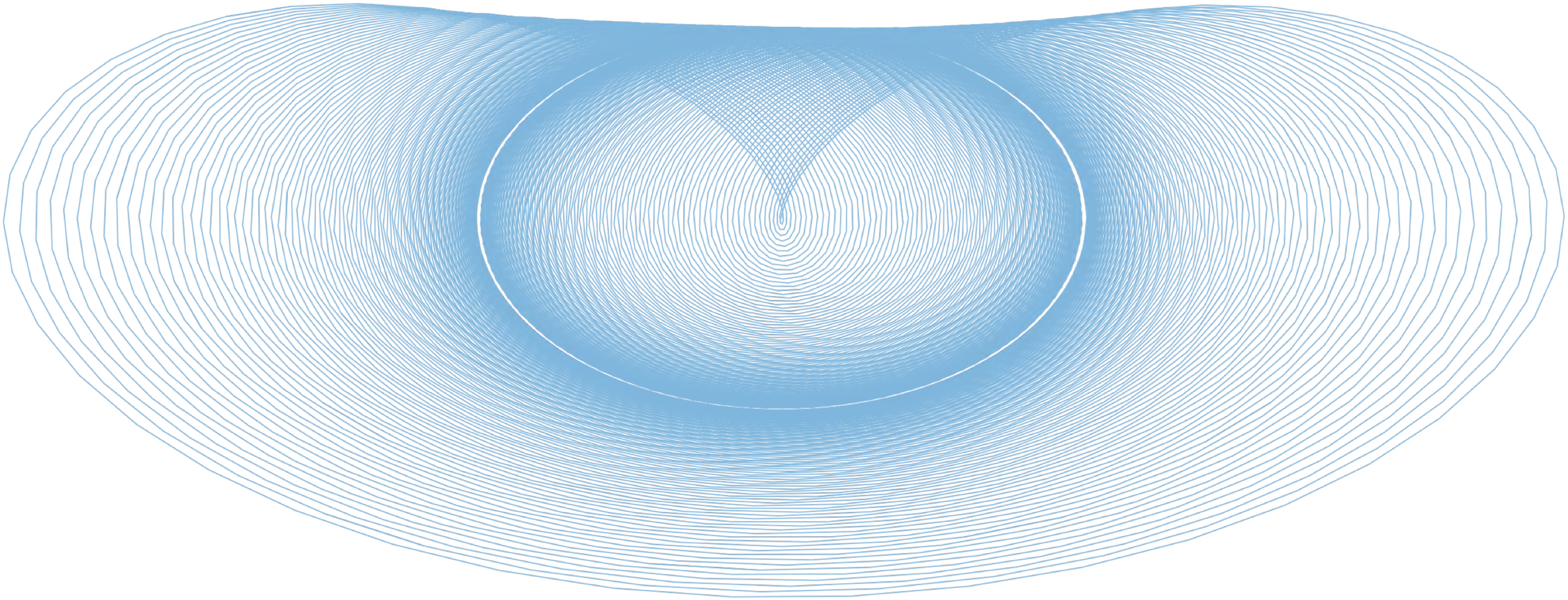
PHYS 1420 (F19)
Physics with Applications to Life Sciences



2019.10.28

Relevant reading:

Kesten & Tauck ch. 7.6, 8.1-8.2

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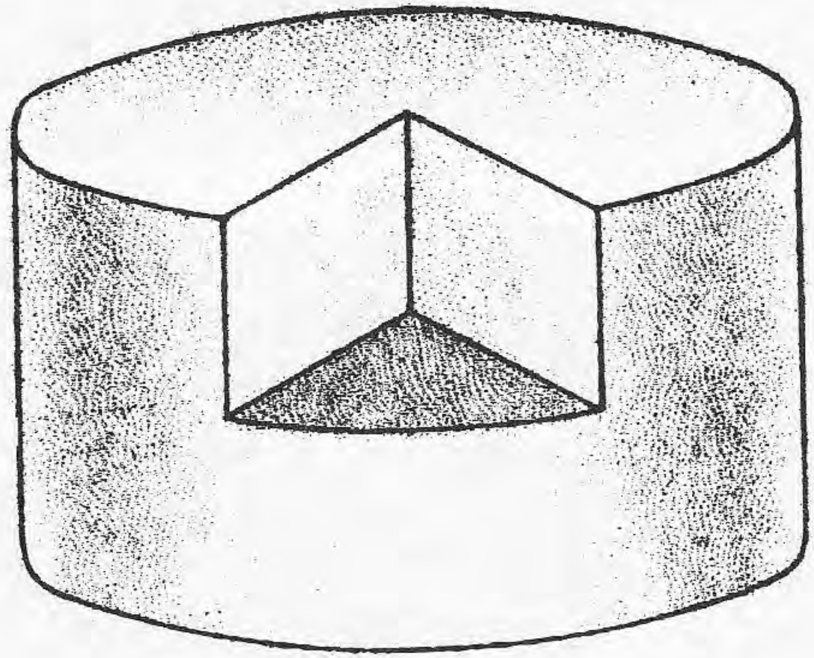
Ref. (re images):
Wolfson (2007), Knight (2017),
M. George, Kesten & Tauck (2012)



This large cheese had a piece cut out. Can you locate the missing portion?

<u>Announcements & Key Concepts</u> (re Today)

<u>Note</u>: There are two sections (one re *energy*, the other re *momentum*)

→ Online HW #6: Posted and due Wednesday (10/30)

→ Midterm exams are (STILL) being graded

→ No tutorial tomorrow Tuesday (10/29)

Some relevant underlying concepts of the day...

- Center of mass (again...)
- Rotational kinetic energy
- Moment of inertia

Finding the center of mass (CM)

Easy case (discrete)

The **center of mass** of a system of n point masses $m_1, m_2, \ldots, m_n$ located at positions $x_1, x_2, \ldots, x_n$ along the x-axis is given by

$$\bar{x} = \frac{\sum x_i m_i}{\sum m_i}.$$

The numerator is the sum of the moments of the masses about the origin; the denominator is the total mass of the system.

- Left-hand term is the vector indicating the center of mass *relative to your chosen coordinate system*

Wolfson notation

$$\vec{r}_{\text{cm}} = \frac{\sum m_i \vec{r}_i}{M}$$

Kesten & Tauck notation

$$x_{\text{CM}} = \frac{1}{M_{\text{tot}}} \sum_{i=1}^{N} m_i x_i$$

✓TIP Choosing the Origin

Choosing the origin at one of the masses here conveniently makes one of the terms in the sum $\sum m_i x_i$ zero. But, as always, the choice of origin is purely for convenience and doesn't influence the actual physical location of the center of mass. Exercise 16 demonstrates this point, repeating Example 9.1 with a different origin.

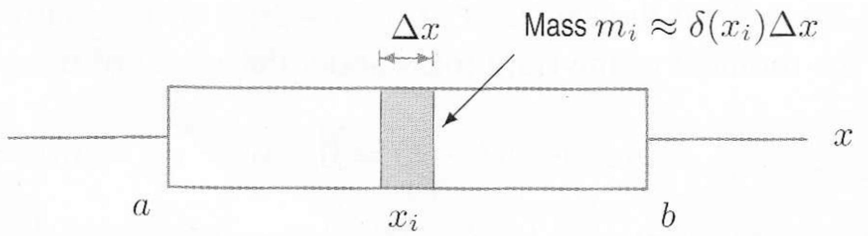
✓TIP Exploit Symmetries

It's no accident that x_{cm} here lies on the vertical line that bisects the triangle; after all, the triangle is symmetric about that line, so its mass is distributed evenly on either side. Exploit symmetry whenever you can; that can save you a lot of computation throughout physics!

Hughes-Hallet et al (2004)
Wolfson

Finding the center of mass (CM)

Finding the center of mass



Harder case (1-D continuous mass distribution)

Continuous Mass Density

Instead of discrete masses arranged along the x-axis, suppose we have an object lying on the x-axis between $x = a$ and $x = b$. At point x, suppose the object has mass density (mass per unit length) of $\delta(x)$. To calculate the center of mass of such an object, divide it into n pieces, each of length Δx. On each piece, the density is nearly constant, so the mass of the piece is given by density times length. See Figure 8.51. Thus, if x_i is a point in the i^{th} piece,

$$\text{Mass of the } i^{\text{th}} \text{ piece, } m_i \approx \delta(x_i)\Delta x.$$

Then the formula for the center of mass, $\bar{x} = \sum x_i m_i / \sum m_i$, applied to the n pieces of the object gives

$$\bar{x} = \frac{\sum x_i \delta(x_i)\Delta x}{\sum \delta(x_i)\Delta x}.$$

In the limit as $n \to \infty$ we have the following formula:

Interdisciplinary connection: Riemann sums and integrals!

The **center of mass** $\bar{x}$ of an object lying along the x-axis between $x = a$ and $x = b$ is

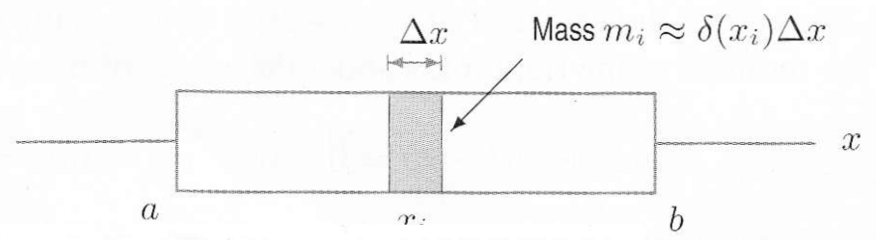
$$\bar{x} = \frac{\int_a^b x\delta(x)\,dx}{\int_a^b \delta(x)\,dx},$$

where $\delta(x)$ is the density (mass per unit length) of the object.

Hughes-Hallet et al (2004)

Finding the center of mass

Harder case (1-D continuous mass distribution)



In the limit as $n \to \infty$ we have the following formula:

> The **center of mass** $\overline{x}$ of an object lying along the x-axis between $x = a$ and $x = b$ is
>
> $$\overline{x} = \frac{\int_a^b x\delta(x)\,dx}{\int_a^b \delta(x)\,dx},$$
>
> where $\delta(x)$ is the density (mass per unit length) of the object.

As in the discrete case, the denominator is the total mass of the object.

Wolfson notation

$$\vec{r}_{\text{cm}} = \lim_{\Delta m_i \to 0} \frac{\sum \Delta m_i \vec{r}_i}{M} = \frac{\int \vec{r}\,dm}{M} \quad \left(\begin{array}{c}\text{center of mass,}\\ \text{continuous matter}\end{array}\right)$$

Hughes-Hallet et al (2004)

TACTICS 9.1 Setting Up an Integral

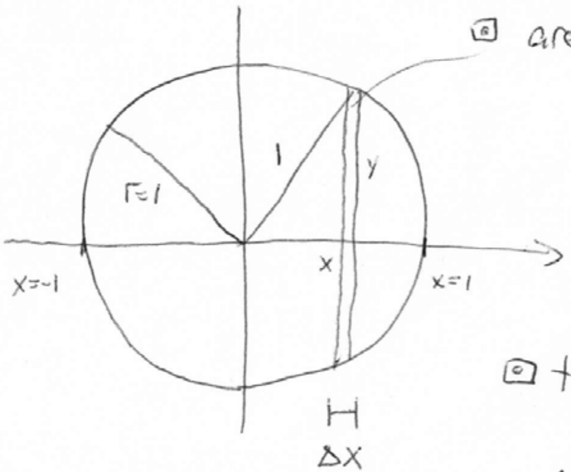
An integral like $\int x\, dm$ can be confusing because you see both x and dm after the integral sign and they don't seem related. But they are, and here's how to proceed:

1. Find a suitable shape for your mass elements, preferably one that exploits any symmetry in the situation. One dimension of the elements should involve an infinitesimal interval in one of the coordinates x, y, or z. In Example 9.3, the mass elements were strips, symmetric about the wing's centerline and with width dx.
2. Find an expression for the infinitesimal area of your mass elements (in a one-dimensional problem it would be the length; in a three-dimensional problem, the volume). In Example 9.3, the infinitesimal area of each mass element was the strip height h multiplied by the width dx.
3. Form ratios that relate the infinitesimal coordinate interval to the physical quantity in the integral—which in Example 9.3 is the mass element dm. Here we formed the ratio of the area of a mass element to the total area, and equated that to the ratio of dm to the total mass M.
4. Solve your ratio statement for the infinitesimal quantity, in this case dm, that appears in your integral. Then you're ready to evaluate the integral.

Sometimes you'll be given a density—mass per volume, per area, or per length—and then in place of steps 3 and 4 you find dm by multiplying the density by the infinitesimal volume, area, or length you identified in step 2.

Although we described this procedure in the context of Example 9.3, it also applies to other integrals you'll encounter in different areas of physics.

Area of a circle via Riemann sums



- area of "strip" ($\equiv A_s$) $\simeq 2 \cdot y \cdot \Delta X = 2\sqrt{1-x^2}\,\Delta X$

  (since $x^2+y^2=1$)

- area of circle (A_c) $\simeq \sum_{x=-1}^{x=1} A_s = \sum 2\sqrt{1-x^2}\,\Delta X$

- taking the limit where our strip get infinitesimally small:

$$A_c = \lim_{\Delta x \to 0} \sum_{x=-1}^{1} 2\sqrt{1-x^2}\,\Delta X = \int_{-1}^{1} 2\sqrt{1-x^2}\,dx$$

- Using a trusty "table of integrals", we find: $\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

- Back to our problem: $A_c = \int_{-1}^{1} 2\sqrt{1-x^2}\,dx = 2\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x\right]_{-1}^{1}$

$$= 1\sqrt{1-1} + \sin^{-1}(1) - \frac{-1\sqrt{1-1}}{1} - \sin^{-1}(-1) = \sin^{-1}(1) - \sin^{-1}(-1)$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi = \pi \cdot 1^2$$

∴

Note: these are the "inverse trig functions" (if $x = \sin y$, then $y = \sin^{-1} x$, i.e. the angle whose sine is x; $\sin^{-1}(1) = \frac{\pi}{2}$ and $\sin^{-1}(-x) = -\sin^{-1} x$)

Finding the center of mass

Note: A basic/important/universal consideration arises here, that these problems can be broken up into a series of independent calculations

Harder-er case (2ff-D continuous mass distribution)

For a system of masses that lies in the plane, the center of mass is a point with coordinates $(\bar{x}, \bar{y})$. In three dimensions, the center of mass is a point with coordinates $(\bar{x}, \bar{y}, \bar{z})$. To compute the center of mass in three dimensions, we use the following formulas in which $A_x(x)$ is the area of a slice perpendicular to the x-axis at x, and $A_y(y)$ and $A_z(z)$ are defined similarly. In two dimensions, we use the same formulas for $\bar{x}$ and $\bar{y}$, but we interpret $A_x(x)$ and $A_y(y)$ as the lengths of strips perpendicular to the x- and y-axes, respectively.

For a region of constant density δ, the center of mass is given by

$$\bar{x} = \frac{\int x\delta A_x(x)\,dx}{\text{Mass}} \qquad \bar{y} = \frac{\int y\delta A_y(y)\,dy}{\text{Mass}} \qquad \bar{z} = \frac{\int z\delta A_z(z)\,dz}{\text{Mass}}.$$

Note: If the density is not constant, finding the CM may require double/triple integrals and multivariable calculus (i.e., beyond the scope of 1st year PHYS 1420!)

Hughes-Hallet et al (2004)

Ex. Find the coordinates of the center of mass of the isosceles triangle in Figure 8.52. The triangle has constant density and mass m.

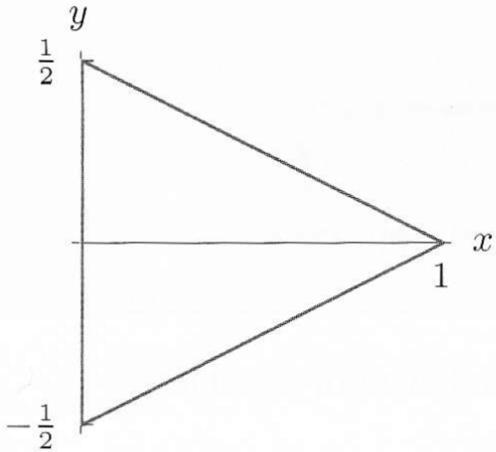


Figure 8.52: Find center of mass of this triangle

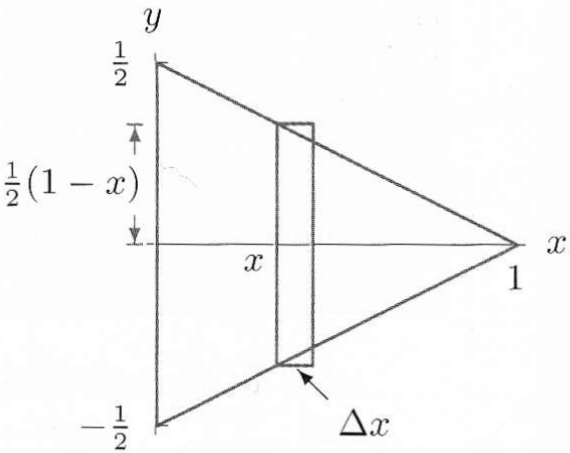


Figure 8.53: Sliced triangle

For a region of constant density δ, the center of mass is given by

$$\bar{x} = \frac{\int x\delta A_x(x)\,dx}{\text{Mass}} \qquad \bar{y} = \frac{\int y\delta A_y(y)\,dy}{\text{Mass}} \qquad \bar{z} = \frac{\int z\delta A_z(z)\,dz}{\text{Mass}}.$$

- Need to determine the density δ [kg/m^2]
- Be careful w/ the units (e.g., [A_x] = m, meaning it is a length and not an area!)

Hughes-Hallet et al (2004)

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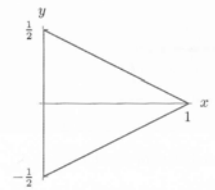

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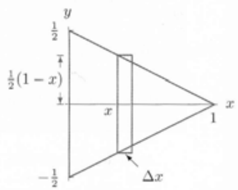


Figure 8.53: Sliced triangle

Wolfson

Ex. (SOL)

Because the mass of the triangle is symmetrically distributed with respect to the x-axis, $\bar{y} = 0$. We expect $\bar{x}$ to be closer to $x = 0$ than to $x = 1$, since the triangle is wider near the origin.

The area of the triangle is $\frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$. Thus Density = Mass/Area = $2m$. If we slice the triangle into strips of width Δx, then the strip at position x has length $A_x(x) = 2 \cdot \frac{1}{2}(1 - x) = (1 - x)$. (See Figure 8.53.) So

$$\text{Area of strip } = A_x(x)\Delta x \approx (1 - x)\Delta x.$$

Since the density is $2m$, the center of mass is given by

$$\bar{x} = \frac{\int x\delta A_x(x)\,dx}{\text{Mass}} = \frac{\int_0^1 2mx(1-x)\,dx}{m} = 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^1 = \frac{1}{3}.$$

So the center of mass of this triangle is at the point $(\bar{x}, \bar{y}) = (1/3, 0)$.

Hughes-Hallet et al (2004)

Ex.

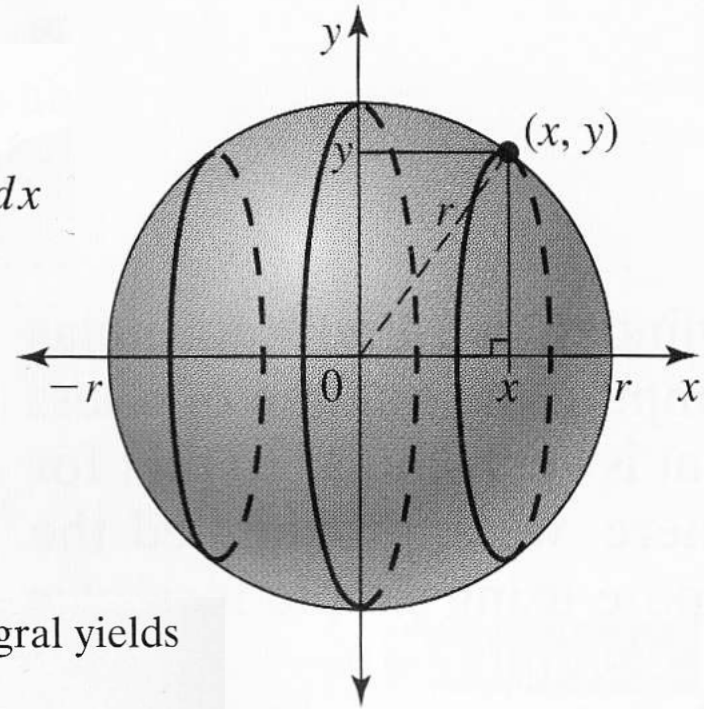
Find the volume of the sphere of radius r centered at the origin.

Neuhauser (2004)

Ex. (SOL)

The cross section at x is perpendicular to the x-axis (see Figure 6.39). It is a disk of radius $y = \sqrt{r^2 - x^2}$ whose area is

$$A(x) = \pi y^2 = \pi (r^2 - x^2)$$



Since the solid is between $-r$ and r, we find

$$\text{Volume} = \int_{-r}^{r} \pi (r^2 - x^2)\, dx$$

The integrand is continuous on $[-r, r]$. Evaluating the integral yields

$$= \pi \left[r^2 x - \frac{1}{3} x^3 \right]_{-r}^{r}$$

$$= \pi \left[\left(r^3 - \frac{1}{3} r^3 \right) - \left(-r^3 + \frac{1}{3} r^3 \right) \right] = \pi \left(\frac{2}{3} r^3 + \frac{2}{3} r^3 \right) = \frac{4}{3} \pi r^3$$

Neuhauser (2004)

Ex.

Find the center of mass of a hemisphere of radius 7 cm and constant density δ.

Kesten & Tauck ch.7 problem

98. •••**Calc** Determine the center of mass of a solid hemisphere of mass M and radius R, relative to the center of the base of the hemisphere.

Hughes-Hallet et al (2004)

Ex. (SOL)

Stand the hemisphere with its base horizontal in the xy-plane, with the center at the origin. Symmetry tells us that its center of mass lies directly above the center of the base, so $\bar{x} = \bar{y} = 0$. Since the hemisphere is wider near its base, we expect the center of mass to be nearer to the base than the top.

To calculate the center of mass, slice the hemisphere into horizontal disks as in Figure 8.9 on page 375. A disk of thickness Δz at height z above the base has

$$\text{Volume of disk} = A_z(z)\Delta z \approx \pi(7^2 - z^2)\Delta z\ \text{cm}^3.$$

So, since the density is δ,

$$\bar{z} = \frac{\int z\delta A_z(z)\,dz}{\text{Mass}} = \frac{\int_0^7 z\delta\pi(7^2 - z^2)\,dz}{\text{Mass}}.$$

Since the total mass of the hemisphere is $\left(\frac{2}{3}\pi 7^3\right)\delta$, we get

$$\bar{z} = \frac{\delta\pi\int_0^7 (7^2 z - z^3)\,dz}{\text{Mass}} = \frac{\delta\pi\left(7^2 z^2/2 - z^4/4\right)\Big|_0^7}{\text{Mass}} = \frac{\frac{7^4}{4}\delta\pi}{\frac{2}{3}\pi 7^3\delta} = \frac{21}{8} = 2.625\ \text{cm}.$$

The center of mass of the hemisphere is 2.625 cm above the center of its base. As expected, it is closer to the base of the hemisphere than its top.

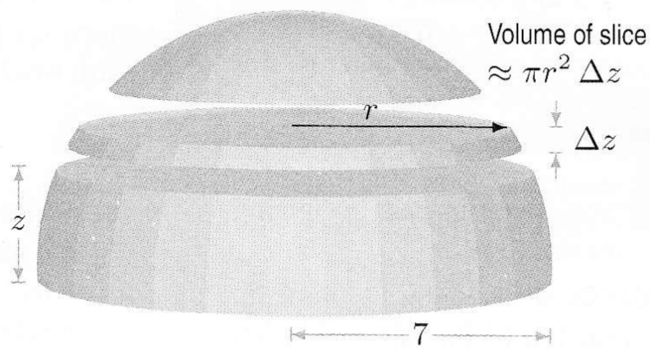
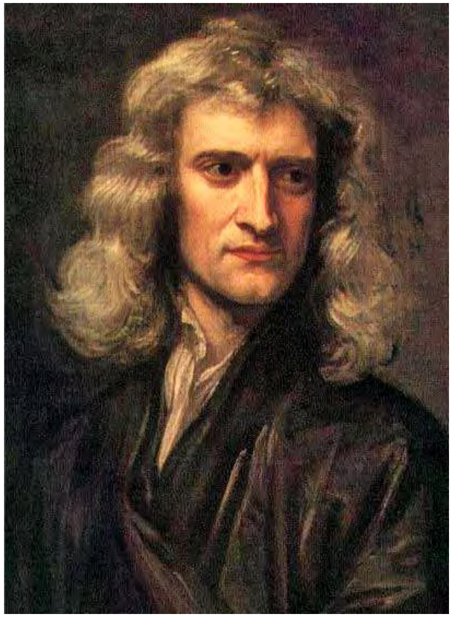


Figure 8.54: Slicing to find the center of mass of a hemisphere

Hughes-Hallet et al (2004)

Stepping back a moment...

I. Newton (1643 -1727)



→ The weight of Newton's contribution should now be a bit more apparent...

$$\mathbf{F}_{12} = -G\frac{m_1 m_2}{|\mathbf{r}_{12}|^2}\,\hat{\mathbf{r}}_{12}$$

where

$\mathbf{F}_{12}$ is the force applied on object 2 due to object 1,

G is the gravitational constant,

m_1 and m_2 are respectively the masses of objects 1 and 2,

$|\mathbf{r}_{12}| = |\mathbf{r}_2 - \mathbf{r}_1|$ is the distance between objects 1 and 2, and

$\hat{\mathbf{r}}_{12} \stackrel{\text{def}}{=} \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|}$ is the unit vector from object 1 to 2.

https://en.wikipedia.org/wiki/Newton's_law_of_universal_gravitation#Vector_form

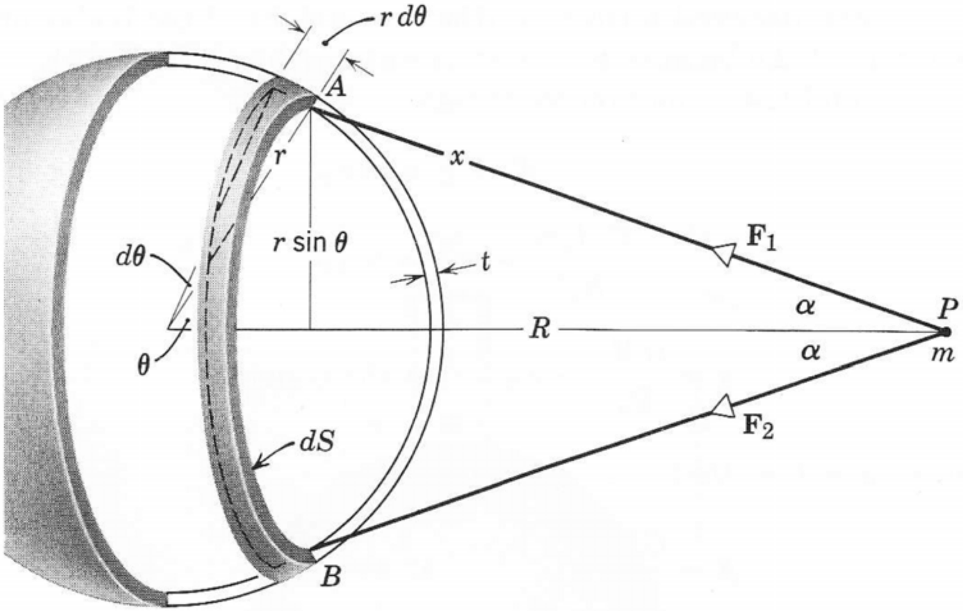
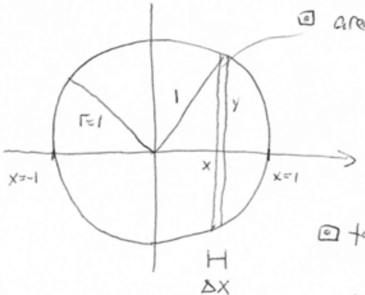


Fig. 16–6 Gravitational attraction of a section dS of a spherical shell of matter on a particle of mass m.

Resnick & Halliday (1966)

Area of a circle via Riemann sums



- area of "strip" ($\equiv A_s$) $\simeq 2 \cdot y \cdot \Delta x = 2\sqrt{1-x^2}\,\Delta x$ (since $x^2+y^2=1$)
- area of circle (A_c) $\simeq \sum_{x=-1}^{x=1} A_s = \sum 2\sqrt{1-x^2}\,\Delta x$
- taking the limit where our strip get infinitesimally small:

$$A_c = \lim_{\Delta x \to 0} \sum_{x=-1}^{1} 2\sqrt{1-x^2}\,\Delta x = \int_{-1}^{1} 2\sqrt{1-x^2}\,dx$$

- Using a trusty "table of integrals", we find: $\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$
- Back to our problem:

$$A_c = \int_{-1}^{1} 2\sqrt{1-x^2}\,dx = 2\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x\right]_{-1}^{1}$$

$$= 1\sqrt{1-1} + \sin^{-1}(1) - \frac{-1\sqrt{1-1}}{1} - \sin^{-1}(-1) = \sin^{-1}(1) - \sin^{-1}(-1)$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi = \pi \cdot 1^2$$

∴

Note: these are the "inverse trig functions" (if $x = \sin y$, then $y = \sin^{-1}x$, i.e. the angle whose sine is x; $\sin^{-1}(1) = \frac{\pi}{2}$ and $\sin^{-1}(-x) = -\sin^{-1}x$)

→ In *some* regards, integral calculus is a bit beyond PHYS 1420, but in others it is not...

Because the mass of the triangle is symmetrically distributed with respect to the x-axis, $\bar{y} = 0$. We expect $\bar{x}$ to be closer to $x = 0$ than to $x = 1$, since the triangle is wider near the origin.

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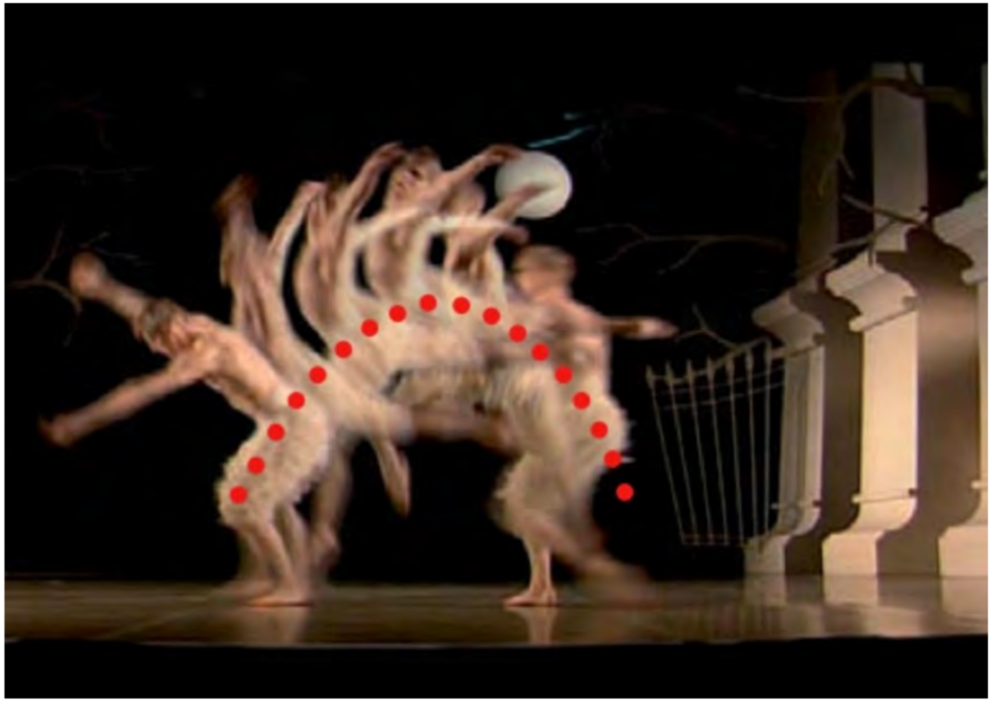
$$\text{Area of strip } = A_x(x)\Delta x \approx (1-x)\Delta x.$$

Since the density is $2m$, the center of mass is given by

$$\bar{x} = \frac{\int x\delta A_x(x)\,dx}{\text{Mass}} = \frac{\int_0^1 2mx(1-x)\,dx}{m} = 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{3}.$$

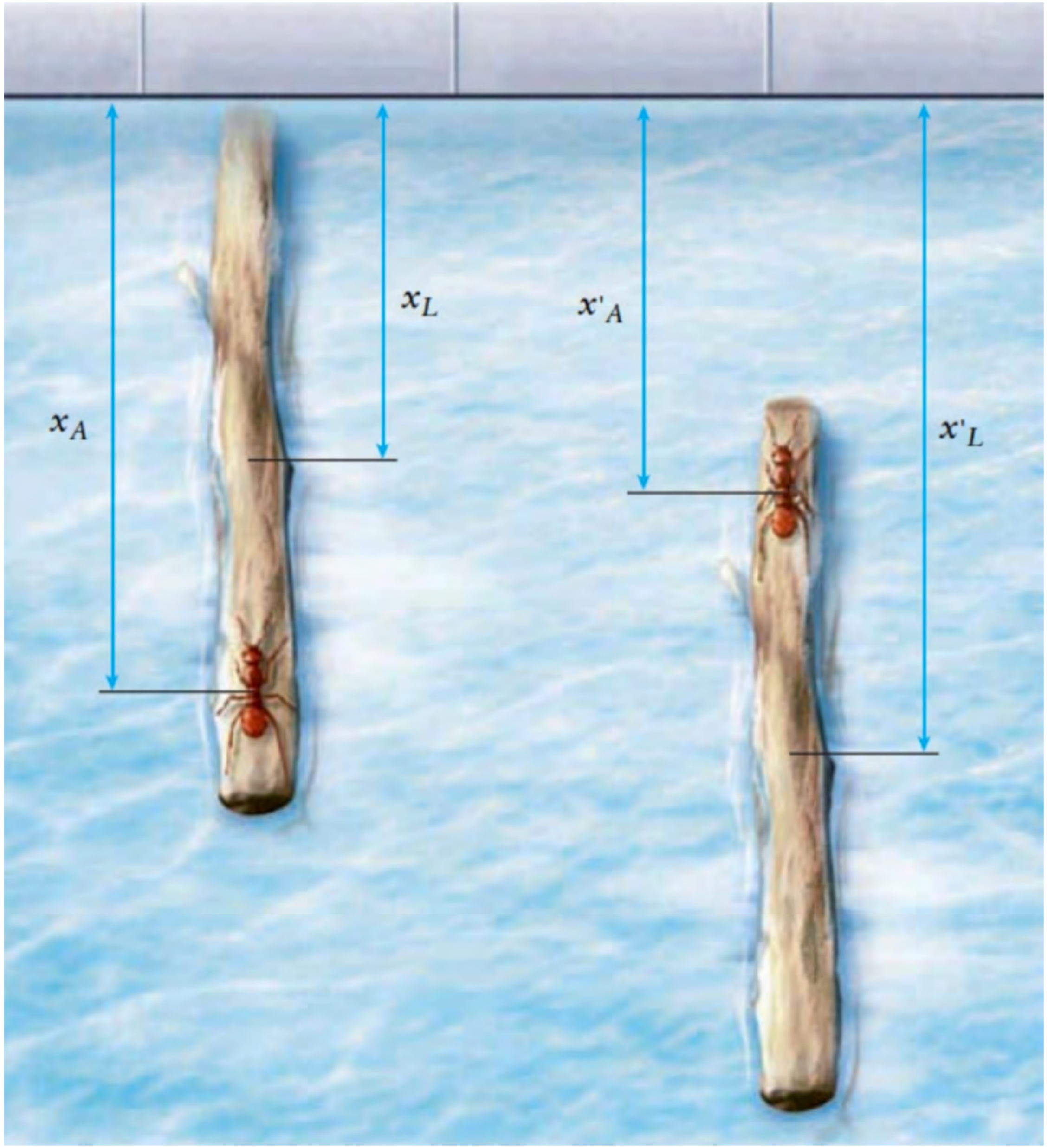
So the center of mass of this triangle is at the point $(\bar{x}, \bar{y}) = (1/3, 0)$.

Motion of the Center of Mass



Kesten & Tauck

Motion of the Center of Mass



Note: If the net force on a system is zero, than the CM does not move, leading to a redistribution of the "particles" inside so to maintain conservation of (linear) momentum

EXAMPLE 9.4 CM Motion: Circus Train

Jumbo, a 4.8-t elephant, stands near one end of a 15-t railcar at rest on a frictionless horizontal track. (Here t is for tonne, or metric ton, equal to 1000 kg.) Jumbo walks 19 m toward the other end of the car. How far does the car move?

INTERPRET We're asked about the car's motion, but we can interpret this problem as being fundamentally about the center of mass. We identify the relevant system as comprising Jumbo and the car. Because there's no net external force acting on the system, its center of mass can't move.

DEVELOP Figure 9.8*a* shows the initial situation. The symmetric car has its CM at its center (here we care only about the *x*-component). Let's take a coordinate system that's fixed to the ground and that has $x = 0$ at this *initial* location of the car's center. After the car moves, its center will be somewhere else! Equation 9.2 applies—here in the simpler one-dimensional, two-object form we used in Example 9.1: $x_{cm} = (m_J x_J + m_c x_c)/M$, where we use the subscripts J and c for Jumbo and the car, respectively, and where $M = m_J + m_c$ is the total mass. We have a before/after situation in which the CM position can't change, so we'll write two versions of this expression, before and after Jumbo's walk. We'll then set them equal to state mathematically that the CM itself doesn't move; that is, we'll write $x_{cm\,i} = x_{cm\,f}$, where the subscripts i and f designate quantities associated with the initial and final states, respectively.

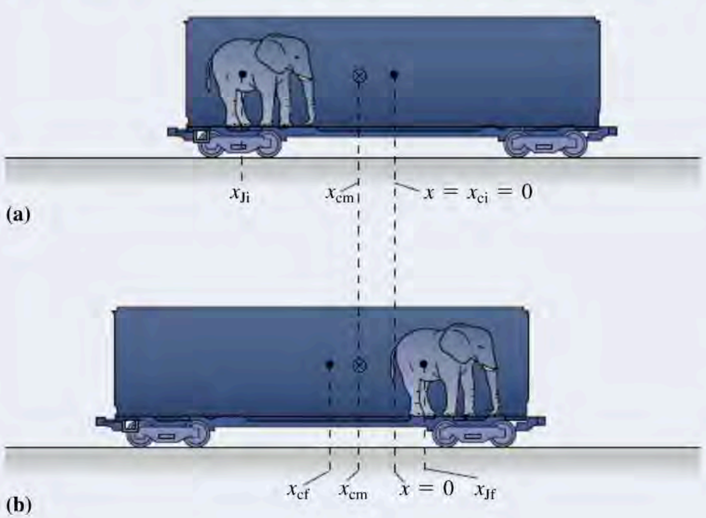


FIGURE 9.8 Jumbo walks, but the center of mass doesn't move.

We chose our coordinate system so that the car's initial position was $x_{ci} = 0$, so our expression for the initial position of the system's center of mass becomes

$$x_{cm\,i} = m_J x_{Ji}/M$$

Our expression for the final center-of-mass position, after Jumbo's walk, is $x_{cm\,f} = (m_J x_{Jf} + m_c x_{cf})/M$. We don't know either of the final coordinates x_{Jf} or x_{cf} here, but we do know that Jumbo walks 19 m *with respect to the car*. The elephant's final position x_{Jf} is therefore 19 m to the right of x_{Ji}, *adjusted for the car's displacement*. Therefore Jumbo ends up at $x_{Jf} = x_{Ji} + 19\text{ m} + x_{cf}$. You might think we need a minus sign because the car moves to the left. That's true, but the sign of x_{cf} will take care of that. Trust algebra! So our expression for the final center-of-mass position is

$$x_{cm\,f} = \frac{m_J x_{Jf} + m_c x_{cf}}{M} = \frac{m_J(x_{Ji} + 19\text{ m} + x_{cf}) + m_c x_{cf}}{M}$$

EVALUATE Finally, we equate our expressions for the initial and final positions of the center of mass. Again, that's because there are no forces external to the elephant–car system acting in the horizontal direction, so the center-of-mass position x_{cm} can't change. Thus we have $x_{cm\,i} = x_{cm\,f}$, or

$$\frac{m_J x_{Ji}}{M} = \frac{m_J(x_{Ji} + 19\text{ m} + x_{cf}) + m_c x_{cf}}{M}$$

The total mass M cancels, so we're left with the equation $m_J x_{Ji} = m_J(x_{Ji} + 19\text{ m} + x_{cf}) + m_c x_{cf}$. We aren't given x_{Ji}, but the term $m_J x_{Ji}$ is on both sides of this equation, so it cancels, leaving $0 = m\,(19\text{ m} + x_{cf}) + m_c x_{cf}$. We solve for the unknown x_{cf} to get

$$x_{cf} = -\frac{(19\text{ m})m_J}{(m_J + m_c)} = -\frac{(19\text{ m})(4.8\text{ t})}{(4.8\text{ t} + 15\text{ t})} = -4.6\text{ m}$$

The minus sign here indicates a displacement to the left, as we anticipated (Fig. 9.8*b*). Because the masses appear only in ratios, we didn't need to convert to kilograms.

ASSESS The car's 4.6-m displacement is quite a bit less than Jumbo's (which is 19 m − 4.6 m, or 14.4 m relative to the ground). That makes sense because Jumbo is considerably less massive than the car. ■

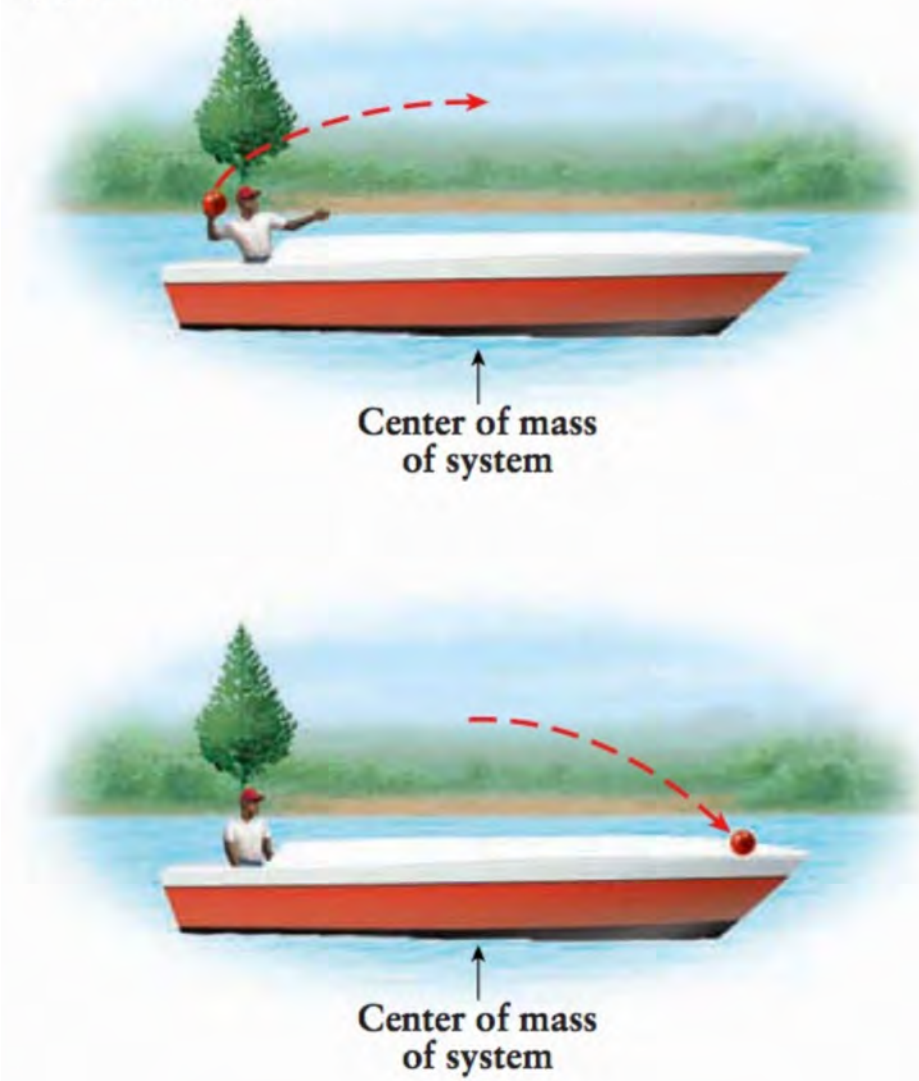
Note that there a few few salient steps here:

- Determine the CM
- Realize the CM does not change
- Figure out how Jumbo's position changes relative to the CM and the railcar

Motion of the Center of Mass

<u>Be careful</u>: What K&T state here, as it is incorrect. The boat doesn't *approximately* move, but move it does (as it must given the stated conservation law!)

(a) **Massive boat**



The center of mass of the system is close to the center of mass of the boat, because the boat is so massive relative to the boy and the ball.

Compare the center of mass position to the fixed position of the tree. The center of mass doesn't move, even after the ball is thrown.

Ex.

A dog, weighing 10.0 lb is standing on a flatboat so that he is 20 ft from the shore. He walks 8.0 ft on the boat toward shore and then halts. The boat weighs 40 lb, and one can assume there is no friction between it and the water. How far is he from the shore at the end of this time? (*Hint:* The center of mass of boat + dog does not move. Why?) The shoreline is also to the left in Fig. 9–15.

Fig. 9–15

Bonus: What breed of dog is Snoopy?

Resnick & Halliday (1966)

Motion of the Center of Mass

- From our definition of center of mass: $$M\mathbf{r}_{\text{cm}} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \cdots + m_n\mathbf{r}_n,$$

- Differentiating w/ respect to t:
  (assuming mass stays const.)

$$M\frac{d\mathbf{r}_{\text{cm}}}{dt} = m_1\frac{d\mathbf{r}_1}{dt} + m_2\frac{d\mathbf{r}_2}{dt} + \cdots + m_n\frac{d\mathbf{r}_n}{dt}$$

$$M\mathbf{v}_{\text{cm}} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \cdots + m_n\mathbf{v}_n,$$

- Differentiating ***again*** w/ respect to t:

$$M\frac{d\mathbf{v}_{\text{cm}}}{dt} = m_1\frac{d\mathbf{v}_1}{dt} + m_2\frac{d\mathbf{v}_2}{dt} + \cdots + m_n\frac{d\mathbf{v}_n}{dt}$$
$$= m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + \cdots + m_n\mathbf{a}_n,$$

$$M\mathbf{a}_{\text{cm}} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n.$$

Hence *the total mass of the group of particles times the acceleration of its center of mass is equal to the vector sum of all the forces acting on the group of particles.*

Resnick & Halliday (1966)

Motion of the Center of Mass

$$M\mathbf{a}_{cm} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n.$$

- Included amongst these forces are *internal* ones, in that from Newton's 3rd Law, they will occur in (equal but opposite) pairs and thereby cancel

$$M\mathbf{a}_{cm} = \mathbf{F}_{ext}.$$

→ So only *external* forces effectively contribute

This states that *the center of mass of a system of particles moves as though all the mass of the system were concentrated at the center of mass and all the external forces were applied at that point.*

Resnick & Halliday (1966)

Additional examples for study...... (some w/ solutions, some w/o)

Ex.

▶ **Example 3.** Consider three particles of different masses acted on by external forces, as shown in Fig. 9–6. Find the acceleration of the center of mass of the system.

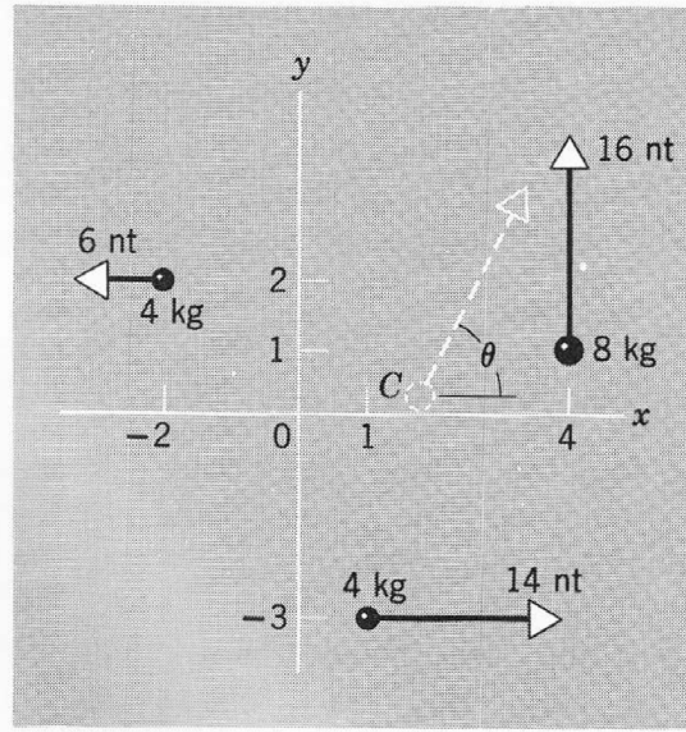


Fig. 9–6 Example 3. Finding the motion of the center of mass of three masses, each subjected to a different force. The forces all lie in the plane defined by the particles. The distances indicated along the axes are in meters.

Resnick & Halliday (1966)

Ex. (SOL)

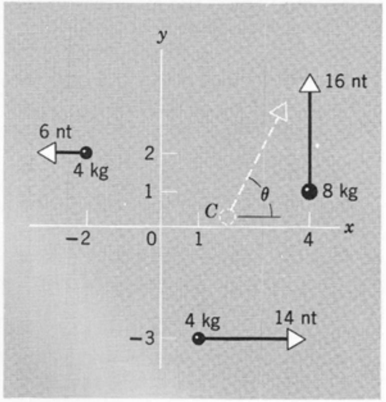


Fig. 9–6 Example 3. Finding the motion of the center of mass of three masses, each subjected to a different force. The forces all lie in the plane defined by the particles. The distances indicated along the axes are in meters.

First we find the coordinates of the center of mass. From Eq. 9–3,

$$x_{cm} = \frac{(8.0 \times 4) + (4.0 \times -2) + (4.0 \times 1)}{16} \text{ meters} = 1.8 \text{ meters},$$

$$y_{cm} = \frac{(8.0 \times 1) + (4.0 \times 2) + (4.0 \times -3)}{16} \text{ meters} = 0.25 \text{ meter}.$$

These are shown as C in Fig. 9–6.

To obtain the acceleration of the center of mass, we first determine the resultant external force acting on the system consisting of the three particles. The x-component of this force is

$$F_x = 14 \text{ nt} - 6.0 \text{ nt} = 8.0 \text{ nt},$$

and the y-component is

$$F_y = 16 \text{ nt}.$$

Hence the resultant external force has a magnitude

$$F = \sqrt{(8.0)^2 + (16)^2} \text{ nt} = 18 \text{ nt},$$

and makes an angle θ with the x-axis given by

$$\tan\theta = \frac{16 \text{ nt}}{8.0 \text{ nt}} = 2.0 \qquad \text{or} \quad \theta = 63°.$$

Then, from Eq. 9–10, the acceleration of the center of mass is

$$a_{cm} = \frac{F}{M} = \frac{18 \text{ nt}}{16 \text{ kg}} = 1.1 \text{ meters/sec}^2,$$

making an angle of 63° with the x-axis.

Although the three particles will change their relative positions as time goes on, the center of mass will move, as shown, with this constant acceleration. ◀

Resnick & Halliday (1966)

Ex.

The density of oil in a circular oil slick on the surface of the ocean at a distance r meters from the center of the slick is given by $\delta(r) = 50/(1 + r)$ kg/m^2.

(a) If the slick extends from $r = 0$ to $r = 10{,}000$ m, find a Riemann sum approximating the total mass of oil in the slick.

(b) Find the exact value of the mass of oil in the slick by turning your sum into an integral and evaluating it.

(c) Within what distance r is half the oil of the slick contained?

Hughes-Hallet et al (2004)

Ex.

If only an external force can change the state of motion of the center of mass of a body, how does it happen that the internal force of the brakes can bring a car to rest?

Resnick & Halliday (1966)

Ex.

A radioactive nucleus, initially at rest, decays by emitting an electron and a neutrino at right angles to one another. The momentum of the electron is 1.2×10^{-22} kg-m/sec and that of the neutrino is 6.4×10^{-23} kg-m/sec. (*a*) Find the direction and magnitude of the momentum of the recoiling nucleus. (*b*) The mass of the residual nucleus is 5.8×10^{-26} kg. What is its kinetic energy of recoil?

Resnick & Halliday (1966)