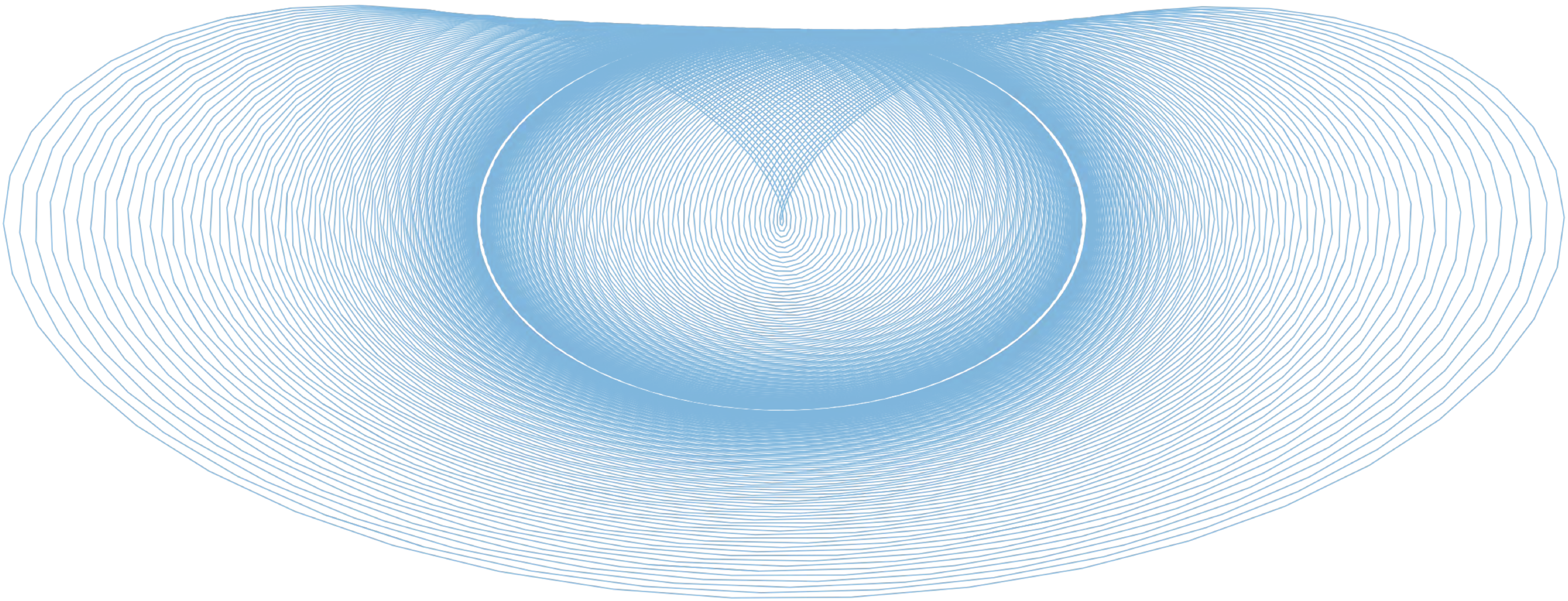
PHYS 1420 (F19)
Physics with Applications to Life Sciences



2019.10.30

Relevant reading:

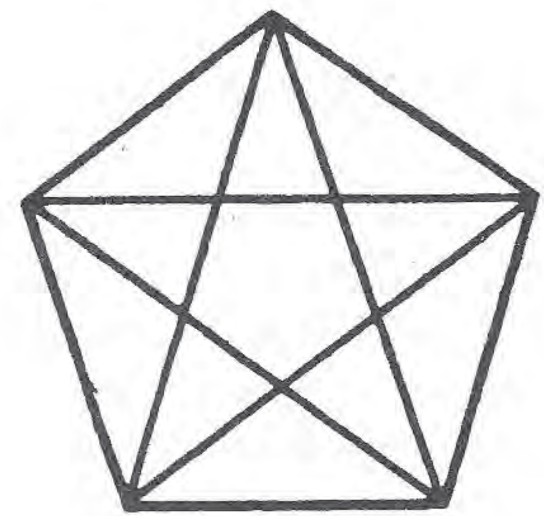
Kesten & Tauck ch. 8.1-8.3

Christopher Bergevin
York University, Dept. of Physics & Astronomy
Office: Petrie 240   Lab: Farq 103
cberge@yorku.ca

Ref. (re images):
Wolfson (2007), Knight (2017),
M. George, Kesten & Tauck (2012)

COUNT!

How many different triangles are there in the figure?



Announcements & Key Concepts (re Today)

→ Online HW #6: Due TODAY!!

→ Online HW #7: Posted and due Monday Nov. 4

→ Midterm exams are graded and...

Some relevant underlying concepts of the day...

- Recap re 2-D center-of-mass (CM) calculations
- CM motion
- Review: Rotational motion
- Rotational kinetic energy
- Moment of inertia

Ex. Find the coordinates of the center of mass of the isosceles triangle in Figure 8.52. The triangle has constant density and mass m.

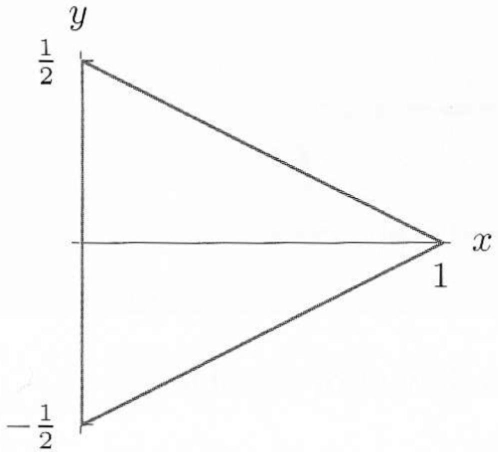


Figure 8.52: Find center of mass of this triangle

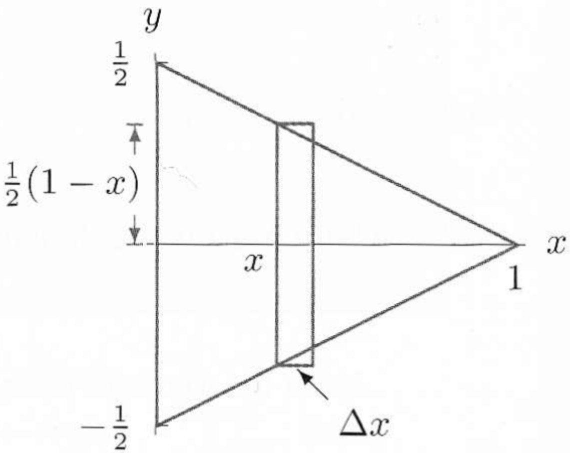


Figure 8.53: Sliced triangle

For a region of constant density δ, the center of mass is given by

$$\bar{x} = \frac{\int x\delta A_x(x)\,dx}{\text{Mass}} \qquad \bar{y} = \frac{\int y\delta A_y(y)\,dy}{\text{Mass}} \qquad \bar{z} = \frac{\int z\delta A_z(z)\,dz}{\text{Mass}}.$$

- Need to determine the density δ [kg/m^2]
- Be careful w/ the units (e.g., [A_x] = m, meaning it is a length and not an area!)

Hughes-Hallet et al (2004)

Ex. (SOL)

Because the mass of the triangle is symmetrically distributed with respect to the x-axis, $\bar{y} = 0$. We expect $\bar{x}$ to be closer to $x = 0$ than to $x = 1$, since the triangle is wider near the origin.

The area of the triangle is $\frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$. Thus Density = Mass/Area = $2m$. If we slice the triangle into strips of width Δx, then the strip at position x has length $A_x(x) = 2 \cdot \frac{1}{2}(1 - x) = (1 - x)$. (See Figure 8.53.) So

$$\text{Area of strip } = A_x(x)\Delta x \approx (1 - x)\Delta x.$$

Since the density is $2m$, the center of mass is given by

$$\bar{x} = \frac{\int x \delta A_x(x)\, dx}{\text{Mass}} = \frac{\int_0^1 2mx(1-x)\, dx}{m} = 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^1 = \frac{1}{3}.$$

So the center of mass of this triangle is at the point $(\bar{x}, \bar{y}) = (1/3, 0)$.

Hughes-Hallet et al (2004)

Motion of the Center of Mass

- From our definition of center of mass: $$M\mathbf{r}_{cm} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \cdots + m_n\mathbf{r}_n,$$

- Differentiating w/ respect to t:
  (assuming mass stays const.)
  $$M\frac{d\mathbf{r}_{cm}}{dt} = m_1\frac{d\mathbf{r}_1}{dt} + m_2\frac{d\mathbf{r}_2}{dt} + \cdots + m_n\frac{d\mathbf{r}_n}{dt}$$

$$M\mathbf{v}_{cm} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \cdots + m_n\mathbf{v}_n,$$

- Differentiating ***again*** w/ respect to t:
  $$M\frac{d\mathbf{v}_{cm}}{dt} = m_1\frac{d\mathbf{v}_1}{dt} + m_2\frac{d\mathbf{v}_2}{dt} + \cdots + m_n\frac{d\mathbf{v}_n}{dt}$$
  $$= m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + \cdots + m_n\mathbf{a}_n,$$

$$M\mathbf{a}_{cm} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n.$$

Hence *the total mass of the group of particles times the acceleration of its center of mass is equal to the vector sum of all the forces acting on the group of particles.*

Resnick & Halliday (1966)

Motion of the Center of Mass

$$M\mathbf{a}_{cm} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n.$$

- Included amongst these forces are *internal* ones, in that from Newton's 3rd Law, they will occur in (equal but opposite) pairs and thereby cancel

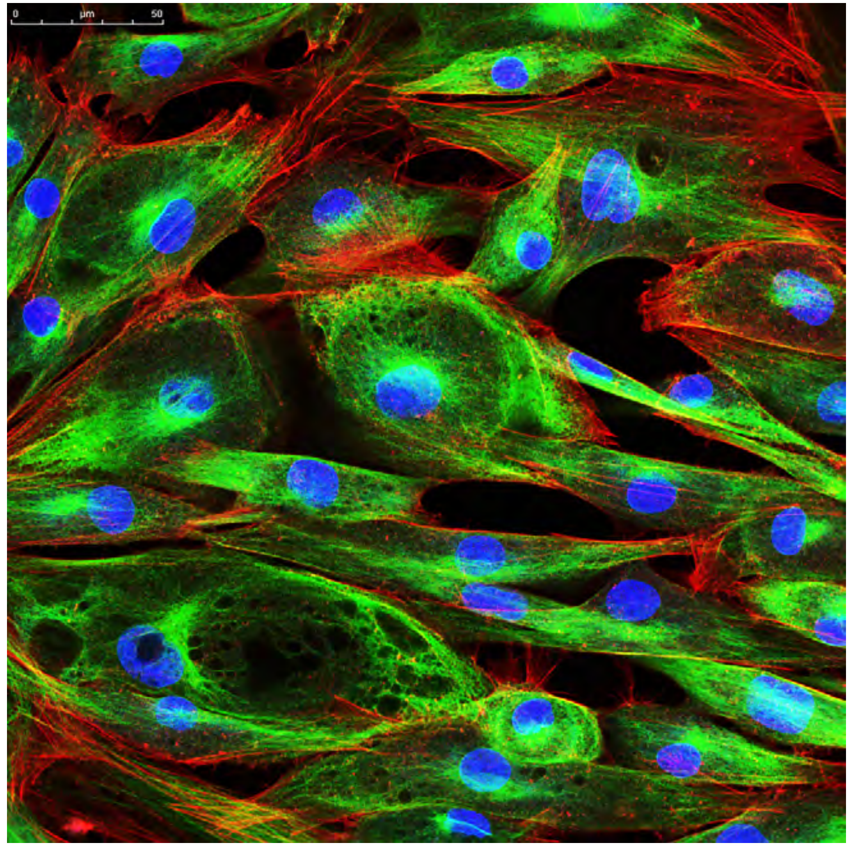
$$M\mathbf{a}_{cm} = \mathbf{F}_{ext}.$$

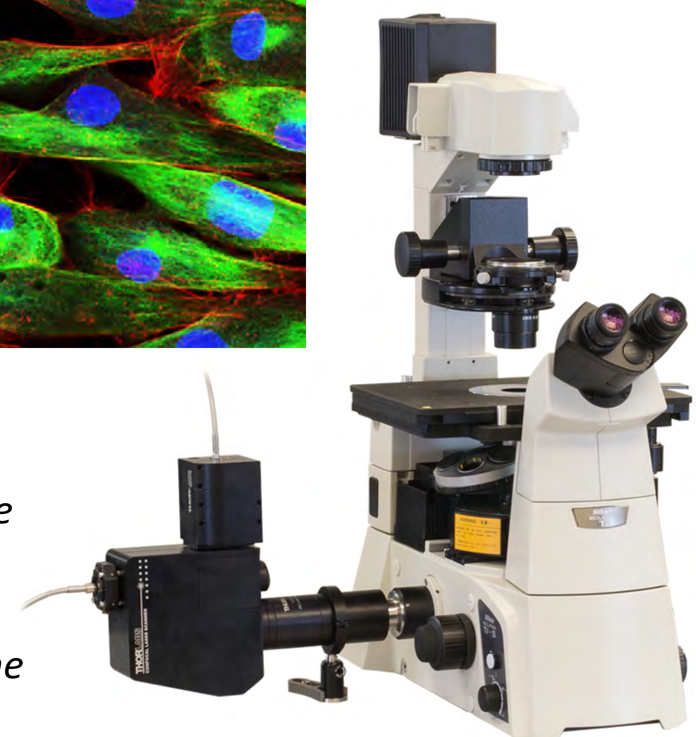
→ So only *external* forces effectively contribute

This states that *the center of mass of a system of particles moves as though all the mass of the system were concentrated at the center of mass and all the external forces were applied at that point.*

Resnick & Halliday (1966)

<u>Review:</u> Rotational Motion





Then, in ***1969****, David Egger and Paul Davidovits of Yale University combined confocal microscopy with lasers to increase the resolution still further. Now, laser confocal microscopy, combined with computers, can produce exquisite three-dimensional images of complex biological material, including the highly amorphous structure of dendritic cells.*

https://www.immunology.org/laser-confocal-microscopy-dendritic-cells-1969

Review: Rotational Motion

From the midterm exam:

3. (*35 points*) Consider the scanning part of a the drive motor and spinning disk of a confocal microscope. Suppose the disk has radius R and is initially at rest. It then speeds up with angular acceleration α.

a. Determine an expression for the tangential velocity after the disk has rotated through an angle $\Delta\phi$.

b. Similarly, determine an expression (in terms of $\Delta\phi$ and any other relevant quantities) for the centripetal acceleration.

Review: Rotational Motion

3. (*35 points*) Consider the scanning part of a the drive motor and spinning disk of a confocal microscope. Suppose the disk has radius R and is initially at rest. It then speeds up with angular acceleration α.

a. Determine an expression for the tangential velocity after the disk has rotated through an angle $\Delta\phi$.

- We know: disk radius (R), const. angular accel. (α) and initially at rest ($\omega_0 = 0$)
- Angular velocity: $\omega = \int \alpha \, dt = \alpha t + \omega_0 = \alpha t$
- Ang. position: $\theta = \int \omega \, dt = \int \alpha t \, dt = \frac{1}{2}\alpha t^2 + \theta_0 = \Delta\phi$ (let this $= 0$ for simplicity)
- So $\Delta\phi = \frac{1}{2}\alpha t^2 \rightarrow t = \sqrt{\frac{2\Delta\phi}{\alpha}}$
- Now $\omega = \alpha t = \alpha\sqrt{\frac{2\Delta\phi}{\alpha}} = \sqrt{2\alpha\Delta\phi}$
- $\omega = \frac{V}{R} \rightarrow$ $V = R\sqrt{2\alpha\Delta\phi}$

b. Similarly, determine an expression (in terms of $\Delta\phi$ and any other relevant quantities) for the centripetal acceleration.

$$a = \frac{V^2}{R} = 2R\alpha\Delta\phi$$

Review (re Circular Motion)

- 1-D kinematics translates directly to circular motion (in polar coords.)

TABLE 4.1 Rotational and linear kinematics for constant acceleration

Rotational kinematics	Linear kinematics
$\omega_f = \omega_i + \alpha\,\Delta t$	$v_{fs} = v_{is} + a_s\,\Delta t$
$\theta_f = \theta_i + \omega_i\,\Delta t + \frac{1}{2}\alpha(\Delta t)^2$	$s_f = s_i + v_{is}\,\Delta t + \frac{1}{2}a_s(\Delta t)^2$
$\omega_f^2 = \omega_i^2 + 2\alpha\,\Delta\theta$	$v_{fs}^2 = v_{is}^2 + 2a_s\,\Delta s$

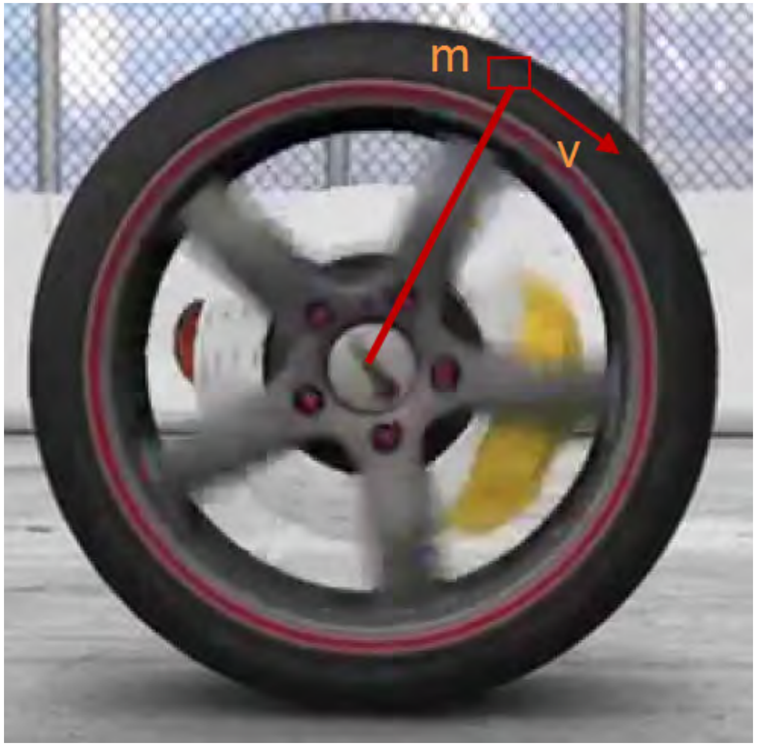
Knight (2013)

→ However, things tend to otherwise get a bit more *complicated* as we will see...

Rotational Kinetic Energy

- Imagine only a small section of the wheel. It has some mass, and it is moving with some velocity. That piece has some kinetic energy.
- Since kinetic energy is a scalar, the KE of all such pieces should add.

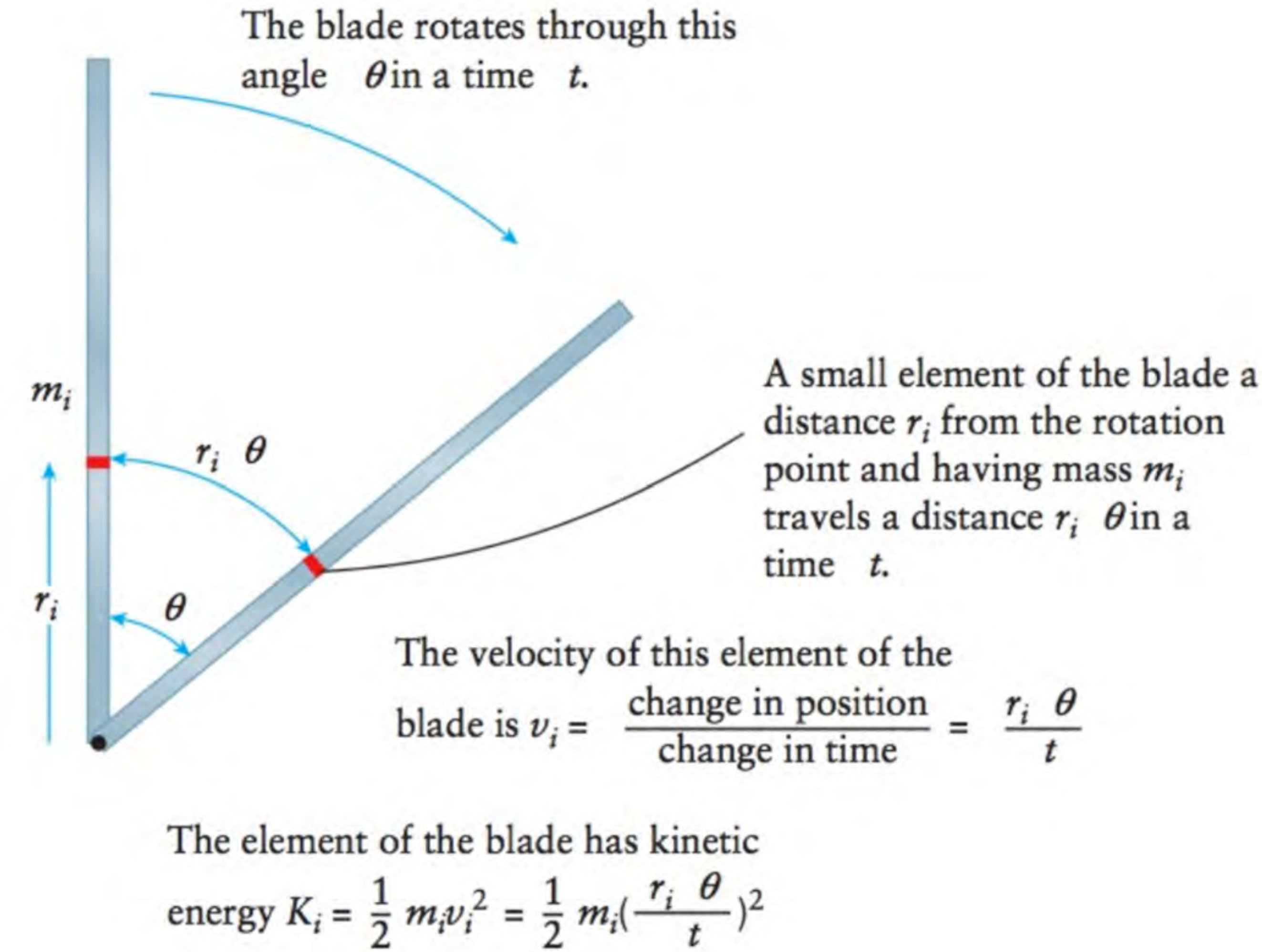
Rotational Kinetic Energy



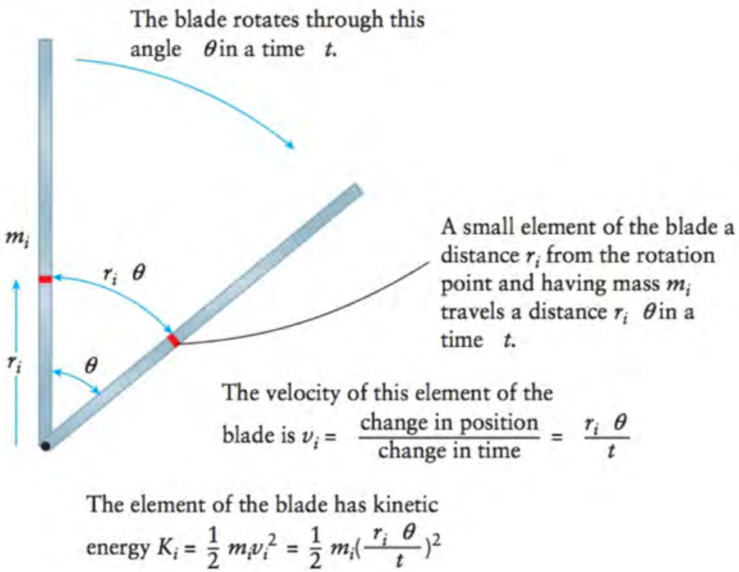
- Something spinning may not have the location of its centre of mass moving. So it would not have any translational kinetic energy.
- But it would have some motional energy, since it takes work to spin an object!

Rotational Kinetic Energy

Consider an element of a rod (with mass m_i) undergoing circular motion



Rotational Kinetic Energy



Velocity of mass element m_i

$$v_i = \frac{r_i \Delta\theta}{\Delta t}$$

Associated kinetic energy

$$K_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i \left(\frac{r_i \Delta\theta}{\Delta t}\right)^2$$

$$= \frac{1}{2} m_i r_i^2 \left(\frac{\Delta\theta}{\Delta t}\right)^2$$

Angular velocity

$$\omega = \frac{\Delta\theta}{\Delta t} \qquad \omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

$$K_i = \frac{1}{2} m_i r_i^2 \omega^2$$

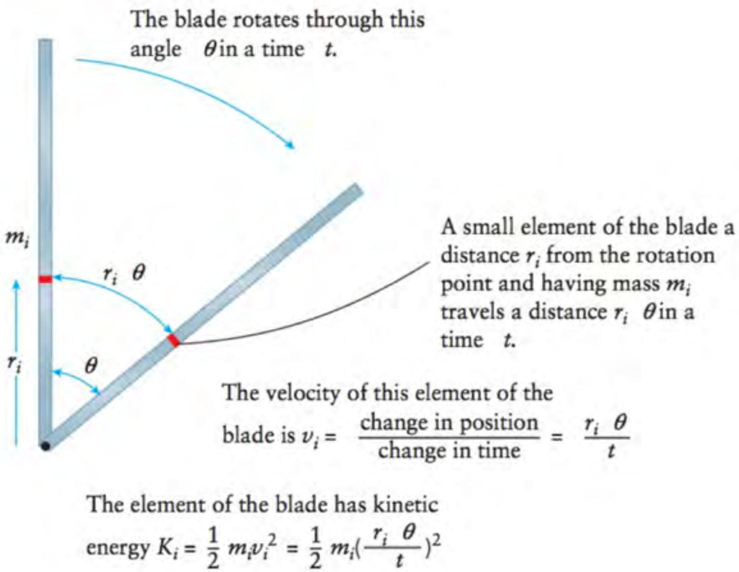
Note: All points on the bar have the same ω

What about the bar as a whole?

$$K = \sum K_i = \sum \frac{1}{2} m_i r_i^2 \omega^2$$

$$K = \frac{1}{2} \left(\sum m_i r_i^2 \right) \omega^2$$

Rotational Kinetic Energy



$$K = \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2$$

Moment of inertia

$$I = \sum m_i r_i^2$$

$$K_{\text{rotational}} = \frac{1}{2} I \omega^2$$

Whereas we interpreted mass as a property of matter that represents the resistance of an object to a change in translational velocity, the moment of inertia represents the resistance of an object to a change in rotational or angular velocity. In the same way that we defined inertia as the tendency of an object to resist a change in translational motion, we can define rotational inertia as the tendency of an object to resist a change in rotational motion.

Rotational vs "Linear" Kinetic Energy

These two wheels have the same mass and size, but the first is a ring with no mass at the center while the other is a uniform disk.

Sliding at the same linear velocity, both have the same linear kinetic energy.

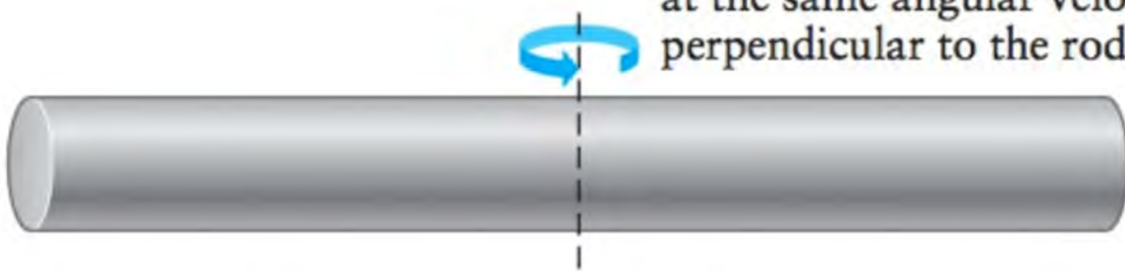
When both rotate at the same angular velocity, the ring has greater rotational kinetic energy. More of its mass is farther from the rotation axis, resulting in a larger moment of inertia around that axis.

Kesten & Tauck

Rotational Kinetic Energy

The kinetic energy of a rod rotating around an axis perpendicular to the rod and through its end is four times larger...

...than the kinetic energy of the rod when it rotates at the same angular velocity around an axis perpendicular to the rod and through its center.



The more mass there is farther from the rotation axis, the larger the moment of inertia, and the larger the rotational kinetic energy for any given angular velocity.

Kesten & Tauck

Inertia

Newton's first law of motion: A body in uniform motion remains in uniform motion, and a body at rest remains at rest, unless acted on by a nonzero net force.

Inertia is the resistance of any physical object to any change in its velocity. This includes changes to the object's speed, or direction of motion. An aspect of this property is the tendency of objects to keep moving in a straight line at a constant speed, when no forces act upon them.

Inertia comes from the Latin word, *iners*, meaning idle, sluggish. Inertia is one of the primary manifestations of mass, which is a quantitative property of physical systems. Isaac Newton defined inertia as his first law in his *Philosophiæ Naturalis Principia Mathematica*, which states:

> The *vis insita*, or innate force of matter, is a power of resisting by which every body, as much as in it lies, endeavours to preserve its present state, whether it be of rest or of moving uniformly forward in a straight line.[1]

Wikipedia (re *inertia*)

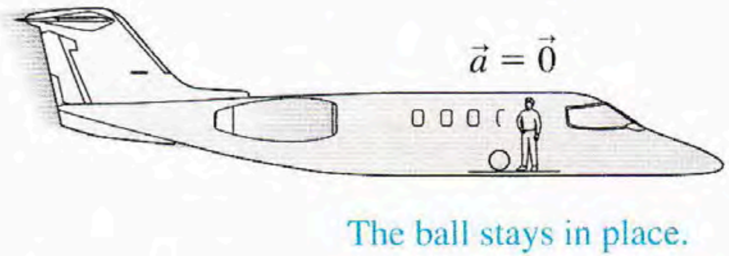
Reminder: Inertial Reference Frames

- Similar to walking along on a (moving) train

- Frame of reference is inertial if it is in uniform motion (which also includes being at rest). That is, the frame itself is not accelerating

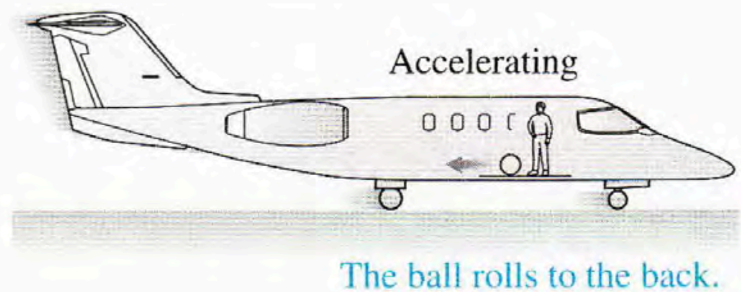
- Strictly speaking, “Earth” is not an inertial frame (due to the rotation of earth)

(a)



A ball with no horizontal forces stays at rest in an airplane cruising at constant velocity. The airplane is an inertial reference frame.

(b)



The ball rolls to the back of the plane during takeoff. An accelerating plane is not an inertial reference frame.

Knight (2013)

→ Newton’s laws only apply in inertial reference frames

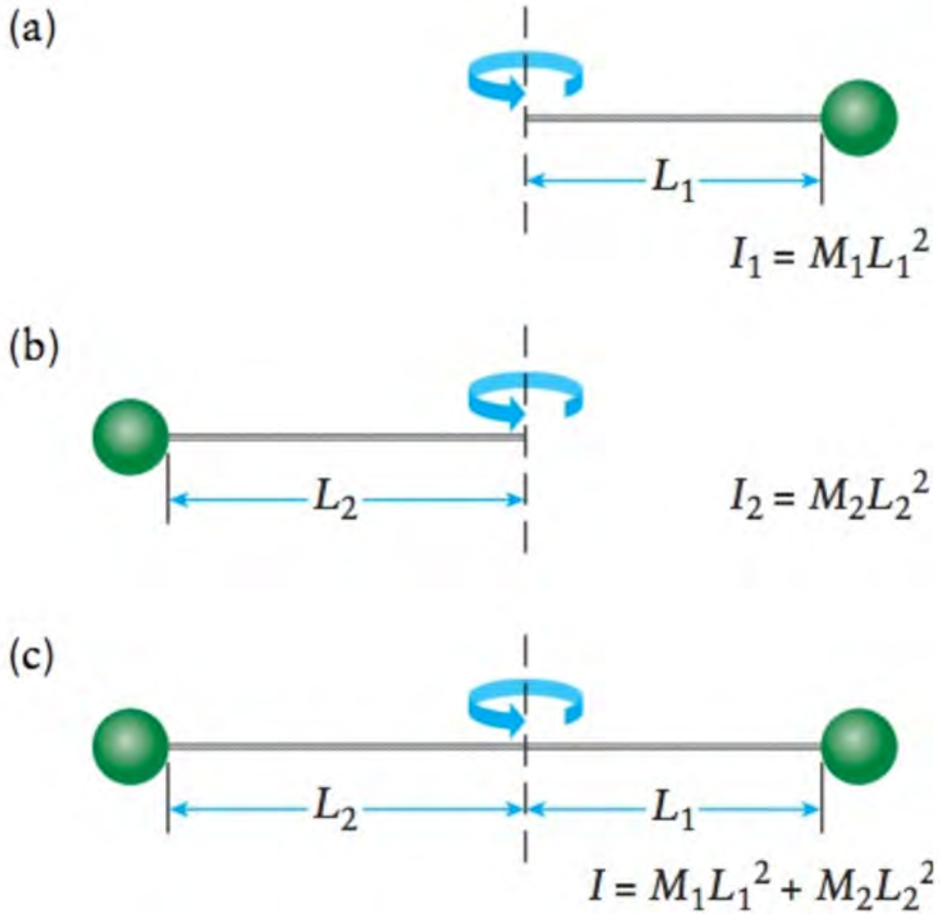
Moment of Inertia

- The moment of inertia, I, tells us how difficult it is to change the angular speed of a rotating (or rotatable) object.

$$K_{rotational} = \frac{1}{2} I \omega^2$$

- This is the same idea as mass quantifying how difficult it is to change the linear speed of an object.
- Moment of inertia depends on the particular sum of $m_i r_i^2$ terms, which depends on the shape of the object, and about which axis the object is rotating.
- In principle, one requires a volume integral to find this sum.
- In practice, for some regularly shaped objects rotating about certain axes, these sums are well known.

Moment of Inertia: Multiple masses



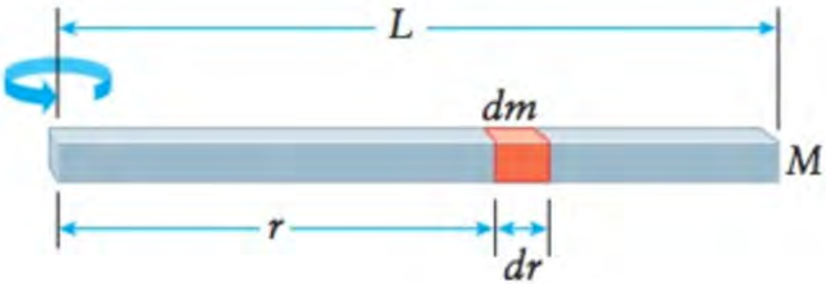
$$I = \sum m_i r_i^2$$

Note the similarity here w/ respect to CM calculations...

$$x_{CM} = \frac{1}{M_{tot}} \sum_{i=1}^{N} m_i x_i$$

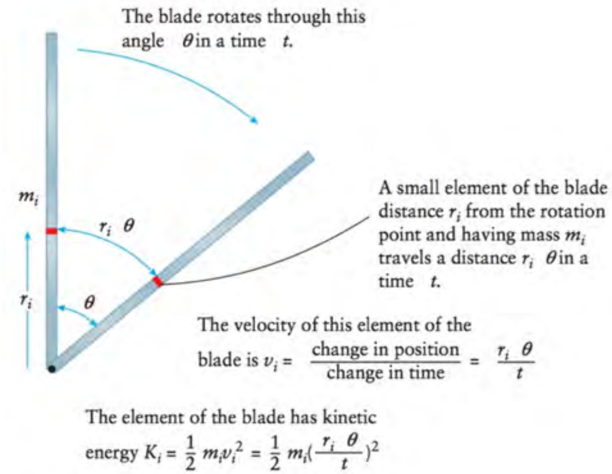
... a key difference being that here things are moving (or more specifically, *rotating*)

Moment of Inertia: Uniform bar rotating about one end



Moment of inertia

$$I = \sum m_i r_i^2$$



Now in integral form, but...

$$I = \int r^2\, dm$$

... how are r and m functionally-related?

$$I = \int r^2 \frac{M}{L} dr = \frac{M}{L} \int r^2\, dr$$

Uniform density!
(equal proportions)

$\Delta m / \Delta r = M/L$

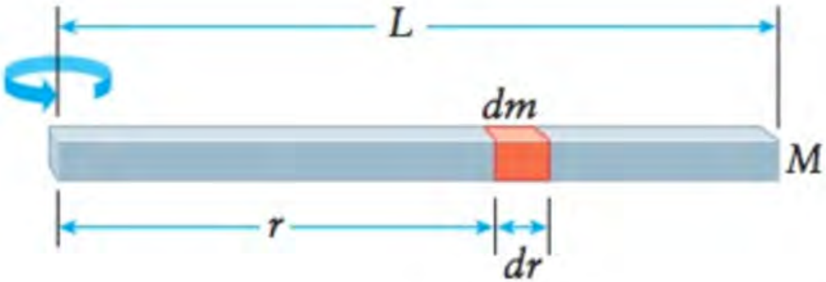
Now in the limit:

$$dm = \frac{M}{L} dr$$

$$I = \frac{M}{L} \frac{r^3}{3} \bigg|_0^L = \frac{M}{L} \left(\frac{L^3}{3} - \frac{0^3}{3} \right) = \frac{ML^2}{3}$$

Kesten & Tauck

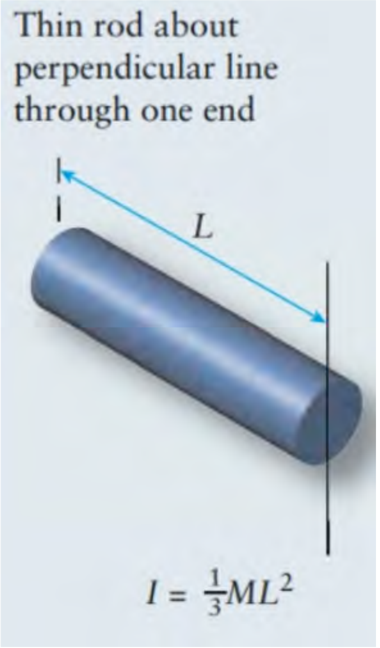
Moment of Inertia: Uniform bar rotating about one end



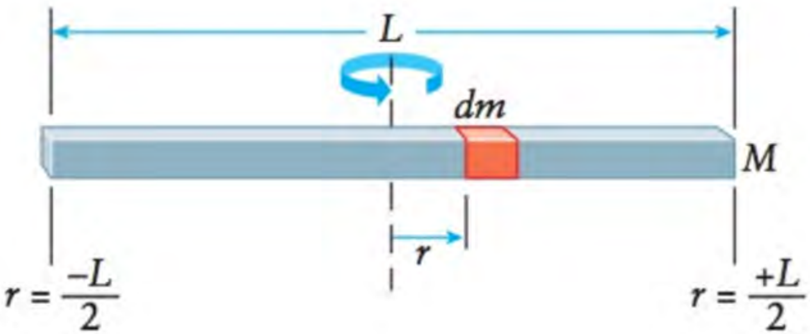
$$I = \frac{M}{L}\frac{r^3}{3}\Big|_0^L = \frac{M}{L}\left(\frac{L^3}{3} - \frac{0^3}{3}\right) = \frac{ML^2}{3}$$

$$I = \frac{1}{3}ML^2$$

Rod about end



Moment of Inertia: Uniform bar rotating about its center



An object does not have "a" moment of inertia. Rather, it has a moment of inertia defined for rotation around *each specific choice* of rotation axis.

$$I = \frac{M}{L}\int_{-L/2}^{+L/2} r^2\, dr$$

Just change the limits of integration!

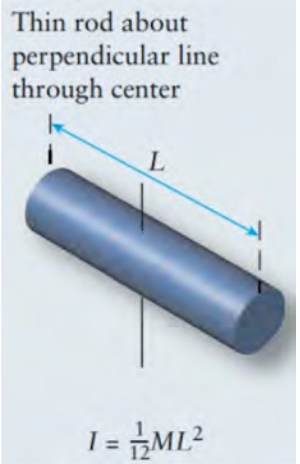
$$= \frac{M}{L}\frac{r^3}{3}\Big|_{-L/2}^{+L/2} = \frac{M}{L}\left(\frac{(+L/2)^3}{3} - \frac{(-L/2)^3}{3}\right)$$

$$= \frac{M}{L}\left(\frac{L^3}{24} - \frac{-L^3}{24}\right) = \frac{ML^2}{12}$$

Recall: Note the similarity here w/ respect to CM calculations...

$$x_{CM} = \frac{1}{M_{tot}}\sum_{i=1}^{N} m_i x_i$$

... a **key difference being that here things are moving** (or more specifically, *rotating*)

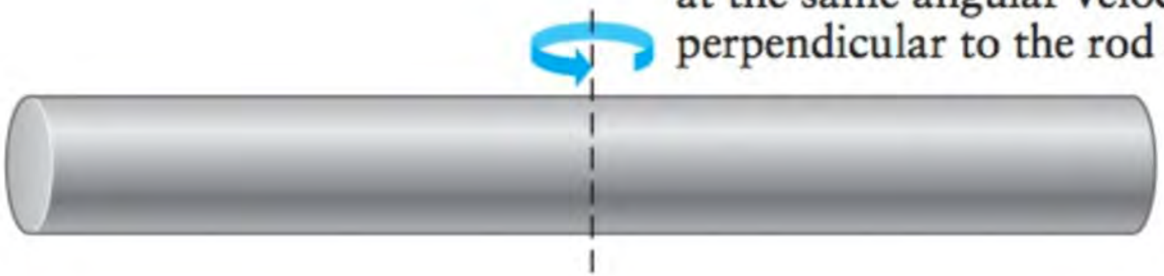


Question: What is a *thin rod* and why is that needed?

Kesten & Tauck

The kinetic energy of a rod rotating around an axis perpendicular to the rod and through its end is four times larger...

...than the kinetic energy of the rod when it rotates at the same angular velocity around an axis perpendicular to the rod and through its center.



The more mass there is farther from the rotation axis, the larger the moment of inertia, and the larger the rotational kinetic energy for any given angular velocity.

Kesten & Tauck