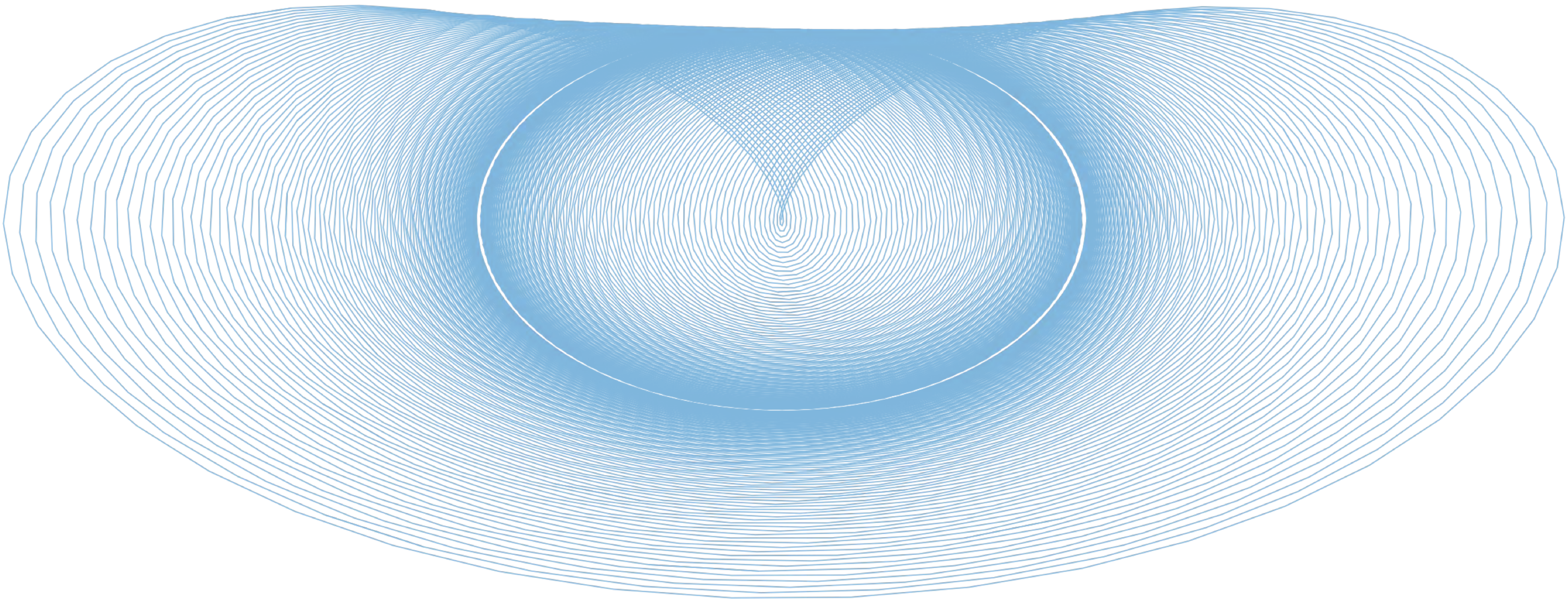
PHYS 1420 (F19)
Physics with Applications to Life Sciences



2019.10.25

Relevant reading:
Kesten & Tauck ch. 7.4-7.6

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Ref. (re images):
Wolfson (2007), Knight (2017)

If you live to be 110, how many times will Uranus orbit the sun in your lifetime?

Announcements & Key Concepts (re Today)

→ Online HW #6: Posted and due next Wednesday (10/30)

Note: There are two sections (one re *energy*, the other re *momentum*)

→ Midterm exams are (STILL) being graded

→ No tutorial next Tuesday (10/29)

Some relevant underlying concepts of the day…

- Notion of a *system*
- Impulse
- Collisions: Inelastic & Elastic
- Center of mass

System?

- Note that many of our "definitions" thus far included the notion of a "system"

This states that *the center of mass of a system of particles moves as though all the mass of the system were concentrated at the center of mass and all the external forces were applied at that point.*

Suppose that the sum of the external forces acting on a system is zero. Then, from Eq. 9–17,

$$\frac{d\mathbf{P}}{dt} = 0 \qquad \text{or} \qquad \mathbf{P} = \text{constant.}$$

Conservation of linear momentum: When the net external force on a system is zero, the total momentum $\vec{P}$ of the system—the vector sum of the individual momenta $m\vec{v}$ of its constituent particles—remains constant.

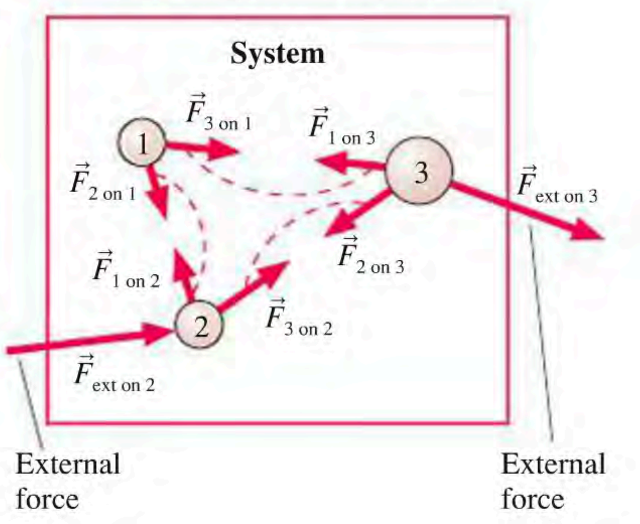
→ But what is a *system*!?!

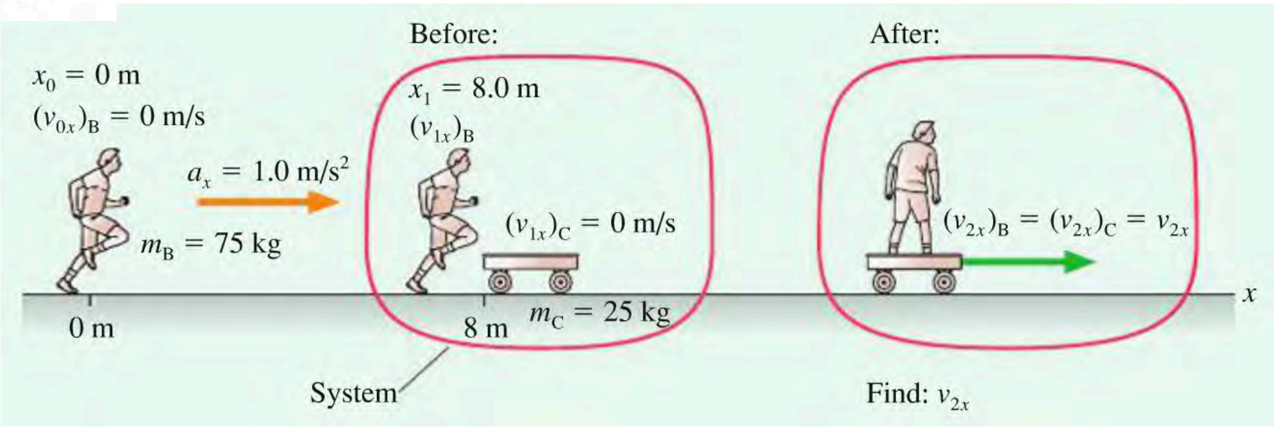
System

"In physics, a physical system is a portion of the physical universe chosen for analysis. Everything outside the system is known as the environment."

https://en.wikipedia.org/wiki/Physical_system

A system of particles.



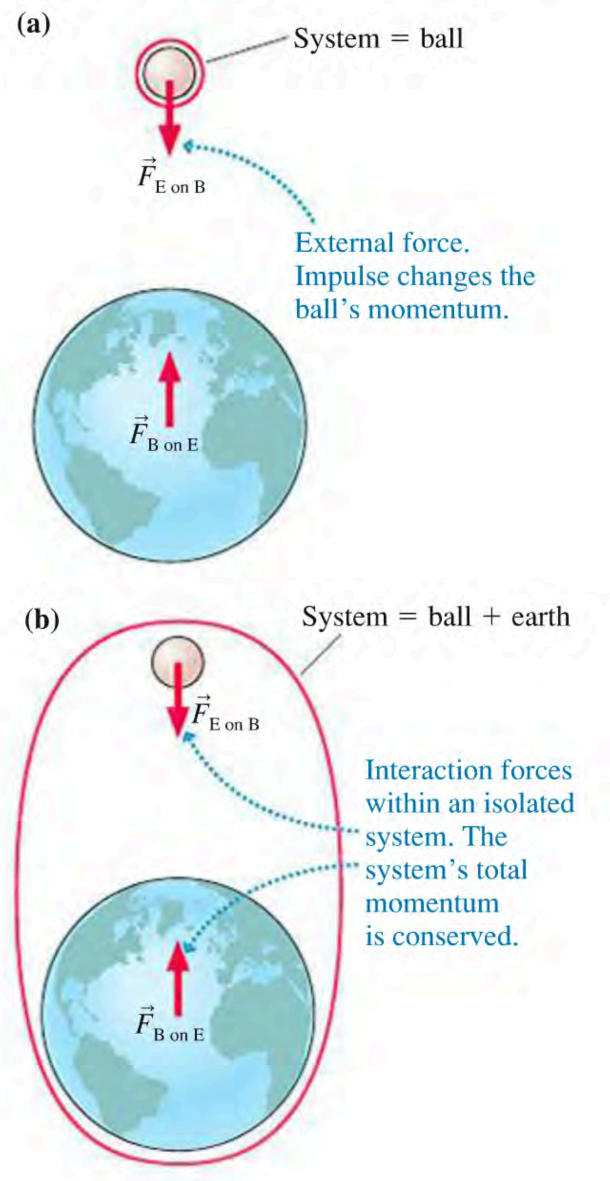


Knight

System

The first step in the problem-solving strategy asks you to clearly define *the system.* This is worth emphasizing because many problem-solving errors arise from trying to apply momentum conservation to an inappropriate system. **The goal is to choose a system whose momentum will be conserved.** Even then, it is the *total* momentum of the system that is conserved, not the momenta of the individual particles within the system.

Whether or not momentum is conserved as a ball falls to earth depends on your choice of the system.



Knight

Impulse

A tennis ball collides with a racket. Notice that the right side of the ball is flattened.

- Recall we showed how a change in momentum, since requiring a change in velocity, required a force to act for a certain time.

$$\int_{t_i}^{t_f} \vec{F}\, dt = \overrightarrow{p_f} - \overrightarrow{p_i}$$

Time difference here (i.e., $t_f - t_i$) is the ***contact time***

- We call this change in momentum an Impulse, $\vec{J}$.

$$\vec{J} = \overrightarrow{p_f} - \overrightarrow{p_i}$$

M. George

Impulse

- Impulse is important in the sense that to change the velocity of an object (say a cliff jumper) from a very fast speed to zero, requires the same Impulse whether or not he lands on water or a hard surface.

- What is different is the time over which the Impulse acts.

- The average force exerted on the object during the impulse is:

$$F_{avg} = \frac{\Delta p}{\Delta t} = \frac{J}{\Delta t}$$

This quantity has a broad range of physical and physiological implications

M. George

Impulse

$$F_{avg} = \frac{\Delta p}{\Delta t}$$

- Therefore, the shorter time of the impulse, the greater the force on the object.
- To minimize the force exerted during a impulse, it is a good idea to increase the time.
- Air bags in cars, padding inside protective equipment, soft foam in your shoes, all serve to increase the time over which an impulse acts, and hence reduce the average force.

- Since Impulse = change in momentum = integral of force with time

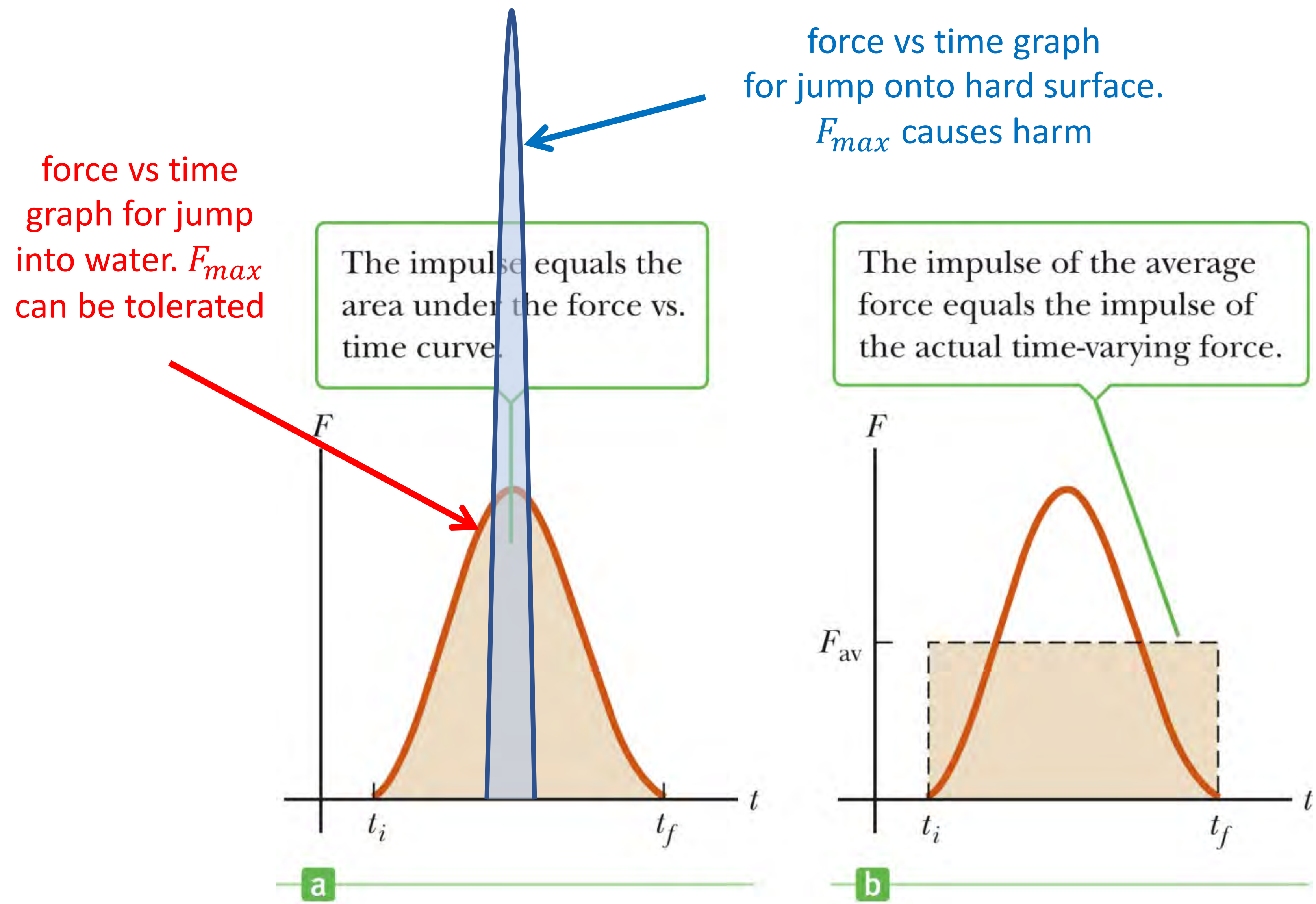
$$\vec{J} = \int_{t_i}^{t_f} \vec{F} dt$$

- And since we know that the area under a curve is like an integral, we can find the Impulse delivered to an object from a Force vs Time graph*.
- Note that for a given Impulse, (cliff jumper into water or onto a hard surface) the area under the force vs time graph must be a constant.

*not to be confused with finding the work done from the area under a force vs distance graph

"Force vs Time Graphs"

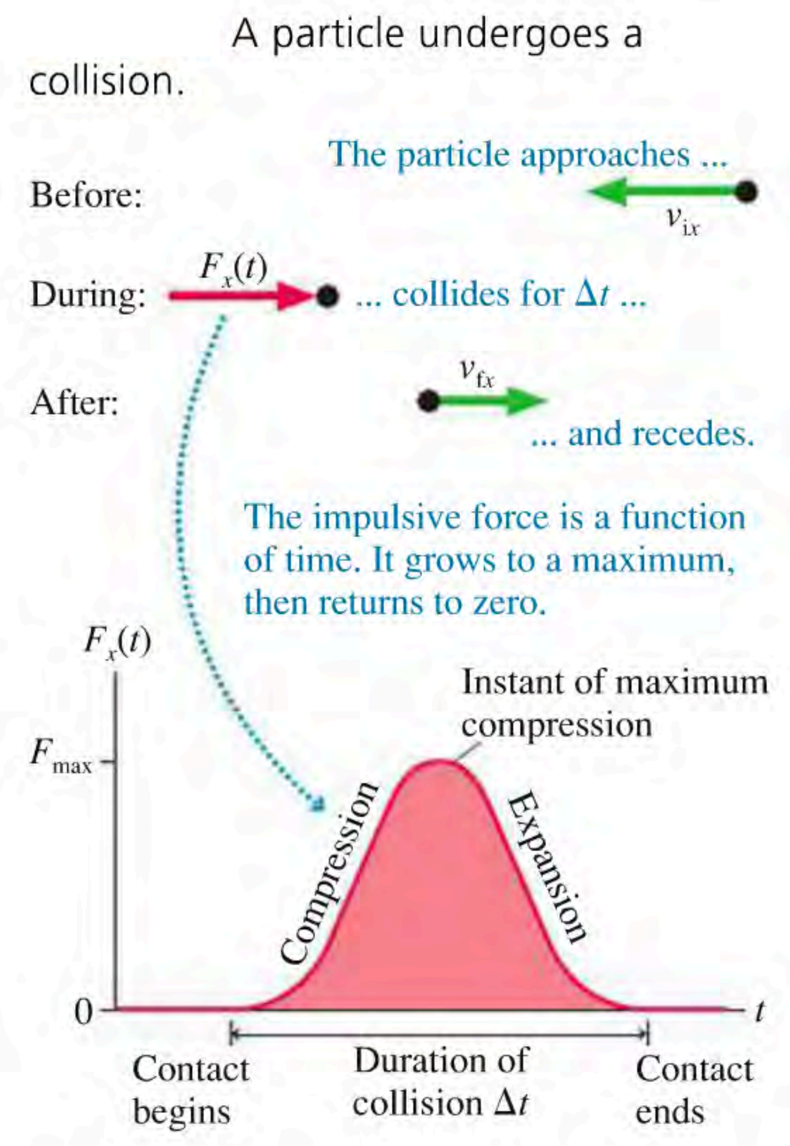
$$\Delta p_x = p_{fx} - p_{ix} = \int_{t_i}^{t_f} F_x(t)\, dt$$



M. George

Impulse

A large force exerted for a small interval of time is called an **impulsive force**.



A tennis ball collides with a racket. Notice that the right side of the ball is flattened.

Newton's 2nd:

$$ma_x = m\frac{dv_x}{dt} = F_x(t)$$

$$m\,dv_x = F_x(t)\,dt$$

Now integrate over the collision interval:

$$m\int_{v_i}^{v_f} dv_x = mv_{fx} - mv_{ix} = \int_{t_i}^{t_f} F_x(t)\,dt$$

Knight

Impulse

$$m\int_{v_i}^{v_f} dv_x = mv_{fx} - mv_{ix} = \int_{t_i}^{t_f} F_x(t)\,dt$$

Now we bring in momentum:

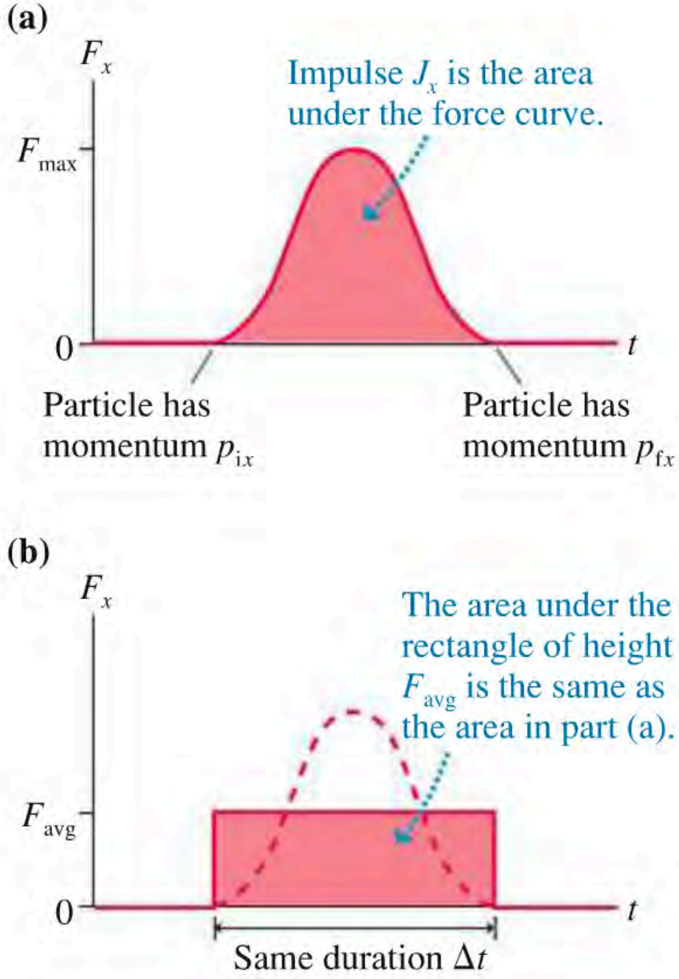
$$\Delta p_x = p_{fx} - p_{ix} = \int_{t_i}^{t_f} F_x(t)\,dt$$

impulse $= J_x \equiv \int_{t_i}^{t_f} F_x(t)\,dt$

$= \text{area under the } F_x(t) \text{ curve between } t_i \text{ and } t_f$

$\Delta p_x = J_x$ (impulse-momentum theorem)

Looking at the impulse graphically.

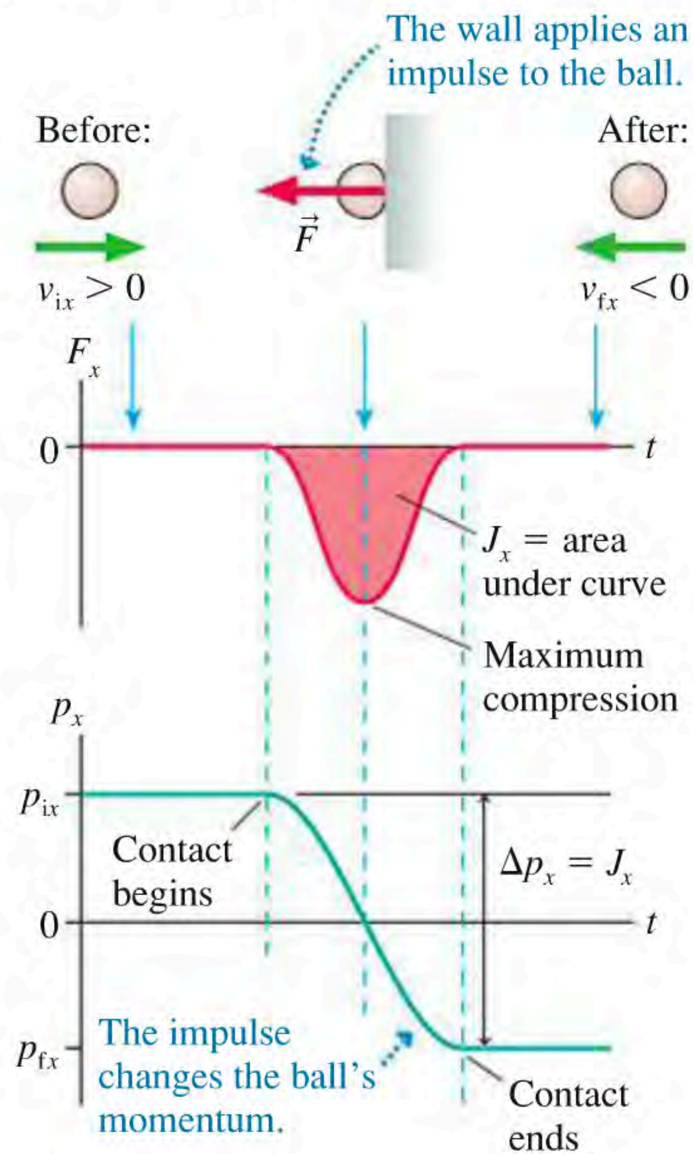


Knight

Impulse

$$\Delta p_x = J_x \qquad \text{(impulse-momentum theorem)}$$

The impulse-momentum theorem helps us understand a rubber ball bouncing off a wall.



Knight

Collisions

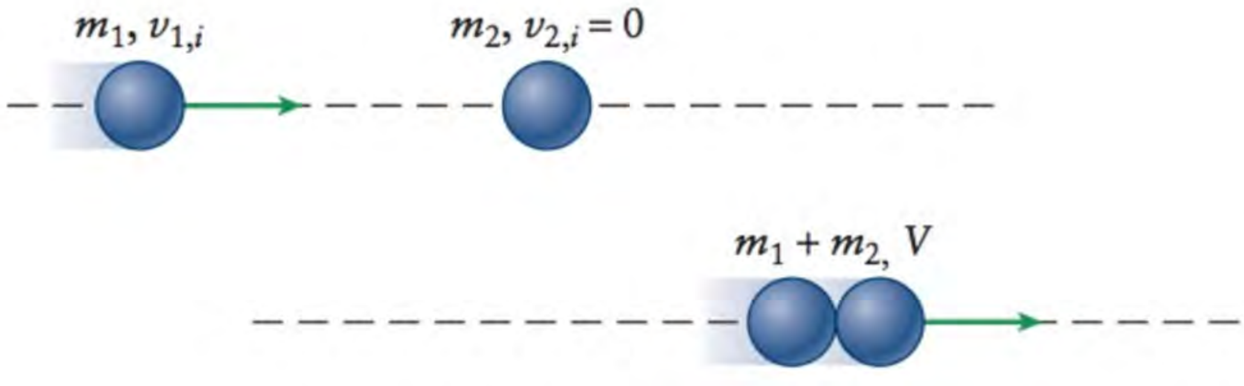
- **Elastic**: objects collide and bounce sharply off one another with no permanent deformation.
  - momentum is conserved
  - mechanical energy is conserved (kinetic+potential)
- **Inelastic**: objects collide and bounce off each other but there is some permanent deformation of the object.
  - momentum is conserved
  - mechanical energy is not conserved (lost to deformation and heat.)
- **Completely Inelastic**: objects collide an stick together, and travel along a common path after the collision.
  - momentum is conserved
  - mechanical energy is not conserved (lost to deformation and heat.)

Inelastic Collisions

Momentum is conserved

Here the two particles "stick together"....

$$m_1\vec{v}_1 = (m_1 + m_2)\vec{V}$$



$$\vec{V} = \frac{m_1}{m_1 + m_2}\vec{v}_1$$

Energy is not conserved

$$K_i = \frac{1}{2}m_1 v_1^2 \qquad K_f = \frac{1}{2}(m_1 + m_2)V^2 \qquad K_f = \left(\frac{m_1}{m_1 + m_2}\right)K_i$$

Useful, but be careful.....

Special Case: The Objects Have the Same Mass

Special Case: The Moving Object Is More Massive than the One at Rest

Special Case: The Moving Object Is Less Massive than the One at Rest

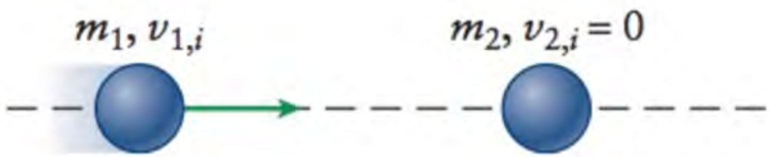
$$V = \frac{m_1}{m_1 + m_2}v_1 \approx \frac{m_1}{m_2}v_1 \approx (0)\,v_1 \approx 0$$

e.g., when precisely is this approximation valid?

Elastic Collisions

For elastic case, consider 2nd particle "at rest"...

$$m_1\vec{v}_{1,i} = m_1\vec{v}_{1,f} + m_2\vec{v}_{2,f}$$



Here, energy is conserved

$$\frac{1}{2}m_1 v_{1,i}^2 = \frac{1}{2}m_1 v_{1,f}^2 + \frac{1}{2}m_2 v_{2,f}^2$$

K&T go on to derive expressions for the resulting velocities:

$$v_{1,f} = v_{1,i}\left(\frac{m_1 - m_2}{m_1 + m_2}\right) \qquad v_{2,f} = v_{1,i}\left(\frac{2m_1}{m_1 + m_2}\right)$$

Be careful though (e.g., what precisely is a "trick"?)

After you've seen expressions such as Equation 7-27 a few times, you'll come to recognize that $v_{1,i}^2 - v_{1,f}^2$ can be factored into $(v_{1,i} + v_{1,f})(v_{1,i} - v_{1,f})$, a "trick" that is often useful in simplifying equations.

Kesten & Tauck

Elastic Collisions

$$v_{2,f} = v_{1,i}\left(\frac{2m_1}{m_1 + m_2}\right) \approx v_{1,i}\left(\frac{2(0)}{0 + m_2}\right)$$

$$v_{2,f} = 0$$

→ There is an untrue statement here!

Those "special cases" again.....

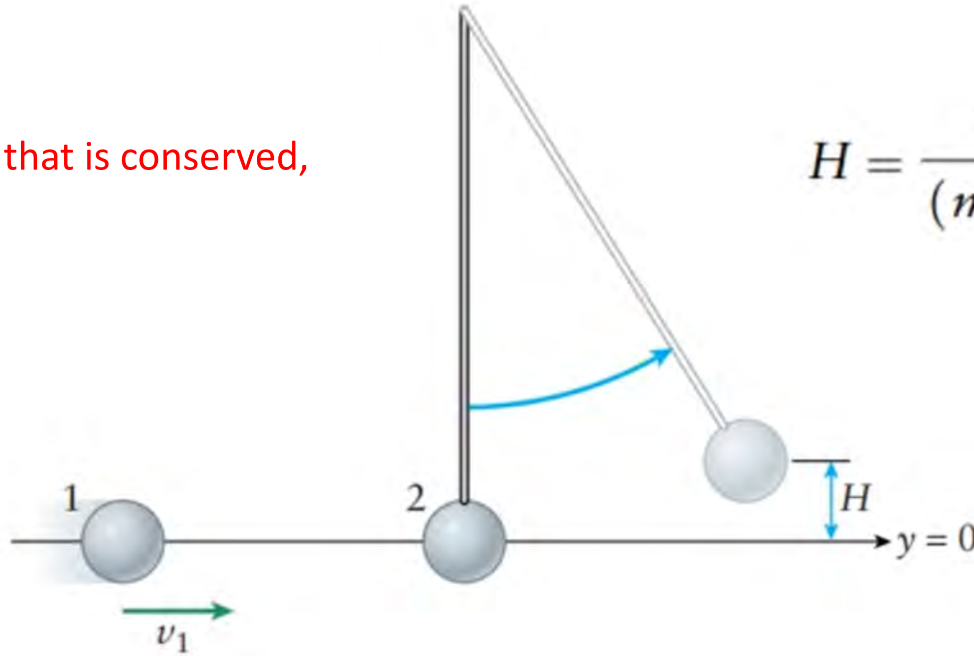
Special Case: The Objects Have the Same Mass

Special Case: The Moving Object Is More Massive than the One at Rest

Special Case: The Moving Object Is Less Massive than the One at Rest

Note also that it is *energy* that is conserved, not just *kinetic* energy....

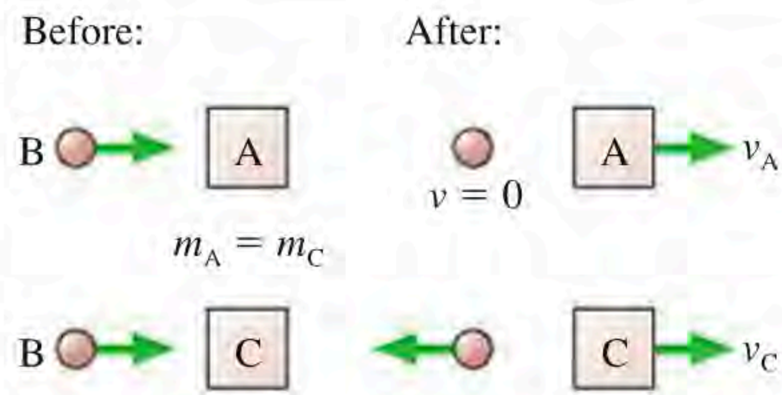
$$H = \frac{2m_1^2 v_1^2}{(m_1 + m_2)^2 g}$$



Kesten & Tauck

Ex.

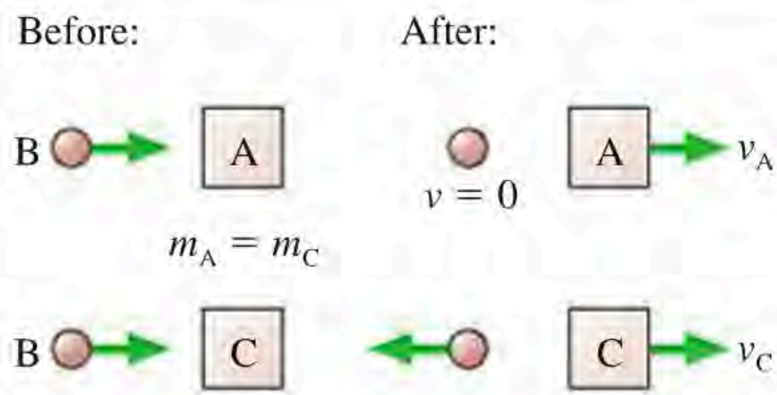
STOP TO THINK 9.3 Objects A and C are made of different materials, with different "springiness," but they have the same mass and are initially at rest. When ball B collides with object A, the ball ends up at rest. When ball B is thrown with the same speed and collides with object C, the ball rebounds to the left. Compare the velocities of A and C after the collisions. Is v_A greater than, equal to, or less than v_C?



Knight

Ex. (SOL)

STOP TO THINK 9.3 Objects A and C are made of different materials, with different "springiness," but they have the same mass and are initially at rest. When ball B collides with object A, the ball ends up at rest. When ball B is thrown with the same speed and collides with object C, the ball rebounds to the left. Compare the velocities of A and C after the collisions. Is v_A greater than, equal to, or less than v_C?



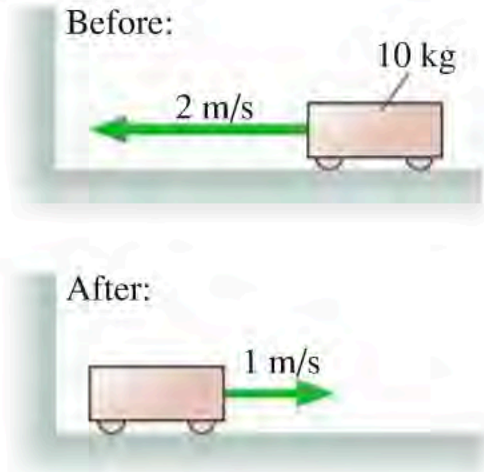
Less than. The ball's momentum $m_B v_B$ is the same in both cases. Momentum is conserved, so the *total* momentum is the same after both collisions. The ball that rebounds from C has *negative* momentum, so C must have a larger momentum than A.

Knight

Ex.

STOP TO THINK 9.1 The cart's change of momentum is

a. −30 kg m/s
b. −20 kg m/s
c. 0 kg m/s
d. 10 kg m/s
e. 20 kg m/s
f. 30 kg m/s

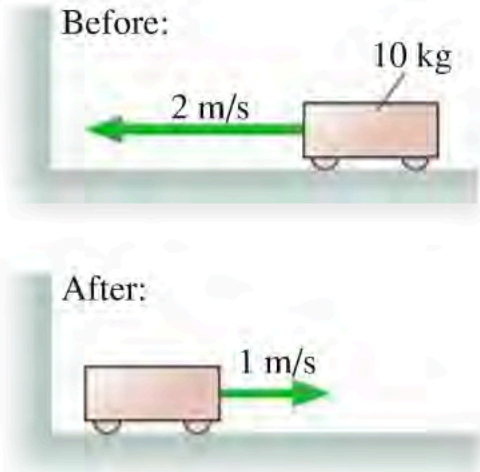


Knight

Ex. (SOL)

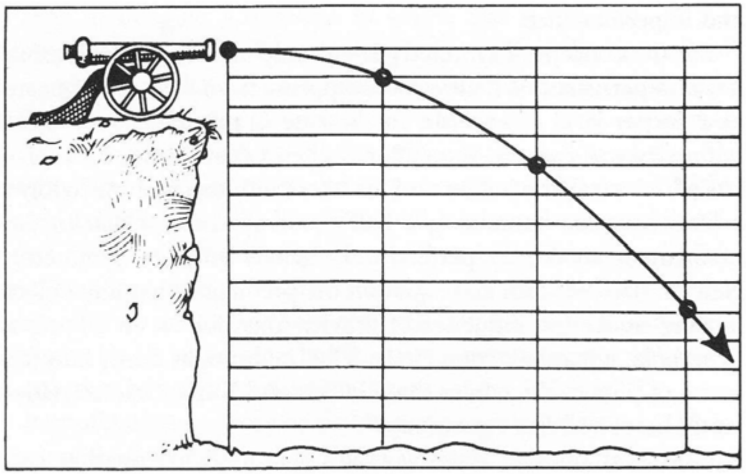
STOP TO THINK 9.1 The cart's change of momentum is

a. −30 kg m/s
b. −20 kg m/s
c. 0 kg m/s
d. 10 kg m/s
e. 20 kg m/s
f. 30 kg m/s



f

<u>More complicated motions....</u>



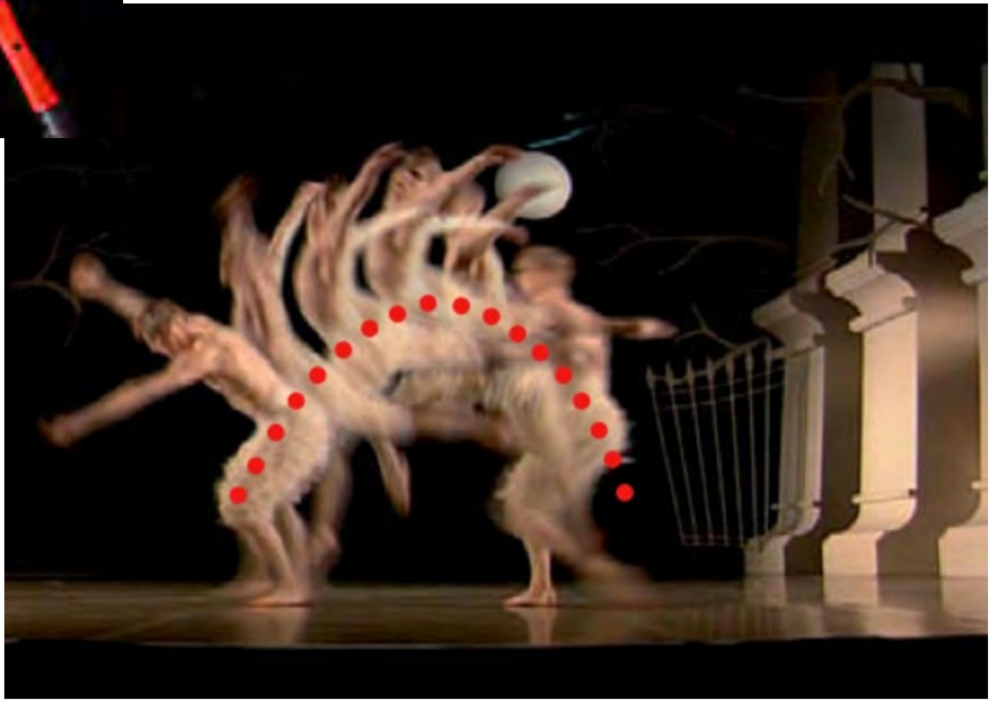
von Baeyer

- Several important points are conveyed here (which will lead us into Kesten & Tauck chapter 8)

  - center-of-mass
  - rotational motion
  - angular momentum

https://cluster31-files.instructure.com/courses/1384~1159/files/1384~240864/course%20files/apb11o/resources/guides/G09-4a.com.htm

Motion of the Center of Mass



Kesten & Tauck

Finding the center of mass (CM)

Easy case (discrete)

The **center of mass** of a system of n point masses $m_1, m_2, \ldots, m_n$ located at positions $x_1, x_2, \ldots, x_n$ along the x-axis is given by

$$\overline{x} = \frac{\sum x_i m_i}{\sum m_i}.$$

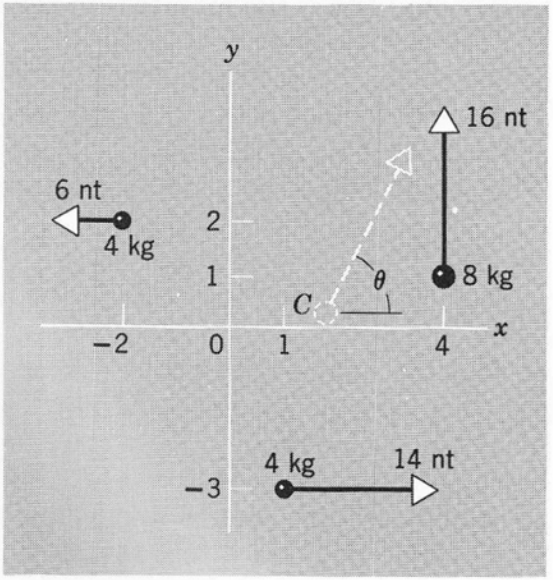


Fig. 9–6 Example 3. Finding the motion of the center of mass of three masses, each subjected to a different force. The forces all lie in the plane defined by the particles. The distances indicated along the axes are in meters.

Resnick & Halliday (1966)

Finding the center of mass (CM)

Easy case (discrete)

The **center of mass** of a system of n point masses $m_1, m_2, \ldots, m_n$ located at positions $x_1, x_2, \ldots, x_n$ along the x-axis is given by

$$\bar{x} = \frac{\sum x_i m_i}{\sum m_i}.$$

The numerator is the sum of the moments of the masses about the origin; the denominator is the total mass of the system.

- Left-hand term is the vector indicating the center of mass *relative to your chosen coordinate system*

Wolfson notation

$$\vec{r}_{\text{cm}} = \frac{\sum m_i \vec{r}_i}{M}$$

Kesten & Tauck notation

$$x_{\text{CM}} = \frac{1}{M_{\text{tot}}} \sum_{i=1}^{N} m_i x_i$$

✓TIP Choosing the Origin

Choosing the origin at one of the masses here conveniently makes one of the terms in the sum $\sum m_i x_i$ zero. But, as always, the choice of origin is purely for convenience and doesn't influence the actual physical location of the center of mass. Exercise 16 demonstrates this point, repeating Example 9.1 with a different origin.

✓TIP Exploit Symmetries

It's no accident that x_{cm} here lies on the vertical line that bisects the triangle; after all, the triangle is symmetric about that line, so its mass is distributed evenly on either side. Exploit symmetry whenever you can; that can save you a lot of computation throughout physics!

Hughes-Hallet et al (2004)
Wolfson

Ex.

Example 2. Find the center of mass of the triangular plate of Fig. 9–5.

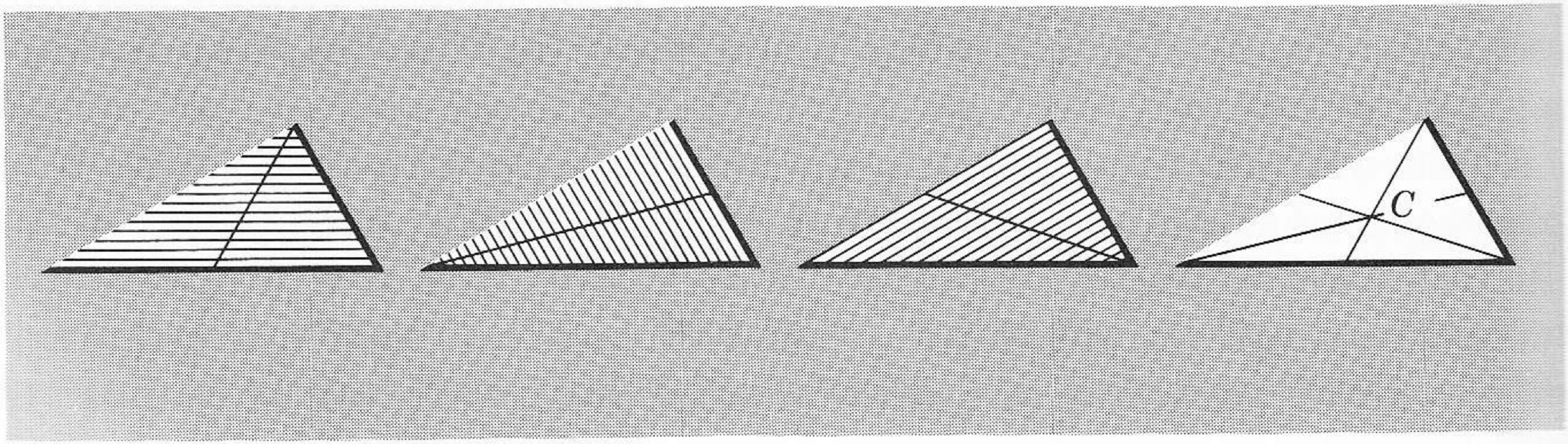


Fig. 9–5 Example 2. Finding the center of mass C of a triangular plate.

Resnick & Halliday (1966)

Ex. (SOL)

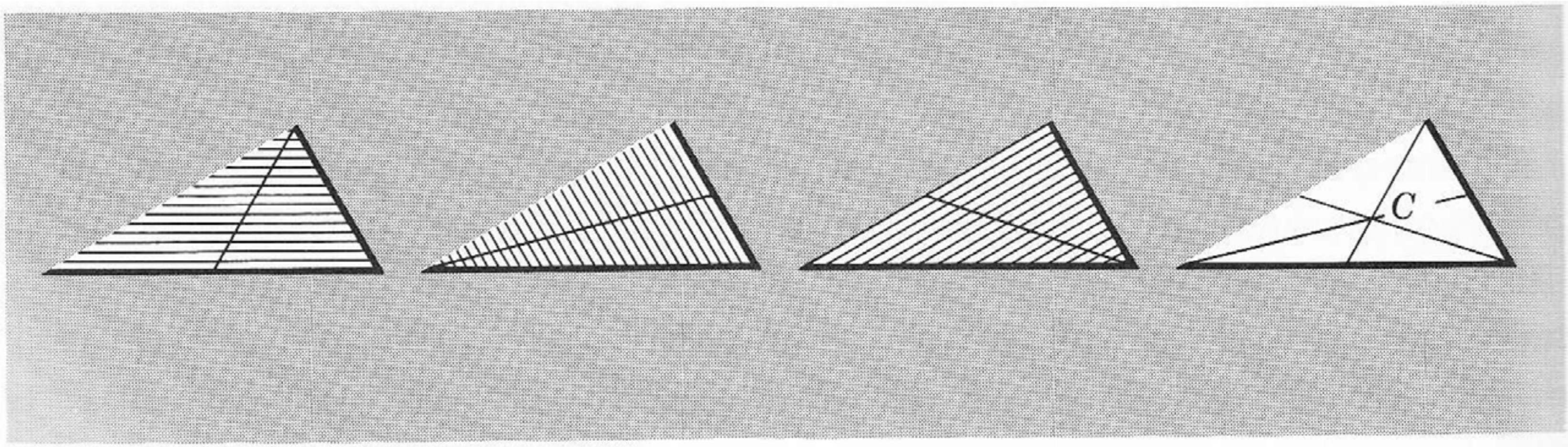


Fig. 9–5 Example 2. Finding the center of mass C of a triangular plate.

- Be smart! Sometimes a simple graphical approach is all you need...

If a body can be divided into parts such that the center of mass of each part is known, the center of mass of the body can usually be found simply. The triangular plate may be divided into narrow strips parallel to one side. The center of mass of each strip lies on the line which joins the middle of that side to the opposite vertex. But we can divide up the triangle in three different ways, using this process for each of three sides. Hence the center of mass lies at the intersection of the three lines which join the middle of each side with the opposite vertices. This is the only point that is common to the three lines. ◀

Resnick & Halliday (1966)

Ex.

A sculptor decides to portray a bird (Fig. 9–13). Luckily the final model is actually able to stand upright. The model is formed of a single sheet of metal of uniform thickness. Of the points shown, which is most likely to be the center of mass?

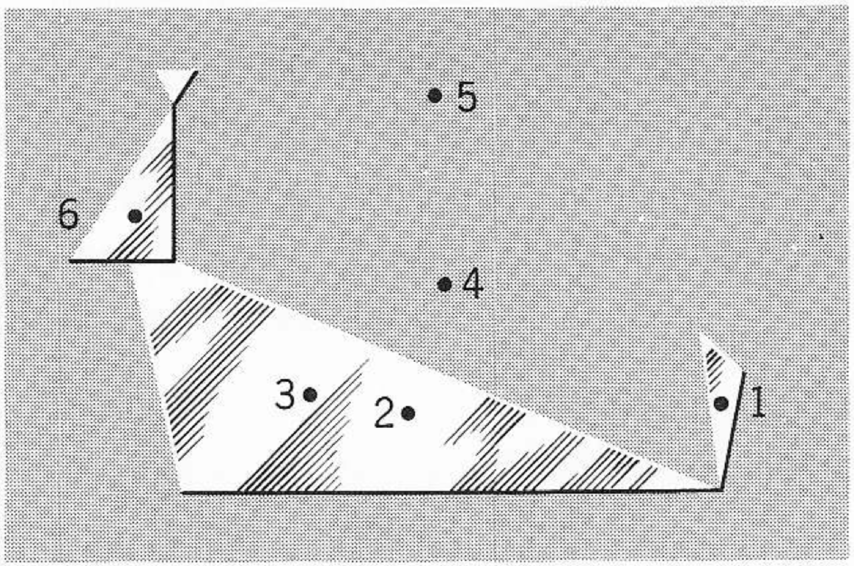
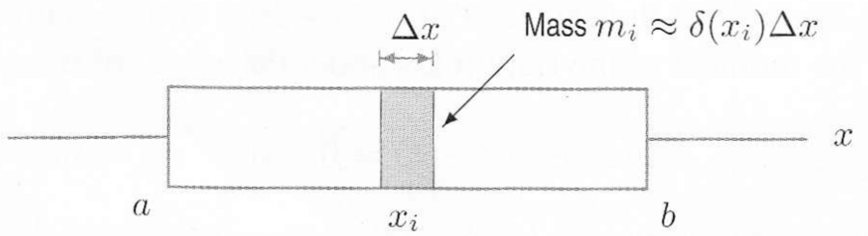


Fig. 9–13

Resnick & Halliday (1966)

Finding the center of mass

Harder case (1-D continuous mass distribution)



Continuous Mass Density

Instead of discrete masses arranged along the x-axis, suppose we have an object lying on the x-axis between $x = a$ and $x = b$. At point x, suppose the object has mass density (mass per unit length) of $\delta(x)$. To calculate the center of mass of such an object, divide it into n pieces, each of length Δx. On each piece, the density is nearly constant, so the mass of the piece is given by density times length. See Figure 8.51. Thus, if x_i is a point in the i^{th} piece,

$$\text{Mass of the } i^{\text{th}} \text{ piece, } m_i \approx \delta(x_i)\Delta x.$$

Then the formula for the center of mass, $\bar{x} = \sum x_i m_i / \sum m_i$, applied to the n pieces of the object gives

$$\bar{x} = \frac{\sum x_i \delta(x_i)\Delta x}{\sum \delta(x_i)\Delta x}.$$

In the limit as $n \to \infty$ we have the following formula:

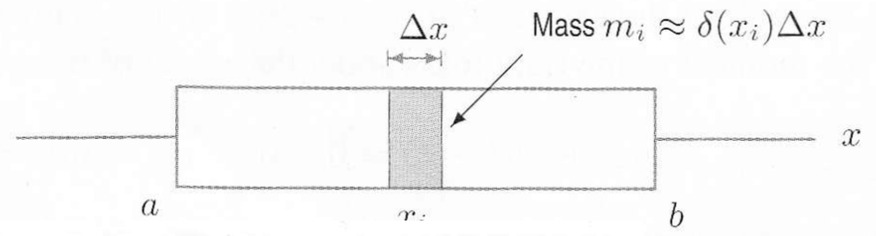
Interdisciplinary connection: Riemann sums and integrals!

> The **center of mass** $\bar{x}$ of an object lying along the x-axis between $x = a$ and $x = b$ is
>
> $$\bar{x} = \frac{\int_a^b x\delta(x)\,dx}{\int_a^b \delta(x)\,dx},$$
>
> where $\delta(x)$ is the density (mass per unit length) of the object.

Hughes-Hallet et al (2004)

Finding the center of mass

Harder case (1-D continuous mass distribution)



In the limit as $n \to \infty$ we have the following formula:

The **center of mass** $\overline{x}$ of an object lying along the x-axis between $x = a$ and $x = b$ is

$$\overline{x} = \frac{\int_a^b x\delta(x)\,dx}{\int_a^b \delta(x)\,dx},$$

where $\delta(x)$ is the density (mass per unit length) of the object.

As in the discrete case, the denominator is the total mass of the object.

Wolfson notation

$$\vec{r}_{\text{cm}} = \lim_{\Delta m_i \to 0} \frac{\sum \Delta m_i \vec{r}_i}{M} = \frac{\int \vec{r}\,dm}{M} \quad \left(\begin{array}{c}\text{center of mass,}\\ \text{continuous matter}\end{array}\right)$$

Hughes-Hallet et al (2004)

TACTICS 9.1 Setting Up an Integral

An integral like $\int x\,dm$ can be confusing because you see both x and dm after the integral sign and they don't seem related. But they are, and here's how to proceed:

1. Find a suitable shape for your mass elements, preferably one that exploits any symmetry in the situation. One dimension of the elements should involve an infinitesimal interval in one of the coordinates x, y, or z. In Example 9.3, the mass elements were strips, symmetric about the wing's centerline and with width dx.
2. Find an expression for the infinitesimal area of your mass elements (in a one-dimensional problem it would be the length; in a three-dimensional problem, the volume). In Example 9.3, the infinitesimal area of each mass element was the strip height h multiplied by the width dx.
3. Form ratios that relate the infinitesimal coordinate interval to the physical quantity in the integral—which in Example 9.3 is the mass element dm. Here we formed the ratio of the area of a mass element to the total area, and equated that to the ratio of dm to the total mass M.
4. Solve your ratio statement for the infinitesimal quantity, in this case dm, that appears in your integral. Then you're ready to evaluate the integral.

Sometimes you'll be given a density—mass per volume, per area, or per length—and then in place of steps 3 and 4 you find dm by multiplying the density by the infinitesimal volume, area, or length you identified in step 2.

Although we described this procedure in the context of Example 9.3, it also applies to other integrals you'll encounter in different areas of physics.

Additional examples for study...... (some w/ solutions, some w/o)

Ex. (+ SOL)

Find the center of mass of a 2-meter rod lying on the x-axis with its left end at the origin if:
(a) The density is constant and the total mass is 5 kg. (b) The density is $\delta(x) = 15x^2$ kg/m.

(a) Since the density is constant along the rod, we expect the balance point to be in the middle, that is, $\bar{x} = 1$. To check this, we compute $\bar{x}$. The density is the total mass divided by the length, so $\delta(x) = 5/2$ kg/m. Then

$$\bar{x} = \frac{\text{Moment}}{\text{Mass}} = \frac{\int_0^2 x \cdot \frac{5}{2}\, dx}{5} = \frac{1}{5} \cdot \frac{5}{2} \cdot \left.\frac{x^2}{2}\right|_0^2 = 1 \text{ meter}.$$

(b) Since more of the mass of the rod is closer to its right end (the density is greatest there), we expect the center of mass to be in the right half of the rod, that is, between $x = 1$ and $x = 2$. We have

$$\text{Total mass } = \int_0^2 15x^2\, dx = 5x^3\Big|_0^2 = 40 \text{ kg}.$$

Thus,

$$\bar{x} = \frac{\text{Moment}}{\text{Mass}} = \frac{\int_0^2 x \cdot 15x^2 dx}{40} = \frac{15}{40} \cdot \left.\frac{x^4}{4}\right|_0^2 = 1.5 \text{ meter}.$$

Hughes-Hallet et al (2004)