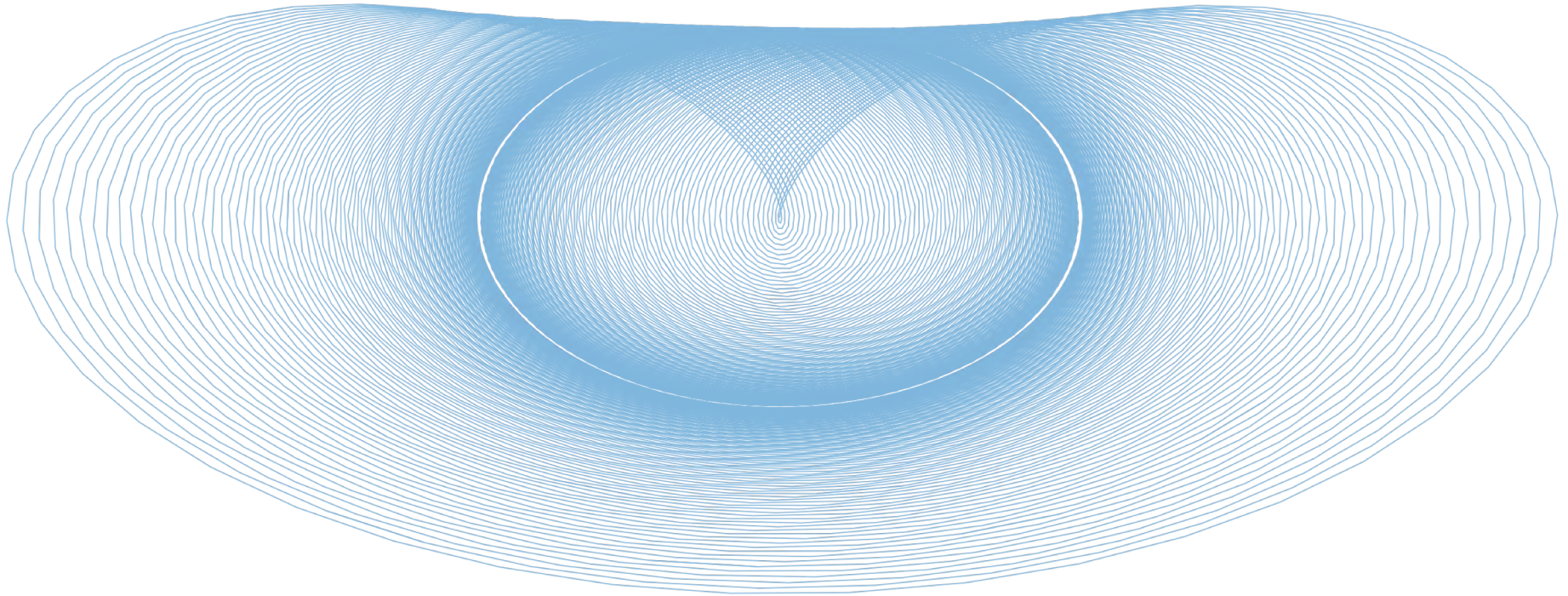


PHYS 1420 (F19)

Physics with Applications to Life Sciences



2019.11.29

Relevant reading:

Kesten & Tauck ch.13.5-13.6

Christopher Bergevin

York University, Dept. of Physics & Astronomy

Office: Petrie 240 Lab: Farq 103

cberge@yorku.ca

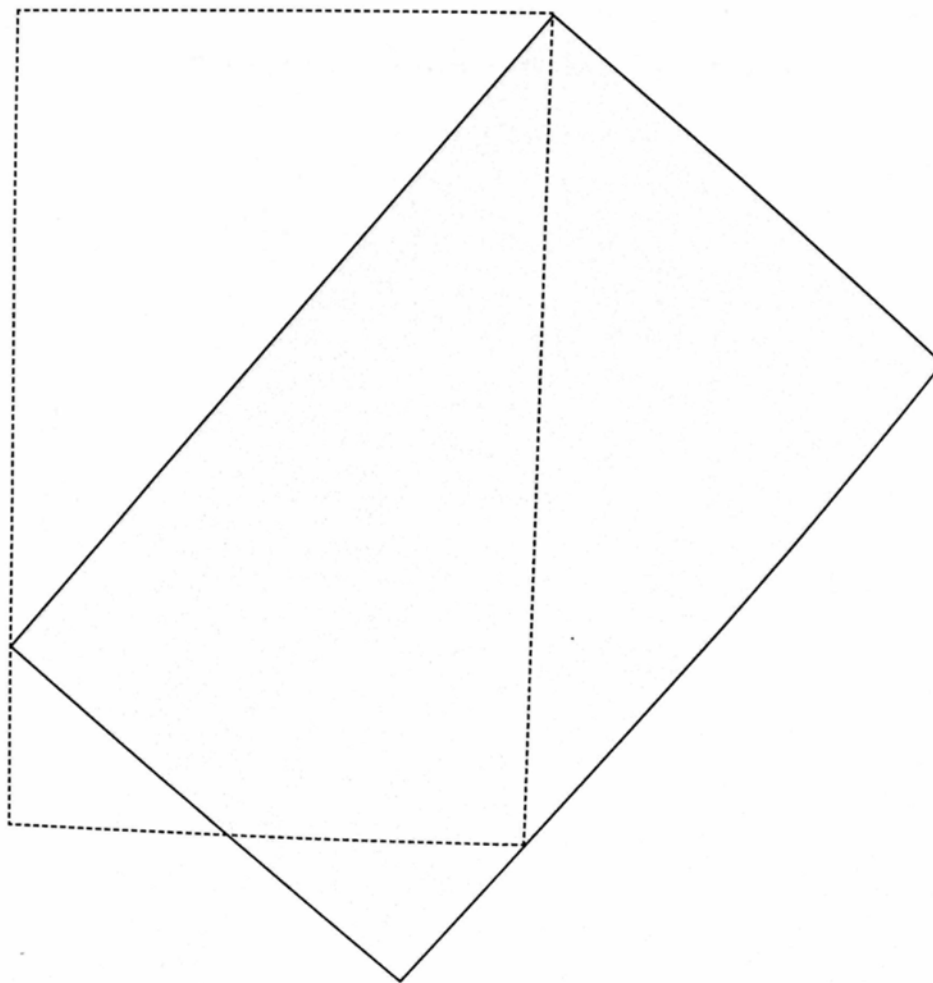
Ref. (re images):

Wolfson (2007), Knight (2017)

Two Sheets of Paper

Two identical rectangular sheets of paper are lying one on top of the other exactly as shown in the diagram.

Which of the two parts of the sheet of paper underneath is bigger, the part that is uncovered, or the part that is covered?



Announcements & Key Concepts (re Today)

- Online HW #9: Posted and due 12/6
- Lectures end Monday 12/3 (→ no tutorial on 12/4)
- Review sessions re final exam to be announced soon

Some relevant underlying concepts of the day...

- How to describe waves
- Wave "math" → Multivariable functions revisited
- PDEs → the "wave equation"
- St

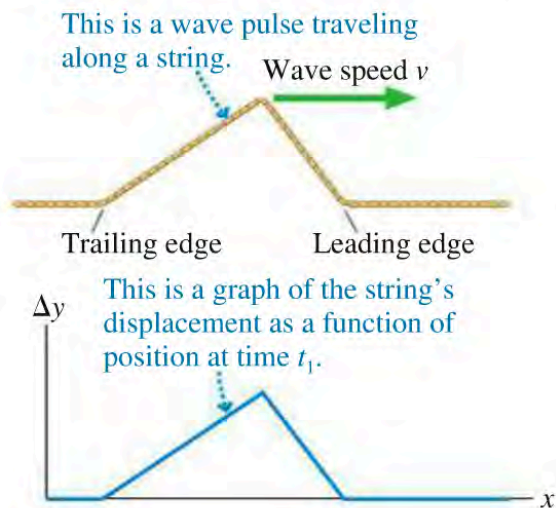
Types of waves

A **transverse wave** is a wave in which the displacement is *perpendicular* to the direction in which the wave travels. For example, a wave travels along a string in a horizontal direction while the particles that make up the string oscillate vertically. Electromagnetic waves are also transverse waves because the electromagnetic fields oscillate perpendicular to the direction in which the wave travels.

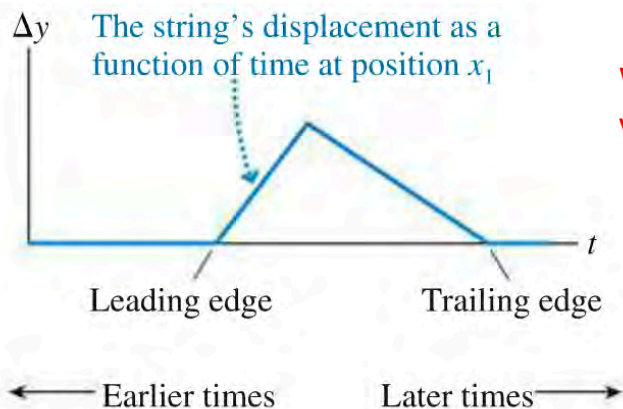
In a **longitudinal wave**, the particles in the medium move *parallel* to the direction in which the wave travels. Here we see a chain of masses connected by springs. If you give the first mass in the chain a sharp push, a disturbance travels down the chain by compressing and expanding the springs. Sound waves in gases and liquids are the most well known examples of longitudinal waves.

How to describe a wave?

A snapshot graph of a wave pulse on a string.



Wave height versus position

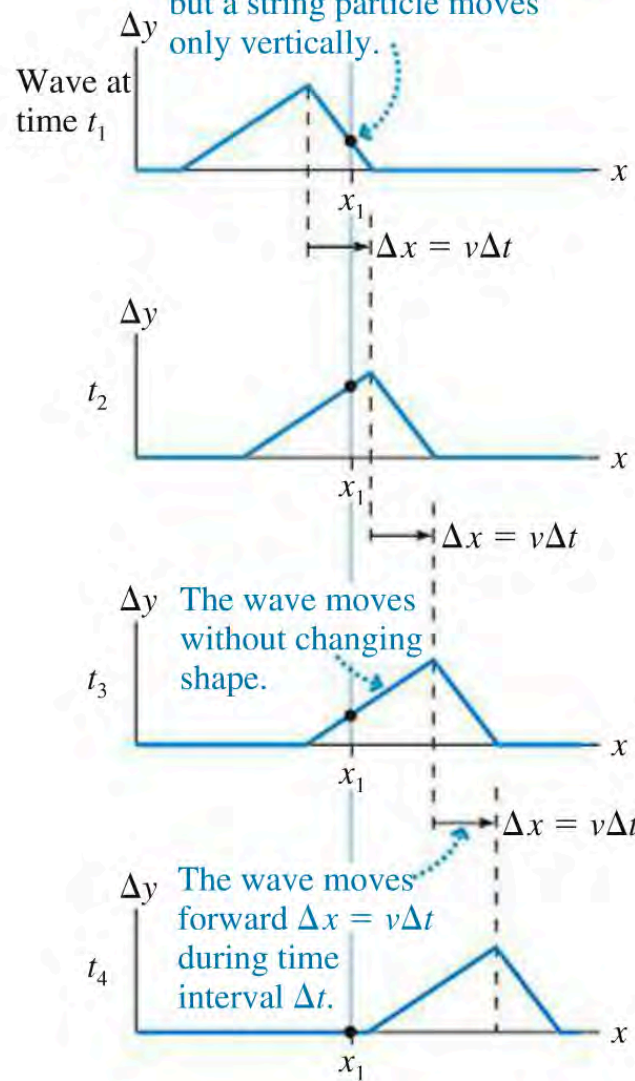


Wave height versus time

Note: Wave shape appears "flipped" between the two...

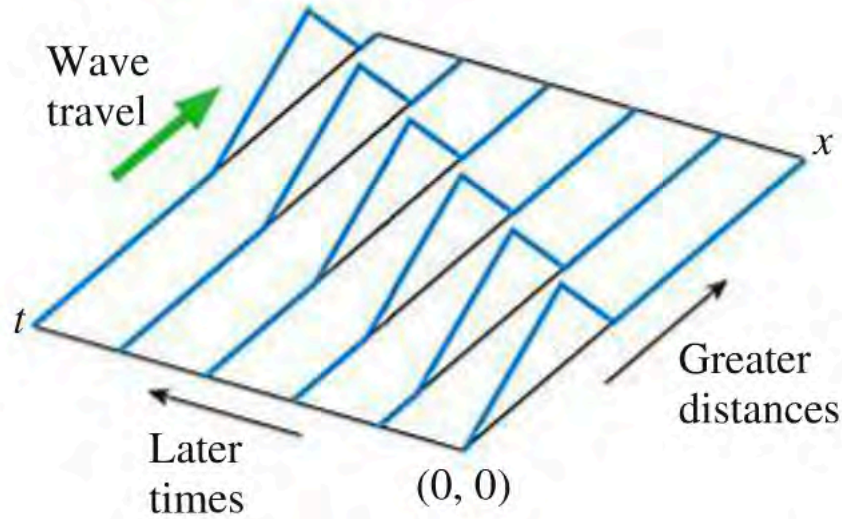
A sequence of snapshot graphs shows the wave in motion.

The wave moves horizontally, but a string particle moves only vertically.

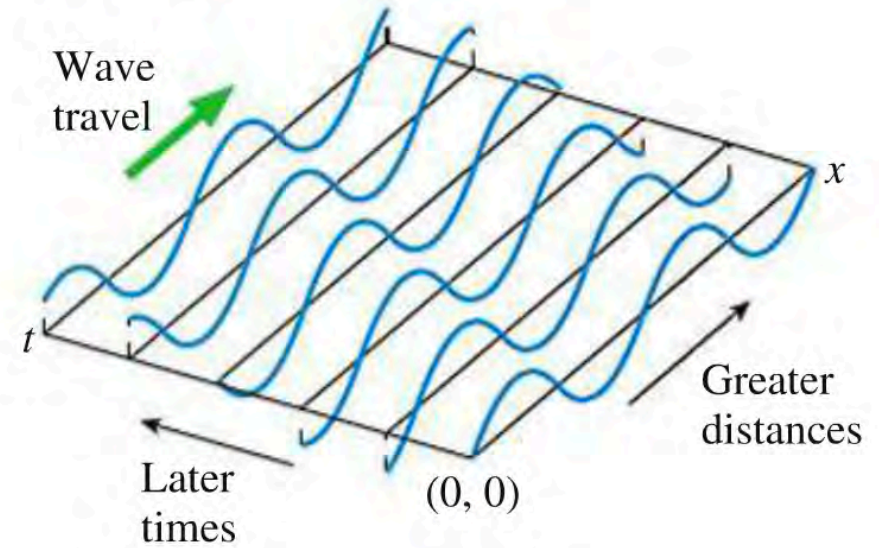


How to describe a wave?

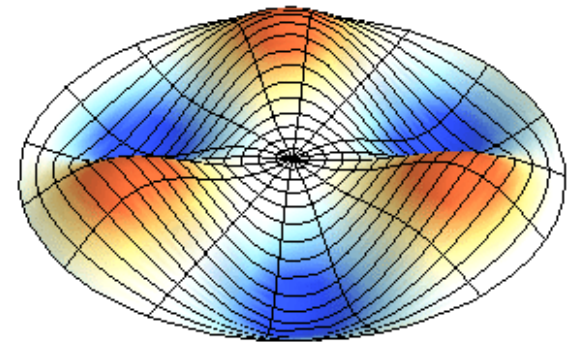
An alternative look at a traveling wave.



A sinusoidal wave moving along the x -axis.



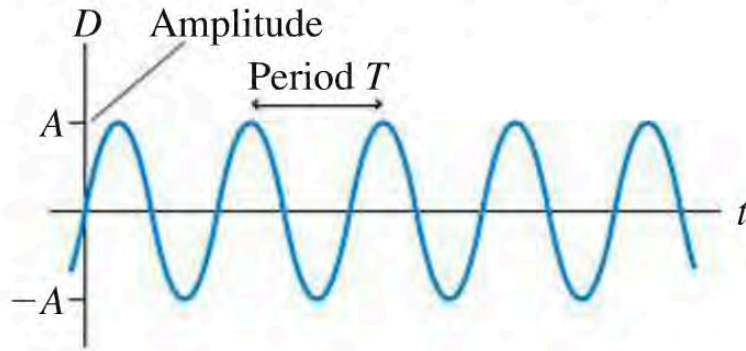
→ Clearly, the displacement of a wave (standing or not) *depends upon both space and time*



“Wave math” → **Multivariable functions**

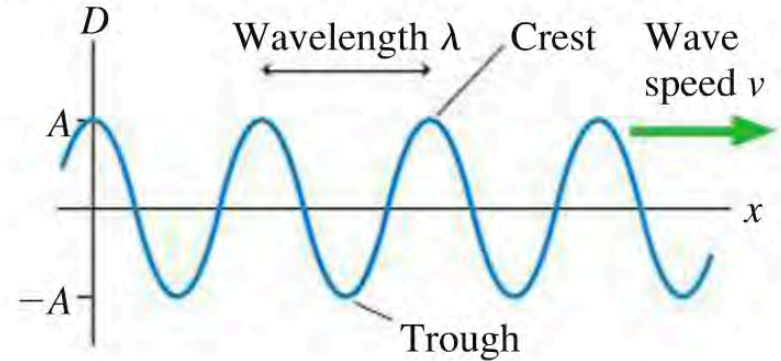
History and snapshot graphs for a sinusoidal wave.

(a) A history graph at one point in space



Time snapshot

(b) A snapshot graph at one instant of time



Space snapshot

➤ Can take “snapshots”, either in time or space

→ We need some additional mathematical tools to deal w/ this new reality....

“Wave math” → Multivariable functions

- Multivariate functions be important for the various types of systems you will see throughout science

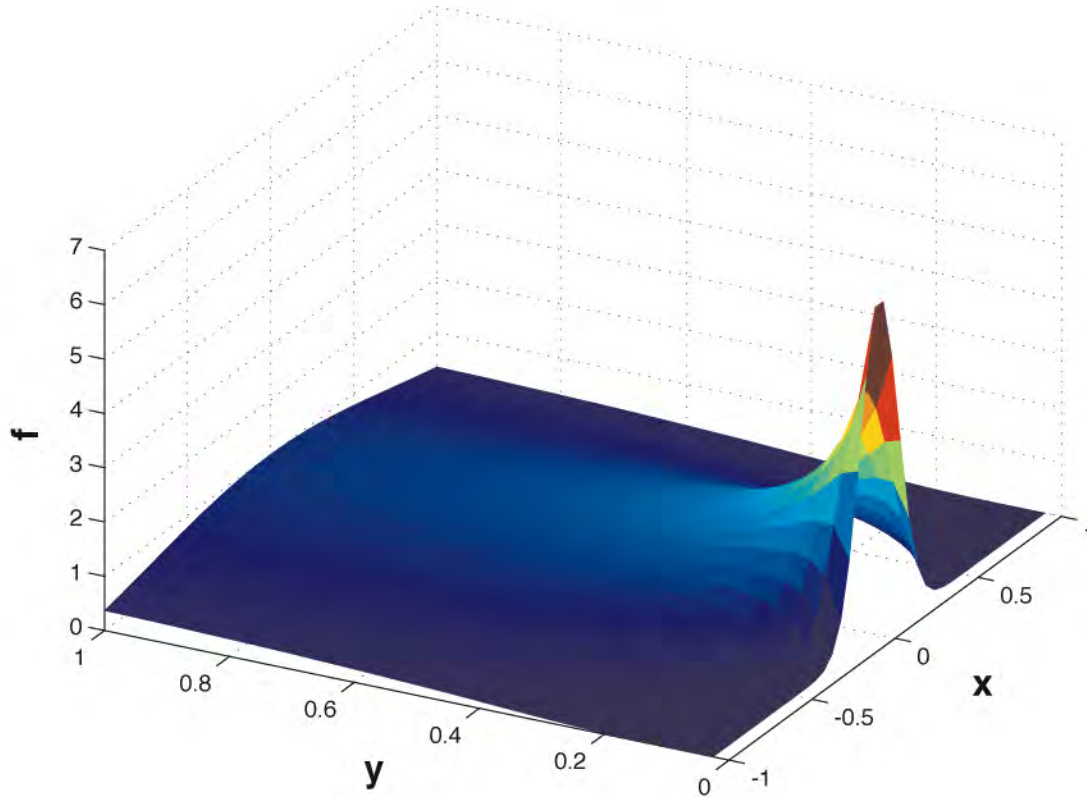
e.g. concentration of a solute in a solution (c) depends upon both spatial location (x) and time (t)

$$f = f(x, y)$$

f - dependent variable

x, y - independent variables

“Wave math” → Multivariable functions



$$f(x, y) = \frac{1}{\sqrt{y}} e^{-x^2/y}$$

Solution to
diffusion equation
(or *heat eqn.*)

Aside → Differentiating multivariable functions

Derivative (definition) for a function of a single variable

$$\frac{dg(x)}{dx} = \lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}$$

For multi-variable function, keep one variable constant

$$\frac{\partial f(x, y)}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

→ “Partial derivative”

note difference in notation

y is effectively held constant here

Aside → Differentiating multivariable functions

- Can take partial derivative with respect to partial derivative

$$\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right)$$

- Simplified notation:

$$\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} (f_x) = (f_x)_y = f_{xy}$$

Aside: Examples

Ex.1 $\frac{\partial z}{\partial x}$ for $z(x, y) = (xy)^{1/2}$

$$\frac{\partial z}{\partial x} = \frac{1}{2} y^{1/2} x^{-1/2}$$

Ex.2 $\frac{\partial}{\partial b} \left(\frac{\partial \phi}{\partial a} \right)$ for $\phi(a, b) = ab^2 + 3a^2e^b$

Notation: $\frac{\partial}{\partial b} \left(\frac{\partial \phi}{\partial a} \right) = \frac{\partial}{\partial b} (\phi_a) = (\phi_a)_b = \phi_{ab}$

First find ϕ_a (assume b is constant): $\phi_a = b^2 + 6ae^b$

$$\frac{\partial}{\partial b} \left(\frac{\partial \phi}{\partial a} \right) = 2b + 6ae^b$$

Now differentiate with respect to b : $\phi_{ab} = \frac{\partial}{\partial b} (\phi_a) = 2b + 6ae^b$

Note: In this case $\phi_{ab} = \phi_{ba}$ which is true for continuous functions (i.e. order of differentiation doesn't matter)

“Wave math” → Partial differential equations

ODE (‘ordinary’): considers a function of one variable and how it changes with respect to that variable

ex. $\frac{dc}{dt} = kc$ where $c = c(t)$ and $k = \text{const.}$

PDE (‘partial’): considers a function of more than one variable and how its various partial derivatives are related

ex. $\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2}$ where $c = c(x, t)$ and $D = \text{const.}$

“Wave math” → The wave equation

- A wave’s dependence upon space and time are interrelated via a PDE commonly referred to as the *wave equation*

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

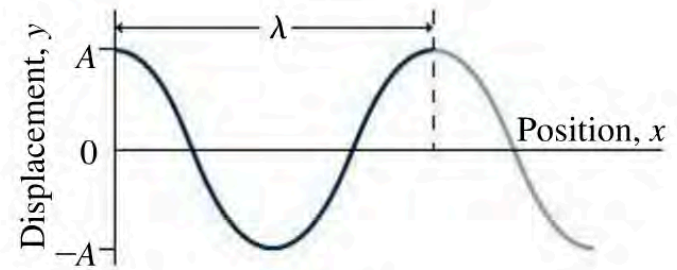
Note: One can readily derive this via combining Newton’s 2nd Law and conservation of mass

$$y(x, t) = A \cos(kx \pm \omega t)$$

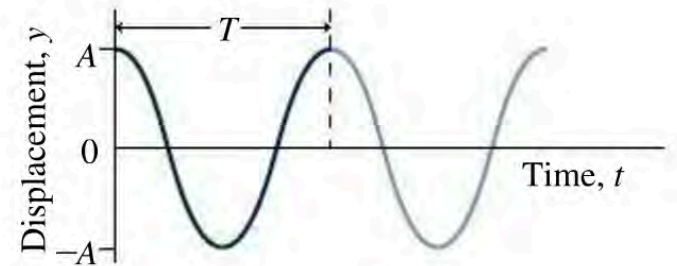
Possible solution to wave eqn. (“sinusoidal wave”)

$$k = \frac{2\pi}{\lambda} \quad (\text{wave number})$$

$$v = \frac{\lambda}{T} = \frac{2\pi/k}{2\pi/\omega} = \frac{\omega}{k}$$



(a)



(b)

“Wave math” → Multivariable functions (REVISITED)

- Let's consider a sinusoidal wave traveling to the right

$$D(x, t = 0) = A \sin\left(2\pi \frac{x}{\lambda} + \phi_0\right)$$

Eventually the wave repeats itself:

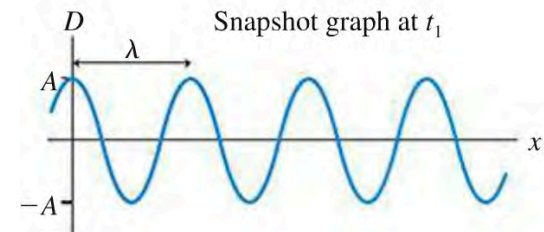
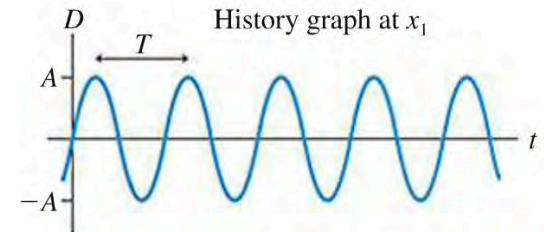
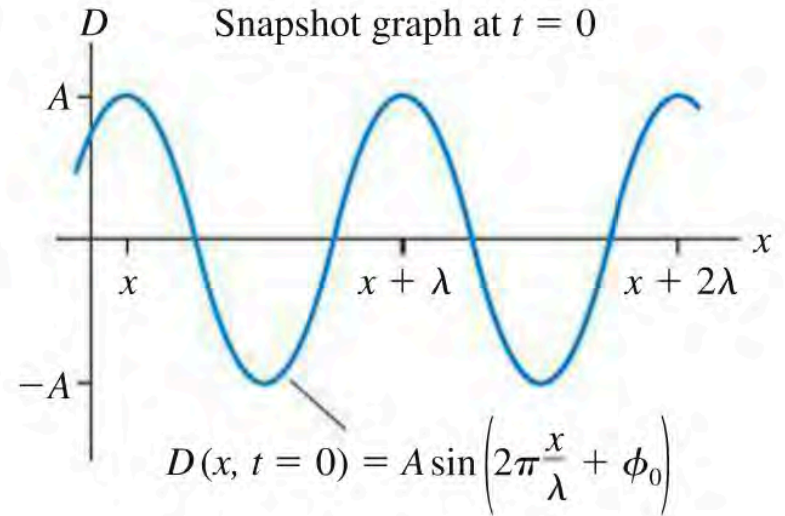
$$\begin{aligned} D(x + \lambda) &= A \sin\left(2\pi \frac{(x + \lambda)}{\lambda} + \phi_0\right) = A \sin\left(2\pi \frac{x}{\lambda} + \phi_0 + 2\pi \text{ rad}\right) \\ &= A \sin\left(2\pi \frac{x}{\lambda} + \phi_0\right) = D(x) \end{aligned}$$

Note:

$$\sin(a + 2\pi \text{ rad}) = \sin a.$$

But the wave is “in motion”, so we can rewrite as:

$$D(x, t) = D(x - vt, t = 0)$$



“Wave math” → Multivariable functions (REVISITED)

$$D(x, t) = D(x - vt, t = 0)$$

So we rewrite as:

$$D(x, t) = A \sin\left(2\pi \frac{x - vt}{\lambda} + \phi_0\right) = A \sin\left(2\pi \left(\frac{x}{\lambda} - \frac{t}{T}\right) + \phi_0\right)$$

$$D(x, t) = A \sin(kx - \omega t + \phi_0)$$

(sinusoidal wave traveling in the positive x -direction)

Relevant derived quantities:

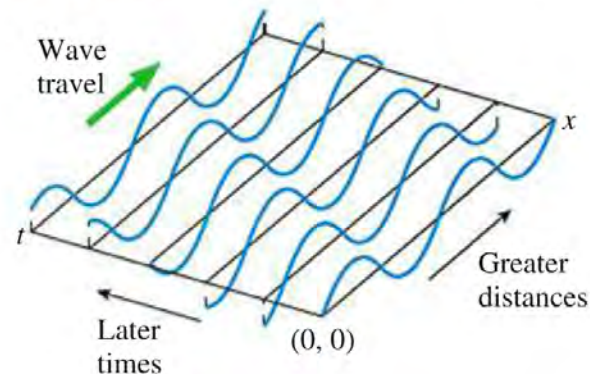
$$v = \lambda f = \lambda/T$$

$$\omega = 2\pi f = \frac{2\pi}{T}$$

$$k = \frac{2\pi}{\lambda}$$

$$v = \lambda f = \frac{2\pi}{k} \frac{\omega}{2\pi} = \frac{\omega}{k}$$

A sinusoidal wave moving along the x -axis.



“Wave math” → Wave speed

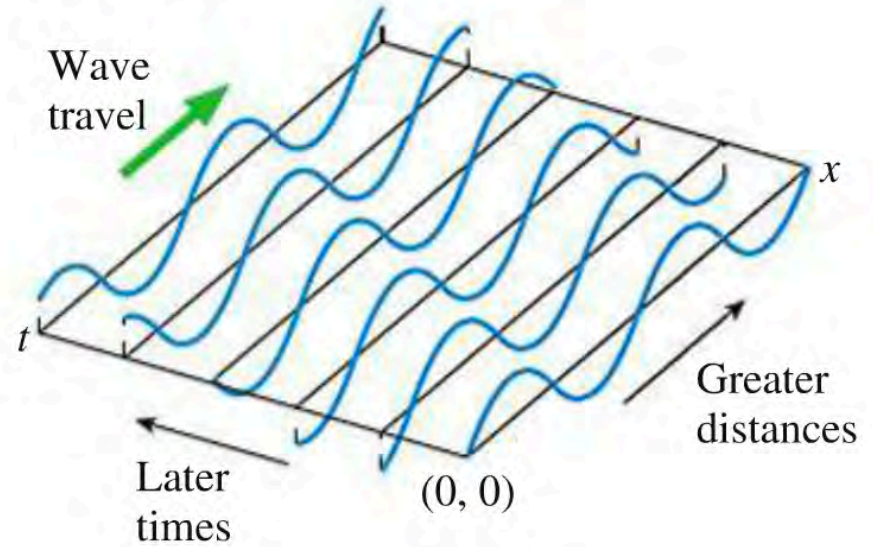
- For sinusoidal waves (w/ period T and wavelength λ), there is a straightforward means to determine the wave's velocity

$$v = \frac{\text{distance}}{\text{time}} = \frac{\lambda}{T}$$

$$v = \lambda f$$

$$\lambda = \frac{v}{f} = \frac{\text{property of the medium}}{\text{property of the source}}$$

A sinusoidal wave moving along the x -axis.



Ex. Try talking after sucking on a helium balloon....

Standing waves

- Consider that in 1-D, there can be two waves on a string: one going *forward* and one going *backward*

The superposition of two sinusoidal waves traveling in opposite directions.

(a)

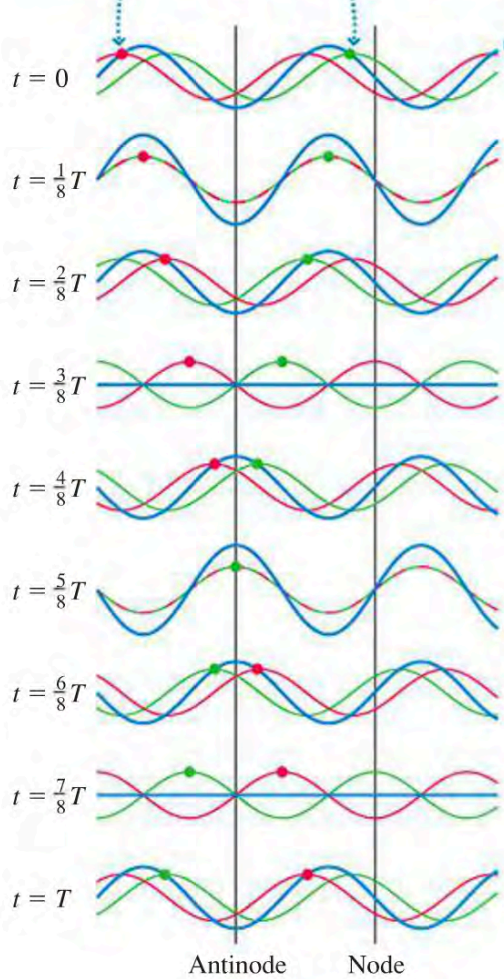
A string is carrying two waves moving in opposite directions.



- Their combination leads to interference (or *superposition*)
- Sometimes the waves interfere (i.e., add up) *constructively*, other times it is *destructively*

Standing waves

(b) The red wave is traveling to the right. The green wave is traveling to the left.



The blue wave is the superposition of the red and green waves.

At this time the waves exactly overlap and the superposition has a maximum amplitude.

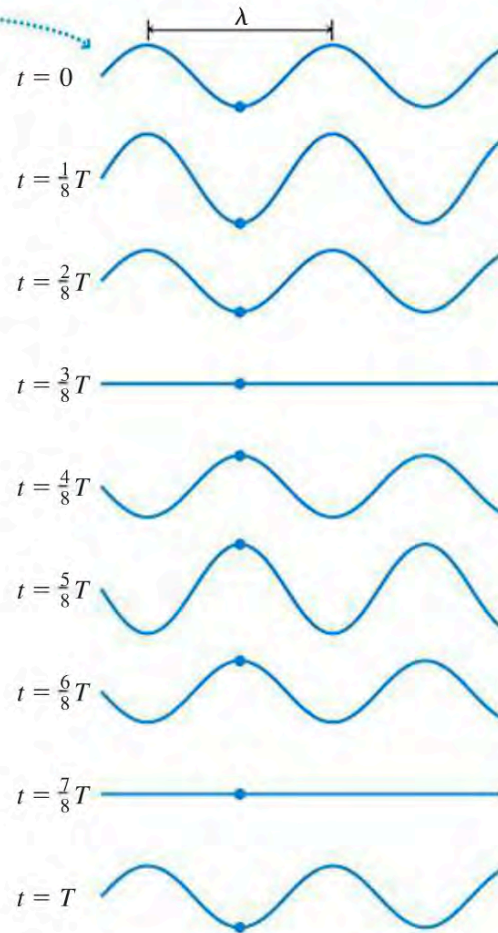
At this time a crest of the red wave meets a trough of the green wave. The waves cancel.

The superposition again reaches a maximum amplitude.

The waves again overlap and cancel.

At this time the superposition has the form it had at $t = 0$.

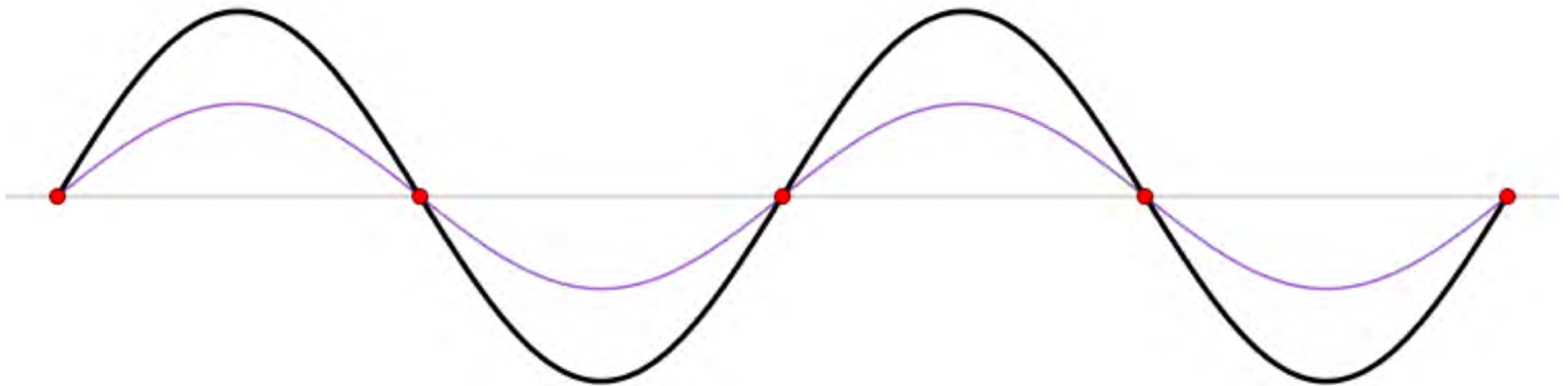
(c) The superposition is a standing wave with the same wavelength as the original waves.



→ A bit hard to see via a static picture....

Standing waves

... but is much more readily apparent via a movie



Blue is the left-going wave

Red is the right-going wave

Black is the sum of the two (i.e., the “standing” wave)

Note: Locations where the amplitude stays zero are called **nodes**

“Wave math” → Standing waves (REVISITED)

$$D_R = a \sin(kx - \omega t) \quad \text{Right-going wave}$$

Note: The difference here is the sign. For “bonus” credit, look up **d’Alembert’s formula**

$$D_L = a \sin(kx + \omega t) \quad \text{Left-going wave}$$

Via superposition:
$$D(x, t) = D_R + D_L = a \sin(kx - \omega t) + a \sin(kx + \omega t)$$

Relevant trig identity:
$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

Rewriting:

$$\begin{aligned} D(x, t) &= a(\sin kx \cos \omega t - \cos kx \sin \omega t) + a(\sin kx \cos \omega t + \cos kx \sin \omega t) \\ &= (2a \sin kx) \cos \omega t \end{aligned}$$

Note: A standing wave is not a traveling wave per se(!)

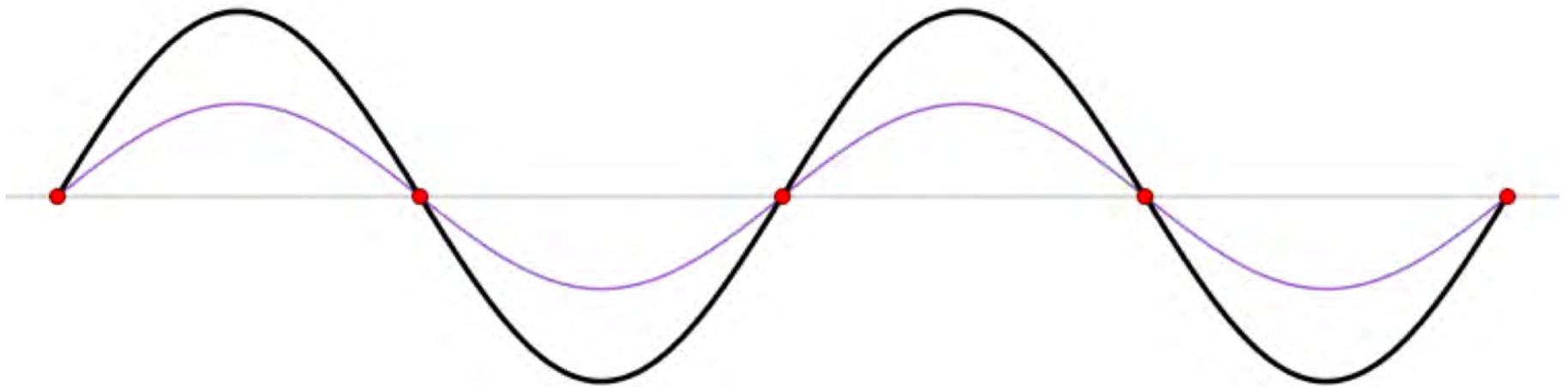
$$D(x, t) = A(x) \cos \omega t$$

$$A(x) = 2a \sin kx$$

Standing Waves (REVISITED)

$$D(x, t) = A(x) \cos \omega t$$

$$A(x) = 2a \sin kx$$



Blue is the left-going wave

Red is the right-going wave

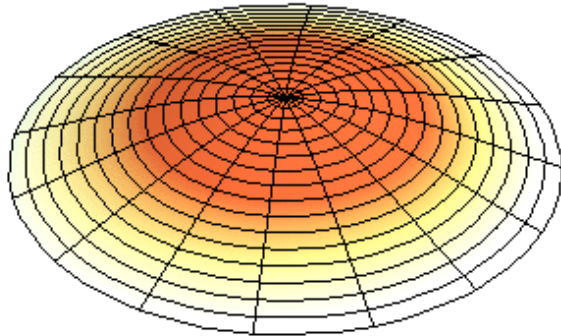
Black is the sum of the two (i.e., the “standing” wave)

Note: Locations where the amplitude stays zero are called **nodes**

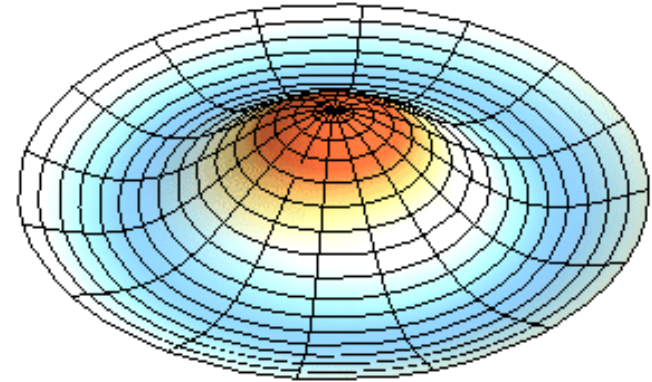
Standing waves

- Standing waves can arise in 2-D as well (e.g., drumhead)

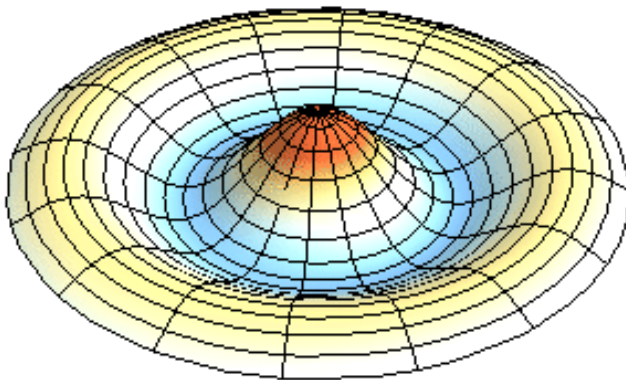
(0,1) mode



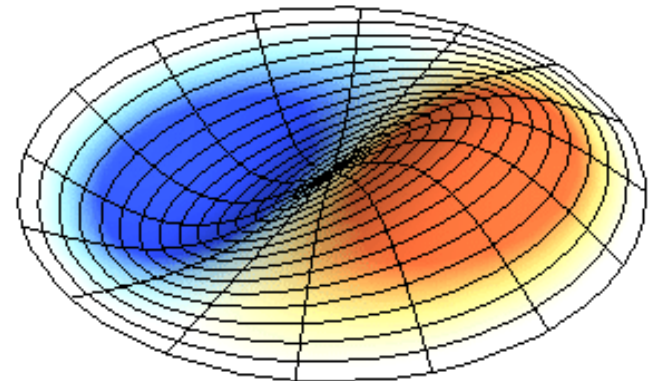
(0,2) mode



(0,3) mode

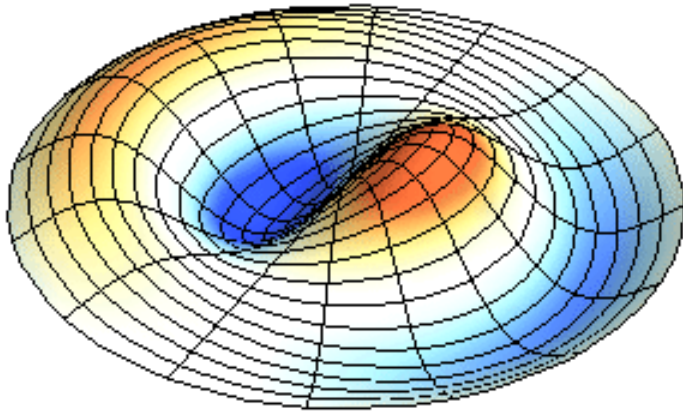


(1,1) mode

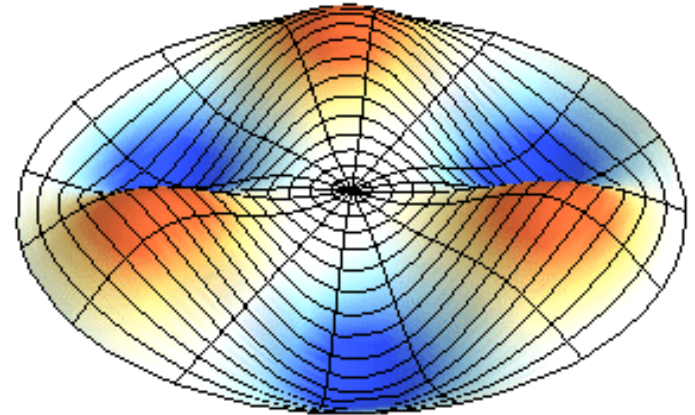


Standing waves

(1,2) mode



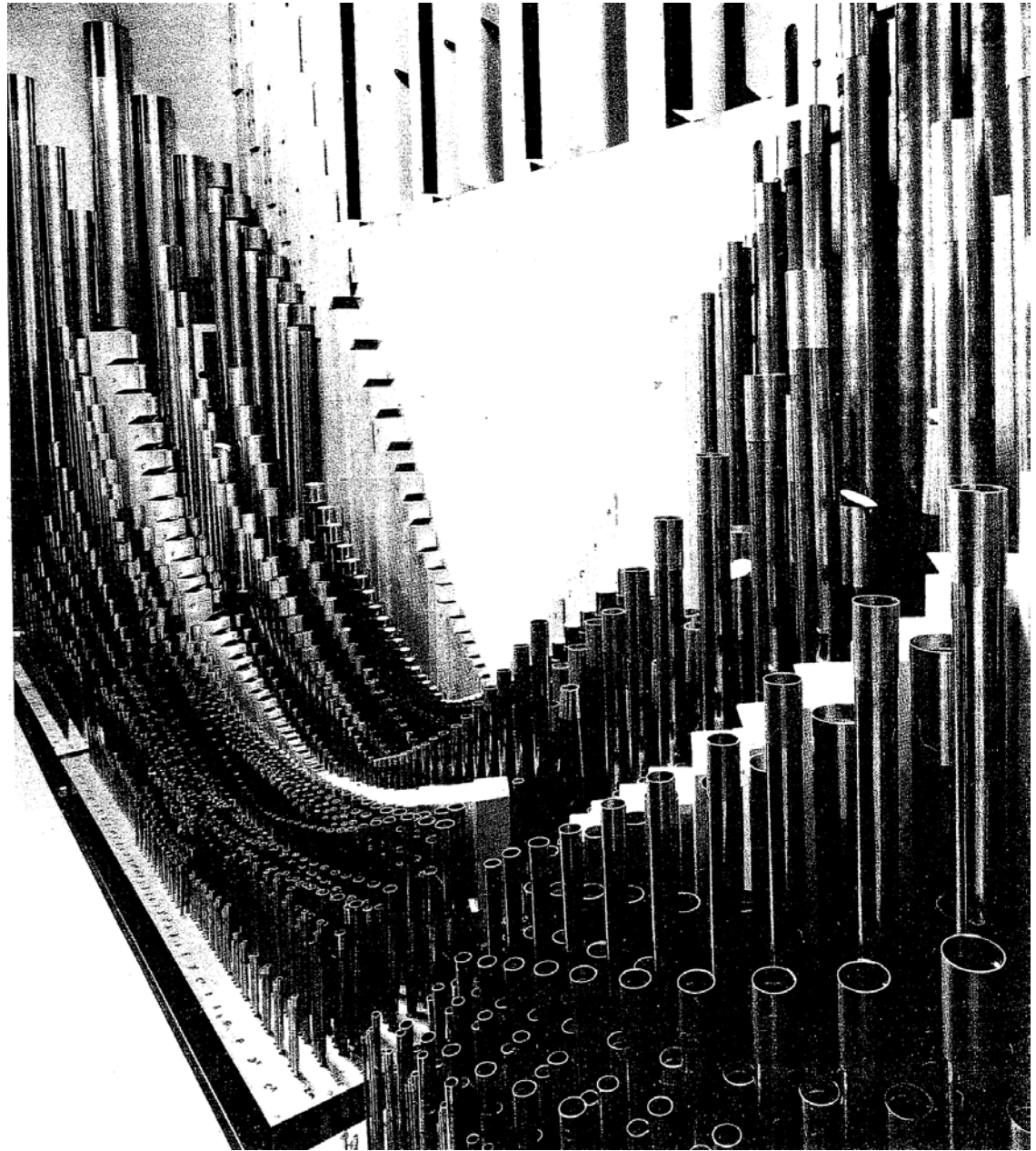
(3,1) mode



→ Note clear presence of nodes (chief characteristic of standing waves)

Standing waves

INTERIOR VIEW OF ORGAN at the Sydney Opera House shows some of its 26 ranks of pipes, most of which are of metal but some of which are of wood. The length of the speaking part of each pipe doubles at every 12th pipe; the pipe diameter doubles at about every 16th pipe. Through long experience master organ builders arrived at the proportions necessary for achieving balanced tone quality.



The Physics of Organ Pipes

The majestic sound of a pipe organ is created by the carefully phased interaction of a jet of air blowing across the mouth of each pipe and the column of air resonating inside the pipe

by Neville H. Fletcher and Suzanne Thwaites

Scientific American (1983)

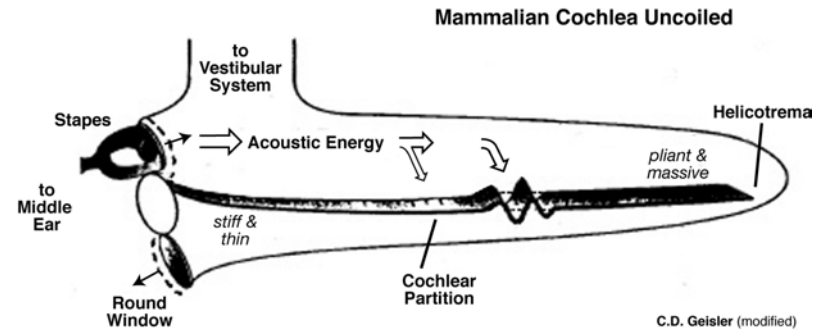
Aside: Cochlear standing waves

Mammalian spontaneous otoacoustic emissions are amplitude-stabilized cochlear standing waves

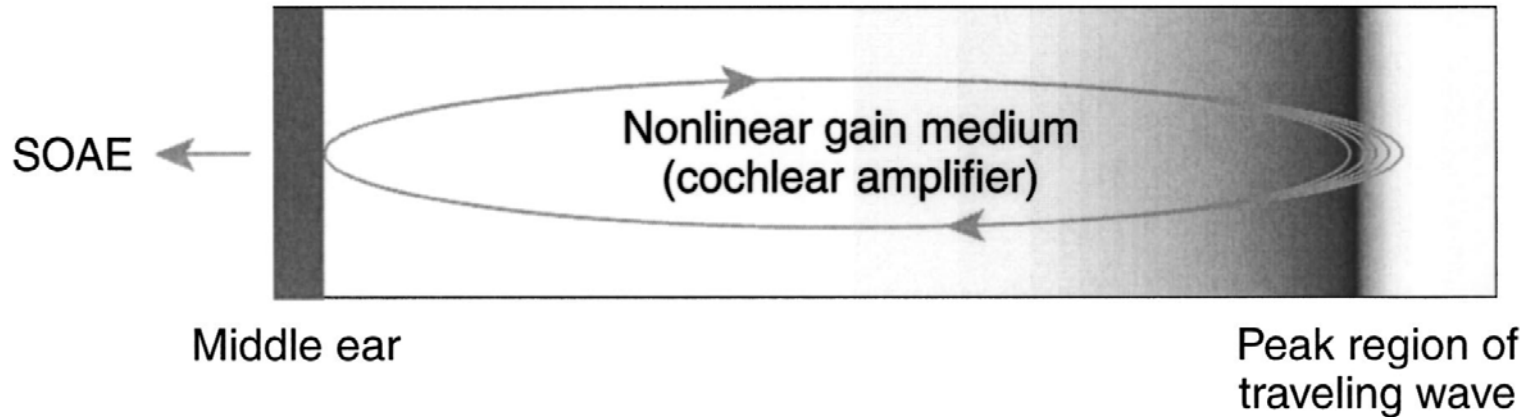
Christopher A. Sera^{a)}

Eaton-Peabody Laboratory of Auditory Physiology, Massachusetts Eye and Ear Infirmary,
243 Charles Street, Boston, Massachusetts 02114 and Department of Otology and Laryngology,
Harvard Medical School, Boston, Massachusetts 02115

J. Acoust. Soc. Am. 114 (1), July 2003



Resonant cavity



“Taken together, the results imply [...] the cochlea acting as a biological, hydromechanical analog of a laser oscillator.”

→ Inner ear acts like a laser!

Further aside:

$$R[1 + RR_{\text{stapes}} + (RR_{\text{stapes}})^2 + \dots] = R \sum_{n=0}^{\infty} (RR_{\text{stapes}})^n$$

→ Reflections back and forth can be described via a *geometric series* (something you'll see soon re integral calculus!)

Looking ahead: Light as a wave....

➤ EM waves are a bit special in that they are not entirely consistent w/ our definition of a wave....

➤ We will need to develop further mathematical tools and physical concepts (e.g., electric fields, magnetism) to properly understand, classically at least, EM waves

Maxwell's equations

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{in}}}{\epsilon_0}$$

Gauss's law

$$\oint \vec{B} \cdot d\vec{A} = 0$$

Gauss's law for magnetism

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_m}{dt}$$

Faraday's law

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{through}} + \epsilon_0 \mu_0 \frac{d\Phi_e}{dt}$$

Ampère-Maxwell law

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \quad (\text{Lorentz force law})$$

Review: A traveling wave is a broad term, but in a general sense can be defined as occurring when a “condition of some kind is transmitted from one place to another by means of a medium, but the medium itself is not transported”

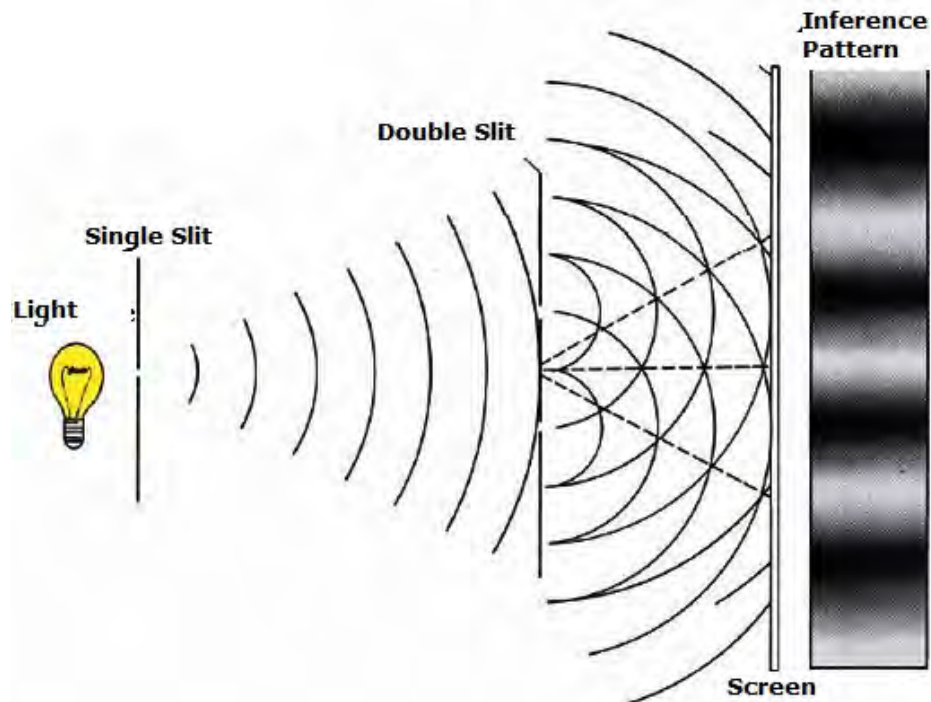
- **Gauss's law:** Charged particles create an electric field.
- **Faraday's law:** An electric field can also be created by a changing magnetic field.
- **Gauss's law for magnetism:** There are no isolated magnetic poles.
- **Ampère-Maxwell law, first half:** Currents create a magnetic field.
- **Ampère-Maxwell law, second half:** A magnetic field can also be created by a changing electric field.
- **Lorentz force law, first half:** An electric force is exerted on a charged particle in an electric field.
- **Lorentz force law, second half:** A magnetic force is exerted on a charge moving in a magnetic field.

→ Buried in all this is an even more basic notion: *Oscillations*

Interference & Diffraction

Review: Sometimes the waves interfere (i.e., add up) *constructively*, other times it is *destructively*

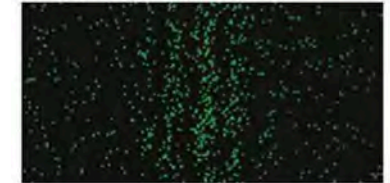
→ Same idea applies here re the “double slit” experiments previously described



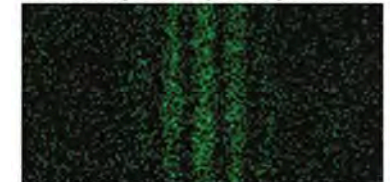
(a) Image after a very short time



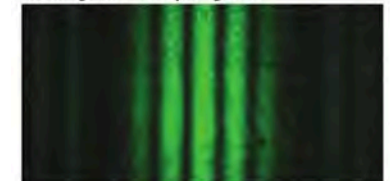
(b) Image after a slightly longer time



(c) Continuing to build up the image



(d) Image after a very long time

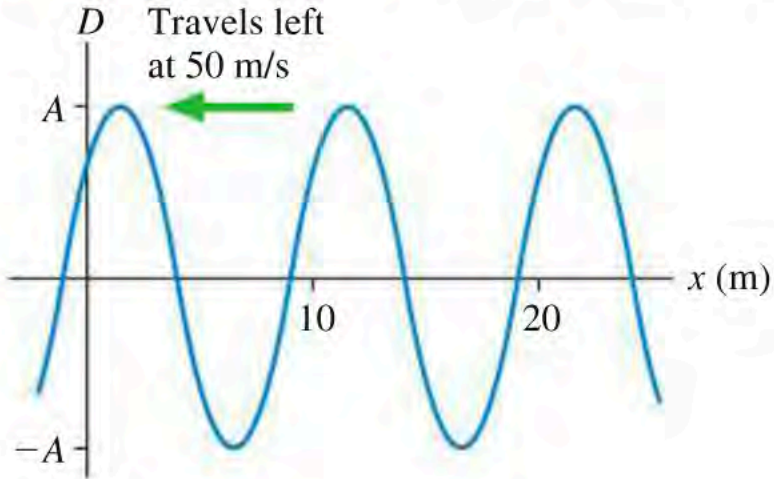


Note: Relevant concept here is known as *Huygens-Fresnel principle*

Question

STOP TO THINK 20.3

What is the frequency of this traveling wave?

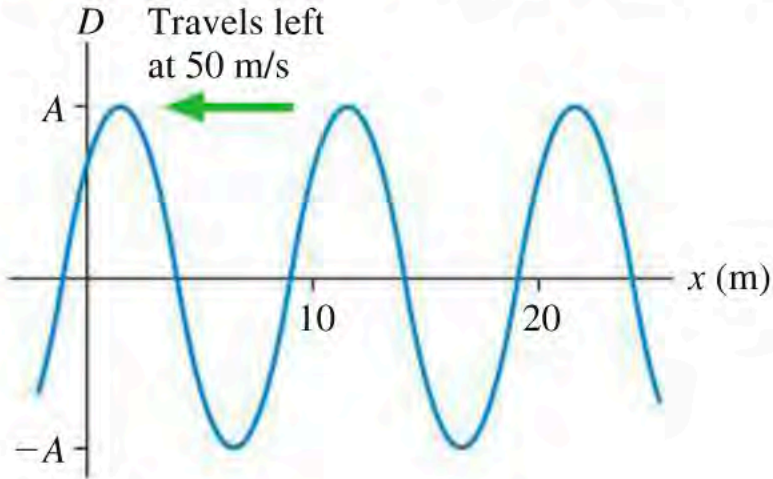


- a. 0.1 Hz
- b. 0.2 Hz
- c. 2 Hz
- d. 5 Hz
- e. 10 Hz
- f. 500 Hz

Question SOL

STOP TO THINK 20.3

What is the frequency of this traveling wave?

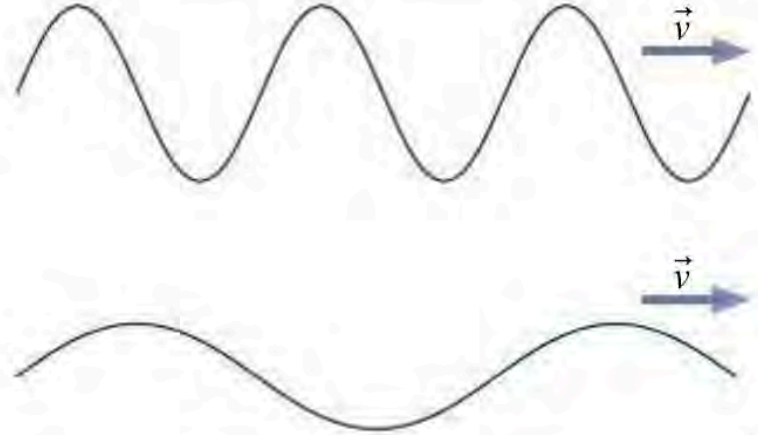


- a. 0.1 Hz
- b. 0.2 Hz
- c. 2 Hz
- d. 5 Hz
- e. 10 Hz
- f. 500 Hz

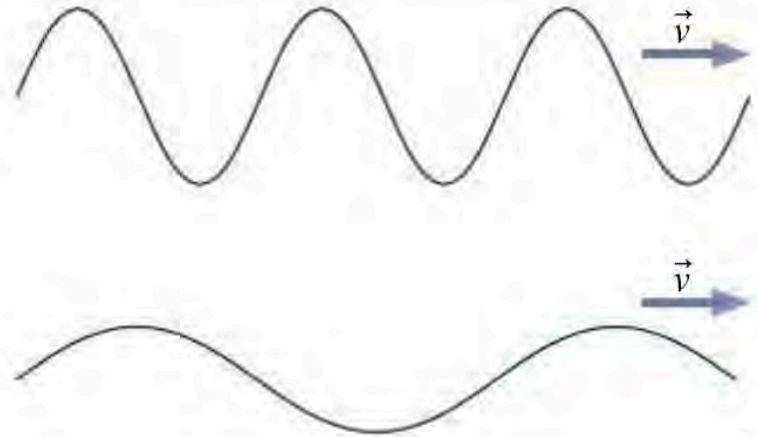
D

Question

GOT IT? 14.2 The figure shows snapshots of two waves propagating with the same speed. Which has the greater (1) amplitude, (2) wavelength, (3) period, (4) wave number, and (5) frequency?



GOT IT? 14.2 The figure shows snapshots of two waves propagating with the same speed. Which has the greater (1) amplitude, (2) wavelength, (3) period, (4) wave number, and (5) frequency?



(1) upper wave; (2) lower; (3) lower; (4) upper; (5) upper

Ex.

|| Ships measure the distance to the ocean bottom with sonar. A pulse of sound waves is aimed at the ocean bottom, then sensitive microphones listen for the echo. **FIGURE P20.45** shows the delay time as a function of the ship's position as it crosses 60 km of ocean. Draw a graph of the ocean bottom. Let the ocean surface define $y = 0$ and ocean bottom have negative values of y . This way your graph will be a picture of the ocean bottom. The speed of sound in ocean water varies slightly with temperature, but you can use 1500 m/s as an average value.

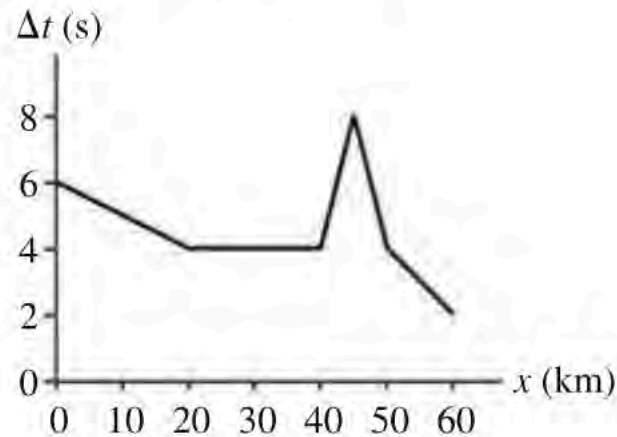


FIGURE P20.45

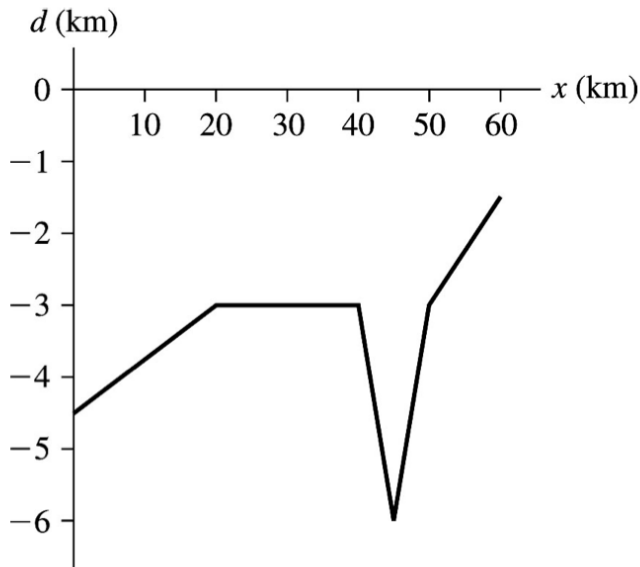
Ex. (SOL)

Solve: Δt is the time the sound wave takes to travel down to the bottom of the ocean and then up to the ocean surface. The depth of the ocean is

$$2d = (v_{\text{sound in water}})\Delta t \Rightarrow d = (750 \text{ m/s})\Delta t$$

Using this relation and the data from Figure P20.45, we can generate the following table for the ocean depth (d) at various positions (x) of the ship.

x (km)	Δt (s)	d (km)
0	6	4.5
20	4	3.0
40	4	3.0
45	8	6.0
50	4	3.0
60	2	1.5



Ex.

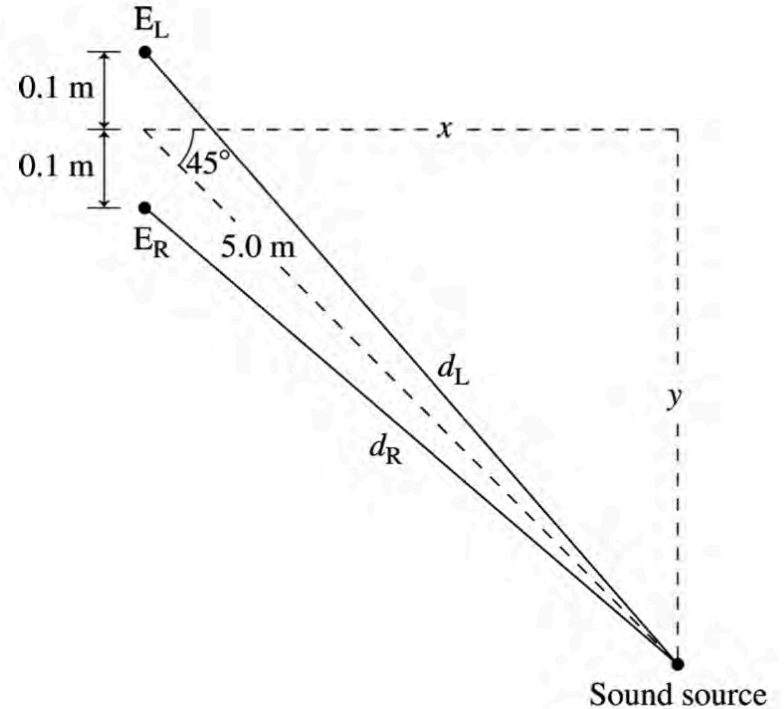
|| One cue your hearing system uses to localize a sound (i.e., to tell where a sound is coming from) is the slight difference in the arrival times of the sound at your ears. Your ears are spaced approximately 20 cm apart. Consider a sound source 5.0 m from the center of your head along a line 45° to your right. What is the difference in arrival times? Give your answer in microseconds.

Hint: You are looking for the difference between two numbers that are nearly the same. What does this near equality imply about the necessary precision during intermediate stages of the calculation?

Ex. (SOL)

Model: Assume a room temperature of 20°C.

Visualize:



Solve: The distance between the source and the left ear (E_L) is

$$d_L = \sqrt{x^2 + (y + 0.1 \text{ m})^2} = \sqrt{[(5.0 \text{ m})\cos 45^\circ]^2 + [(5.0 \text{ m})\sin 45^\circ + 0.1 \text{ m}]^2} = 5.0712 \text{ m}$$

Similarly $d_R = 4.9298 \text{ m}$. Thus,

$$d_L - d_R = \Delta d = 0.1414 \text{ m}$$

For the sound wave with a speed of 343 m/s, the difference in arrival times at your left and right ears is

$$\Delta t = \frac{\Delta d}{343 \text{ m/s}} = \frac{0.1414 \text{ m}}{343 \text{ m/s}} = 410 \mu\text{s}$$

Ex.

- || A wave on a string is described by $D(x, t) = (3.0 \text{ cm}) \times \sin[2\pi(x/(2.4 \text{ m}) + t/(0.20 \text{ s}) + 1)]$, where x is in m and t is in s.
- In what direction is this wave traveling?
 - What are the wave speed, the frequency, and the wave number?
 - At $t = 0.50 \text{ s}$, what is the displacement of the string at $x = 0.20 \text{ m}$?

Ex. (SOL)

Model: This is a sinusoidal wave.

Solve: (a) The displacement of a wave traveling in the positive x -direction with wave speed v must be of the form $D(x, t) = D(x - vt)$. Since the variables x and t in the given wave equation appear together as $x + vt$, the wave is traveling toward the left, that is, in the $-x$ direction.

(b) The speed of the wave is

$$v = \frac{\omega}{k} = \frac{2\pi/0.20 \text{ s}}{2\pi \text{ rad}/2.4 \text{ m}} = 12 \text{ m/s}$$

The frequency is

$$f = \frac{\omega}{2\pi} = \frac{2\pi \text{ rad}/0.20 \text{ s}}{2\pi} = 5.0 \text{ Hz}$$

The wave number is

$$k = \frac{2\pi \text{ rad}}{2.4 \text{ m}} = 2.6 \text{ rad/m}$$

(c) The displacement is

$$D(0.20 \text{ m}, 0.50 \text{ s}) = (3.0 \text{ cm}) \sin \left[2\pi \left(\frac{0.20 \text{ m}}{2.4 \text{ m}} + \frac{0.50 \text{ s}}{0.20 \text{ s}} + 1 \right) \right] = -1.5 \text{ cm}$$