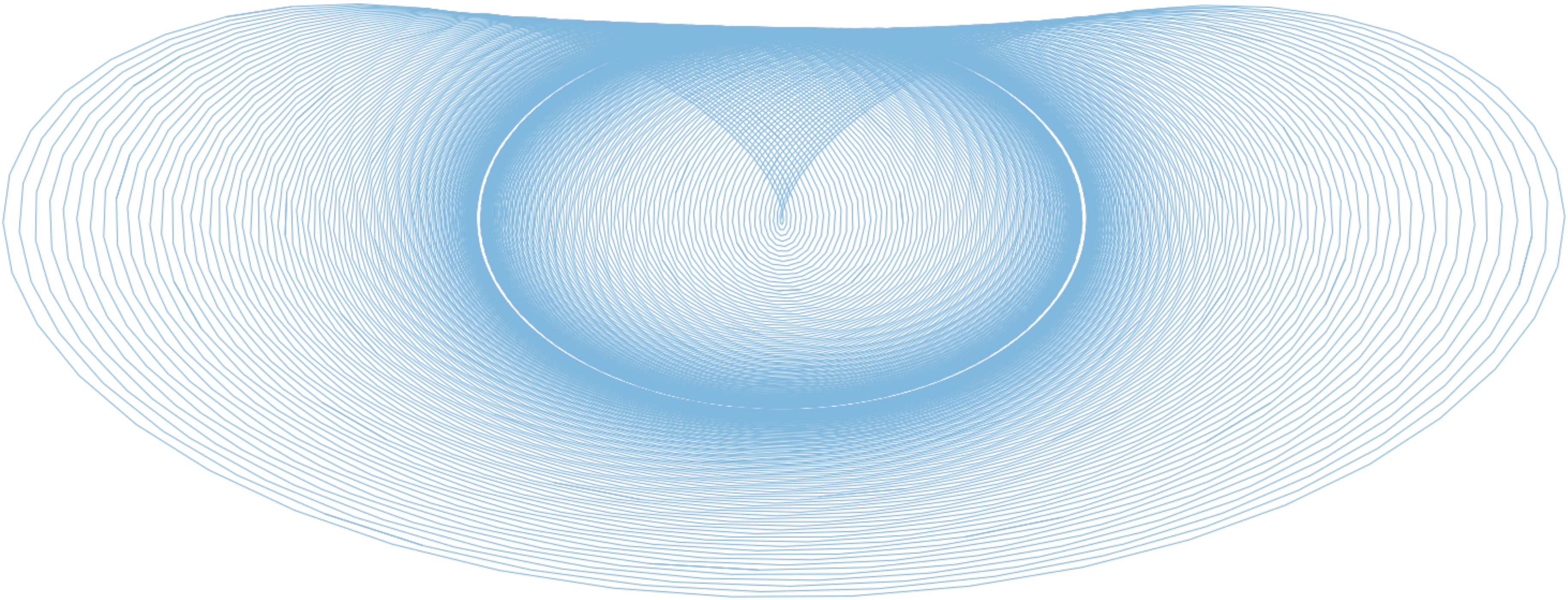
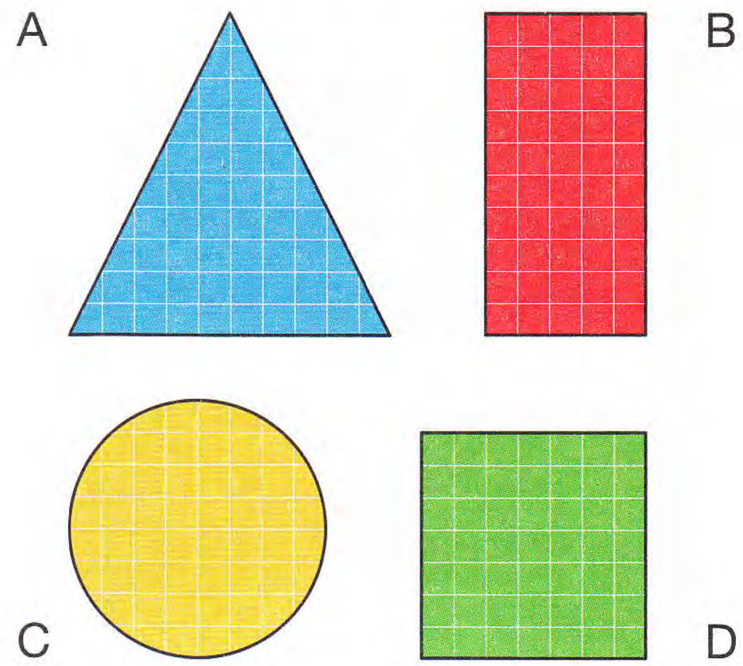
PHYS 1420 (F19)
Physics with Applications to Life Sciences



2019.09.06
Relevant reading:
Kesten & Tauck ch.2.1-2.2

Christopher Bergevin
York University, Dept. of Physics & Astronomy
Office: Petrie 240   Lab: Farq 103
cberge@yorku.ca

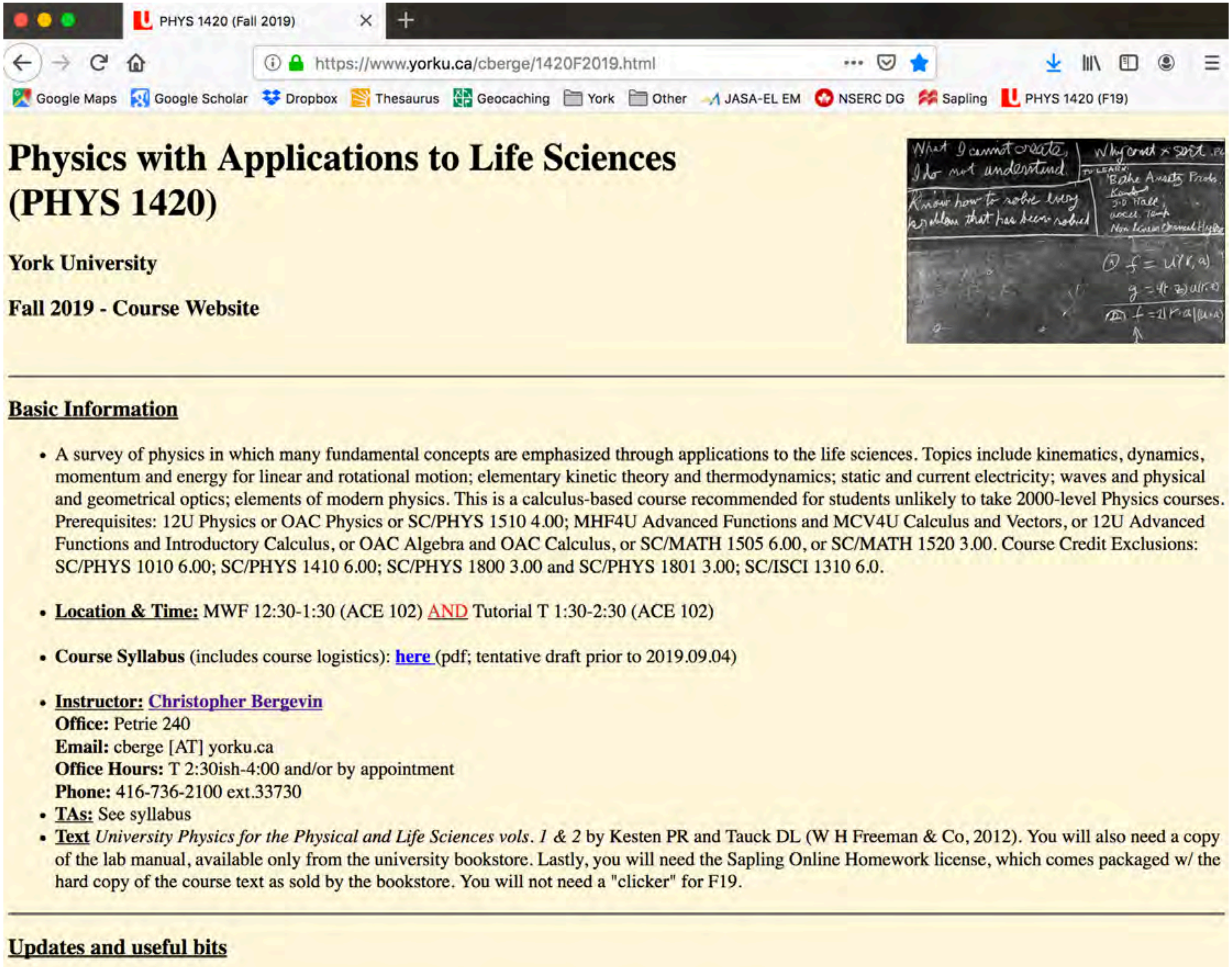
70. Basic Shapes



Which shape has the greatest area?

A B C D

PHYS 1420 Webpage



Physics with Applications to Life Sciences (PHYS 1420)

York University

Fall 2019 - Course Website

Basic Information

- A survey of physics in which many fundamental concepts are emphasized through applications to the life sciences. Topics include kinematics, dynamics, momentum and energy for linear and rotational motion; elementary kinetic theory and thermodynamics; static and current electricity; waves and physical and geometrical optics; elements of modern physics. This is a calculus-based course recommended for students unlikely to take 2000-level Physics courses. Prerequisites: 12U Physics or OAC Physics or SC/PHYS 1510 4.00; MHF4U Advanced Functions and MCV4U Calculus and Vectors, or 12U Advanced Functions and Introductory Calculus, or OAC Algebra and OAC Calculus, or SC/MATH 1505 6.00, or SC/MATH 1520 3.00. Course Credit Exclusions: SC/PHYS 1010 6.00; SC/PHYS 1410 6.00; SC/PHYS 1800 3.00 and SC/PHYS 1801 3.00; SC/ISCI 1310 6.0.

- **Location & Time:** MWF 12:30-1:30 (ACE 102) AND Tutorial T 1:30-2:30 (ACE 102)

- **Course Syllabus** (includes course logistics): here (pdf; tentative draft prior to 2019.09.04)

- **Instructor:** Christopher Bergevin
  Office: Petrie 240
  Email: cberge [AT] yorku.ca
  Office Hours: T 2:30ish-4:00 and/or by appointment
  Phone: 416-736-2100 ext.33730
- **TAs:** See syllabus
- **Text** *University Physics for the Physical and Life Sciences vols. 1 & 2* by Kesten PR and Tauck DL (W H Freeman & Co, 2012). You will also need a copy of the lab manual, available only from the university bookstore. Lastly, you will need the Sapling Online Homework license, which comes packaged w/ the hard copy of the course text as sold by the bookstore. You will not need a "clicker" for F19.

Updates and useful bits

https://www.yorku.ca/cberge/1420F2019.html

Key Topics & Concepts (re Today)

→ Class reps needed (email me if you want to volunteer)

Some relevant underlying concepts of the day…

- Problem solving (revisited) & **quantitative reasoning**

- Mechanical motivations….

- *Modeling* & notion of differential equations

Review: algebra, geometry, coordinate systems, differential calculus, integral calculus, differential equations, etc….

Types of “problems”

Some types of problems we’ll deal with:

- Those w/ known answers (e.g., yes/no, ¾, etc.....)

Note: For the most part, these will be the type of problems you encounter on HW and exams

- Open-ended (i.e., answers typically lead to more questions)

e.g., Why is the sky blue?

- Lateral thinking (i.e., “outside the box”)

It's not an easy life being a zoo keeper. I can't begin to imagine how they tell all those penguins apart.

To see if you should apply to your local zoo for a job, I'll supply an initial aptitude test. Here is a pair of pictures of a panda, and there are two clearly visible differences between them.

Identify both of them.

Polya

How To Solve It

A New Aspect of Mathematical Method

G. POLYA

Stanford University

How To Solve It was originally published by Princeton University Press in 1945. The Anchor Books edition is published by arrangement with Princeton University Press.

Polya

1. UNDERSTAND THE PROBLEM

- **First.** You have to *understand* the problem.
- What is the unknown? What are the data? What is the condition?
- Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?
- Draw a figure. Introduce suitable notation.
- Separate the various parts of the condition. Can you write them down?

Polya

2. DEVISING A PLAN

- **Second.** Find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should obtain eventually a *plan* of the solution.
- Have you seen it before? Or have you seen the same problem in a slightly different form?
- *Do you know a related problem?* Do you know a theorem that could be useful?
- *Look at the unknown!* Try to think of a familiar problem having the same or a similar unknown.
- *Here is a problem related to yours and solved before. Could you use it?* Could you use its result? Could you use its method? Should you introduce some auxiliary element in order to make its use possible?
- Could you restate the problem? Could you restate it still differently? Go back to definitions.
- If you cannot solve the proposed problem, try to solve first some related problem. Could you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem? Could you solve a part of the problem? Keep only a part of the condition, drop the other part; how far is the unknown then determined, how can it vary? Could you derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown or data, or both if necessary, so that the new unknown and the new data are nearer to each other?
- Did you use all the data? Did you use the whole condition? Have you taken into account all essential notions involved in the problem?

Polya

3. CARRYING OUT THE PLAN

- **Third.** *Carry out* your plan.
- Carrying out your plan of the solution, *check each step.* Can you see clearly that the step is correct? Can you prove that it is correct?

4. LOOKING BACK

- **Fourth.** *Examine* the solution obtained.
- Can you *check the result*? Can you check the argument?
- Can you derive the solution differently? Can you see it at a glance?
- Can you use the result, or the method, for some other problem?

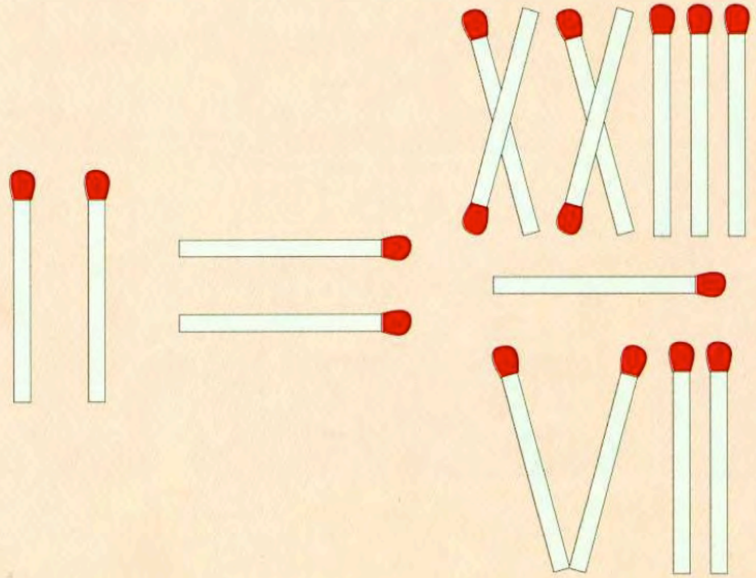
Two people were discussing the matchstick puzzle shown below. The idea was to move only one match to make the statement correct.

"I reckon it's impossible to make it exactly correct," claimed Kevin, "unless you cheat by turning the equals sign into a 'not equal to' sign."

"Maybe," said Mandy, "but I can make it approximately correct." She made her move and the result was only a few percent from being exactly correct.

"Aha!" exclaimed Kevin. "I can move the same match as you've just moved to make the statement even closer to being exactly correct."

Can you work out what the two solutions were?



Two grandmasters played five games of chess. Each won the same number of games and lost the same number of games. None of the games ended in a draw. How could this be?

Types of "problems"

Some types of problems we'll deal with:

- Those w/ known answers (e.g., yes/no, ¾, etc.....)

- Open-ended (i.e., answers typically lead to more questions)

  e.g., Why is the sky blue?

- Lateral thinking (i.e., "outside the box")

- Fermi problems (e.g., estimate an order of magnitude)

Ex.

52. Estimate the number of (a) atoms and (b) cells in your body.

Ex.

52. Estimate the number of (a) atoms and (b) cells in your body.

Be thoughtful/careful w/ units!

- I weigh ~200 lbs. (that's ~91 kg)
- Assume I'm made up entirely of water. Density of water is 1000 kg/m^3
- Avogadro's # (6.02×10^{23}/mol) tells me # of particles in one mole
- Molecular mass of water is ~18 (i.e., one mole of water has a mass of about 18 grams) → I'm made up of ~91000/18 = 5060 mols
- Thus there are ~5056 × 6.02×10^{23} = 3×10^{27} atoms in my body

→ But is this even right!?!

SOL

52. **INTERPRET** This problem calls for a rough estimate, instead of a precise numerical answer. We are asked to estimate the number of atoms and cells in the human body.

DEVELOP Human tissue is mostly water, so for a rough estimate we can consider the human body to contain about as many atoms as an equivalent mass of water. One mole of water (H_2O) is 18 g and contains Avogadro's number of molecules ($N_A = 6\times 10^{23}$). Because H_2O has 3 atoms per molecule, the number n of atoms per kg water is about

$$n = \frac{3\times 6\times 10^{23}}{18\ \cancel{g}}\overbrace{\left(\frac{10^3\ \cancel{g}}{\text{kg}}\right)}^{=1} = 1\times 10^{26}\ \text{kg}^{-1}$$

To estimate the number of cells in the body, take the size of a red blood cell (= 10 μm, see Table 1.2) as a typical cell. Assuming again that the human body is mostly water, the volume V_H of a human body can be estimated by dividing its mass m = 60 kg by the density $\rho = 1$ gm/cm^3 of water, or

$$V_H = \frac{m}{\rho} = \left(\frac{60\ \cancel{\text{kg}}}{1\ \cancel{g}/\text{cm}^3}\right)\overbrace{\left(\frac{10^3\ \cancel{g}}{\cancel{\text{kg}}}\right)}^{=1} = 6\times 10^4\ \text{cm}^3$$

EVALUATE **(a)** Assuming that the mass of the average human is m = 60 kg, we find that the number N_a of atoms in the human body is approximately

$$N_a = mn = (60\ \text{kg})(1\times 10^{26}\ \text{kg}^{-1}) = 6\times 10^{27}$$

Still not clear if either of us are right(!!)

(b) To estimate the number N_c of cells in the human body we divide the volume V_H by the volume $V_b = (10\ \mu\text{m})^3$ of a red blood cell. This gives

$$N_c = \frac{V_H}{V_b} = \frac{6\times 10^4\ \cancel{\text{cm}^3}}{(10\ \cancel{\mu\text{m}})^3}\overbrace{\left(\frac{10^4\ \cancel{\mu\text{m}}}{\cancel{\text{cm}}}\right)^3}^{=1} = 6\times 10^{13}$$

ASSESS Conventional wisdom claims that the human body has about 10^{13} cells, so our estimate for part **(b)** seems reasonable.

DOWNTIME

Irrational Numbers

On the pleasure of procrastinating with absurd back-of-the-envelope calculations.

By HARRY KENNARD

SEPT 03, 2019 • 10:00 AM

At its best, what starts out as a little harmless procrastination can end up inverting your entire worldview. The physicist Richard Feynman called it "the pleasure of finding things out." Simple calculations, straightforwardly communicated, have the power to be understood by almost anyone. Like the very best of rabbit holes, thinking about the physical world is a potentially infinite adventure. The only hope you have of not falling deeper in is to simplify your assumptions, but where's the fun in that?

https://slate.com/human-interest/2019/09/back-envelope-math-science-rabbit-holes.html

Approaches to Problem Solving

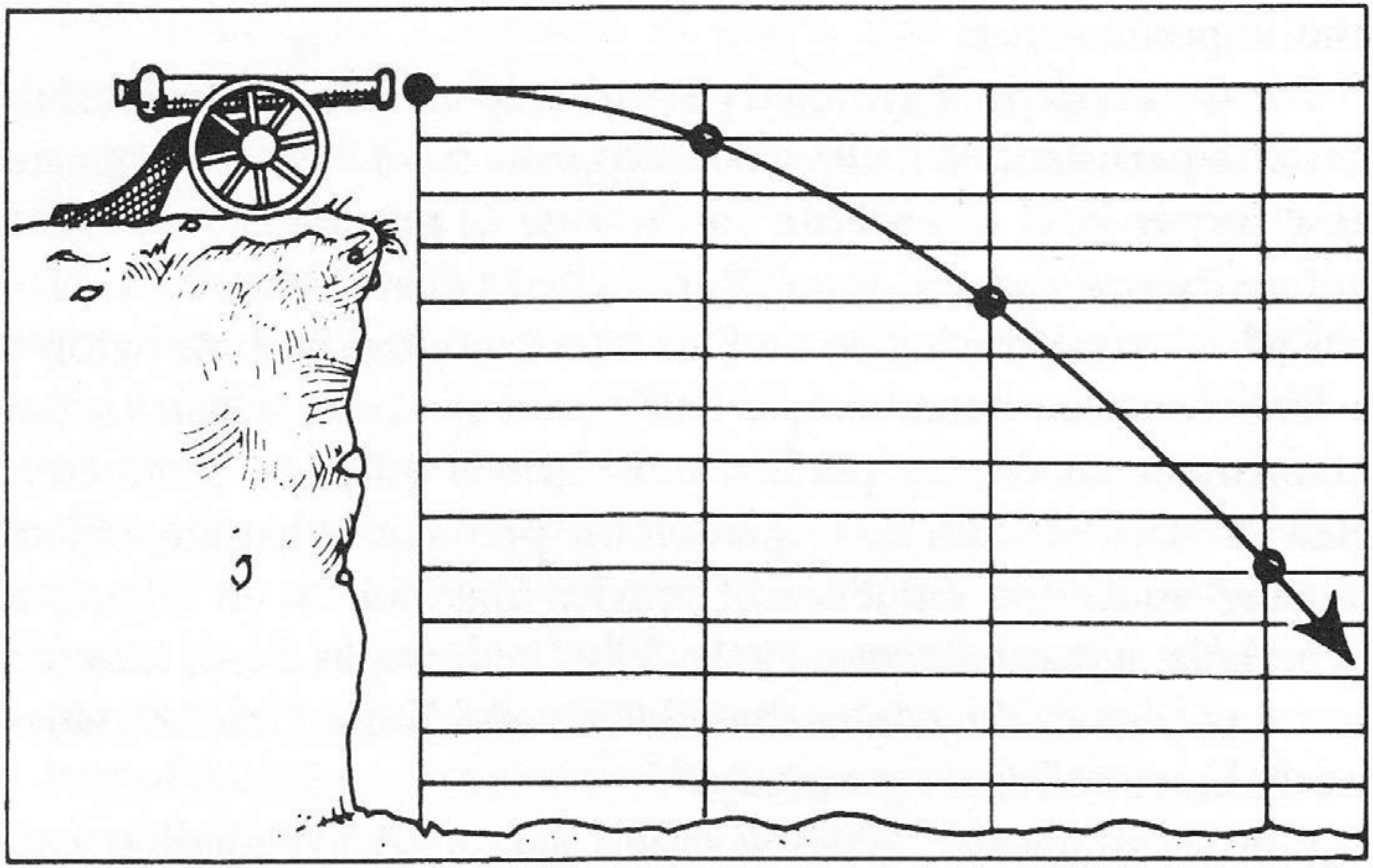
How To Solve It

A New Aspect of Mathematical Method

G. POLYA

Stanford University

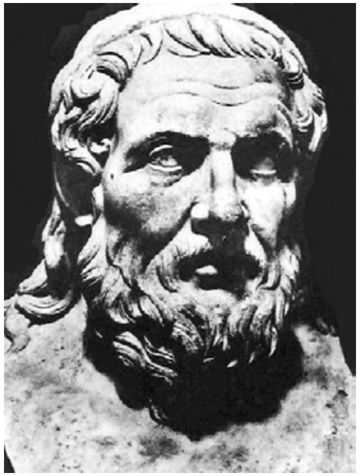
→ This is a reasonable starting point. But keep in mind that ideally you'll find an approach/method that works best for you!



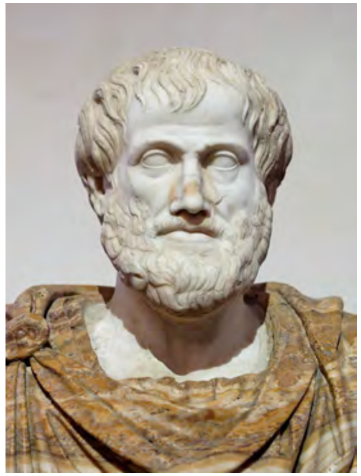
von Baeyer

Mechanics

Apollonius of Perga

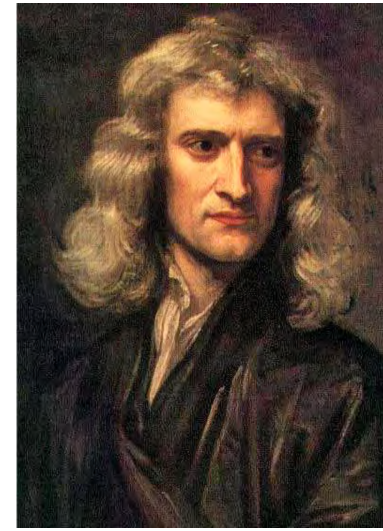


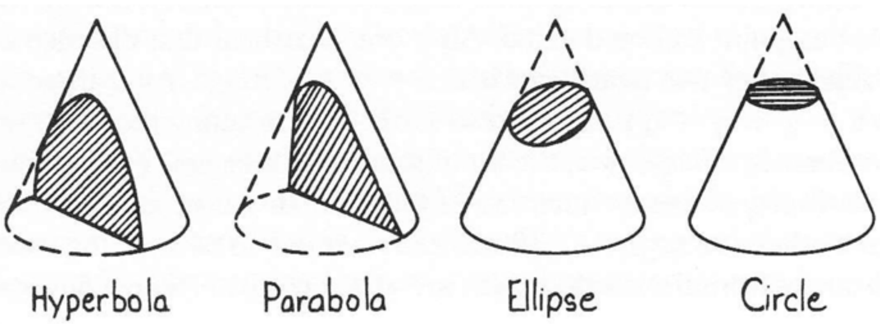
Aristotle

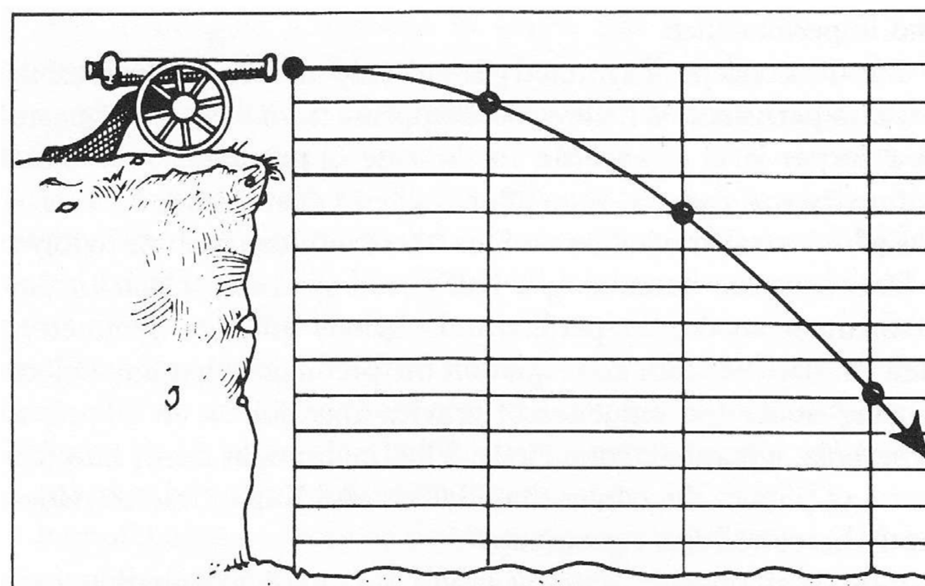


G. Galileo

I. Newton







von Baeyer

Jumping ahead slightly for the moment....

4.1 The Wrong Question

Actually, "What keeps things moving?" is the wrong question. In the early 1600s, Galileo Galilei did experiments that convinced him that a moving object has an intrinsic "quantity of motion" and needs no push to keep it moving (Fig. 4.1). Instead of answering "What keeps things moving?," Galileo declared that the question needs no answer. In so doing, he set the stage for centuries of progress in physics, beginning with the achievements of Issac Newton and culminating in the work of Albert Einstein.

The Right Question

Our first question—about why the spacecraft keeps moving—is the wrong question. So what's the right question? It's the second one, about why the baseball's motion *changed*. Dynamics isn't about what causes motion itself; it's about what causes *changes* in motion. Changes include starting and stopping, speeding up and slowing down, and changing direction. Any *change* in motion begs an explanation, but motion itself does not. Get used to this important idea and you'll have a much easier time with physics. But if you remain a "closet Aristotelian," secretly looking for causes of motion itself, you'll find it difficult to understand and apply the simple laws that actually govern motion.

→ The notion of *change* is a lynchpin of physics....

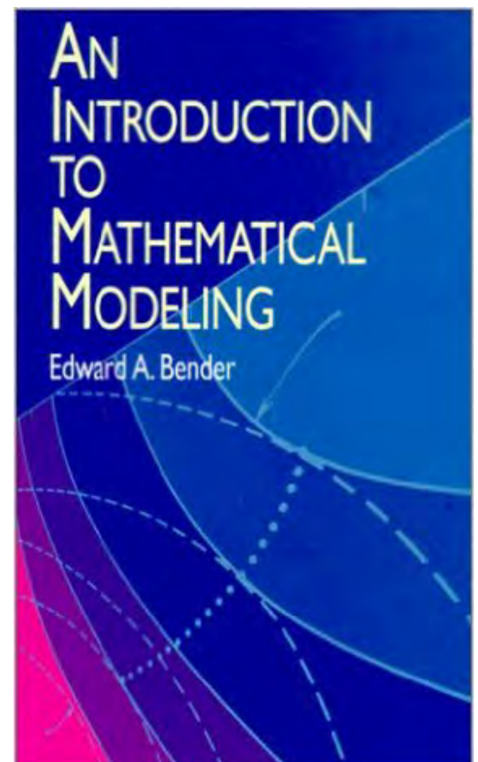
“Modeling”

- To help put some context in place for the physics ahead, let’s take a slight detour....
- Calculus provides wonderful tools to help study *change*
- In particular, a very useful extension of calculus is known as *differential equations*

Table 2.1 Equations of Motion for Constant Acceleration

Equation	Contains	Number
$v = v_0 + at$	v, a, t; no x	2.7
$x = x_0 + \frac{1}{2}(v_0 + v)t$	x, v, t; no a	2.9
$x = x_0 + v_0 t + \frac{1}{2}at^2$	x, a, t; no v	2.10
$v^2 = v_0^2 + 2a(x - x_0)$	x, v, a; no t	2.11

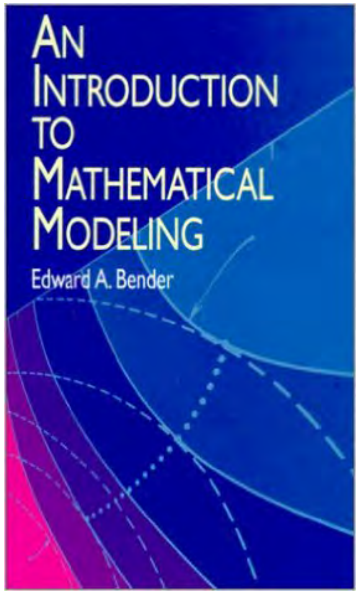
Whether you realize it or not, you have already been dealing with DEs in some fashion....



- Here comes the fun part: Many problems fall under the purview of *mathematical modeling*

"Mathematical Modeling"

From the preface (1978)



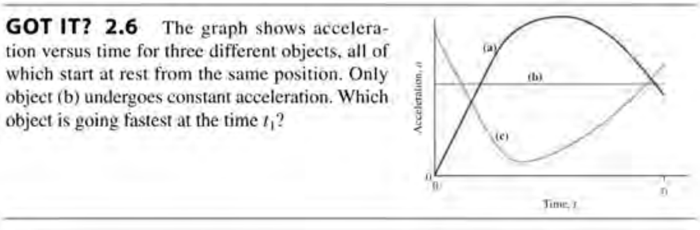
"This book is designed to teach students how to apply mathematics by formulating, analyzing, and criticizing models."

"The first part of the book requires only elementary calculus and, in one chapter, basic probability theory."

"Although the level of mathematics required is not high, this is not an easy text: Setting up and manipulating models requires thought, effort, and usually discussion."

"Often problems have no single best answer, because different models can illuminate different facets of a problem. Discussion of homework in class by the students is an integral part of the learning process; in fact, my classes have spent about half the time discussing homework.."

"I'd appreciate hearing about any errors...."



Bender

"The theoretical and scientific study of a situation centers around a model, that is, something that mimics relevant features of the situation being studied. For example, a road map, a geological map, and a plant collection are all models that mimic different aspects of a portion of the earth's surface."

"The ultimate test of a model is how well it performs when it is applied to the problems it was designed to handle. (You cannot reasonably criticize a geological map if a major highway is not marked on it; however, this would be a serious deficiency in a road map.) When a model is used, it may lead to incorrect predictions. The model is often modified, frequently discarded, and sometimes used anyway because it is better than nothing. *This is the way science develops*."

"Here we are concerned exclusively with mathematical models, that is, models that mimic reality by using the language of mathematics. [...] What makes mathematical models useful? If we "speak in mathematics, then:

1. We must formulate our ideas precisely and so are less likely to let implicit assumptions slip by.
2. We have a concise "language" which encourages manipulation.
3. We have a large number of potentially useful theorems available.
4. We have high speed computers available for carrying out calculations.

Bender

"Modeling" & Differential equations (DEs)

→ A very common/useful tool in our toolbox....

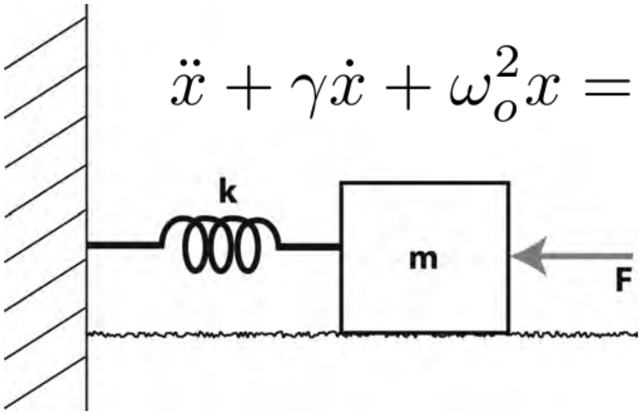
Wave equation

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}$$

Note: Though DEs pervade much of 1420 material, you are not expected to become adept at solving them for 1420

Harmonic oscillator

$$\ddot{x} + \gamma \dot{x} + \omega_o^2 x = 0$$



Note: This just a specific case of Newton's 2nd law (F=ma)!

Laplace's equation

$$\Delta f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

Maxwell's equations

$$\nabla \cdot \mathbf{E} = \frac{\rho_v}{\varepsilon} \quad \text{(Gauss' Law)}$$

$$\nabla \cdot \mathbf{H} = 0 \quad \text{(Gauss' Law for Magnetism)}$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad \text{(Faraday's Law)}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \quad \text{(Ampere's Law)}$$

Several basic flavors apparent:

- Ordinary (ODE)
- Partial (PDE)
- Scalar vs. Vector

“Modeling” & Differential equations

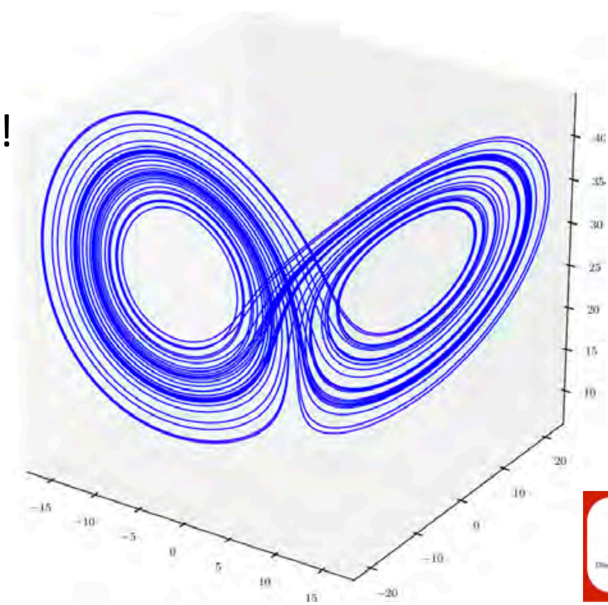
Lorenz equations

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = rx - y - xz$$

$$\frac{dz}{dt} = xy - bz$$

→ Chaos!



SIR model
(‘compartmental’ model in epidemiology)

S = the number of *susceptibles*, the people who are not yet sick but who could become sick

I = the number of *infecteds*, the people who are currently sick

R = the number of *recovered*, or *removed*, the people who have been sick and can no longer infect others or be reinfected.

$$\frac{dS}{dt} = -\beta IS$$

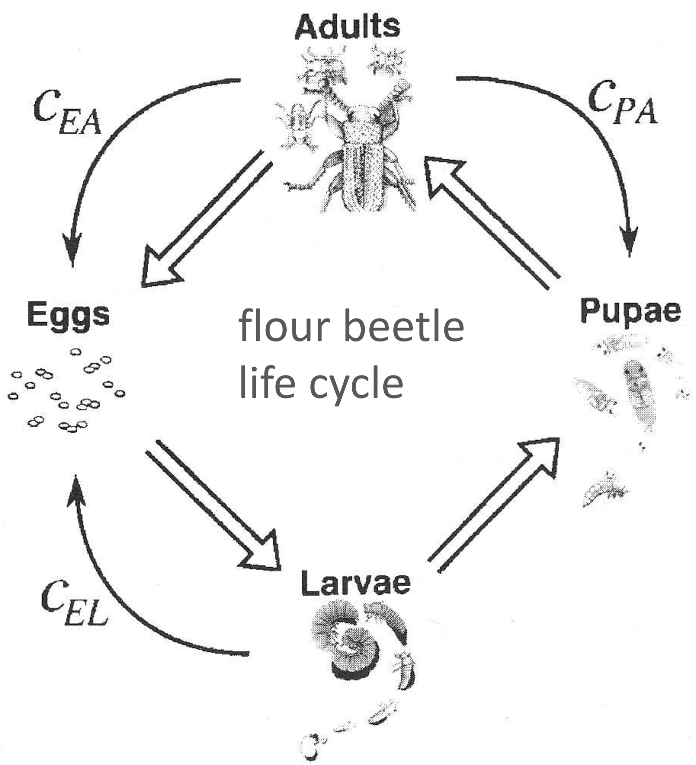
$$\frac{dI}{dt} = \beta IS - \gamma I$$

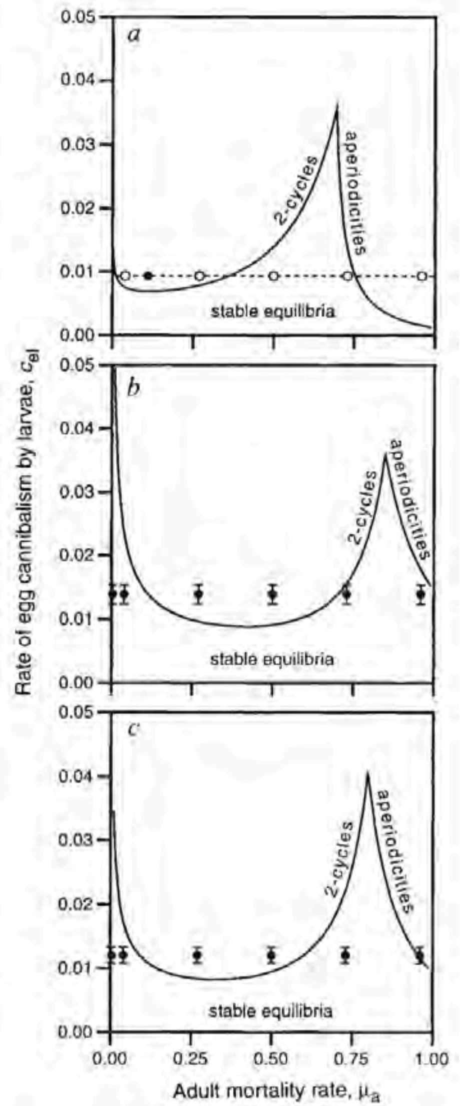
$$\frac{dR}{dt} = \gamma I$$

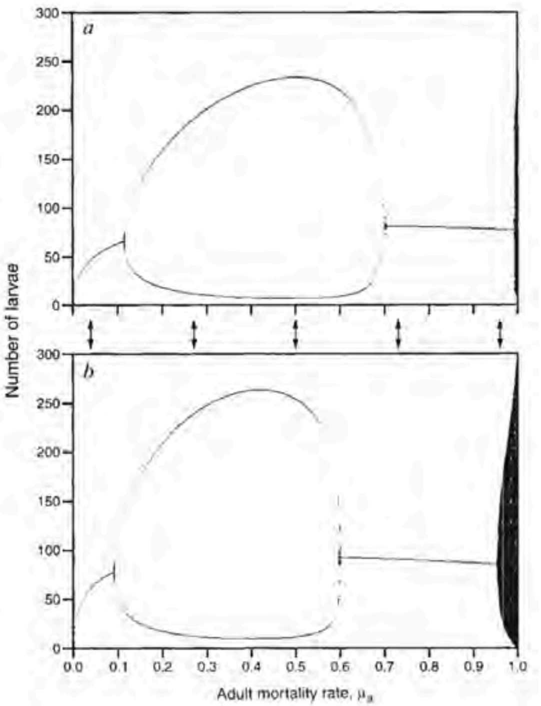
“Modeling” & Differential equations

Discrete LPA Model (Cushing, Costantino, et al.)

$$L_{t+1} = bA_t \exp(-c_{el}L_t - c_{ea}A_t)$$
$$P_{t+1} = (1-\mu_l)L_t$$
$$A_{t+1} = P_t \exp(-c_{pa}A_t) + (1-\mu_a)A_t$$







→ Chaos! In a jar!

Costatino et al. (1995)

Solving DEs (+ a computational aside)

→ Differential equations are very common/useful tool in our toolbox....

In a nutshell, we are very good at describing how things *change*....

$$\frac{d\mathbf{y}}{dt} = f(t, \mathbf{y}) \tag{7.1.1}$$

... but less good at finding solutions to the the corresponding equations (though a variety of analytic methods certainly are in place)

Table 2.1 Equations of Motion for Constant Acceleration

Equation	Contains	Number
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$v^2 = v_0^2 + 2a(x - x_0)$	x, v, a; no t	2.11

Idea: Since equations tell us how things change, numerically integrate to find solution(s)

$$\frac{d\mathbf{y}}{dt} = f(t, \mathbf{y}) \quad \Rightarrow \quad \frac{\mathbf{y}_{n+1} - \mathbf{y}_n}{\Delta t} \approx f(t_n, \mathbf{y}_n). \tag{7.1.4}$$

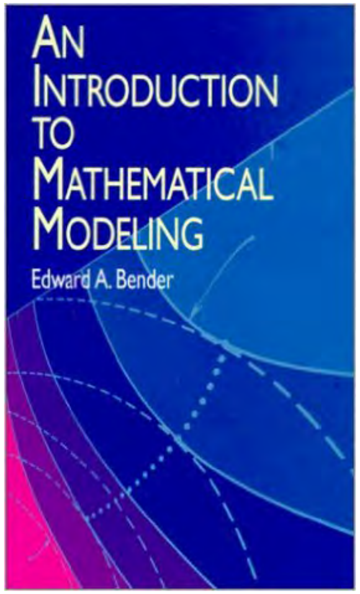
Approximation:

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \Delta t \cdot f(t_n, \mathbf{y}_n). \tag{7.1.5}$$

→ This is called Euler's method (very basic, but a bit beyond the scope of 1^{st} year PHYS 1420)

Kutz (2013)

“Mathematical Modeling”



“Mathematics and physical science each had important effects on the development of the other. Mathematics is starting to play a greater role in the development of the life and social sciences, and these sciences are starting to influence the development of mathematics.”

“We begin with a definition based on the previous discussion: A mathematical model is an abstract, simplified, mathematical construct related to a part of reality and created for a particular purpose. [...] As far as a model is concerned the world can be divided into three parts:

1. Things whose effects are neglected.
2. Things that affect the model but whose behavior the model is not designed to study.
3. Things the model is designed to study the behavior of.

Two key ingredients should be apparent here:

- Figuring out what question you want to try to answer
- What assumptions you are willing to make

Bender

“Mathematical Modeling”

How To Solve It

A New Aspect of Mathematical Method

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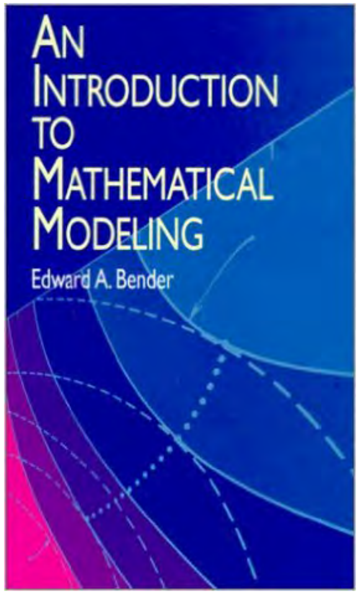


Just as Polya suggests a means to approach solving problems, so does Bender re modeling....

“Model building involves imagination and skill. Giving rules for doing it is like listing rules for being an artist; at best this provides a framework around which to build skills and develop imagination. *It may be impossible to teach imagination*.”

1. Formulate the problem
2. Outline the model
3. Is it useful?
4. Test the model

→ This sounds a lot easier than it is. So let us jump in by looking at some examples and trying it ourselves....

Bender

Ex.

Question: How fast does a person learn?

Ex.

Question: How fast does a person learn?

(very) Simple model: Rate a person learns = Percentage of task not yet learned

y is the percentage learned as a function of time t

$$\frac{dy}{dt} = 100 - y$$

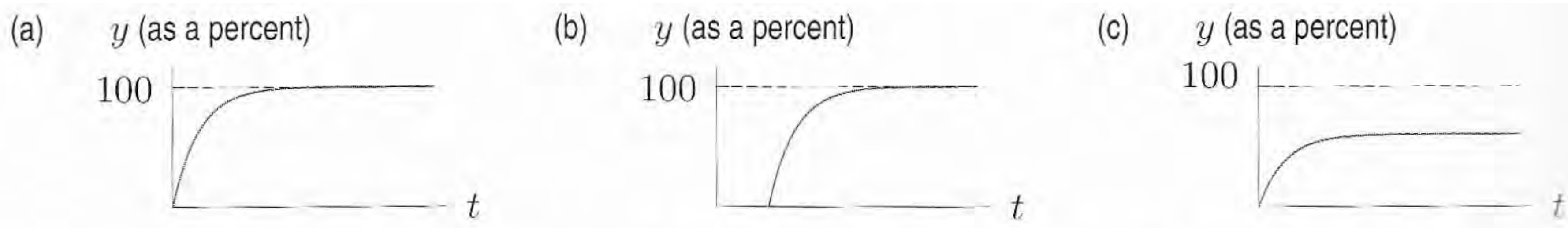


Figure 11.1: Possible graphs showing percentage of task learned, y, as a function of time, t

Solution
(e.g., via "separation of variables")

$$y(t) = 100 - Ce^{-t}$$

Hughes-Hallett et al. (2005)

Ex.

$$\frac{dy}{dt} = 100 - y$$

$$y(t) = 100 - Ce^{-t}$$

→ Note that our 'model' (redundantly) allows for y greater than 100

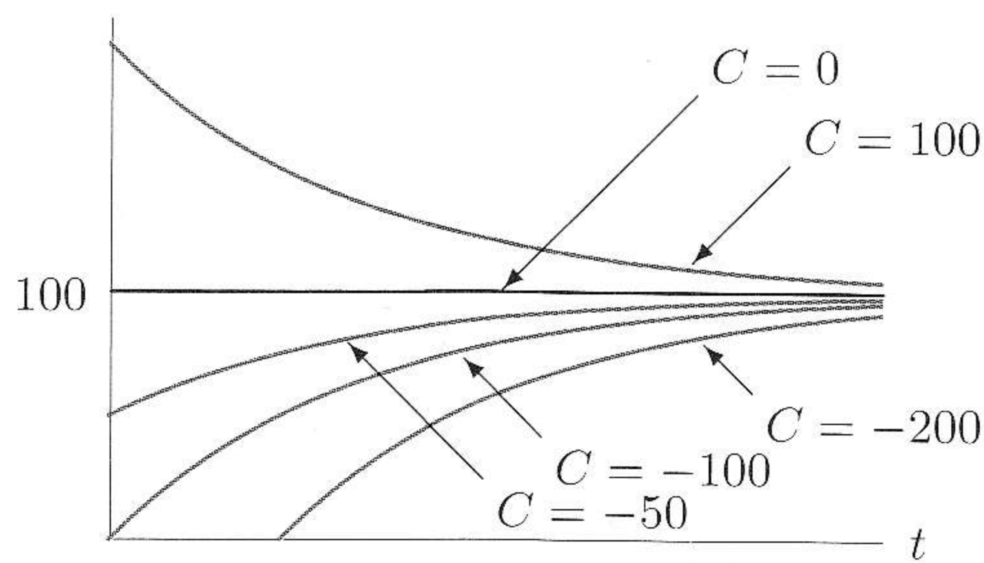


Figure 11.2: Solution curves for $dy/dt = 100 - y$: Members of the family $y = 100 + Ce^{-t}$

- Equilibrium points?

  Values of $y(t)$ where $dy/dt = 0$

  $y(t) = 100$

- Stability?

  Do solutions move towards or away from the equilibrium if starting nearby?

  stable (solution move towards $y(t) = 100$ with increasing t)

- What determines the value of C?

  initial conditions (→ E&U theorem!)

Hughes-Hallett et al. (2005)

<u>Some further common examples</u>

Solution

Exponential growth/decay

$$\frac{dP}{dt} = kP$$

$$P = P_0 e^{kt}$$

e.g., Nuclear decay, 1st order chemical reaction, bacterial growth

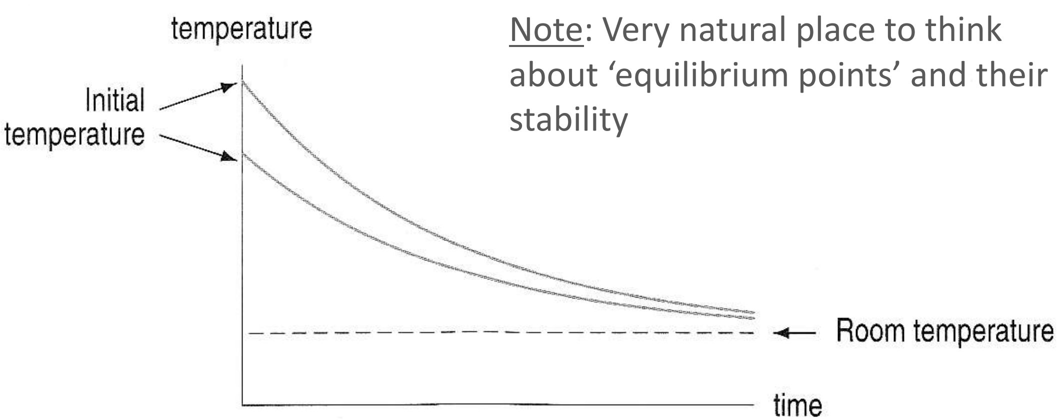
Newton's law of heating/cooling

"Newton proposed that the temperature of a hot object decreases at a rate proportional to the difference between its temperature and that of its surroundings. Similarly, a cold object heats up at a rate proportional to the temperature difference between the object and its surroundings."

$$\frac{dT}{dt} = \alpha(T_o - T)$$

Solution

$$T(t) = T_0 + Ce^{-\alpha t}$$



<u>Note</u>: Very natural place to think about 'equilibrium points' and their stability

Hughes-Hallett et al. (2005)

Stability

Newton's law of heating/cooling

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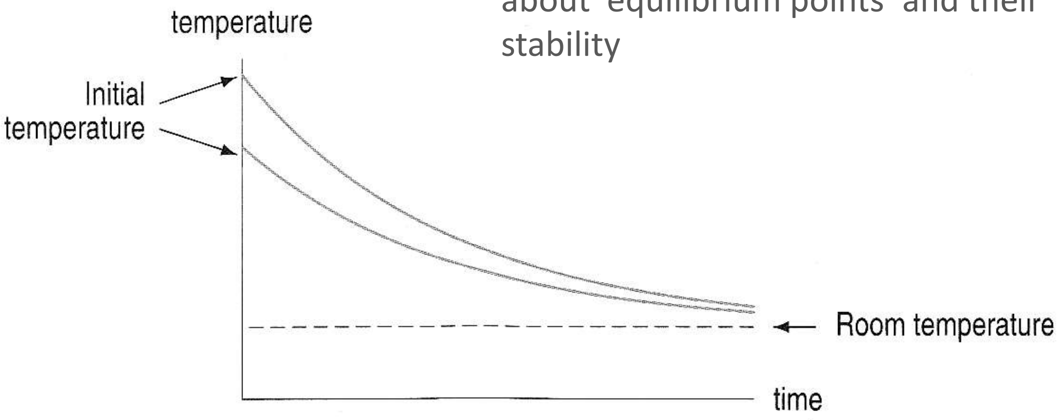
Solution

$$T(t) = T_0 + Ce^{-\alpha t}$$



Note: Very natural place to think about 'equilibrium points' and their stability

- An **equilibrium solution** is constant for all values of the independent variable. The graph is a horizontal line.
- An equilibrium is **stable** if a small change in the initial conditions gives a solution which tends toward the equilibrium as the independent variable tends to positive infinity.
- An equilibrium is **unstable** if a small change in the initial conditions gives a solution curve which veers away from the equilibrium as the independent variable tends to positive infinity.

Hughes-Hallett et al. (2005)

Some further common examples

Note: We will come back to this in more detail a bit later in the semester…

Falling body: Terminal velocity

Assume air resistance is proportional to velocity, the Newton's 2nd Law leads to:

$$m\frac{dv}{dt} = mg - kv$$

$$\frac{dv}{dt} = -\frac{k}{m}\left(v - \frac{mg}{k}\right)$$

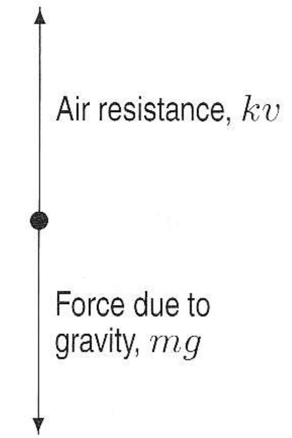


Figure 11.44: Forces acting on a falling object

Solution $$v = \frac{mg}{k}\left(1 - e^{-kt/m}\right)$$

Hughes-Hallett et al. (2005)

<u>A word of caution....</u> (Part II)

Don't Drink and Derive

$$\frac{\partial^2 u}{\partial t^2} + \frac{1}{c^2}\frac{\partial^2 u}{\partial x^2} = 0$$

$$E = 4\pi\varepsilon_0 \frac{q\hat{r}}{r^2}$$

$$\vec{p} = \vec{m}v$$

$$F_g = G\frac{m_1 m_2 m_3}{r^3}$$

$$f(x) = \int_{-\infty}^{\infty} dk\, g(k)\, e^{\pi i x}$$

$$\nabla \cdot E = \frac{1}{\mu_0}\rho$$

$$\nabla \cdot B = 4\pi$$

$$E = mc^3$$

$$p = \frac{mv}{\sqrt{1 - \frac{c^2}{v^2}}}$$

$$V = I - R$$

$$\frac{-\hbar^2}{2m}\nabla\Psi + V(\vec{r}) = i\hbar\frac{\partial^2\Psi}{\partial t^2}$$

$$\frac{\partial \mathcal{L}}{\partial p_i} = \dot{q}_i$$

$$PV = n + R + T$$

$$n_a \sin\varphi_b = n_b \sin\varphi_a$$

$$n\lambda = 2d\tan\theta$$

$$F = \sqrt{ma}$$

$$v = \ddot{x}$$

$$x = \frac{1}{2}at^2 + v_0 t^2 + x_0 t^2$$

A word of caution.... (Part II)

Some further common examples

Falling body: Terminal velocity

$$m\frac{dv}{dt} = mg - kv \qquad v = \frac{mg}{k}\left(1 - e^{-kt/m}\right)$$

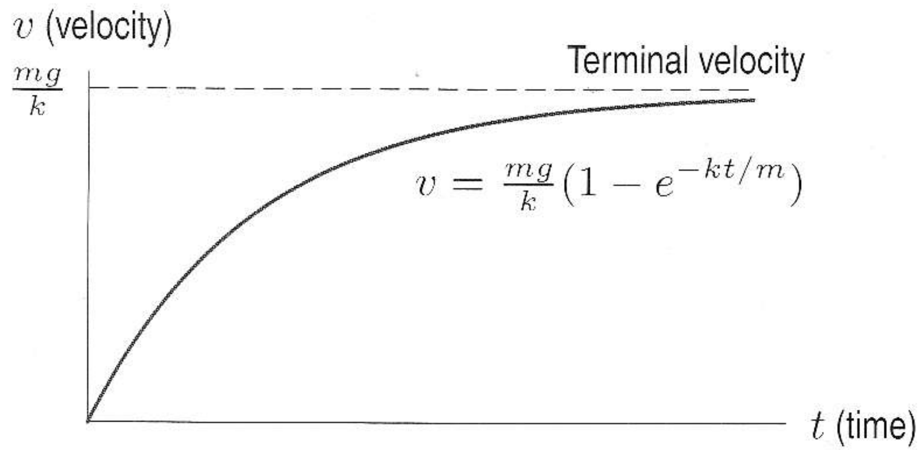
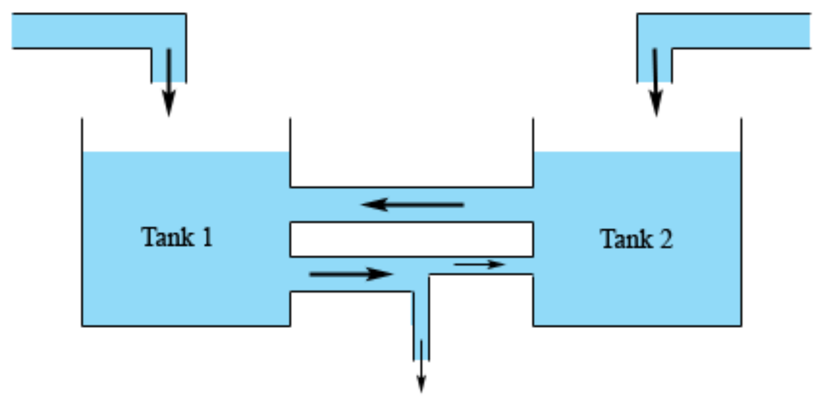


Figure 11.45: Velocity of falling dust particle assuming that air resistance is kv

e.g., speed of falling raindrop

Compartmental models (e.g., salt in a reservoir)



http://tutorial.math.lamar.edu/

Some further common examples

Compartmental models (e.g., salt in a reservoir)

"A water reservoir holds 100 million gallons of water and supplies a city with 1 million gallons a day. The reservoir is partly refilled by a spring which provides 0.9 million gallons a day, and the rest of the water, 0.1 million gallons a day, comes from run-off from the surrounding land. The spring is clean, but the run-off contains salt with a concentration of 0.0001 pound per gallon. There was no salt in the reservoir initially and the water is well mixed (that is, the out-flow contains the concentration of salt in the tank at that instant)."

Hughes-Hallett et al. (2005)

Some further common examples

Compartmental models (e.g., salt in a reservoir)

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Think about units!

It is important to distinguish between the total quantity, Q, of salt in pounds, and the concentration, C, of salt, in pounds/gallon, where

$$\text{Concentration} = C = \frac{\text{Quantity of salt}}{\text{Volume of water}} = \frac{Q}{100\,\text{million}} \left(\frac{\text{lb}}{\text{gal}}\right).$$

$$\begin{array}{c}\text{Rate of change of}\\ \text{quantity of salt}\end{array} = \text{Rate salt entering} - \text{Rate salt leaving}.$$

$$\text{Rate salt entering} = \text{Concentration} \cdot \text{Volume per day}$$

$$\frac{dQ}{dt} = 10 - \frac{Q}{100}$$

Hughes-Hallett et al. (2005)

Ex. ("The Ladder Problem")

You are a burglar. You want to break into a certain house, but it is unfortunately surrounded by a fence that you can not climb directly (see figure). So you are going to need a ladder. But ladders are expensive! So you need to figure out what is the shortest ladder (of length L) you would need to get over the fence to the house. Your answer should depend upon d and h. **Make sure to clearly explain your thinking/methodology!**

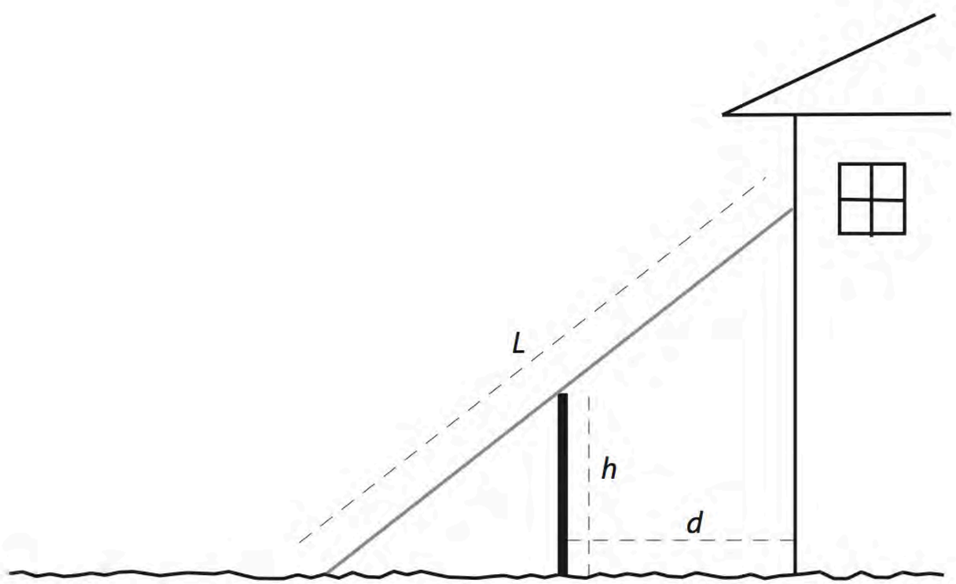

Figure 1: Schematic showing nature of problem. Note that fence is of height h and is at a distance d from the house. The ladder only need touch the house (i.e., the height of the window is irrelevant).

```
% ### LadderOPT.m ###   5.21.09

% simple code to numerically verify solution to Neuhaser's ladder
% optimization problem

clear
% ----------
d= 20;  % length of space between fence and house
h= 13;  % height of fence
% ---------

theta= linspace(0,pi/2,1000);       % create range of possible angles

% length of ladder as a function of theta (this is the function to minimize)
L= d./cos(theta) + h./sin(theta);
Lprime= d*(sin(theta))./(cos(theta).^2) - h*(cos(theta))./(sin(theta).^2); % derivative of above
function

% analytically derived solution for optimal angle; should match where Lprime = 0
thetaOPT= atan( (h/d)^(1/3) );
Lopt = h/sin(thetaOPT) + d/cos(thetaOPT);   % corresponding minimum ladder length

% ====
figure(1); clf;
h1= plot(theta,L); grid on; hold on;
h2= plot(theta,Lprime,'r-');
axis([-0.1 pi/2+0.1 -5 100]);
ylabel('Length or deriv. [arb]'); xlabel('angle re fence [deg]');
title('Numerical check re ladder optimization problem');
%h3= plot(thetaOPT,Lopt,'kx','MarkerSize',8,'LineWidth',2);
h3= stem(thetaOPT,Lopt,'k--')
legend([h1 h2 h3],'ladder length','deriv. of length','shortest length','Location','SouthWest')
```

Reference

Problem 1. A variable $n(t)$ is described by a first-order linear differential equation with constant coefficients

$$\tau \frac{dn(t)}{dt} + n(t) = n_\infty$$

where τ and n_∞ are constants. Let $n(0) = n_0$.

a) For $t \geq 0$, determine an expression for $n(t)$ in terms of τ, n_∞, and n_0.

b) Plot $n(t)$ versus t for the following two cases and explain the difference between the two plots:
   i) $n_0 = 0, \quad n_\infty = 10, \quad \tau = 1,$
   ii) $n_0 = 0, \quad n_\infty = 10, \quad \tau = 10,$

c) Plot $n(t)$ versus t for the following two cases and explain the difference between the two plots:
   iii) $n_0 = 10, \quad n_\infty = 0, \quad \tau = 1,$
   iv) $n_0 = 10, \quad n_\infty = 10, \quad \tau = 1,$

d) Plot $n(t)$ versus t for the following two cases and explain the difference between the two plots:
   v) $n_0 = 10, \quad n_\infty = 0, \quad \tau = 1,$
   vi) $n_0 = -10, \quad n_\infty = 10, \quad \tau = 1.$

Note: This is essentially the same form of eqn. as others we saw earlier (e.g., Newton's Law of Cooling)

$$\frac{dT}{dt} = \alpha(T_o - T)$$

Reference (SOL)

Problem 1. A first-order, linear differential equation with constant coefficients and a constant inhomogeneous (drive or input) term has an exponential solution. Therefore, the solution can be written in the form

$$n(t) = n_\infty + \left(n_0 - n_\infty\right) e^{-t/\tau},$$

where $n_0 = n(0)$ is the initial value of $n(t)$ and $n_\infty = \lim_{t\to\infty} n(t)$ is the final value of $n(t)$. The form of this solution can be verified by evaluating $n(t)$ at $t = 0$ and $t \to \infty$. Substitution into the differential equation shows that this solution satisfies the differential equation. The solutions for cases i-vi are shown in Figure 1. The solutions for part a (i and ii) have the same initial and final values but different time constants (by $t = 10$ s, curve ii is just above 6 and has not yet reached its final value of 10). The solutions for part b (iii and iv) have the same initial values and different final values. Although curve iv was calculated with the same time constant as in iii, it doesn't make sense to compare the time constants of the curves, since curve iv isn't changing. The solutions for part c (v and vi) have different initial and final values and the same time constants.

Note: The solution is essentially the same too, just written in a more general way

$$T(t) = T_0 + Ce^{-\alpha t}$$

Reference (SOL)

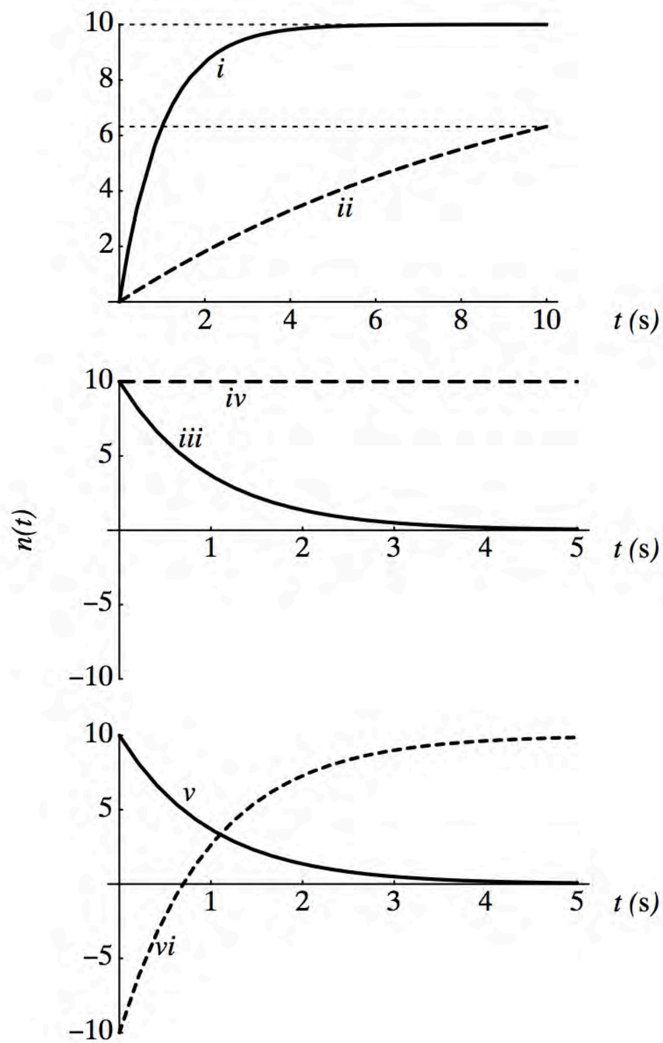

Figure 1. Solutions to parts i-vi. In the upper panel, horizontal dotted lines are shown at the final value of 10 and for the value of $n(t)$ at $t = \tau$, i.e., the line is at $10(1 - e^{-1})$.