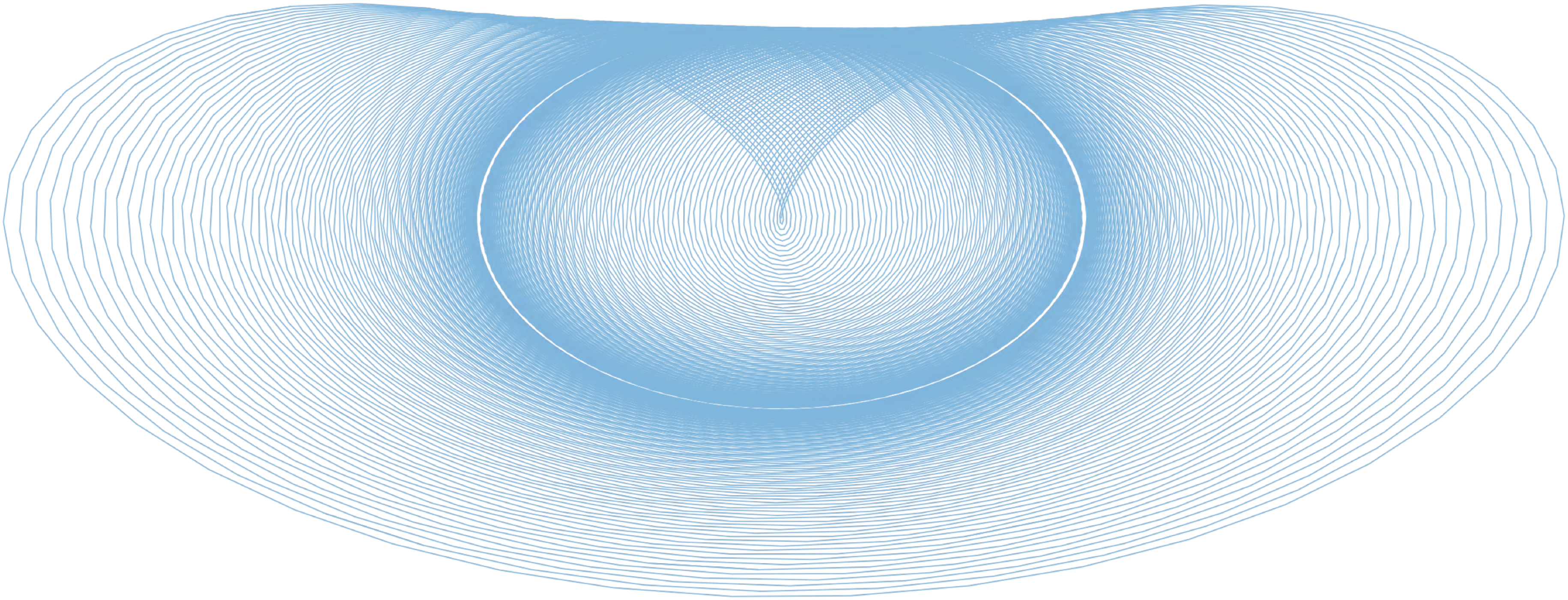
PHYS 1420 (F19)
Physics with Applications to Life Sciences



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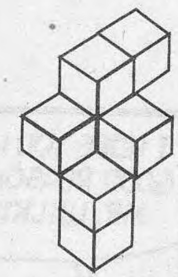
STICKELERS BY TERRY STICKELS

Below is a 7-cube configuration (one cube is hidden from this view).



If you were to pick it up and look at it from every angle, how many faces would you find?

2017.11.29 Toronto Star

1. •Not all oscillatory motion is simple harmonic, but simple harmonic motion is always oscillatory. Explain this statement and give an example to support your explanation. SSM

12.1 Oscillatory motion not only includes simple harmonic motion, but it also includes circular motion, decaying oscillations, and oscillations that have shapes other than pure sine waves.

2. •The fundamental premise of simple harmonic motion is that a force must be proportional to an object's displacement. Is anything else required?

12.2 Simple harmonic motion also requires that the displacement point be in a direction that is opposite to the force.

3. •List several examples of simple harmonic motion that you have observed in everyday life.

12.3 Examples of simple harmonic motion include a child on a swing and any resonant musical instrument.

4. •Explain the difference between a simple pendulum and a physical pendulum.

12.4 A simple pendulum is made from a long, thin string that is tied to a small mass. We can treat it as a point mass attached to the end of a string. A physical pendulum has a massive connecting structure from the pivot point to the end; that is, there is a continuously varying mass spread over the length of the pendulum.

5. •If the rise and fall of your lungs is considered to be simple harmonic motion, how would you relate the period of the motion to your breathing rate (breaths per minute)? SSM

12.5 Breathing rate (breaths per minute) is a frequency. The period is its reciprocal.

6. •Explain how *either* a cosine or a sine function will satisfy the force equation for simple harmonic motion.

12.6 Sine is essentially the same mathematical function as cosine; it is just shifted to a different starting point.

7. •(a) What are the units of ω? (b) What are the units of ωt?

12.7 Part a) The SI units for ω are radians/second.

Part b) The SI units for ωt are radians.

8. •Galileo was one of the first scientists to observe that the period of a simple harmonic oscillator is independent of its amplitude. Explain what it means that the period is independent of the amplitude. Be sure to mention how the requirement that simple harmonic motion undergo small oscillations is affected by this supposition.

12.8 The time it takes a pendulum to repeat its motion is the same regardless of the initial starting angle. This only works if the angle is "small" (less than 15 degrees) so that the small angle approximation is valid.

9. •Compare $x(t) = A \cos \omega t$ to $x(t) = A \cos(\omega t + \phi)$. What is the phase angle ϕ and how does it change the solution to simple harmonic motion?

12.9 It is an offset of where in the oscillation cycle we choose to set the zero time. If, for example, the phase angle is π rather than zero, the solution starts at $-A$ instead of $+A$. In either case, however, the initial velocity is zero and the motion is the same.

10. •What are three factors that can help you distinguish between a simple pendulum and a physical pendulum?

12.10 (1) The mass is concentrated at the bob for a simple pendulum, whereas it is spread over the entire length for the physical pendulum. (2) The simple pendulum has a period that is larger than the period of a physical pendulum of the same length. (3) The mass of the string can be neglected with a simple pendulum.

11. •Explain how you could do an experiment to measure the elevation of your location through the use of a simple pendulum. SSM

12.11 The force of gravity depends on elevation and determines the period of the pendulum. Measuring the period and length of the pendulum will yield enough information to estimate the strength of gravity; doing so very precisely would enable useful comparison to established values.

12. •In the case of the damped harmonic oscillator, what are the units of the damping constant, b?

12.12 For a mass on a spring, b has SI units of kg/s.

13. •The application of an external force on a simple pendulum can create many different outcomes, depending on how frequently the force is applied. Explain what will happen to the amplitude of the motion if an external force is applied to a simple pendulum at the same frequency as the natural frequency of the pendulum.

12.13 The amplitude will increase over time until whatever damping there is in the system is sufficient to counteract the driving force.

14. •Starting from the full description of an oscillating system,

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F(t)$$

under what physical and mathematical circumstances will you arrive at the expression describing the basic case of simple harmonic motion?

12.14 The basic case of simple harmonic motion occurs when $b = 0$ and $F(t)$ = constant (especially when this is zero).

15. •Explain the difference between the frequency of the driving force and the natural frequency of an oscillator.

12.15 The frequency of the driving force is just how often the applied force repeats. That depends on things outside the oscillator. The natural frequency of the oscillator is the frequency at which it oscillates most readily or the frequency at which it will oscillate if displaced from equilibrium and released.

81. ••Calc Starting with the force equation for a damped harmonic oscillator, show that a solution of the form $x(t) = A\,e^{-(b/2m)t}\sin\omega_1 t$ works. The differential equation and the lightly damped oscillation frequency are

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \quad \text{and} \quad \omega_1 = \sqrt{\omega_0^2 - \frac{b^2}{4m^2}}.$$

12.81

SET UP

We are asked to explicitly show that $x(t) = Ae^{-\frac{b}{2m}t}\sin(\omega_1 t)$ is a solution for the damped harmonic oscillator, where $\omega_1 = \sqrt{\omega_0^2 - \frac{b^2}{4m^2}}$. The differential equation describing the damped harmonic oscillator is $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$. We will need to invoke both the chain rule and the product rule.

SOLVE

$$x(t) = Ae^{-\frac{b}{2m}t}\sin(\omega_1 t)$$

$$\frac{dx}{dt} = \frac{d}{dt}[Ae^{-\frac{b}{2m}t}\sin(\omega_1 t)] = A\left[\left(-\frac{b}{2m}\right)e^{-\frac{b}{2m}t}\sin(\omega_1 t) + \omega_1 e^{-\frac{b}{2m}t}\cos(\omega_1 t)\right]$$

$$\frac{d^2x}{dt^2} = A\frac{d}{dt}\left[\left(-\frac{b}{2m}\right)e^{-\frac{b}{2m}t}\sin(\omega_1 t) + \omega_1 e^{-\frac{b}{2m}t}\cos(\omega_1 t)\right]$$

$$= A\left[\left(-\frac{b}{2m}\right)^2 e^{-\frac{b}{2m}t}\sin(\omega_1 t) + 2\left(-\frac{b}{2m}\right)\omega_1 e^{-\frac{b}{2m}t}\cos(\omega_1 t) - \omega_1^2 e^{-\frac{b}{2m}t}\sin(\omega_1 t)\right]$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx \overset{?}{=} 0$$

$$m\left(A\left[\left(-\frac{b}{2m}\right)^2 e^{-\frac{b}{2m}t}\sin(\omega_1 t) + 2\left(-\frac{b}{2m}\right)\omega_1 e^{-\frac{b}{2m}t}\cos(\omega_1 t) - \omega_1^2 e^{-\frac{b}{2m}t}\sin(\omega_1 t)\right]\right)$$

$$+ b\left(A\left[\left(-\frac{b}{2m}\right)e^{-\frac{b}{2m}t}\sin(\omega_1 t) + \omega_1 e^{-\frac{b}{2m}t}\cos(\omega_1 t)\right]\right) + k(Ae^{-\frac{b}{2m}t}\sin(\omega_1 t)) \overset{?}{=} 0$$

$$A\left[\left(\frac{b^2}{4m}\right)e^{-\frac{b}{2m}t}\sin(\omega_1 t) - b\omega_1 e^{-\frac{b}{2m}t}\cos(\omega_1 t) - m\omega_1^2 e^{-\frac{b}{2m}t}\sin(\omega_1 t)\right]$$

$$+ A\left[\left(-\frac{b^2}{2m}\right)e^{-\frac{b}{2m}t}\sin(\omega_1 t) + b\omega_1 e^{-\frac{b}{2m}t}\cos(\omega_1 t)\right] + k\left(Ae^{-\frac{b}{2m}t}\sin(\omega_1 t)\right) \stackrel{?}{=} 0$$

$$A\left[\left(\frac{b^2}{4m}\right)e^{-\frac{b}{2m}t}\sin(\omega_1 t) - m\omega_1^2 e^{-\frac{b}{2m}t}\sin(\omega_1 t) - \left(\frac{b^2}{2m}\right)e^{-\frac{b}{2m}t}\sin(\omega_1 t) + ke^{-\frac{b}{2m}t}\sin(\omega_1 t)\right] \stackrel{?}{=} 0$$

$$Ae^{-\frac{b}{2m}t}\sin(\omega_1 t)\left[-\left(\frac{b^2}{4m}\right) - m\omega_1^2 + k\right] \stackrel{?}{=} 0$$

$$Ae^{-\frac{b}{2m}t}\sin(\omega_1 t)\left[-\left(\frac{b^2}{4m}\right) - m\omega_1^2 + m\omega_0^2\right] \stackrel{?}{=} 0$$

$$Ae^{-\frac{b}{2m}t}\sin(\omega_1 t)\left[-m\omega_1^2 + m\left(\omega_0^2 - \left(\frac{b^2}{4m^2}\right)\right)\right] \stackrel{?}{=} 0$$

$$Ae^{-\frac{b}{2m}t}\sin(\omega_1 t)[-m\omega_1^2 + m\omega_1^2] = 0$$

89. ••Calc (a) By taking derivatives, show that the following function for $x(t)$ satisfies the complete differential equation for oscillating systems:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F(t)$$

$$F(t) = F_0\cos(\omega t); \quad x(t) = A\sin(\omega t + \phi)$$

(b) Find the values of A and ϕ.

12.89

SET UP

The differential equation for a damped driven harmonic oscillator is $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F(t)$. We need to show that $x(t) = A\sin(\omega t + \phi)$ is a valid solution given that $F(t) = F_0\cos(\omega t)$ by taking derivatives and simplifying the equation. In order for $x(t)$ to be a solution, it must satisfy the differential equation for all time. We can find expressions for both A and ϕ by enforcing this condition.

SOLVE

Part a)

Taking derivatives:

$$x(t) = A\sin(\omega t + \phi)$$

$$\frac{dx}{dt} = \frac{d}{dt}[A\sin(\omega t + \phi)] = A\omega\cos(\omega t + \phi)$$

$$\frac{d^2x}{dt^2} = \frac{d}{dt}[A\omega\cos(\omega t + \phi)] = -A\omega^2\sin(\omega t + \phi)$$

Differential equation:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx \stackrel{?}{=} F(t)$$

$$m(-A\omega^2\sin(\omega t + \phi)) + b(A\omega\cos(\omega t + \phi)) + k(A\sin(\omega t + \phi)) \stackrel{?}{=} F_0\cos(\omega t)$$

$$-A\omega^2\sin(\omega t + \phi) + A\frac{b\omega}{m}\cos(\omega t + \phi) + A\frac{k}{m}\sin(\omega t + \phi) \stackrel{?}{=} \frac{F_0}{m}\cos(\omega t)$$

$$-A\omega^2\sin(\omega t+\phi)+A\frac{b\omega}{m}\cos(\omega t+\phi)+A\omega_0^2\sin(\omega t+\phi)\overset{?}{=}\frac{F_0}{m}\cos(\omega t)$$

$$A(\omega_0^2-\omega^2)\sin(\omega t+\phi)+A\frac{b\omega}{m}[\cos(\omega t)\cos(\phi)-\sin(\omega t)\sin(\phi)]\overset{?}{=}\frac{F_0}{m}\cos(\omega t)$$

$$A(\omega_0^2-\omega^2)[\sin(\omega t)\cos(\phi)+\cos(\omega t)\sin(\phi)]+A\frac{b\omega}{m}[\cos(\omega t)\cos(\phi)-\sin(\omega t)\sin(\phi)]\overset{?}{=}\frac{F_0}{m}\cos(\omega t)$$

$$\left[A(\omega_0^2-\omega^2)\sin(\phi)+A\frac{b\omega}{m}\cos(\phi)-\frac{F_0}{m}\right]\cos(\omega t)+A\left[(\omega_0^2-\omega^2)\cos(\phi)-\frac{b\omega}{m}\sin(\phi)\right]\sin(\omega t)\overset{?}{=}0$$

This will only be true for all time if the coefficients to $\cos(\omega t)$ and $\sin(\omega t)$ are equal to zero.

Part b)

$$(\omega_0^2-\omega^2)\cos(\phi)-\frac{b\omega}{m}\sin(\phi)=0$$

$$\tan(\phi)=\frac{\omega_0^2-\omega^2}{\left(\frac{b\omega}{m}\right)}$$

$$\boxed{\phi=\arctan\left(\frac{\omega_0^2-\omega^2}{\left(\frac{b\omega}{m}\right)}\right)}$$

$$A(\omega_0^2-\omega^2)\sin(\phi)+A\frac{b\omega}{m}\cos(\phi)-\frac{F_0}{m}=0$$

$$A\left[(\omega_0^2-\omega^2)\sin(\phi)+\frac{b\omega}{m}\cos(\phi)\right]=\frac{F_0}{m}$$

$$A\left[(\omega_0^2-\omega^2)\left(\frac{\tan(\phi)}{\sqrt{1+\tan^2(\phi)}}\right)+\frac{b\omega}{m}\left(\frac{1}{\sqrt{1+\tan^2(\phi)}}\right)\right]=\frac{F_0}{m}$$

$$\frac{A}{\sqrt{1+\tan^2(\phi)}}\left[(\omega_0^2-\omega^2)\tan(\phi)+\frac{b\omega}{m}\right]=\frac{F_0}{m}$$

$$\frac{A}{\sqrt{1+\left(\frac{\omega_0^2-\omega^2}{\left(\frac{b\omega}{m}\right)}\right)^2}}\left[(\omega_0^2-\omega^2)\left(\frac{\omega_0^2-\omega^2}{\left(\frac{b\omega}{m}\right)}\right)+\frac{b\omega}{m}\right]=\frac{F_0}{m}$$

$$\frac{A}{\left(\frac{b\omega}{m}\right)\sqrt{1+\left(\frac{\omega_0^2-\omega^2}{\left(\frac{b\omega}{m}\right)}\right)^2}}\left[(\omega_0^2-\omega^2)^2+\left(\frac{b\omega}{m}\right)^2\right]=\frac{F_0}{m}$$

$$\frac{A}{\sqrt{\left(\frac{b\omega}{m}\right)^2 + (\omega_0^2 - \omega^2)^2}} = \frac{F_0}{m\left[(\omega_0^2 - \omega^2)^2 + \left(\frac{b\omega}{m}\right)^2\right]}$$

$$\boxed{A = \frac{F_0}{m\sqrt{(\omega_0^2 - \omega^2)^2 + \left(\frac{b\omega}{m}\right)^2}}}$$

89. ••**Calc** (a) By taking derivatives, show that the following function for $x(t)$ satisfies the complete differential equation for oscillating systems:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F(t)$$

$$F(t) = F_0 \cos(\omega t); \quad x(t) = A \sin(\omega t + \phi)$$

(b) Find the values of A and ϕ.

Purpose : Use complex exponentials to solve (steady-state) damped driven HO.

☐ Eqn. of motion : $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m} e^{i\omega t}$ where $i \equiv \sqrt{-1}$.

Note that from Euler's identity ($Ae^{i\theta} = A[\cos\theta + i\sin\theta]$), we could just take the "real part" to get $F_0/m \cos\omega t$ as the sinusoidal drive

☐ Assume a solution of the form $x(t) = A(\omega)e^{i[\omega t + \delta(\omega)]}$

$$= Ae^{i\delta}e^{i\omega t} \quad \left(\text{since } e^{a+b} = e^a \cdot e^b\right)$$

- Now taking derivatives of exponentials is easy! (remember that i is simply a const.)

$$\dot{x} = \frac{dx}{dt} = \frac{d}{dt}\left(Ae^{i\delta}e^{i\omega t}\right) = i\omega Ae^{i\delta}e^{i\omega t} = i\omega x$$

$$\ddot{x} = \frac{d^2x}{dt^2} = \frac{d}{dt}\left(i\omega Ae^{i\delta}e^{i\omega t}\right) = -\omega^2 Ae^{i\delta}e^{i\omega t} = -\omega^2 x$$

- Now plug back into the ODE:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}e^{i\omega t} = -\omega^2 x + i\omega\gamma x + \omega_0^2 x = Ae^{i\delta}e^{i\omega t}\left[-\omega^2 + i\omega\gamma + \omega_0^2\right]$$

$$\rightarrow \boxed{Ae^{i\delta} = \frac{F_0/m}{-\omega^2 + i\omega\gamma + \omega_0^2}} = \frac{F_0/m}{(\omega_0^2 - \omega^2) + i\omega\gamma}$$

☐ Now we have the complex # on the left in polar form, but the one on the right in an oddball Cartesian form (i.e. the complex bit is in the denominator!). We can note a few properties of complex #s here to apply (so to simplify):

- $(a+ib)(a-ib) = a^2 - (ib)^2 = a^2 - (-b^2) = a^2+b^2$
- $\dfrac{1}{a+ib} = \dfrac{1}{a+ib} \cdot \dfrac{a-ib}{a-ib} = \dfrac{a-ib}{a^2+b^2} = \dfrac{a}{a^2+b^2} - i\,\dfrac{b}{a^2+b^2}$
- $Ae^{i\theta} = a+ib \longrightarrow A = |a+ib| = \sqrt{a^2+b^2} = (a^2+b^2)^{\frac{1}{2}}, \quad \theta = \tan^{-1}\left(\dfrac{b}{a}\right)$

- Now let's put these to work:

$$Ae^{i\delta} = \frac{F_0/m}{(\omega_0^2-\omega^2)+i\omega\gamma} = \frac{F_0/m}{(\omega_0^2-\omega^2)^2+(\omega\gamma)^2} \cdot (\omega_0^2-\omega^2)+i\omega\gamma$$

$$\rightarrow A = \sqrt{\left[\frac{F_0/m}{(\omega_0^2-\omega^2)^2+(\omega\gamma)^2}(\omega_0^2-\omega^2)\right]^2 + \left[\frac{F_0/m}{(\omega_0^2-\omega^2)^2+(\omega\gamma)^2}\cdot\omega\gamma\right]^2}$$

$$= \frac{F_0/m}{(\omega_0^2-\omega^2)^2+(\omega\gamma)^2} \cdot \sqrt{(\omega_0^2-\omega^2)^2-(\omega\gamma)^2}$$

$$= \boxed{\frac{F_0/m}{\left[(\omega_0^2-\omega^2)^2+(\omega\gamma)^2\right]^{\frac{1}{2}}} = A}$$

$$\rightarrow \boxed{\delta = \tan^{-1}\left(\frac{\omega\gamma}{\omega_0^2-\omega^2}\right)} \quad \therefore$$

89. ••**Calc** (a) By taking derivatives, show that the following function for $x(t)$ satisfies the complete differential equation for oscillating systems:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F(t)$$

$$F(t) = F_0\cos(\omega t); \quad x(t) = A\sin(\omega t + \phi)$$

(b) Find the values of A and ϕ.

- Assume a solution of the form $x(t) = A(\omega)e^{i[\omega t + \delta(\omega)]}$

$$= Ae^{i\delta}e^{i\omega t} \quad \left(\text{since } e^{a+b} = e^{a}\cdot e^{b}\right)$$

→ What if instead we had a negative sign in front of the i in the argument of the exponent in the assumed form of the solution (as per the class notes)?

90. ••The *quality factor* Q is a parameter that specifies the width and height of the resonant peak when an oscillator is driven by a sinusoidal external force. It is defined as follows:

$$Q = \frac{m\omega_0}{b}$$

The "width" of the resonant peak is $\Delta\omega = \omega_0/Q$. Although different definitions exist, we'll use the FWHM ("full-width half-max") concept for this width. FWHM is basically the width of the peak at the point that is one-half of the maximum value. For a driven oscillator that has a mass of 100 g, a damping constant of 0.2 kg/s, a peak force of 7.5 N, and a spring constant 25 N/m, calculate the quality factor Q and the FWHM of the resonant peak.

12.90

SET UP

The quality factor $Q = \frac{m\omega_0}{b}$ is a parameter that describes the width and height of the resonant peak for a driven harmonic oscillator. The "width" of the resonant peak, which we'll call the full-width half-max (FWHM), is given by $\Delta\omega = \frac{\omega_0}{Q}$. The natural frequency of a mass–spring system is given by $\omega_0 = \sqrt{\frac{k}{m}}$. The numerical values given are $m = 0.100$ kg, $b = 0.2$ kg/s, $F_0 = 7.5$ N, and $k = 25$ N/m.

SOLVE

Quality factor:

$$Q = \frac{m\omega_0}{b} = \frac{m}{b}\sqrt{\frac{k}{m}} = \frac{\sqrt{km}}{b} = \frac{\sqrt{\left(25\frac{\text{N}}{\text{m}}\right)(0.100\text{ kg})}}{\left(0.2\frac{\text{kg}}{\text{s}}\right)} = \boxed{7.91}$$

Full-width half-max:

$$\text{FWHM} = \Delta\omega = \frac{\omega_0}{Q} = \frac{1}{Q}\sqrt{\frac{k}{m}} = \frac{1}{7.91}\sqrt{\frac{\left(25\frac{\text{N}}{\text{m}}\right)}{0.100\text{ kg}}} = \boxed{2.00\frac{\text{rad}}{\text{s}}}$$

REFLECT

A large Q means the effects of damping are small and it will take longer for the oscillations to die out.