 + \frac{1}{\gamma})s_m^2\right]$ and its volatility is $\sigma_c = \frac{s_m}{\gamma}$.

- The consumption growth rate increases with riskfree rate, the Sharpe ratio for the market, decreases with the rate of time preference and the risk-aversion parameter;

- The consumption volatility is not affected by either the riskfree rate or the rate of time preference; A high risk-aversion parameter γ induces a low consumption volatility while a high Sharpe ratio of the market portfolio induces a high consumption volatility.

Important features of optimal portfolio holding strategy:

- The optimal policy confirms the two-fund separation results. That is, the agent will only hold the market portfolio and the riskfree asset.

- More specifically, the agent will hold a constant portion of his wealth in the market portfolio. This constant portion is positively related to the market's Sharpe ratio s_m, negatively related to the agent's risk-aversion parameter γ and the volatility of the market portfolio σ_m.

- Stated differently, the agent will invest heavily in the market portfolio if he is less risk-averse or when the market's Sharpe ratio is high.

10.1.4 N-asset Case

Now, we extend the two-asset economy to a N-asset economy. The pure discount bond price still evolves as

$$\frac{dB}{B} = rdt.$$

Furthermore, we assume the returns of the N assets are characterized by the following system:

$$\frac{dS_i}{S_i} = (\mu_i - q_i)dt + \sigma_i dz_i,$$

where the correlation coefficient matrix of $\mathbf{z} = \begin{pmatrix} z_1 & ... & z_N \end{pmatrix}'$ is

$$\Sigma = \begin{pmatrix} 1 & \rho_{12}.... & \rho_{1N} \\ \rho_{12} & 1..... & \rho_{2N} \\ : & : & : \\ \rho_{1N} & \rho_{2N} & 1 \end{pmatrix},$$

and the cum-dividend expected returns, dividend yields, volatilities and Sharpe-Ratios are

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ : \\ \mu_N \end{pmatrix}, \qquad \mathbf{q} = \begin{pmatrix} q_1 \\ : \\ q_N \end{pmatrix}, \qquad V = \boldsymbol{\sigma}\Sigma\boldsymbol{\sigma}$$

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_1 & 0.... & 0 \\ 0 & \sigma_2..... & 0 \\ : & : & : \\ 0 & 0 & \sigma_N \end{pmatrix} \qquad and \qquad s = \begin{pmatrix} \frac{\mu_1 - r}{\sigma_1} \\ : \\ \frac{\mu_N - r}{\sigma_N} \end{pmatrix}.$$

We first solve the consumption-investment problem with N risky assets. To this end, we define the portfolio wealth is

$$W_t = \theta_{B_t} B_t + \sum_{n=1}^{N} \theta_{n_t} S_{n_t}.$$

Denote the percentages of his total wealth invested in the risky assets as $\mathbf{x}_t = (x_{1t}, x_{2t}, ... \, x_{Nt})'$, then we have

$$x_{1t} = \frac{\theta_{1_t} S_{1_t}}{W_t}, \qquad x_{2t} = \frac{\theta_{2_t} S_{2_t}}{W_t}, \quad \text{... and} \quad x_{Nt} = \frac{\theta_{N_t} S_{N_t}}{W_t}.$$

Similar to Merton (1971), the budget constraint is

$$\frac{dW_t}{W_t} = \left(r + \mathbf{x}_t'(\boldsymbol{\mu} - r) - \frac{c_t}{W_t} \right) dt + \mathbf{x}_t' \boldsymbol{\sigma} d\mathbf{z}_t.$$

We solve the optimal policy in the following way (which is similar to Merton 1969). Define the value function J and apply the optimal control rule:

$$J(W, S, t) = \max_{\mathbf{x},c} E_t \left[\int_t^T U(c(\tau), \tau) \mathrm{d}\tau \right],$$

$$\Psi(\mathbf{x}, c; W, \mathbf{S}, t) = U(c(t), t) + \mathfrak{L}[J],$$

where $\mathfrak{L}[J]$ is the differential generator of J associated with its control function:

$$\mathfrak{L}[J] = \frac{\partial J}{\partial t} + \frac{\partial J}{\partial W}\left(\sum_{i=1}^{N} x_i(\mu_i - r)W + rW - c\right) + \sum_{i=1}^{N} \mu_i S_i \frac{\partial J}{\partial S_i}$$

$$+\tfrac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j \sigma_{ij} W^2 \frac{\partial^2 J}{\partial W^2} + +\tfrac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} S_i S_j \sigma_{ij}\frac{\partial^2 J}{\partial S_i \partial S_j}$$

$$+\sum_{i=1}^{N}\sum_{j=1}^{N} S_i W x_j \sigma_{ij}\frac{\partial^2 J}{\partial W \partial S_i}$$

For simplicity, we assume the bequest function at the finite horizon T is 0. The optimal consumption, c_t^* and the optimal weights, $\mathbf{x}_t^*$ are solved by maximizing $\Psi(\mathbf{x}, c; W, S, t)$. The first order conditions are:

$$\frac{\partial \Psi}{\partial c} = \frac{\partial U}{\partial c} - \frac{\partial J}{\partial W} = 0, \tag{1}$$

$$\frac{\partial \Psi}{\partial x_n} = \frac{\partial J}{\partial W}W(\mu_n - r) + \frac{\partial^2 J}{\partial W^2}W^2\sum_{i=1}^{N} x_i \sigma_{ni} + \sum_{i=1}^{N}\frac{\partial^2 J}{\partial W \partial S_i}\sigma_{in} S_n W = 0 \qquad \forall \qquad n = 1, ...N \tag{2}$$

together with

$$\Psi(x^*, c^*; W, M, S, I, t) = U(c^*(t) + w, t) + \mathfrak{L}[J] = 0, \tag{3}$$

$$J(W, S, T) = 0 \tag{4}$$

Tedious exercise will lead to the following solutions:

$$\mathbf{x} = \frac{1}{\gamma}\sigma^{-1}\Sigma^{-1}\sigma^{-1}(\boldsymbol{\mu} - r).$$

and

$$c_t^* = f(a, t, \overline{T})W_t \qquad and \qquad J(W, t, f) = e^{-\phi t} f(a, t, \overline{T})^{-\gamma}\tfrac{W_t{}^{1-\gamma}}{1-\gamma} \qquad \forall \qquad T < t < \overline{T},$$

$$a = \tfrac{\phi}{\gamma} - \tfrac{1-\gamma}{\gamma}\left(r + \tfrac{(\boldsymbol{\mu}-r)'(\sigma\Sigma\sigma)^{-1}(\boldsymbol{\mu}-r)}{2\gamma}\right) = \tfrac{\phi}{\gamma} - \tfrac{1-\gamma}{\gamma}\left(r + \tfrac{s_m^2}{2\gamma}\right), \qquad f(a, t, \overline{T}) = \tfrac{a}{1-e^{-a(\overline{T}-t)}}.$$

Note that this optimal weights are propositional to the market portfolio weights. That is

$$\mathbf{x} = \frac{1}{\gamma}(\sigma\Sigma\sigma)^{-1}(\boldsymbol{\mu} - r) = \frac{e'(\boldsymbol{\sigma\Sigma\sigma})^{-1}(\boldsymbol{\mu} - r)}{\gamma}\frac{(\sigma\Sigma\sigma)^{-1}(\boldsymbol{\mu} - r)}{e'(\boldsymbol{\sigma\Sigma\sigma})^{-1}(\boldsymbol{\mu} - r)} = \frac{e'(\boldsymbol{\sigma\Sigma\sigma})^{-1}(\boldsymbol{\mu} - r)}{\gamma}w^* = \frac{s_m}{\gamma\sigma_m}w^*.$$

Properties of the Optimal Portfolio

Recall that, in the discrete-time framework, we have derived the market portfolio as follows:

Denote the portfolio weights as

$$w = \begin{pmatrix} w_1 \\ : \\ w_N \end{pmatrix}.$$

The market portfolio is obtained by solving the following maximization problem

$$\max_{w} \frac{\mu_m - r}{\sigma_m}$$

s.t.

$$\begin{aligned} w'e &= 1, \qquad e' = \begin{pmatrix} 1 & ... & 1 \end{pmatrix}, \\ w'\boldsymbol{\mu} &= \mu_m, \qquad w'\boldsymbol{\sigma}\boldsymbol{\Sigma}\boldsymbol{\sigma}w = \sigma_m. \end{aligned}$$

The optimal solution is

$$\begin{aligned} w^* &= \frac{(\boldsymbol{\sigma}\boldsymbol{\Sigma}\boldsymbol{\sigma})^{-1}(\boldsymbol{\mu} - r)}{e'(\boldsymbol{\sigma}\boldsymbol{\Sigma}\boldsymbol{\sigma})^{-1}(\boldsymbol{\mu} - r)}, \\ \mu_m &= r + \frac{(\boldsymbol{\mu} - r)'(\boldsymbol{\sigma}\boldsymbol{\Sigma}\boldsymbol{\sigma})^{-1}(\boldsymbol{\mu} - r)}{e'(\boldsymbol{\sigma}\boldsymbol{\Sigma}\boldsymbol{\sigma})^{-1}(\boldsymbol{\mu} - r)}, \\ \sigma_m^2 &= \frac{(\boldsymbol{\mu} - r)'(\boldsymbol{\sigma}\boldsymbol{\Sigma}\boldsymbol{\sigma})^{-1}(\boldsymbol{\mu} - r)}{\left[e'(\boldsymbol{\sigma}\boldsymbol{\Sigma}\boldsymbol{\sigma})^{-1}(\boldsymbol{\mu} - r)\right]^2}, \\ s_m &= \frac{\mu_m - r}{\sigma_m} = \sqrt{(\boldsymbol{\mu} - r)'(\boldsymbol{\sigma}\boldsymbol{\Sigma}\boldsymbol{\sigma})^{-1}(\boldsymbol{\mu} - r)} \\ Cov(R_m, R) &= \frac{\boldsymbol{\mu} - r}{e'(\boldsymbol{\sigma}\boldsymbol{\Sigma}\boldsymbol{\sigma})^{-1}(\boldsymbol{\mu} - r)} \\ Corr(R_m, R) &= \frac{s}{\sigma_m e'(\boldsymbol{\sigma}\boldsymbol{\Sigma}\boldsymbol{\sigma})^{-1}(\boldsymbol{\mu} - r)}. \end{aligned}$$

Given this endogenized market portfolio, we can easily derive the following:

$$\boldsymbol{\mu} = re + \boldsymbol{\beta}(\mu_m - r)$$

where

$$\boldsymbol{\beta} = \frac{(\boldsymbol{\sigma}\boldsymbol{\Sigma}\boldsymbol{\sigma})^{-1}(\boldsymbol{\mu} - r)}{\sigma_m^2 e'(\boldsymbol{\sigma}\boldsymbol{\Sigma}\boldsymbol{\sigma})^{-1}(\boldsymbol{\mu} - r)}.$$

The above results indicate that, the investor holds $\frac{e'(\boldsymbol{\sigma}\boldsymbol{\Sigma}\boldsymbol{\sigma})^{-1}(\boldsymbol{\mu}-r)}{\gamma}$ percent in the market portfolio and $1 - \frac{e'(\boldsymbol{\sigma}\boldsymbol{\Sigma}\boldsymbol{\sigma})^{-1}(\boldsymbol{\mu}-r)}{\gamma}$ in the riskfree bond.

Therefore, we confirm the two-fund separation with N assets in a continuous-time setting.

Optimal Consumption

The optimal consumption evolves according to:

$$\begin{aligned}
\frac{dc}{c} &= \mu_c dt + \sigma_c dz_c = \frac{1}{\gamma}\left[r - \phi + \frac{1}{2}(1+\frac{1}{\gamma})s_m^2\right] dt + \mathbf{x}'\boldsymbol{\sigma}\mathbf{dz} \\
\sigma_c dz_c &= \mathbf{x}'\boldsymbol{\sigma}\mathbf{dz} \\
\mu_c &= \frac{1}{\gamma}\left[r - \phi + \frac{1}{2}(1+\frac{1}{\gamma})s_m^2\right] \\
\sigma_c^2 &= \frac{(\boldsymbol{\mu} - r)'(\sigma\Sigma\sigma)^{-1}(\boldsymbol{\mu} - r)}{\gamma^2} = \frac{s_m^2}{\gamma^2}.
\end{aligned}$$

Since the employee's optimal consumption is proportional to his total wealth which consists of investments in the risk-free asset and the market portfolio, the uncertainty in his consumption is completely induced by that of the market portfolio. The consumption growth rate $\mu_c = \frac{1}{\gamma}\left(r - \phi + \frac{1}{2}(1+\frac{1}{\gamma})s_m^2\right)$ and its volatility $\sigma_c = \frac{s_m}{\gamma}$ are negatively related to the risk-aversion parameter γ. As a result, a less risk-averse employee will consume at a higher rate and endure a higher consumption volatility.

10.1.5 N-asset Case Revisited

We introduce the market portfolio into the N-asset economy. The market portfolio is formed based on optimizing the Sharpe ratio. In this case, the portfolio choice set for investors includes the riskfree asset, N risky assets and the market portfolio. Please verify the two-fund separation result in the following exercise:

Exercise 10.1.1 *Consider an economy where there are N risky assets and a riskfree bond. In addition, there is a market portfolio which is formed with the N assets with the highest Sharpe ratio. The prices of the N risky assets are governed by the following set of stochastic processes*

$$\frac{dS_i}{S_i} = \mu_i dt + \sigma_i dz_i, \qquad for \qquad i = 1, 2, ..., N.$$

The market portfolio follows

$$\frac{dS^M}{S^M} = \mu_m dt + \sigma_m dz_m.$$

The correlation between dz_m and $d\mathbf{z}_i$ is ρ_{mi} and among the risky assets are ρ_{ij}. Because of the formation of the market portfolio, the returns of the N risky assets satisfy the following relationship:

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 = r + \beta_1(\mu_m - r) \\ : \\ \mu_i = r + \beta_i(\mu_m - r) \\ : \\ \mu_I = r + \beta_I(\mu_m - r) \end{pmatrix}.$$

Derive the optimal consumption and portfolio holdings for an agent whose utility function is the CRRA used in this section. Verify the two-fund separation results.

10.2 Competitive Equilibrium and Equilibrium Asset Valuation

10.2.1 Structure of the Economy

Consider an economy where there is a single good for consumption purpose. The financial market is frictionless. Agents in the economy can trade a riskfree asset and a risky stock, which represents the ownership of the productive technology for the single good. The total supply of this risky stock is normalized to one. Denote its price at time t as S_t^M and the dividend as D_t^M. The dividend stream $\{D_t^M\}$ can be understood as aggregate dividends in this economy, which are exogenously given as:

$$\frac{\mathrm{d}D^M}{D^M} = \mu_m \mathrm{d}t + \sigma_m \mathrm{d}z_m, \tag{10.3}$$

where $\mathrm{d}z_m$ is a one-dimensional Gauss-Wiener process characterizing the only source of uncertainty for the economy. μ_m and σ_m are the expected growth rate and volatility for the aggregate dividends.

In addition to this risky stock, there are many contingent claims written on this stock.

10.2.2 The Agent's Optimization Problem

There are many identical risk-averse agents in the economy whose lifetime horizon is infinite. A representative agent's information structure is given by the filtration $\mathcal{F}_t \equiv \sigma(D_\tau^M; 0 \le \tau \le t)$. His period utility at time t is described as $u(c_t, t)$, where $u(\cdot, t) : \mathcal{R}_+ \to \mathcal{R}$ is increasing and strictly concave. Then, his intertemporal utility is

$$U(c) = E_0 \int_0^\infty u(c_t, t)\mathrm{d}t.$$

Initially, the agent is endowed with one share of the risky stock. His consumption over time is financed by a continuous trading strategy $\{\theta_{st}, \theta_{Bt}, \theta_{At}, \forall\ t \ge 0\}$, where θ_{st} is the holding in the risky asset, θ_{Bt} is the holding in the riskfree bond and θ_{At} are the holdings of contingent claims at time t. Denote the prices of all financial assets at time t by a vector $X_t = (S_t^M,\ B_t,\ x_t')'$ and the corresponding vector of dividends by q_t. The cumulative dividends up to t are defined as $D_t = \int_0^t q_\tau \mathrm{d}\tau$. At any point $\tau \ge 0$, the agent's wealth is $W_\tau = \theta_\tau \cdot X_\tau$ and the flow budget constraint is

$$c_\tau \mathrm{d}\tau = \theta_\tau^X \cdot (\mathrm{d}D_\tau + \mathrm{d}X_\tau) - \mathrm{d}W_\tau. \tag{10.4}$$

This constraint intuitively states that the sum of the wealth increase ($\mathrm{d}W_\tau$) and consumption flow ($c_\tau \mathrm{d}\tau$) is bounded by the dividend and capital gain from the portfolio θ_τ.

With this flow budget constraint, we can transform it into an integrated one (see Duffie, 1992 for a similar formulation on page 110):

$$\int_0^t c_\tau \mathrm{d}\tau = \theta_0^X \cdot X_0 + \int_0^t \theta_\tau^X \cdot (\mathrm{d}D_\tau + \mathrm{d}X_\tau) - \theta_t^X \cdot X_t. \tag{10.5}$$

The agent chooses an optimal portfolio trading strategy $\{\theta_t, \forall\, t \geq 0\}$ so as to maximize his expected lifetime utility. Precisely, he solves:

$$\max_{\{c_t,\theta_t\}} E \int_0^\infty u(c_t,t)\mathrm{d}t \qquad \text{s.t. (10.5) holds.}$$

The expectation is taken with respect to the filtration specified earlier. The Euler equations are:

$$X_t = E_t\left(\int_t^\infty \frac{u_c(c_\tau,\tau)}{u_c(c_t,t)}\mathrm{d}D\tau\right). \tag{10.6}$$

The price of any other asset equals the expected discounted sum of dividends, with the stochastic state price deflator being the marginal rate of substitution between consumption at different dates.

All agents take the assets' prices as given and solve for the optimal consumption and portfolio strategy, as shown in the previous subsection. In this subsection, we focus on the competitive equilibrium and solve for the equilibrium assets' prices. As stated in Chapter 8, although each agent is insignificant, but their joint action affect the demand of assets and consumption good. The equilibrium assets' prices are defined by setting the market clearing conditions for the good market and the financial market, as shown below.

10.2.3 Competitive Equilibrium and Equilibrium Asset Prices

A competitive equilibrium can be defined as a contingent plan $\{c_t,\theta_t\}_{t=0}^\infty$, a value function $J(\theta, D^M)$ and a sequence of price functions $\{S_t(D_t^M)\}_{t=0}^\infty$ such that

(i) given the price sequence, the contingent plan solves the agent's expected utility maximization problem;

(ii) the value function satisfies the Hamilton-Jacobi-Bellman equation;

(iii) goods and share markets clear, i.e., for each state (θ_t, D_t^M) in period t and for each t, $c_t = D_t^M$ and $\theta_t = 1$.

In other words, equilibrium prices are such that the representative agent holds neither the bonds nor any other contingent claims, because the net supply of each such asset is zero. Also, the equilibrium price for the risky stock is such that the representative agent holds one share the asset. On the other hand, the goods market is cleared by setting the supply (D_t^m) equal to the demand (c_t).

Example 1 *Based on the above description of the economy, if the agent's period utility function is*

$$u(c_t,t) = e^{-\phi t}\frac{{c_t}^{1-\gamma}}{1-\gamma},$$

the price for the risky stock is determined as

$$\begin{aligned}
S_t^M &= E_t\left(\int_t^\infty \frac{u_{c_T}}{u_{c_t}}\cdot D_T^M\, dT\right) = E_t\left(\int_t^\infty e^{-\phi(T-t)}\frac{c_t^\gamma D_T^M}{c_T^\gamma}\, dT\right)\\
&= E_t\left(\int_t^\infty e^{-\phi(T-t)}\frac{(D_t^M)^\gamma D_T^M}{(D_T^M)^\gamma}\, dT\right) = E_t\left(\int_t^\infty e^{-\phi(T-t)}\frac{(D_t^M)^\gamma}{(D_T^M)^{\gamma-1}}\right) dT\\
&= D_t^M E_t\left(\int_t^\infty e^{-\phi(T-t)}\left(\frac{D_T^M}{D_t^M}\right)^{1-\gamma}\right) dT.
\end{aligned}$$

Given the aggregate dividend process, we know the conditional density $y_T = \ln \frac{D_T^M}{D_t^M}$ *has a normal distribution with mean* $(\mu_m - \frac{1}{2}\sigma_m^2)(T-t)$ *and variance* $\sigma_m^2(T-t)$. *That is*

$$h(y_T) = \frac{1}{\sqrt{2\pi\sigma_m^2(T-t)}} \exp\left(-\frac{[y_T - (\mu_m - \frac{1}{2}\sigma_m^2)(T-t)]^2}{2\sigma_m^2(T-t)}\right)$$

Tedious exercise shows that

$$\begin{aligned} S_t^M &= D_t^M E_t\left(\int_t^\infty e^{-\phi(T-t)}\left(\frac{D_T^M}{D_t^M}\right)^{1-\gamma}\right) dT \\ &= D_t^M \int_t^\infty e^{-\phi(T-t)} E_t\left(e^{y_T^{1-\gamma}}\right) dT \\ &= D_t^M \int_t^\infty e^{-\phi(T-t)} e^{(1-\gamma)(\mu_m - \frac{1}{2}\sigma_m^2)(T-t) + \frac{1}{2}(1-\gamma)^2\sigma_m^2(T-t)} dT \\ &= D_t^M \int_t^\infty e^{-[\phi - (1-\gamma)(\mu_m - \frac{1}{2}\sigma_m^2) - \frac{1}{2}(1-\gamma)^2\sigma_m^2](T-t)} dT \\ &= \frac{D_t^M}{\phi - (1-\gamma)(\mu_m - \frac{1}{2}\sigma_m^2) - \frac{1}{2}(1-\gamma)^2\sigma_m^2}. \end{aligned}$$

Example 2 *Following the above example, the price for the pure discount riskfree bond with maturity* T *is determined as*

$$\begin{aligned} B(t,T) &= E_t\left(\frac{u_{c_T}}{u_{c_t}} \cdot D_T^B\right) = E_t\left(e^{-\phi(T-t)}\frac{c_t^\gamma}{c_T^\gamma}\right) = E_t\left(e^{-\phi(T-t)}\frac{(D_t^M)^\gamma}{(D_T^M)^\gamma}\right) \\ &= e^{-\phi(T-t)} E_t\left(e^{-\gamma y_T}\right) \\ &= e^{-\phi(T-t)} e^{-\gamma(\mu_m - \frac{1}{2}\sigma_m^2)(T-t) + \frac{1}{2}\gamma^2\sigma_m^2(T-t)} \\ &= e^{-[\phi + \gamma(\mu_m - \frac{1}{2}\sigma_m^2) - \frac{1}{2}\gamma^2\sigma_m^2](T-t)}. \end{aligned}$$

The implies that the riskfree rate is

$$r = \phi + \gamma(\mu_m - \frac{1}{2}\sigma_m^2) - \frac{1}{2}\gamma^2\sigma_m^2.$$

Given this riskfree rate, we can rewrite the price for the risky stock as

$$\begin{aligned} S_t^M &= \frac{D_t^M}{\phi - (1-\gamma)(\mu_m - \frac{1}{2}\sigma_m^2) - \frac{1}{2}(1-\gamma)^2\sigma_m^2} \\ &= \frac{D_t^M}{r - \mu_m + \gamma\sigma_m^2}. \end{aligned}$$

The derived equilibrium price for the risky stock indicates that

$$D_t^M = (r - \mu_m + \gamma\sigma_m^2)S_t^M.$$

Using the dividend yield definition, we know that the dividend yield for this stock is $q = r - \mu_m + \gamma\sigma_m^2$.

To examine the dynamics followed by the equilibrium price, we apply Ito's Lemma to obtain the following price process for the risky stock

$$\frac{\mathrm{d}S^M}{S^M} = (\mu - q)\mathrm{d}t + \sigma \mathrm{d}z = \mu_m \mathrm{d}t + \sigma_m \mathrm{d}z_m.$$

This suggests that the expected growth rate for the stock is $\mu = \mu_m + q = r + \gamma\sigma_m^2$ and its volatility is $\sigma = \sigma_m$. In other words, the dynamics followed by the stock price mimics that of the dividend process. The return on this stock has a normal distribution. The risk premium on this stock is $\mu - r = \gamma\sigma_m^2$, which is proportional to the risk aversion parameter and the volatility of the aggregate dividend.

Recall that we derived the individual's optimal consumption and portfolio holding under an exogenous price process for the market portfolio. The equilibrium price process supports the exogenous process we assumed for the individual optimal problem.

Example 3 *Also, we can price any derivative security written on the risky asset. Let us consider a forward contract on the risky asset with a delivery price K and a maturity T. The value of this contract is determined as*

$$\begin{aligned}
f(S_t, K, t, T) &= E_t\left(\frac{u_{c_T}}{u_{c_t}}(S_T^M - K, 0)\right) = E_t\left(e^{-\phi(T-t)}\frac{c_t^\gamma}{c_T^\gamma}(S_T^M - K, 0)\right) \\
&= e^{-\phi(T-t)}E_t\left[\left(\frac{D_T^M}{D_t^M}\right)^{-\gamma}\left(\frac{D_T^M}{r - \mu_m + \gamma\sigma_m^2} - K, 0\right)\right] \\
&= e^{-\phi(T-t)}\int_{-\infty}^{\infty}\left(\frac{D_t^M}{r - \mu_m + \gamma\sigma_m^2}e^{(1-\gamma)y_T} - Ke^{-\gamma y_T}\right)h(y_T)\,dy_T \\
&= e^{-\phi(T-t)}\int_{-\infty}^{\infty}\left(S_t e^{(1-\gamma)y_T}) - Ke^{-\gamma y_T}\right)h(y_T)\,dy_T \\
&= S_t^M e^{-\phi(T-t)}\int_{-\infty}^{\infty}e^{(1-\gamma)y_T}h(y_T)\,dy_T - Ke^{-\phi(T-t)}\int_{-\infty}^{\infty}e^{-\gamma y_T}h(y_T)\,dy_T \\
&= S_t^M e^{-q(T-t)} - Ke^{-r(T-t)}.
\end{aligned}$$

Since the value of the contract is set to be 0 at the initiation of the contract, therefore, the futures price (or the delivery price for this contract at initiation) is

$$F_t(S_t^M, T) = K = S_t^M e^{(r-q)(T-t)}.$$

The above results are the same as those under the no-arbitrage condition (see Hull's text book).

Example 4 *Second, we can price a call option written on the risky asset. Assume that the call option is written on the risky asset with a strike K and a maturity T.*

Its price is determined as

$$
\begin{aligned}
C(S_t, K, t, T) &= E_t\left(\frac{u_{c_T}}{u_{c_t}}\max(S_T^M - K, 0)\right) = E_t\left(e^{-\phi(T-t)}\frac{c_t^\gamma}{c_T^\gamma}\max(S_T^M - K, 0)\right) \\
&= e^{-\phi(T-t)}E_t\left[\left(\frac{D_T^M}{D_t^M}\right)^{-\gamma}\max(\frac{D_T^M}{r-\mu_m+\gamma\sigma_m^2} - K, 0)\right] \\
&= e^{-\phi(T-t)}\int_{\ln K/S_t}^{\infty}\left(\frac{D_t^M}{r-\mu_m+\gamma\sigma_m^2}e^{(1-\gamma)y_T} - Ke^{-\gamma y_T}\right)h(y_T)\,dy_T \\
&= e^{-\phi(T-t)}\int_{\ln K/S_t}^{\infty}\left(S_t e^{(1-\gamma)y_T}) - Ke^{-\gamma y_T}\right)h(y_T)\,dy_T \\
&= S_t^M e^{-\phi(T-t)}\int_{\ln K/S_t}^{\infty} e^{(1-\gamma)y_T}h(y_T)\,dy_T \\
&\quad -Ke^{-\phi(T-t)}\int_{\ln K/S_t}^{\infty} e^{-\gamma y_T}h(y_T)\,dy_T.
\end{aligned}
$$

Define

$$
\begin{aligned}
d_1 &= \frac{\ln S_t^M/K + (r - q + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\
d_2 &= d_1 - \sigma\sqrt{T-t},
\end{aligned}
$$

we have

$$
\begin{aligned}
C(S_t, K, t, T) &= S_t^M e^{-\phi(T-t)}\int_{-d_1}^{\infty} e^{(1-\gamma)(\mu_m-\frac{1}{2}\sigma_m^2)(T-t)+\frac{1}{2}(1-\gamma)^2\sigma_m^2)(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{w^2}{2}}\,dw \\
&\quad -Ke^{-\phi(T-t)}\int_{-d_2}^{\infty} e^{-\gamma(\mu_m-\frac{1}{2}\sigma_m^2)(T-t)+\frac{1}{2}\gamma^2\sigma_m^2)(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{w^2}{2}}\,dw \\
&= S_t^M e^{-q(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2).
\end{aligned}
$$

This result is the same as the Black-Scholes's call price, which is derived under no-arbitrage condition. This example shows that no-arbitrage valuation is consistent with equilibrium valuation. We will discuss this point further in the next section.

Exercise 10.2.1 *Show the corresponding put option on the risky stock has the following equilibrium price*

$$P(S_t, K, t, T) = Ke^{-r(T-t)}N(-d_2) - S_t^M e^{-q(T-t)}N(-d_1).$$

10.3 Equivalent Martingale Pricing Principle

Recall the pricing equations for the risky asset

$$S_t^M = E_t\left[\int_t^{\infty} D_T^M\frac{u'(D_T^M)}{u'(D_t^M)}dT\right]$$

and for the riskfree asset

$$B(t,T) = E_t\left[\frac{u'(D_T^M)}{u'(D_t^M)}\right].$$

We can rewrite the above equation as

$$1 = E_t\left[\frac{1}{B(t,T)}\frac{u'(D_T^M)}{u'(D_t^M)}\right] = \int_{P-measure}\frac{1}{B(t,T)}\frac{u'(D_T^M)}{u'(D_t^M)}\cdot dP(\cdot\mid\mathfrak{F}_t).$$

Since $\int_{P-measure}\frac{1}{B(t,T)}\frac{u'(D_T^M)}{u'(D_t^M)}\cdot dP(\cdot\mid\mathfrak{F}_t) = 1$ and $\frac{1}{B(t,T)}\frac{u'(D_T^M)}{u'(D_t^M)}\cdot dP(\cdot\mid\mathfrak{F}_t) > 0$. We can treat $\frac{1}{B(t,T)}\frac{u'(D_T^M)}{u'(D_t^M)}\cdot dP(\cdot\mid\mathfrak{F}_t)$. as a conditional probability (the $Q-measure$) with respect to the filtration $\mathfrak{F}_t$. That is, we use the rigorous concept of **Radon-Nikodym derivative** to define the $Q-measure$ as

$$dQ(\cdot\mid\mathfrak{F}_t) = \frac{1}{B(t,T)}\frac{u'(D_T^M)}{u'(D_t^M)}\cdot dP(\cdot\mid\mathfrak{F}_t)$$

Then the equilibrium price can be written as

$$\begin{aligned}
S_t^M &= E_t\left[\int_t^\infty D_T^M\frac{u'(D_T^M)}{u'(D_t^M)}dT\right] \\
&= \int_t^\infty E_t\left[D_T^M\frac{u'(D_T^M)}{u'(D_t^M)}\right]dT \\
&= \int_t^\infty\left[\int_{P-measure} D_T^M\frac{B(t,T)}{B(t,T)}\frac{u'(D_T^M)}{u'(D_t^M)}\cdot dP(\cdot\mid\mathfrak{F}_t)\right]dT \\
&= \int_t^\infty\left[\int_{P-measure} B(t,T)D_T^M\frac{1}{B(t,T)}\frac{u'(D_T^M)}{u'(D_t^M)}\cdot dP(\cdot\mid\mathfrak{F}_t)\right]dT \\
&= \int_t^\infty\left[\int_{Q-measure} B(t,T)D_T^M\cdot dQ(\cdot\mid\mathfrak{F}_t)\right]dT \\
&= \int_t^\infty E_t^Q[B(t,T)D_T^M]dT \\
&= E_t^Q\left(\int_t^\infty B(t,T)D_T^M dT\right).
\end{aligned}$$

Define the yield-to-maturity for the riskfree bond (in the continuous compounding sense) as

$$B(t,T) = e^{-R(t,T)(T-t)}.$$

We can rewrite the risky stock price equation as

$$S_t^M = E_t^Q\left(\int_t^\infty e^{-R(t,T)(T-t)}D_T^M dT\right).$$

We have just shown the Equivalent Martingale Pricing Principle. In fact, this result holds for any asset. It basically states that **an asset price at time t is equal to the**

expected sum of future dividends discounted at the riskfree rates where the expectation is calculated under the equivalent probability $Q-measure$. Such a result is also shown in Cox, Ingersoll and Ross (1985a).

Now we use the previous example to illustrate how to determine the equivalent Martingale measure from the actual probability measure. The previous example shows that the equilibrium price of the risky stock follows a lognormal process as

$$\begin{aligned}
\frac{\mathrm{d}S^M}{S^M} &= (\mu - q)\mathrm{d}t + \sigma \mathrm{d}z = (r + \gamma\sigma_m^2 - q)\mathrm{d}t + \sigma_m \mathrm{d}z_m \\
&= (r - q)\mathrm{d}t + \sigma_m \mathrm{d}z_m + \gamma\sigma_m^2 \mathrm{d}t \\
&= (r - q)\mathrm{d}t + \sigma_m(\mathrm{d}z_m + \gamma\sigma_m \mathrm{d}t) \\
&= (r - q)\mathrm{d}t + \sigma_m \mathrm{d}z_m^*
\end{aligned}$$

where $\mathrm{d}z_m^*$ (the equivalent Martingale uncertainty) is defined as

$$\mathrm{d}z_m^* = \gamma\sigma_m \mathrm{d}t + \mathrm{d}z_m.$$

This precise relation links the equivalent Martingale probability measure to the actual probability. The transformation is equivalent to the counterpart we discussed for the discrete case in Chapter 10.

Exercise 10.3.1 *Derive the conditional density under Q-measure. (Hint: start with*

$$Q(\cdot \mid \mathfrak{F}_t) = \frac{1}{B(t,T)} \frac{u'(D_T^M)}{u'(D_t^M)} \cdot P(\cdot \mid \mathfrak{F}_t)$$

with

$$\begin{aligned}
B(t,T) &= e^{-r(T-t)} \qquad \textit{and} \qquad r = \phi + \gamma(\mu_m - \frac{1}{2}\sigma_m^2) - \frac{1}{2}\gamma^2\sigma_m^2 \\
\frac{u'(D_T^M)}{u'(D_t^M)} &= \left(e^{-\phi(T-t)}\frac{D_T^M}{D_t^M}\right)^{-\gamma} = e^{-\gamma\phi(T-t)-\gamma y_T} \qquad \textit{where} \qquad y_T = \ln\frac{D_T^M}{D_t^M} \\
P(\cdot \mid \mathfrak{F}_t) = h(y_T) &= \frac{1}{\sqrt{2\pi\sigma_m^2(T-t)}} \exp\left(-\frac{[y_T - (\mu_m - \frac{1}{2}\sigma_m^2)(T-t)]^2}{2\sigma_m^2(T-t)}\right).
\end{aligned}$$

10.4 Pricing Derivatives by No-Arbitrage

10.4.1 Black-Scholes Analysis

The Black-Scholes analysis is based on a partial equilibrium approach where the asset's price is taken as exogenous. The standard Black-Scholes assumption on the price process is as follows:

$$\frac{\mathrm{d}S}{S} = (\mu - q)\mathrm{d}t + \sigma \mathrm{d}z.$$

Denote $f(S,t)$ as the price for any derivative security whose value depends on the underlying asset S, t. Using Ito's Lemma, we have

$$\mathrm{d}f = \left[(\mu - q)S\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right]\mathrm{d}t + \sigma S \frac{\partial f}{\partial S}\mathrm{d}z.$$

Consider a portfolio consisting of

$$\begin{aligned} -1: &\quad \text{derivative security,} \\ +\tfrac{\partial f}{\partial S}: &\quad \text{underlying asset.} \end{aligned}$$

Then the value of the portfolio is

$$\Pi = -f + S\frac{\partial f}{\partial S}.$$

The change in the portfolio value in time dt is given by

$$\begin{aligned} \mathrm{d}\Pi &= -\left[(\mu - q)S\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right]\mathrm{d}t - \sigma S\frac{\partial f}{\partial S}\mathrm{d}z \\ &\quad + (\mu - q)S\frac{\partial f}{\partial S}\mathrm{d}t + \sigma S\frac{\partial f}{\partial S}\mathrm{d}z \\ &= -\left[\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right]\mathrm{d}t. \end{aligned}$$

However, the holder of the portfolio earns capital gains equal to $\mathrm{d}\Pi$ plus dividends on the stock. That is, the change in wealth is

$$\mathrm{d}W = \mathrm{d}\Pi + qS\frac{\partial f}{\partial S}\mathrm{d}t.$$

Since the change in wealth doesn't involve any uncertainty, the return on the portfolio must equal the risk-free rate. Hence

$$\mathrm{d}W = -\left[\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right]\mathrm{d}t + qS\frac{\partial f}{\partial S}\mathrm{d}t = r\Pi\mathrm{d}t.$$

That is,

$$-\left[\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right] + qS\frac{\partial f}{\partial S} = r(S\frac{\partial f}{\partial S} - f).$$

Rearranging terms, we have

$$(r - q)S\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf.$$

This is the Black-Scholes fundamental Partial Differential Equation (PDE) satisfied by any Derivative Security.

- The above PDE is satisfied by any security whose price is dependent on the stock price;
- The value of a particular security is determined by a specific boundary condition;
- The variable μ does not appear in the Black-Scholes equation!

- The equation is independent of all variables affected by risk preference;
- The solution to the PDE is therefore the same in a risk-free world as it is in the real world;
- This leads to the principle of risk-neutral valuation.

10.4.2 Applying Risk-Neutral Valuation to Options

According to the Black-Scholes fundamental PDE, we can assume that the expected return from the stock price is the risk-free rate, i.e., the risk-neutral process as

$$\frac{\mathrm{d}S}{S} = (r-q)\mathrm{d}t + \sigma \mathrm{d}z^*,$$

where the riskfree rate r and the stock volatility σ are constant. Then, the conditional distribution $y_T^* = \ln S_T/S_t$ has a normal distribution with mean $(r-q-\frac{1}{2}\sigma^2)(T-t)$ and variance $\sigma^2(T-t)$.

Any derivative security can be value as

$$f_t(S_t) = E_t^Q(e^{-r(T-t)} f_T(S_T)).$$

Example 5 *Consider a call option written on the risky asset with a strike K and a maturity T. Its price is computed as*

$$\begin{aligned} C(S_t, K, t, T) &= E_t^Q\left(e^{-r(T-t)} \max(S_T - K, 0)\right) \\ &= e^{-r(T-t)} \int_{\ln \frac{K}{S_t}}^{\infty} (S_T - K)\, h(y_T^*)\, dy_T^* \\ &= S_t e^{-r(T-t)} \int_{\ln \frac{K}{S_t}}^{\infty} e^{y_T} h(y_T^*)\, dy_T^* - K e^{-r(T-t)} \int_{\ln K}^{\infty} h(y_T^*)\, dy_T^*. \end{aligned}$$

Define

$$\begin{aligned} d_1 &= \frac{\ln S_t/K + (r-q+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= d_1 - \sigma\sqrt{T-t}, \end{aligned}$$

we have

$$\begin{aligned} C(S_t, K, t, T) &= S_t e^{-r(T-t)} \int_{-d_1}^{\infty} e^{(r-q-\frac{1}{2}\sigma^2)(T-t)+\frac{1}{2}\sigma^2)(T-t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}}\, dw \\ &\quad - K e^{-r(T-t)} \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}}\, dw \\ &= S_t e^{-q(T-t)} N(d_1) - K e^{-r(T-t)} N(d_2). \end{aligned}$$

Exercise 10.4.1 *Consider the corresponding put option written on the risky asset with a strike K and a maturity T. Its price is computed as*

$$P(S_t, K, t, T) = E_t^Q \left(e^{-r(T-t)} \max(K - S_T \right).$$

Use the risk-neutral approach to show that

$$P(S_t, K, t, T) = Ke^{-r(T-t)} N(-d_2) - S_t e^{-q(T-t)} N(-d_1).$$

Summary 1 *In this chapter we discuss continuous-time financial models covering the following main topics:*
(i) Intertemporal maximization for individual agents over consumption and portfolio decision;
(ii) Competitive equilibrium and asset prices in equilibrium;
(iii) Equivalent Martingale Pricing Principle;
(iv) Derivative security valuation by no-arbitrage.

10.4.3 Properties of Option Prices

Based on the option pricing formulas derived above, we can further examine the properties of option prices. Take a call option as an example, we can derive the process followed by the call value. Given the following actual process for the underlying asset (assuming dividend yield $q = 0$),

$$\frac{\mathrm{d}S}{S} = \mu \mathrm{d}t + \sigma \mathrm{d}z^*,$$

we have

$$\begin{aligned}
\frac{\partial c}{\partial S} &= N(d_1), \qquad \frac{\partial^2 c}{\partial S^2} = \frac{n(d_1)}{S\sigma\sqrt{T-t}} \qquad \textit{with} \qquad n(d_1) = \frac{1}{\sqrt{2\pi}} e^{-d_1^2/2}, \\
\frac{\partial c}{\partial t} &= -\frac{S\sigma n(d_1)}{2\sqrt{T-t}} - rKe^{-r(T-t)} N(d_2), \qquad \frac{\partial c}{\partial K} = -e^{-r(T-t)} N(d_2).
\end{aligned}$$

it is easy to show the process for a call price with Ito's Lemma

$$\frac{\mathrm{d}c}{c} = \left[r + (\mu - r) \frac{SN(d_1)}{c} \right] \mathrm{d}t + \frac{SN(d_1)}{c} \sigma \mathrm{d}z^*.$$

Therefore, the instantaneous return on a call option is

$$\begin{aligned}
\mu_c &= r + (\mu - r)\frac{SN(d_1)}{c} = r + \beta(\mu_m - r)\frac{SN(d_1)}{c} = r + \beta_c(\mu_m - r) \\
& \quad \textit{with} \\
\beta_c &= \beta \frac{SN(d_1)}{c}.
\end{aligned}$$

Properties:

1. The beta for a call option is bigger than the beta for the underlying asset. That is, $\beta_c > \beta$ since $\frac{SN(d_1)}{c} > 1$.

2. The call option return increases with the strike price. That is, $\frac{\partial \mu_c}{\partial K} > 0$.

   Proof:
   We first obtain
   $$\frac{\partial \mu_c}{\partial K} = \beta \frac{Se^{-r(T-t)N(d_1)N(d_2)}}{c^2} \frac{\mu - r}{\sigma\sqrt{T-t}} \left[\frac{n(d_1)}{N(d_1)} + d_1 - \frac{n(d_2)}{N(d_2)} - d_2\right].$$

To show $\frac{\partial \mu_c}{\partial K} > 0$ is equivalent to show that
$$\frac{n(d_1)}{N(d_1)} + d_1 - \frac{n(d_2)}{N(d_2)} - d_2 > 0.$$

To do so, let us define
$$f(x) = \frac{n(x)}{N(x)} + x.$$

Given that $d_1 > d_2$, it is equivalent to show that
$$f_x = \frac{df}{dx} = 1 - \left(x + \frac{n(x)}{N(x)}\right) \frac{n(x)}{N(x)} > 0.$$

To show $f_x > 0$, we need to verify the following:

1. f_x is continuous (which is obvious since $n(x)$ and $N(x)$ are);
2. In the domain $[-M, M]$, $f_x(-M) > 0$;
3. For all $x_0 \in [-M, M]$, s.t. $f_x(x_0) = 0$ we have $f_{xx}(x_0) > 0$.

Since the domain for x is $(-\infty, \infty)$ for $f(x)$, we have $f_x(-\infty) = 1 - \lim_{x\to-\infty} \left(x + \frac{n(x)}{N(x)}\right) \frac{n(x)}{N(x)} = 1 > 0$. For all $x_0 \in (-\infty, \infty)$ s.t. $f_x(x_0) = 0$, we have
$$f_x(x_0) = 1 - \left(x_0 + \frac{n(x_0)}{N(x_0)}\right) \frac{n(x_0)}{N(x_0)} = 0.$$

That is,
$$x_0 + \frac{n(x_0)}{N(x_0)} = \frac{N(x_0)}{n(x_0)}.$$

We know that
$$f_{xx} = f_x(x - 2f) + f.$$

Therefore
$$f_{xx}(x_0) = f_x(x_0)[x_0 - 2f(x_0)] + f(x_0) = x_0 + \frac{n(x_0)}{N(x_0)} = \frac{N(x_0)}{n(x_0)} > 0.$$

Hence,
$$f_x > 0.$$

Exercise 10.4.2 *Derive the following properties for put options: 1) the beta of a put option; 2) the relationship between the return of a put option and its strike price.*