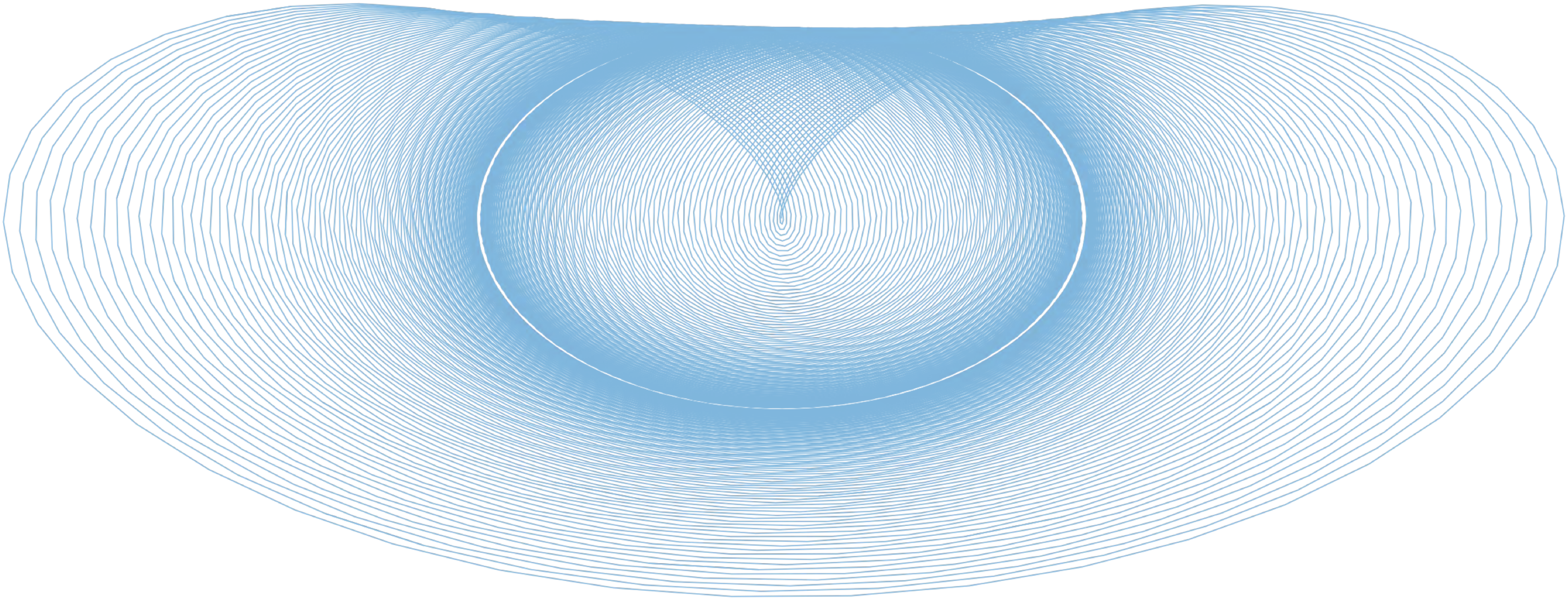
PHYS 1420 (F19)
Physics with Applications to Life Sciences



2019.11.04

Relevant reading:

Kesten & Tauck ch. 8.6, 8.8

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Ref. (re images):
Wolfson (2007), Knight (2017),
M. George, Kesten & Tauck (2012)
Resnick & Halliday (1966)

Not including the words in the question, or the page number, there are **six** words on this page. What are they?

RED

BOLD

REVERSE

NARROW

ITALIC

<u>Announcements & Key Concepts</u> (re Today)

→ Online HW #7: Posted and due today (Monday Nov. 4)

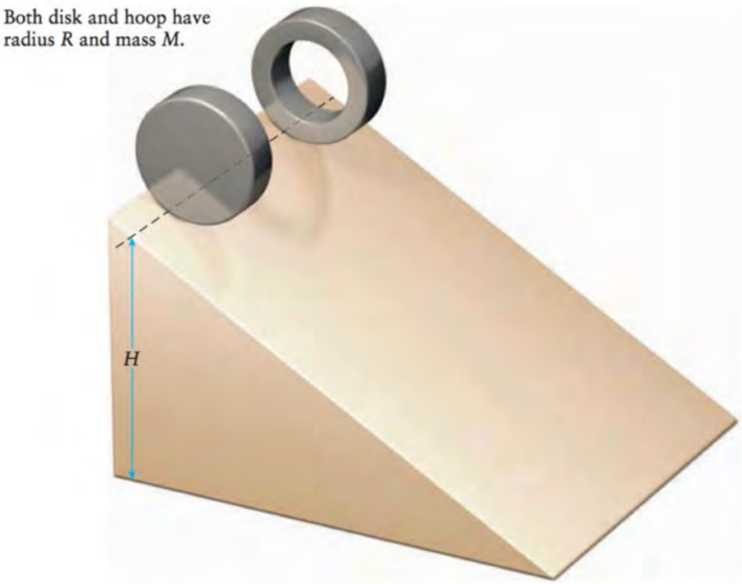
→ Midterm exams are graded and (hopefully) to be handed back tomorrow at tutorial

Some relevant underlying concepts of the day…

- Torque
- Cross-product
- Vector nature of angular quantities

Conservation of Energy (Revisited)

When the smoke clears....



$$v_{\text{disk, f}} = \sqrt{\frac{4}{3} gH}$$

$$v_{\text{hoop, f}} = \sqrt{gH}$$

The "disk" wins the race. Perhaps not very intuitive at first....

... until you keep in mind a key principle at play here:
Energy can be stored in a variety of ways

Note: We will revisit this calculation during tutorial (there is a bit more here than meets the eye...)

Torque

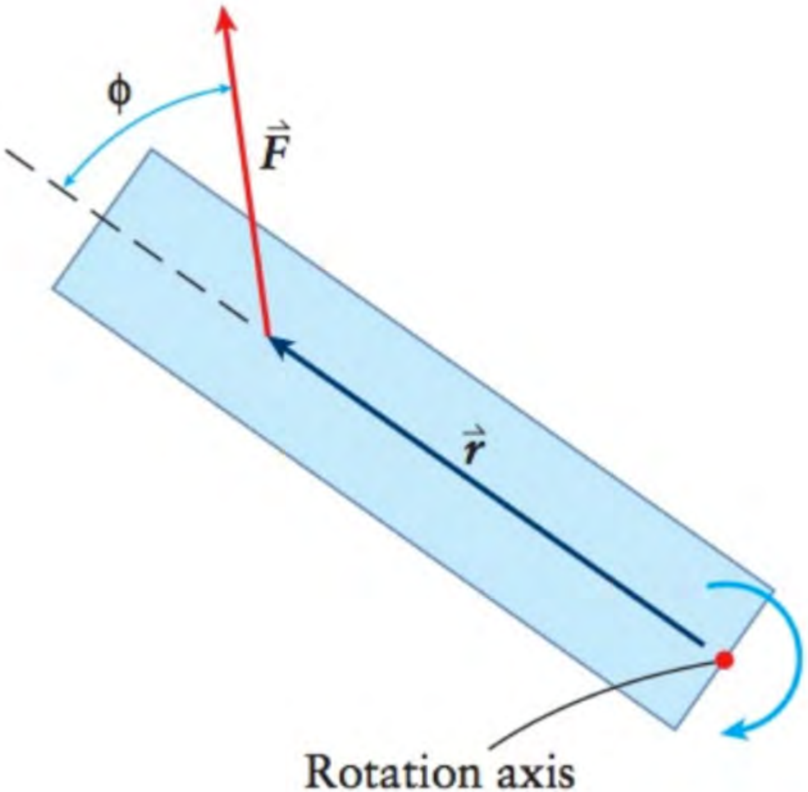
- But how does one achieve a particular angular speed? What cause angular acceleration?

- For translational motion, it was a force that caused changes in velocity.

- For rotational motion, it is the torque, $\vec{\tau}$, (tau) that causes changes in angular speed. Torque can be thought of as the angular force.

Torque

Figure 8-24 Torque τ is the rotational analog of force and takes into account the distance r between where a force F is applied and the rotation axis. Torque also takes into account the angle φ between the force vector $\vec{F}$ and the $\vec{r}$ vector that points from the rotation axis to the point at which the force is applied.



$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\tau = rF \sin\varphi$$

Units:

$$[\tau] = [r][F][\sin\varphi] = \mathrm{m} \cdot \mathrm{N}$$

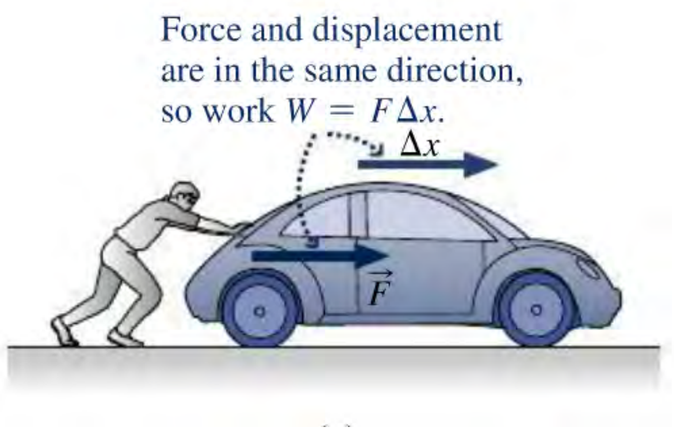
Scalar version (figure above motivates where this comes from...)

<u>Review</u>: Vector Algebra

The "dot product" (re work)

$$W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}$$

$$\vec{A} \cdot \vec{B} = AB\cos\theta$$



Wolfson

→ Another means to multiply vectors (the "*cross product*") arises as we deal w/ torque...

$$\vec{\tau} = \vec{r} \times \vec{F} = |\vec{r}|\,|\vec{F}| sin\phi$$

where $\vec{r}$ is the distance from the axis of rotation

$\vec{F}$ is the applied force

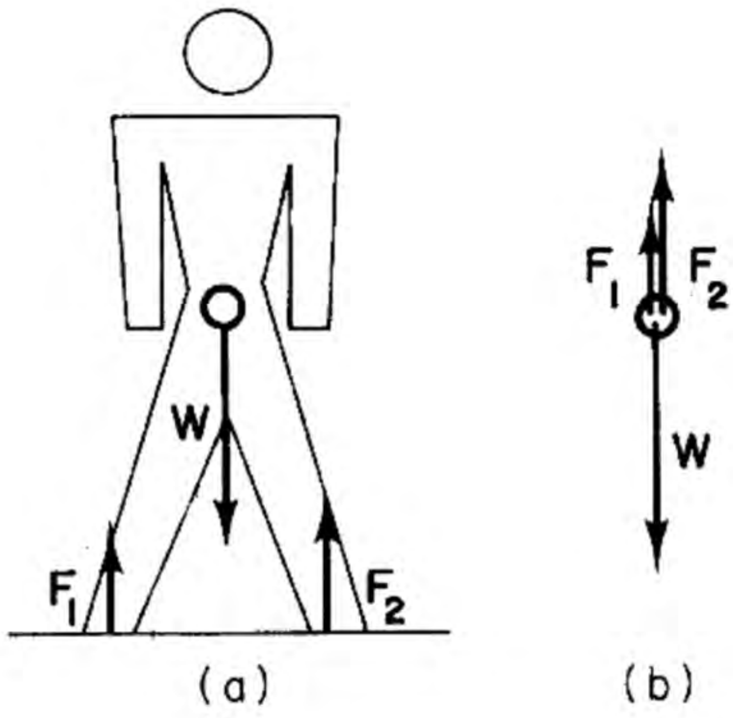
ϕ is the angle between $\vec{r}$ and $\vec{F}$

We'll come back to the vector aspect of things in a bit....

Biomechanical example

Let us consider a "rotational" context, but under equilibrium conditions (i.e., things are "balanced" so they do not actually move)

$$\sum_i \tau_i = \sum_i r_i F_i = 0.$$



Assuming no net torques, you don't "tip over" as you stand

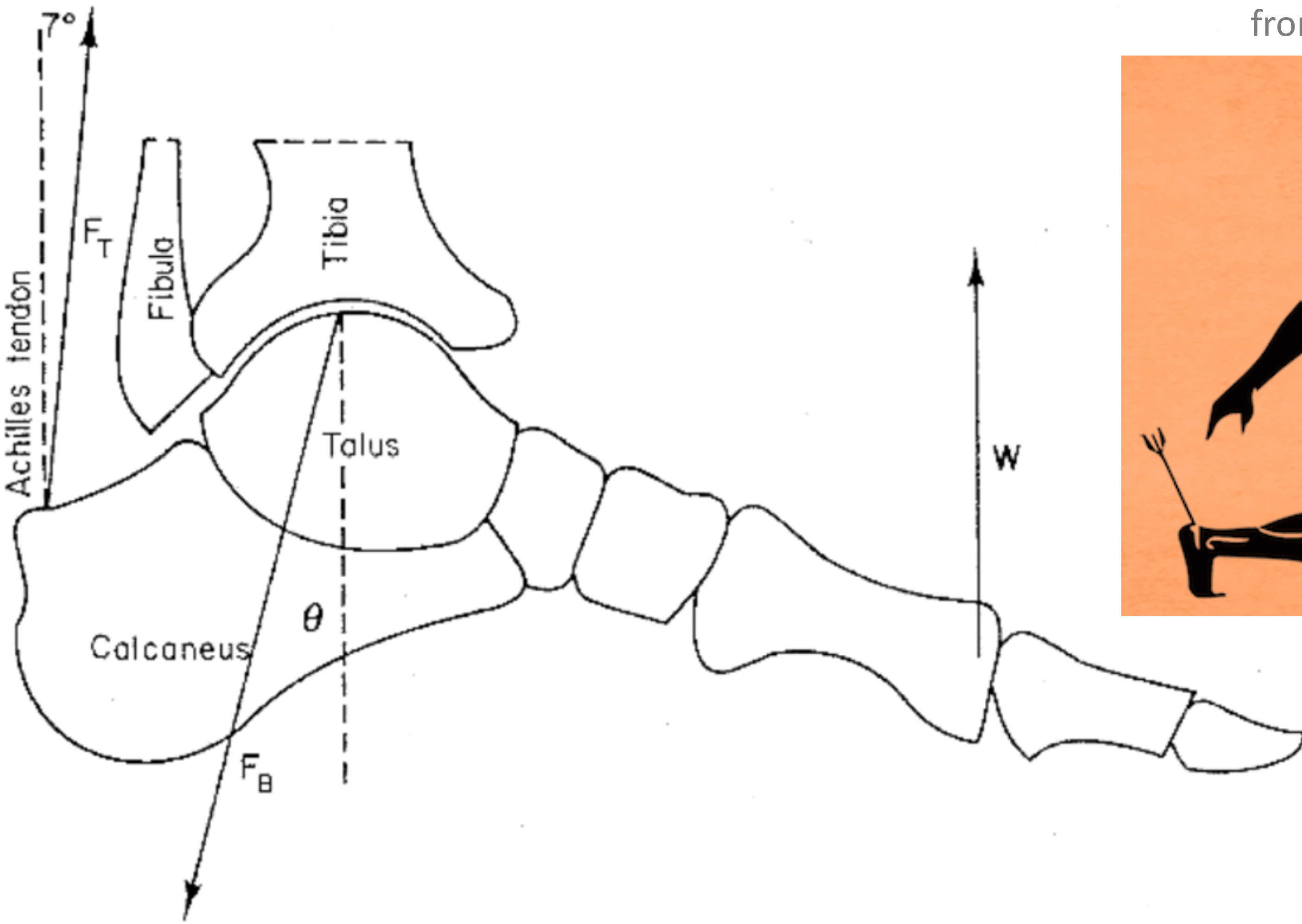
FIGURE 1.4. A person standing. (a) The forces on the person. (b) A free-body or force diagram.

Hobbie & Roth

Biomechanical example

Now consider standing on the "balls" of your feet. We can estimate the force associated with your Achille's tendon...

"The Achilles tendon connects the calf muscles (the gastrocnemius and the soleus) to the calcaneus at the back of the heel"

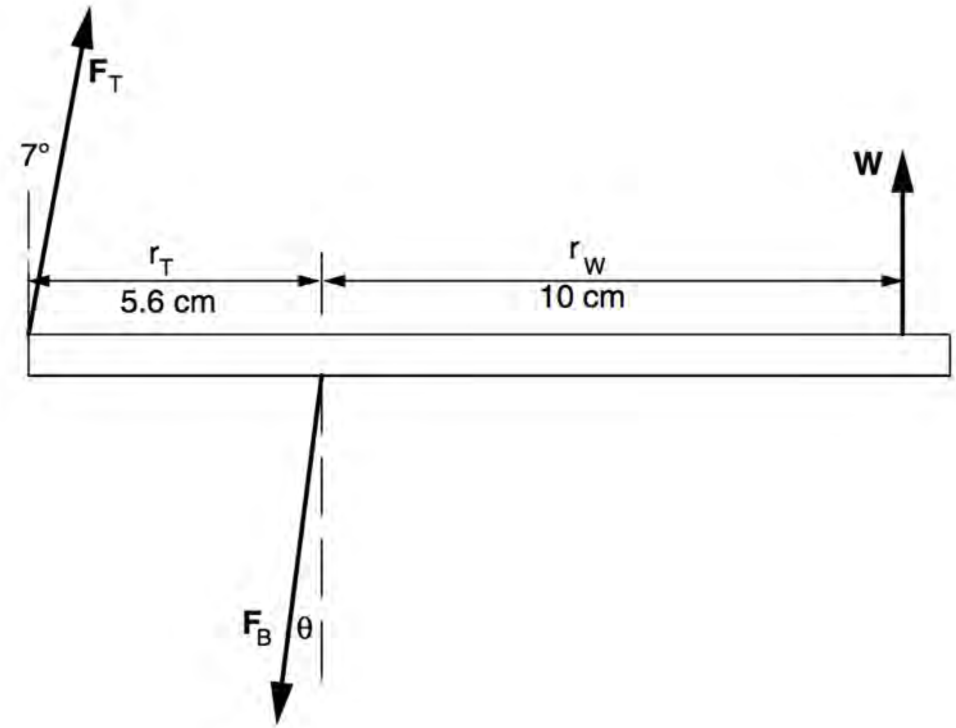


from Homer's *The Illiad*

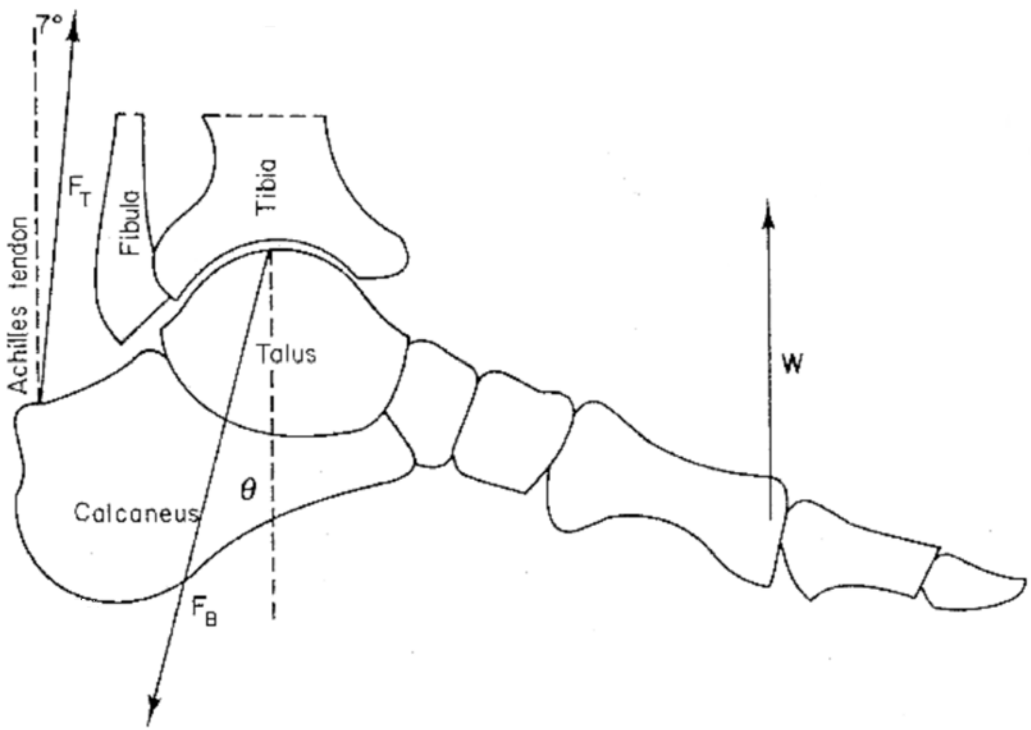
Hobbie & Roth

Biomechanical example

Now consider standing on the "balls" of your feet. We can estimate the force associated with your Achille's tendon...



- **F_T** – force exerted by tendon on foot
- **F_B** – force of leg bones (tibia & fibula) on foot
- **W** – normal force re weight of body



Assumptions

- At equilibrium
- Ignore mass of foot
- Treat foot as rigid body
- Ignore horizontal torque contributions

Hobbie & Roth

Biomechanical example

Now consider standing on the "balls" of your feet. We can estimate the force associated with your Achille's tendon...

- F_T – force exerted by tendon on foot
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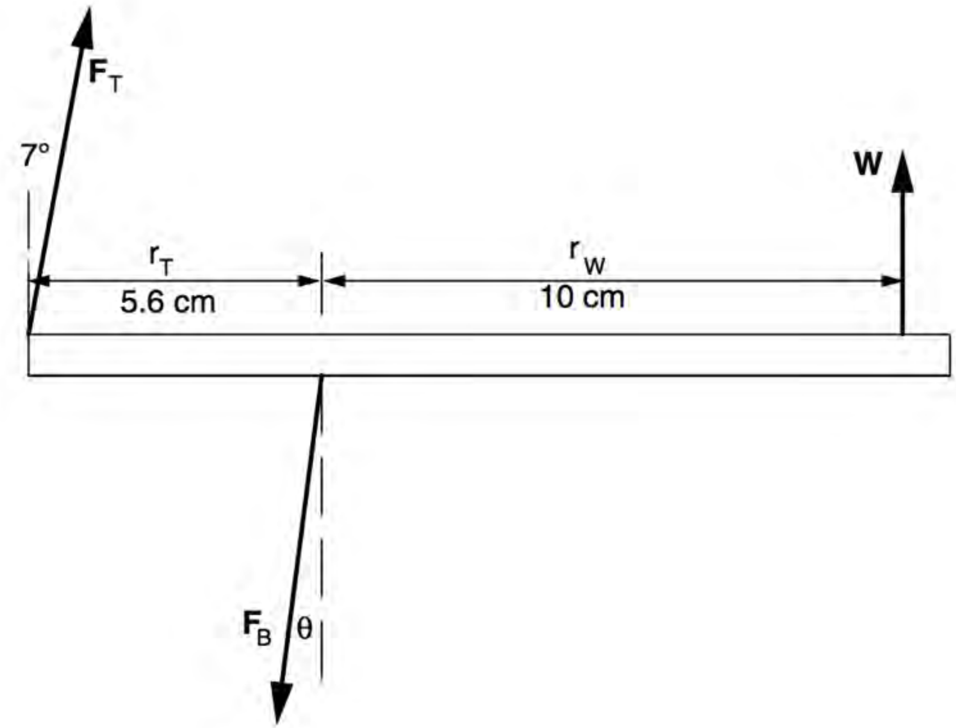
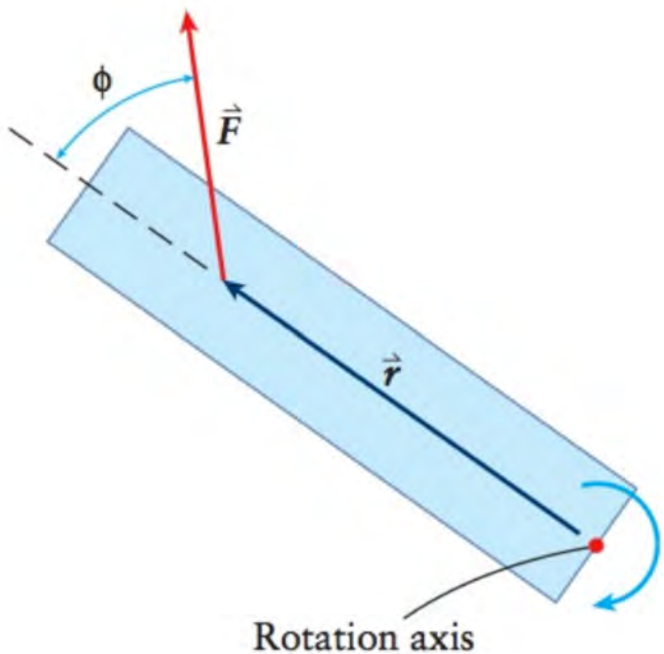


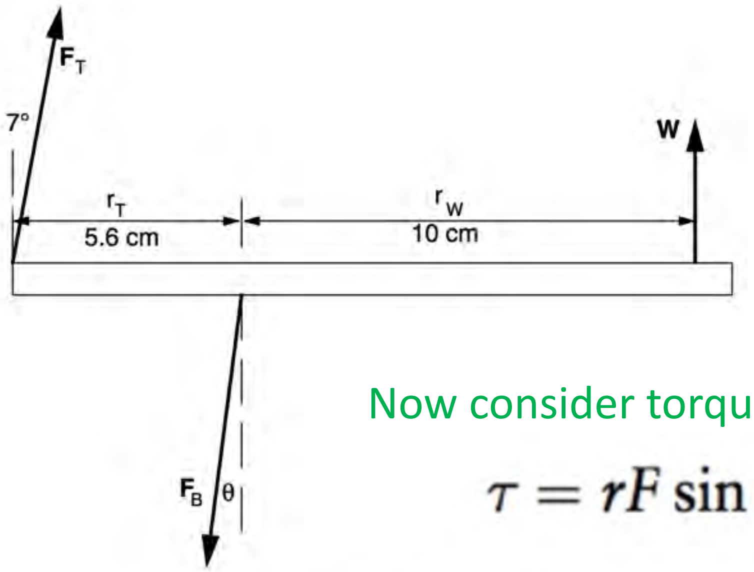


Note the change in how the angle is defined here!

Hobbie & Roth

<u>Biomechanical example</u>

Now consider standing on the "balls" of your feet. We can estimate the force associated with your Achille's tendon...



Translational equilibrium requires:

$$F_T \cos(7^\circ) + W - F_B \cos\theta = 0,$$
$$F_T \sin(7^\circ) - F_B \sin\theta = 0.$$

Now consider torques:

$$\tau = rF \sin\varphi$$

$$\sum_i \tau_i = \sum_i r_i F_i = 0.$$

Rotational equilibrium requires:

$$10W - 5.6 F_T \cos 7^\circ = 0.$$

$$F_T = \frac{10W}{5.6 \cos 7^\circ} = 1.8W.$$

Solving for the other parts:

$$(1.8)(W)(0.993) + W = F_B \cos\theta,$$
$$2.8W = F_B \cos\theta.$$

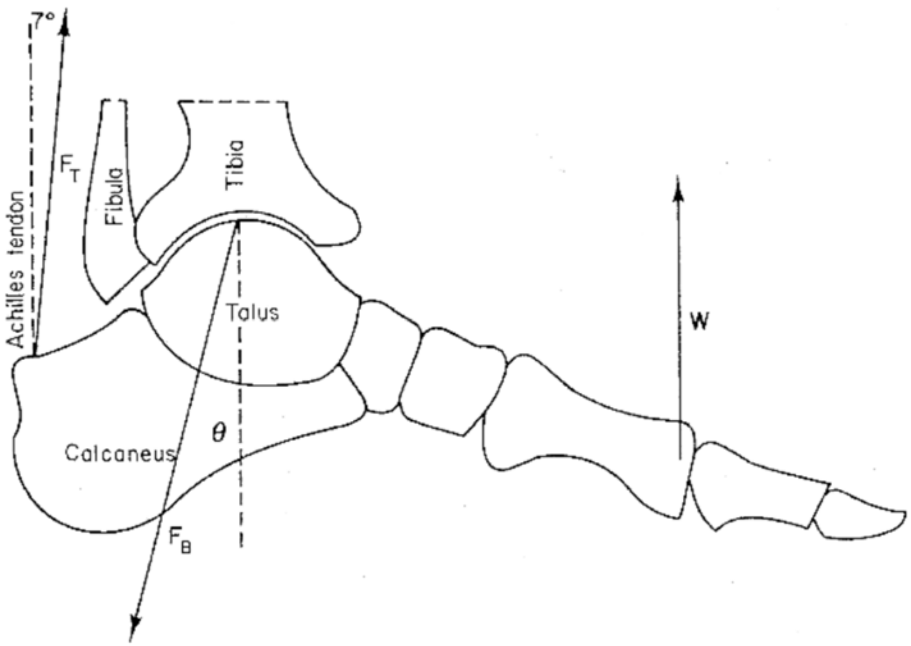
$$(1.8)(W)(0.122) = F_B \sin\theta,$$
$$0.22W = F_B \sin\theta.$$

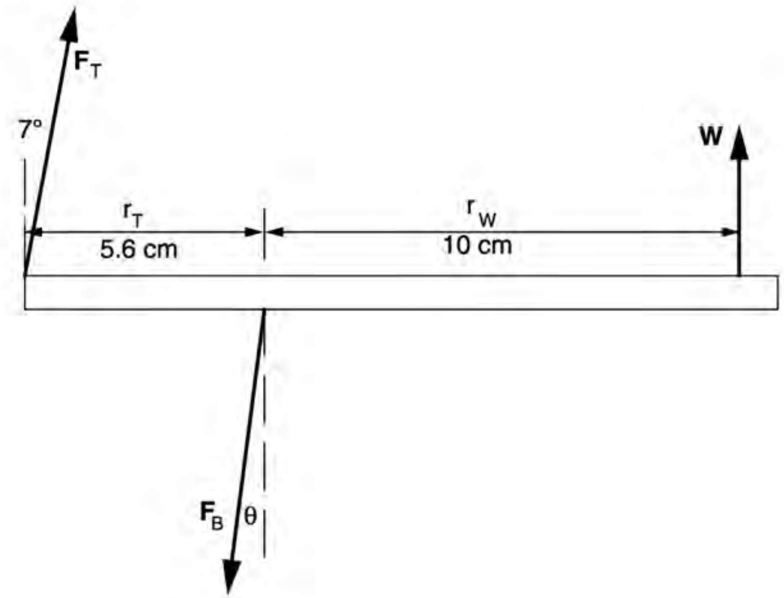
$$\tan\theta = \frac{0.22}{2.8} = 0.079,$$
$$\theta = 4.5^\circ.$$

Hobbie & Roth

<u>Biomechanical example</u>

Now consider standing on the "balls" of your feet. We can estimate the force associated with your Achille's tendon...



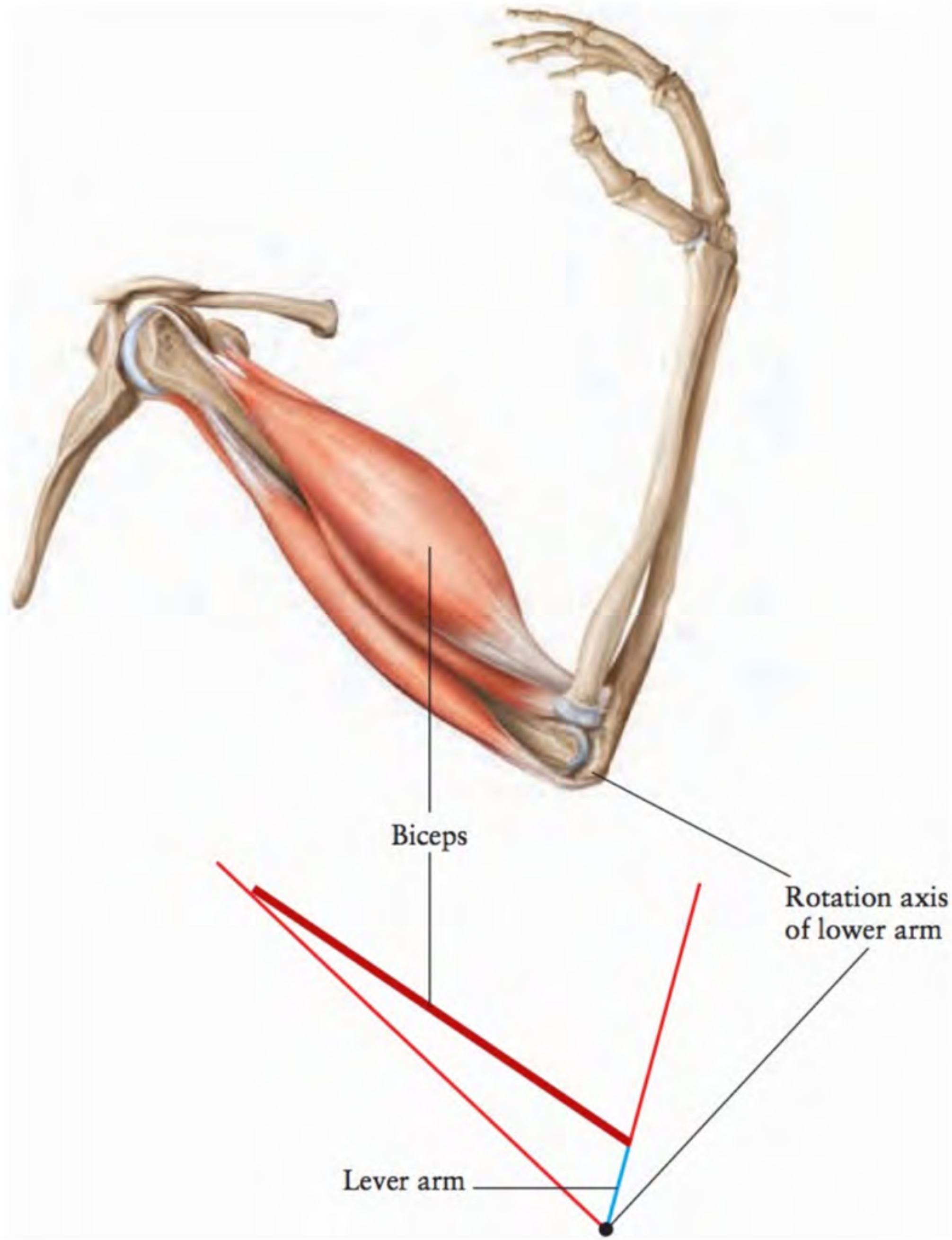


$$F_T = \frac{10W}{5.6 \cos 7^\circ} = 1.8W.$$

"The tension in the Achilles tendon is nearly twice the person's weight, while the force exerted on the leg by the talus is nearly three times the body weight. One can understand why the tendon might rupture."

Hobbie & Roth

Biomechanical considerations....



This sort of analysis can be extended to a wide variety of problems to get a 1st order estimate of things....

Kesten & Tauck

Reminder: Circular "kinematics"

- 1-D kinematics translates directly to circular motion (in polar coords.)

TABLE 4.1 Rotational and linear kinematics for constant acceleration

Rotational kinematics	Linear kinematics
$\omega_f = \omega_i + \alpha \Delta t$	$v_{fs} = v_{is} + a_s \Delta t$
$\theta_f = \theta_i + \omega_i \Delta t + \frac{1}{2}\alpha(\Delta t)^2$	$s_f = s_i + v_{is} \Delta t + \frac{1}{2} a_s(\Delta t)^2$
$\omega_f^2 = \omega_i^2 + 2\alpha \Delta\theta$	$v_{fs}{}^2 = v_{is}{}^2 + 2a_s \Delta s$

Knight (2013)

→ But wait, if the quantities on the right are all vectors, shouldn't the ones on the left also be too?

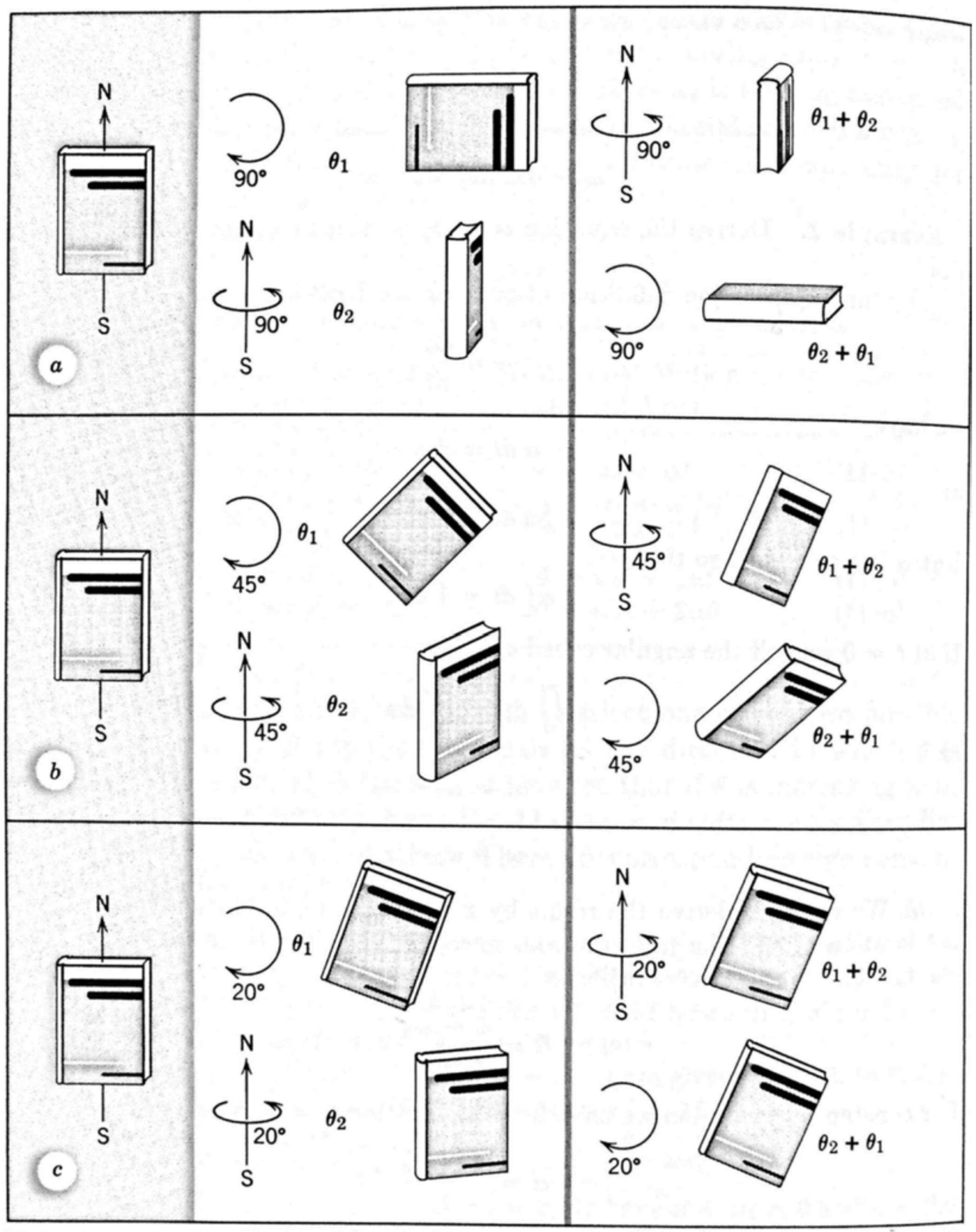


Fig. 11–6 (*a*) A book rotated θ_1 (90° as shown about a vertical axis) and then θ_2 (90° as shown about a north-south axis) has a different final orientation than if rotated first through θ_2 and then θ_1. This property is called the noncommutivity of finite angles under addition: $\theta_1 + \theta_2 \neq \theta_2 + \theta_1$. (*b*) The middle group is the same except that the angular displacements are smaller, being 45°. Although the final orientations still differ, they are much nearer each other. (*c*) The lower group repeats the experiment for 20° displacements. We see here that $\theta_1 + \theta_2 \cong \theta_2 + \theta_1$. As $\theta_1, \theta_2 \rightarrow 0$, the final positions approach each other. Finite angles under addition tend to commute as the angles become very small. Infinitesimal angles *do* commute under addition, making it possible to treat them as vectors.

Consider rotating a book 90^o in 3-D

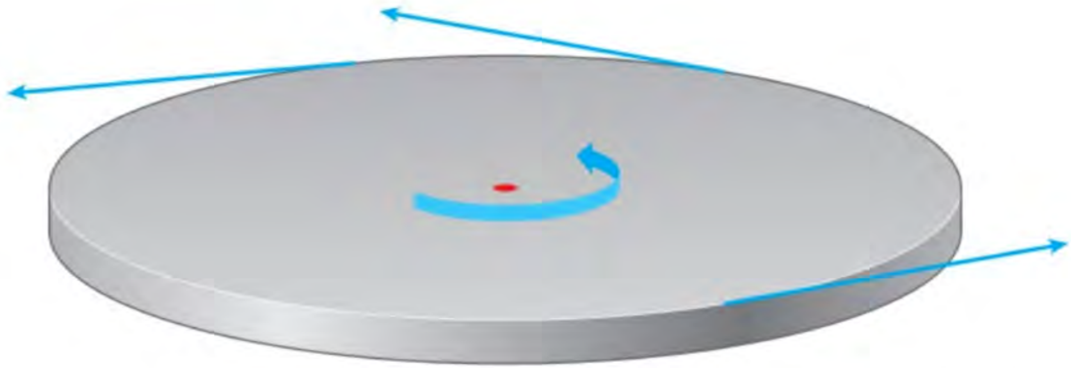
The order and direction you do it in matters re the final orientation of the book

→ Motivates that angular quantities need to be treated like vectors as well...

Resnick & Halliday

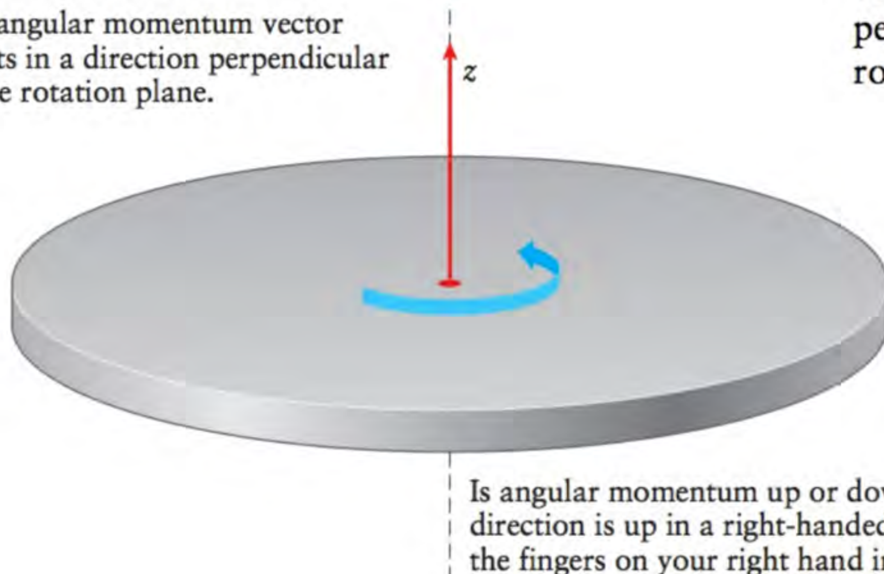
Vector nature of angular quantities

(a)



Which one vector represents the rotational motion? (None of them)

(b) The angular momentum vector points in a direction perpendicular to the rotation plane.

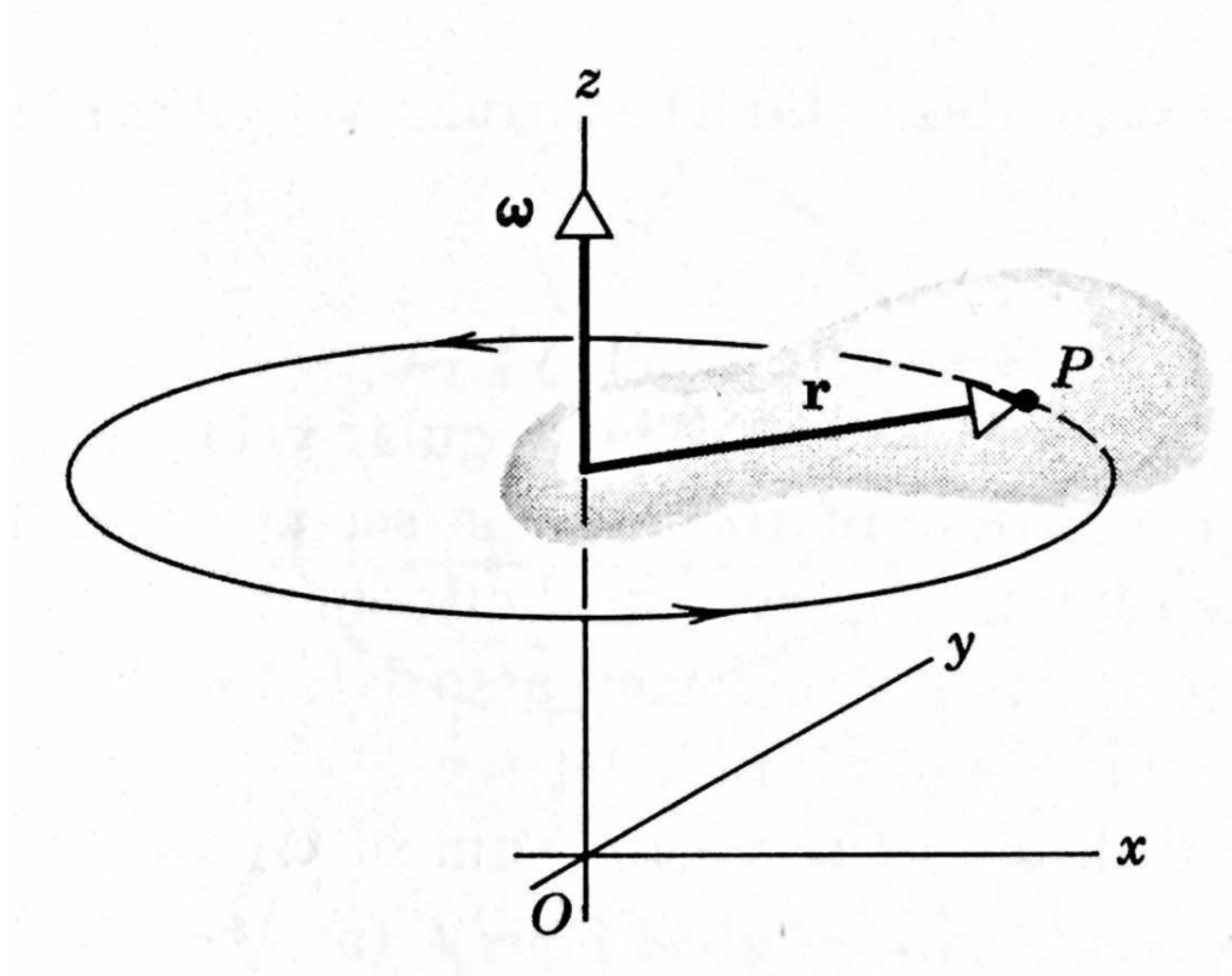


Is angular momentum up or down? The direction is up in a right-handed sense. Curl the fingers on your right hand in the direction of motion and stick your thumb straight out; your thumb points in the direction of the angular momentum vector.

Figure 8-35 (a) No single vector that lies in the plane of rotation indicates the direction of rotation. (b) The angular momentum vector $\vec{L}$ points in a direction perpendicular to the plane of rotation.

Vector nature of angular quantities

We choose a out-of-plane convention



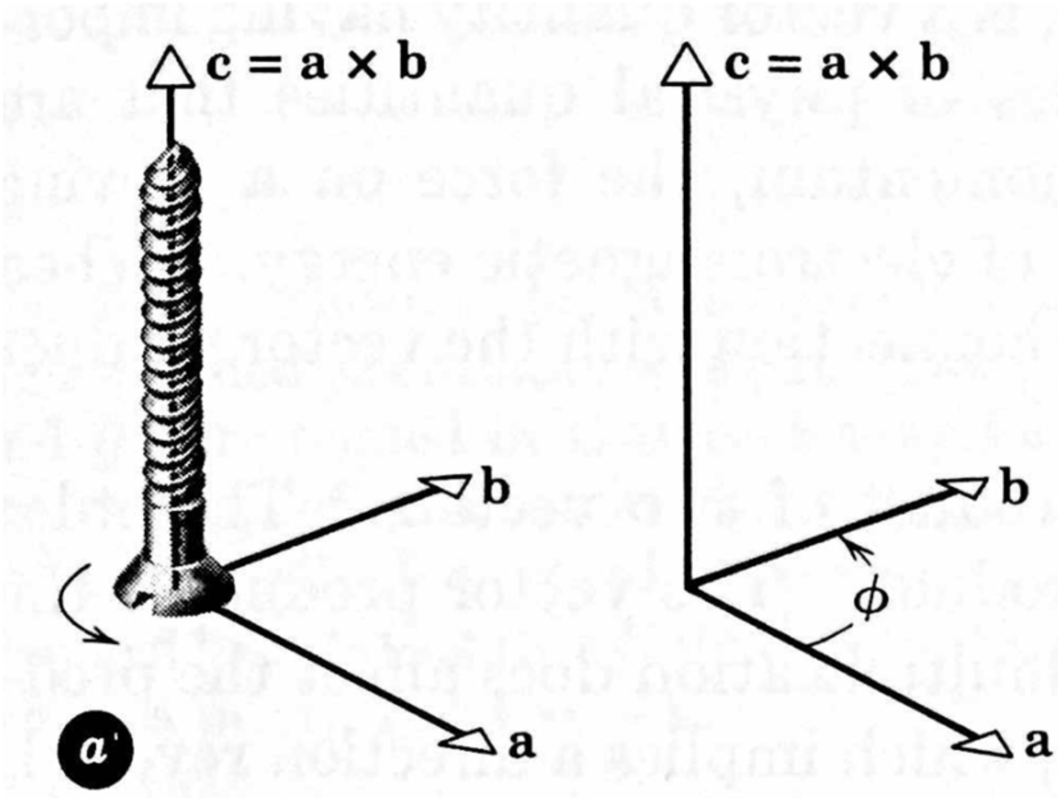
Which leads us back to the *cross product*

$$\vec{\tau} = \vec{r} \times \vec{F}$$

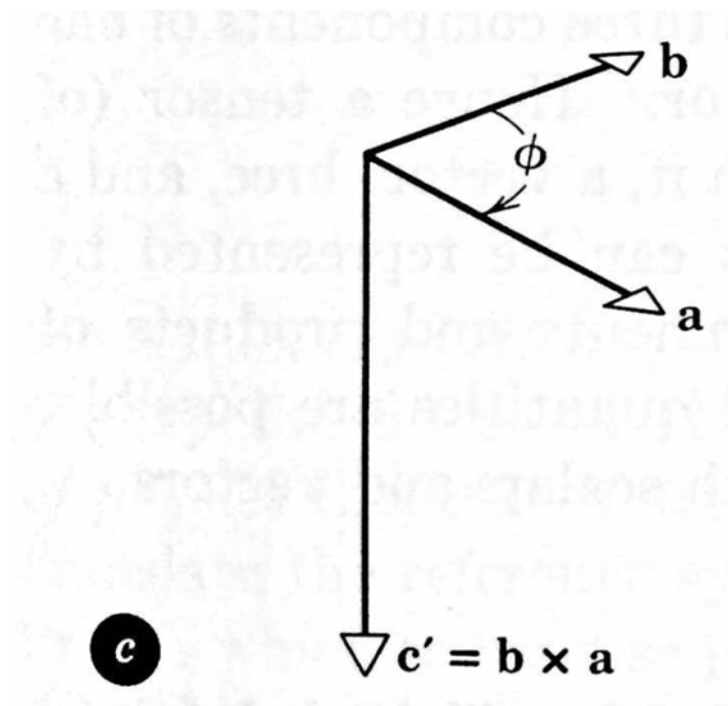
$$\tau = rF \sin\varphi$$

Resnick & Halliday

Cross Product



Order matters!

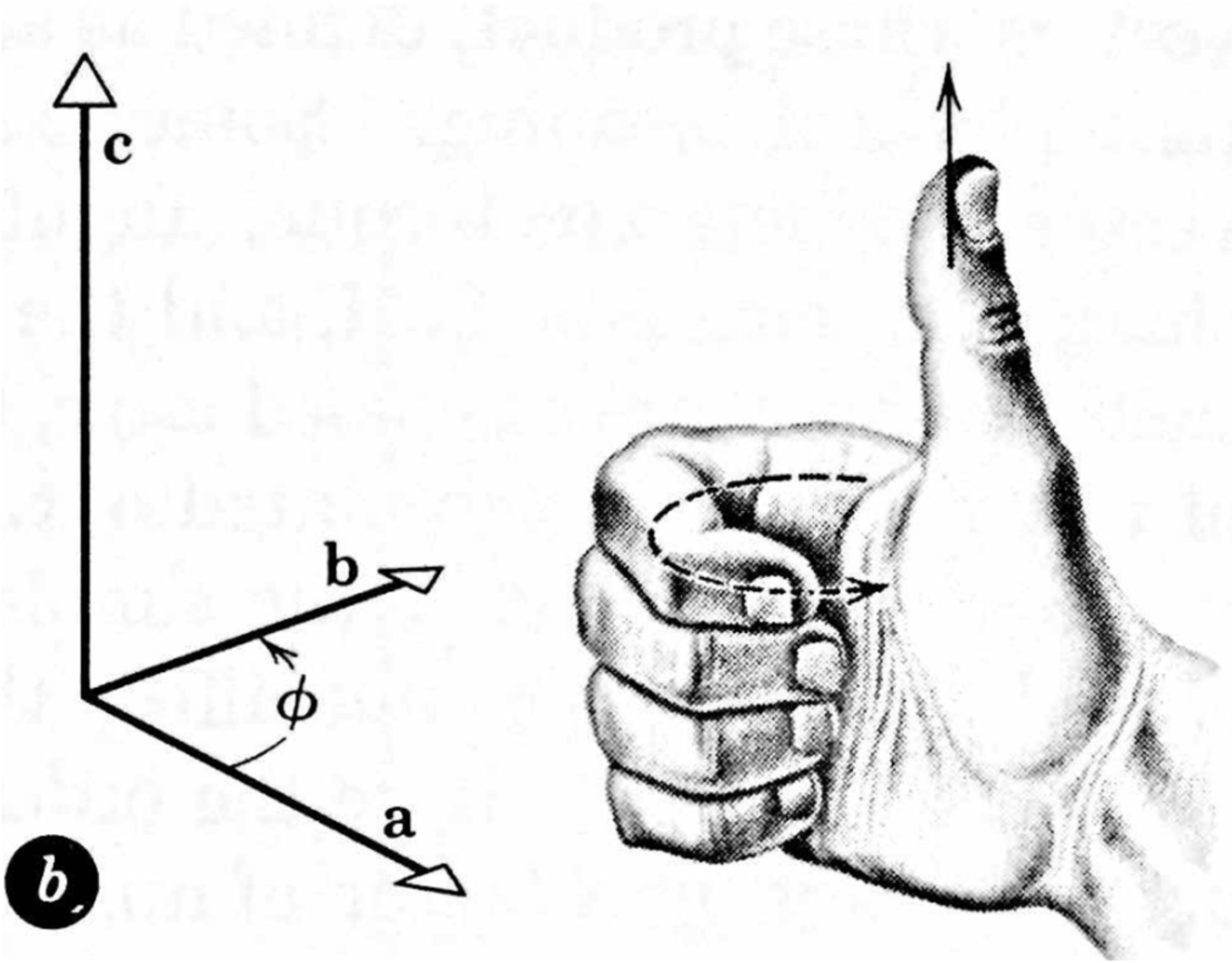


Ultimately, this is a convention....

Resnick & Halliday

Right-hand Rule (RHR)

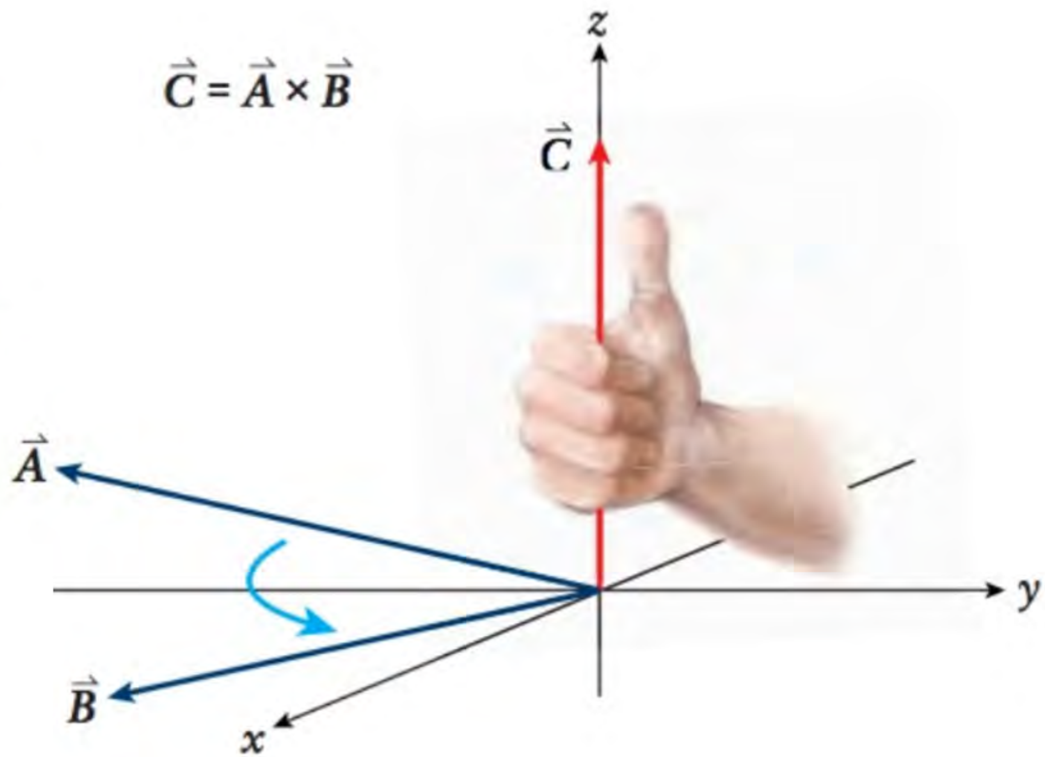
Key aspect here to get correct is that your fingers turn **a** towards **b** through the *smaller angler*



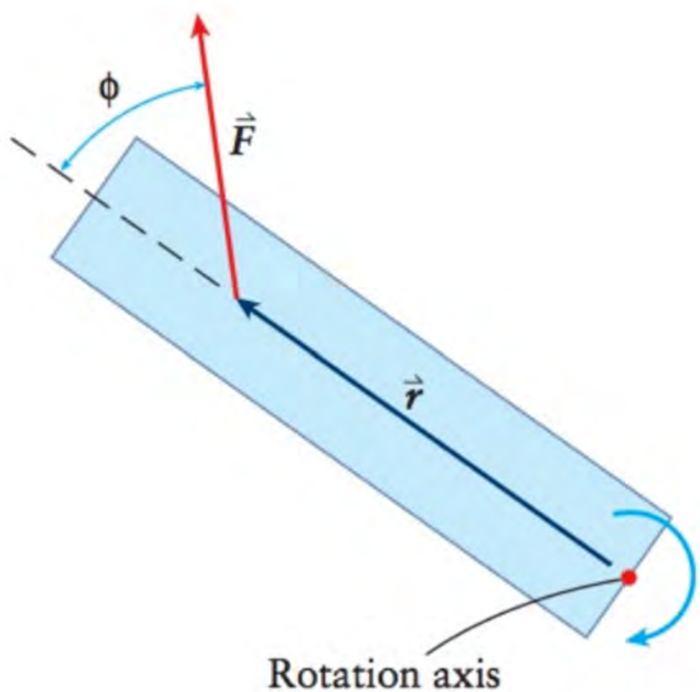
Resnick & Halliday

Right-hand Rule (RHR)

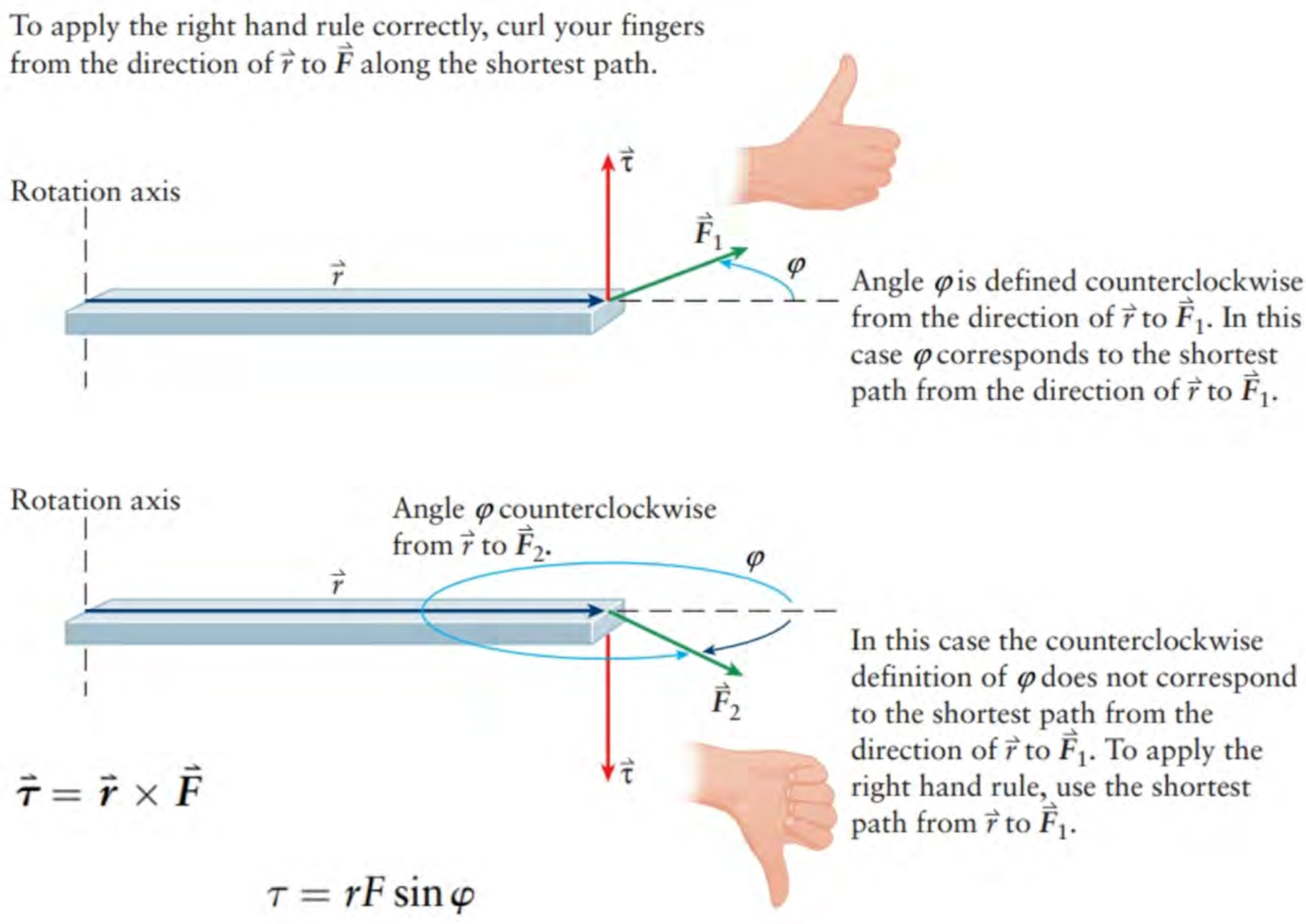
Right hand rule: Curl the fingers on your right hand from $\vec{A}$ to $\vec{B}$ along the closest path. Stick out your thumb; it points in the direction of $\vec{C}$, the result of the cross product $\vec{A} \times \vec{B}$.



$\vec{A}$ and $\vec{B}$ lie in the xy plane in this example.
$\vec{C}$ points in the positive z direction.



<u>Torque</u> (Revisited)



Kesten & Tauck

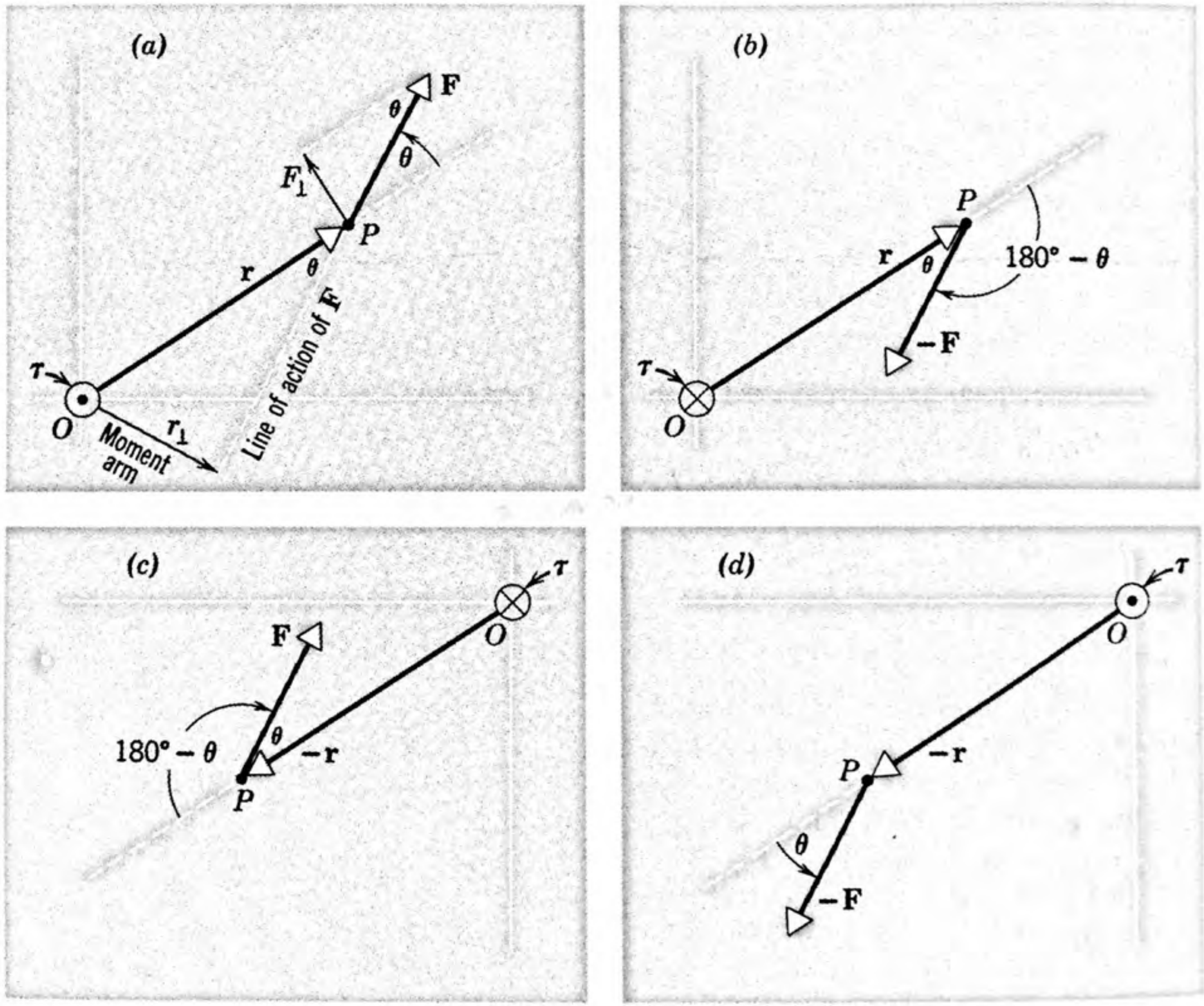


Fig. 12–2 The plane shown is that defined by **r** and **F** in Fig. 12–1. (*a*) The magnitude of **τ** is given by $Fr_\perp$ (Eq. 12–2*b*) or by $rF_\perp$ (Eq. 12–2*c*). (*b*) Reversing **F** reverses the direction of **τ**. (*c*) Reversing **r** reverses the direction of **τ**. (*d*) Reversing **F** and **r** leaves the direction of **τ** unchanged. The directions of **τ** are represented by ⊙ (perpendicularly out of the figure, the symbol representing the tip of an arrow) and by ⊗ (perpendicularly into the figure, the symbol representing the tail of an arrow).

Note the visual convention re "*tip of an arrow*" versus "*tail of an arrow*" when plotting in 2-D *(located at the point of rotation axis)*

Resnick & Halliday

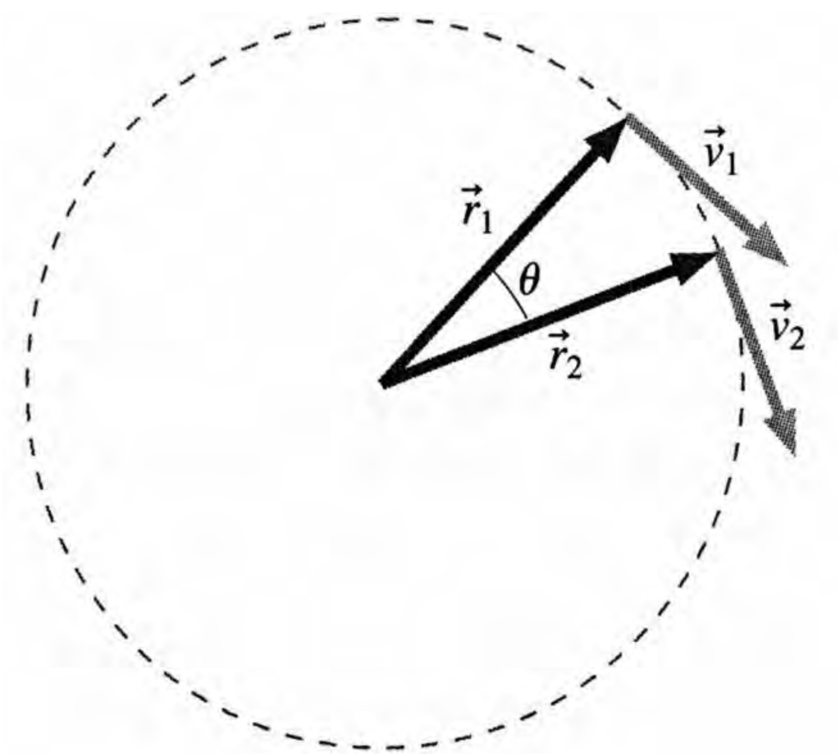
Wait a sec....

What about that centripetal acceleration we derived awhile back?!

$$a = \frac{v^2}{r} \qquad \text{(uniform circular motion)}$$

Wolfson

- v is the tangential velocity
- a is the radial (or centripetal) acceleration



Things are getting a tad complicated. So let's slow down for a moment (pun!) and take a step back to see how various quantities of interest (inter-)relate...

Knight (2013)

Interrelationships: Linear vs Angular Kinematics

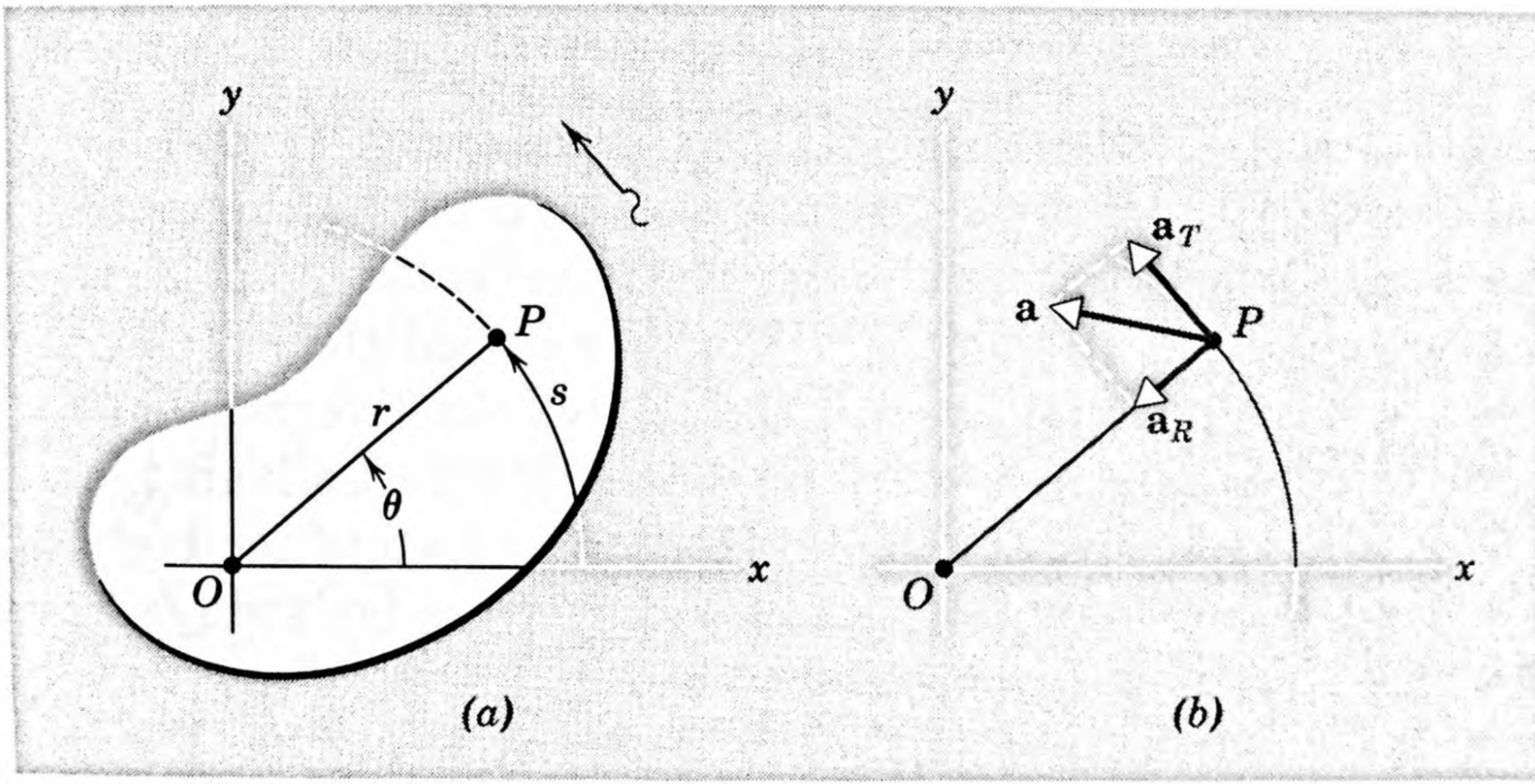


Fig. 11–9 (*a*) A rigid body rotates about a fixed axis through O perpendicular to the page. The point P sweeps out an arc s which subtends an angle θ. (*b*) The acceleration $\mathbf{a}$ of point P has components $\mathbf{a}_T$ (tangential) where $a_T = \alpha r$ and $\mathbf{a}_R$ (radial) where $a_R = v^2/r = \omega^2 r$ (ω = angular speed).

Resnick & Halliday

Interrelationships: Linear vs Angular Kinematics

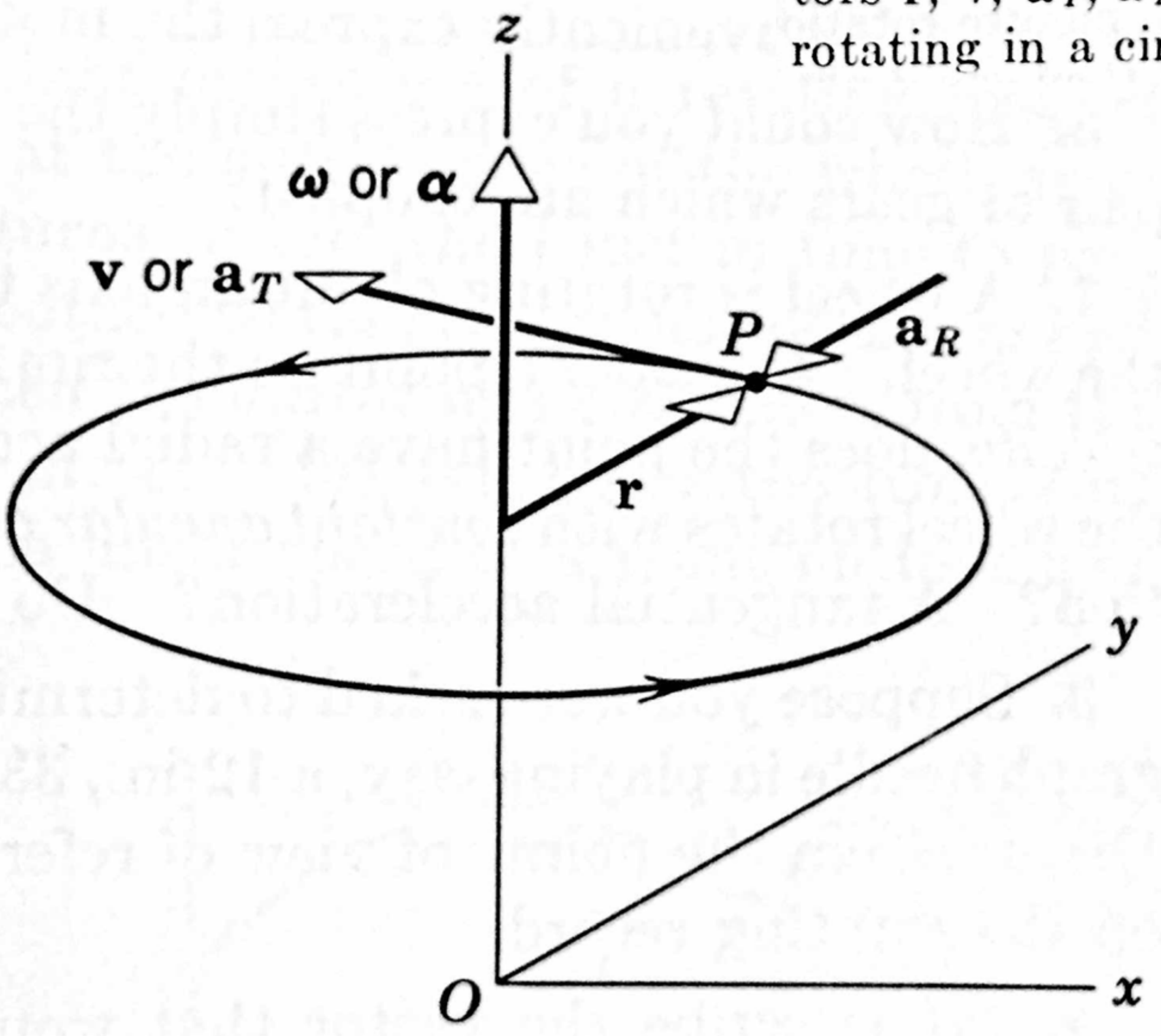
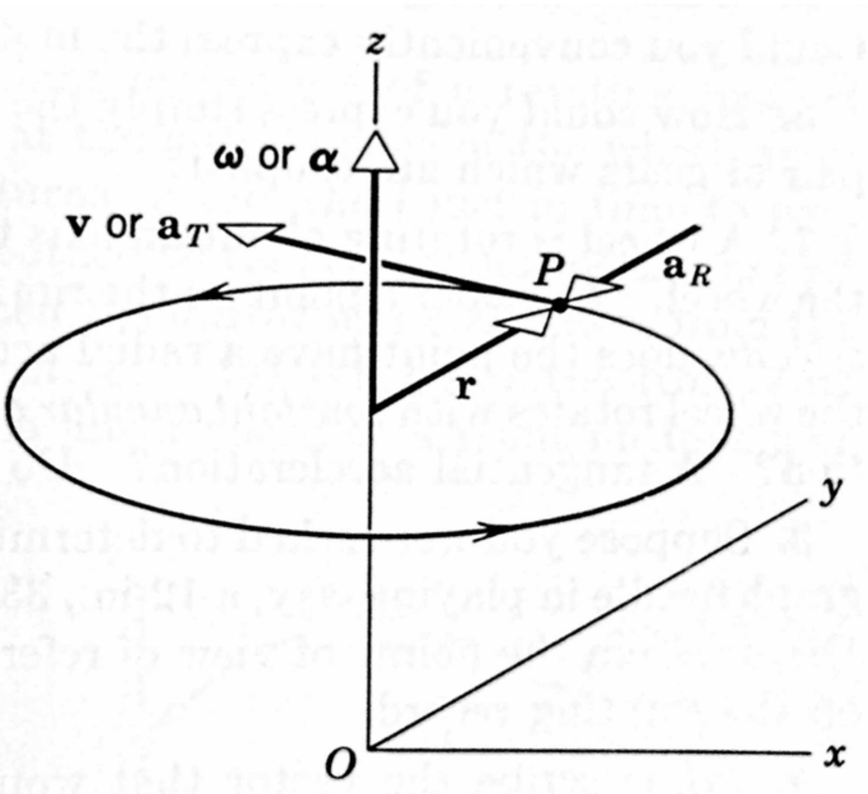


Fig. 11–11 The directions of the vectors **r**, **v**, $\mathbf{a}_T$, $\mathbf{a}_R$, **ω** and **α** for a particle rotating in a circle about the z-axis.

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

$$\mathbf{a} = \mathbf{a}_T + \mathbf{a}_R,$$

Interrelationships: Linear vs Angular Kinematics



$$\mathbf{a} = \mathbf{a}_T + \mathbf{a}_R,$$

$$\mathbf{a} = \mathbf{u}_\theta \alpha r - \mathbf{u}_r \omega^2 r.$$

$$a_T = \alpha r$$ Tangential accel.

$$a_R = \omega^2 r = v^2/r.$$ Radial accel.

$$\mathbf{a}_T = \boldsymbol{\alpha} \times \mathbf{r}$$

$$\mathbf{a}_R = \boldsymbol{\omega} \times \mathbf{v}$$

Note that a_R IS NOT α
(the angular accel. $\alpha = d\omega/dt$)

Resnick & Halliday

Comparison summary

TABLE 4.1 Rotational and linear kinematics for constant acceleration

Rotational kinematics	Linear kinematics
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$\theta_f = \theta_i + \omega_i \Delta t + \frac{1}{2}\alpha(\Delta t)^2$	$s_f = s_i + v_{is} \Delta t + \frac{1}{2}a_s(\Delta t)^2$
$\omega_f^2 = \omega_i^2 + 2\alpha \Delta\theta$	$v_{fs}^2 = v_{is}^2 + 2a_s \Delta s$

Rectilinear Motion		Rotation about a Fixed Axis	
Displacement	x	Angular displacement	θ
Velocity	$v = \frac{dx}{dt}$	Angular velocity	$\omega = \frac{d\theta}{dt}$
Acceleration	$a = \frac{dv}{dt}$	Angular acceleration	$\alpha = \frac{d\omega}{dt}$
Mass	M	Rotational inertia	I
Force	$F = Ma$	Torque	$\tau = I\alpha$
Work	$W = \int F\,dx$	Work	$W = \int \tau\,d\theta$
Kinetic energy	$\frac{1}{2}Mv^2$	Kinetic energy	$\frac{1}{2}I\omega^2$
Power	$P = Fv$	Power	$P = \tau\omega$
Linear momentum	Mv	Angular momentum	$I\omega$

→ Now we come to that last bit: *Angular momentum*

Resnick & Halliday

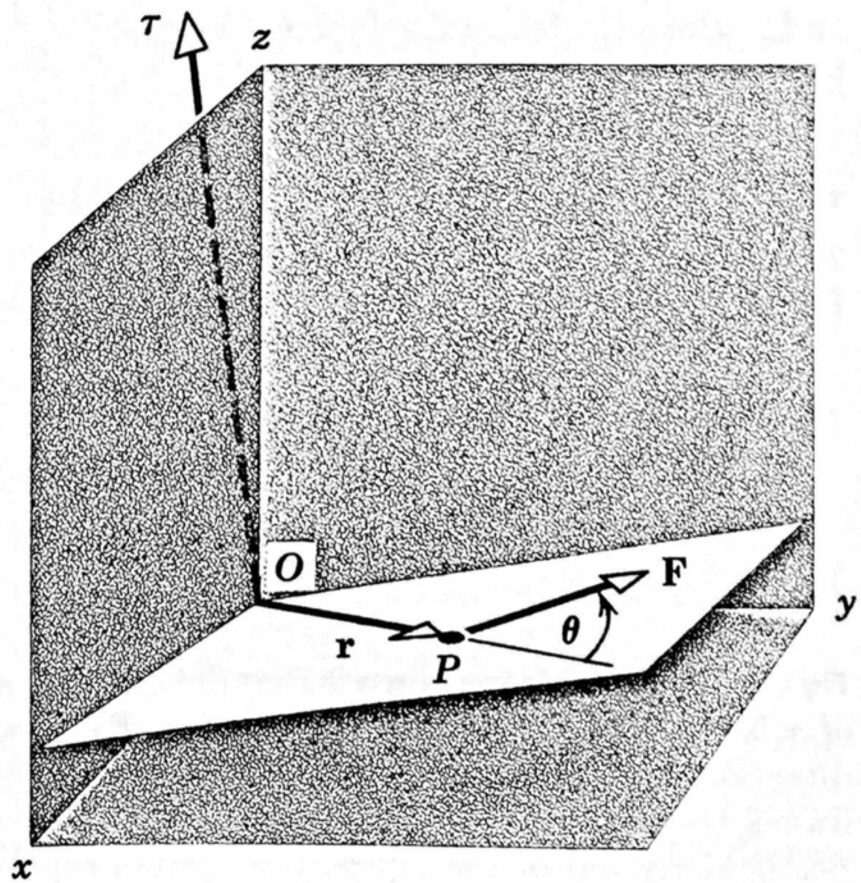
Rotational Dynamics

Torque = "Force analog"

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Now what if all the torques don't add up to zero?

$$\sum_i \tau_i = \sum_i r_i F_i = 0.$$



Think along the lines of rotational analog of Newton's 2nd

$$\vec{F}_{\text{net}} = m\vec{a}$$

Net force: the vector sum of all real, physical forces acting on an object

Product of object's mass and its acceleration; not a force.

Equal sign indicates that the two sides are mathematically equal — but that doesn't mean they're the same physically. Only $\vec{F}_{\text{net}}$ involves physical forces.

Fig. 12–1 A force **F** is applied to a particle P, displaced **r** relative to the origin. The force vector makes an angle θ with the radius vector **r**. The torque **τ** about O is shown. Its direction is perpendicular to the plane formed by **r** and **F** with the sense given by the right-hand rule.

Resnick & Halliday