Modeling Cochlear Dynamics

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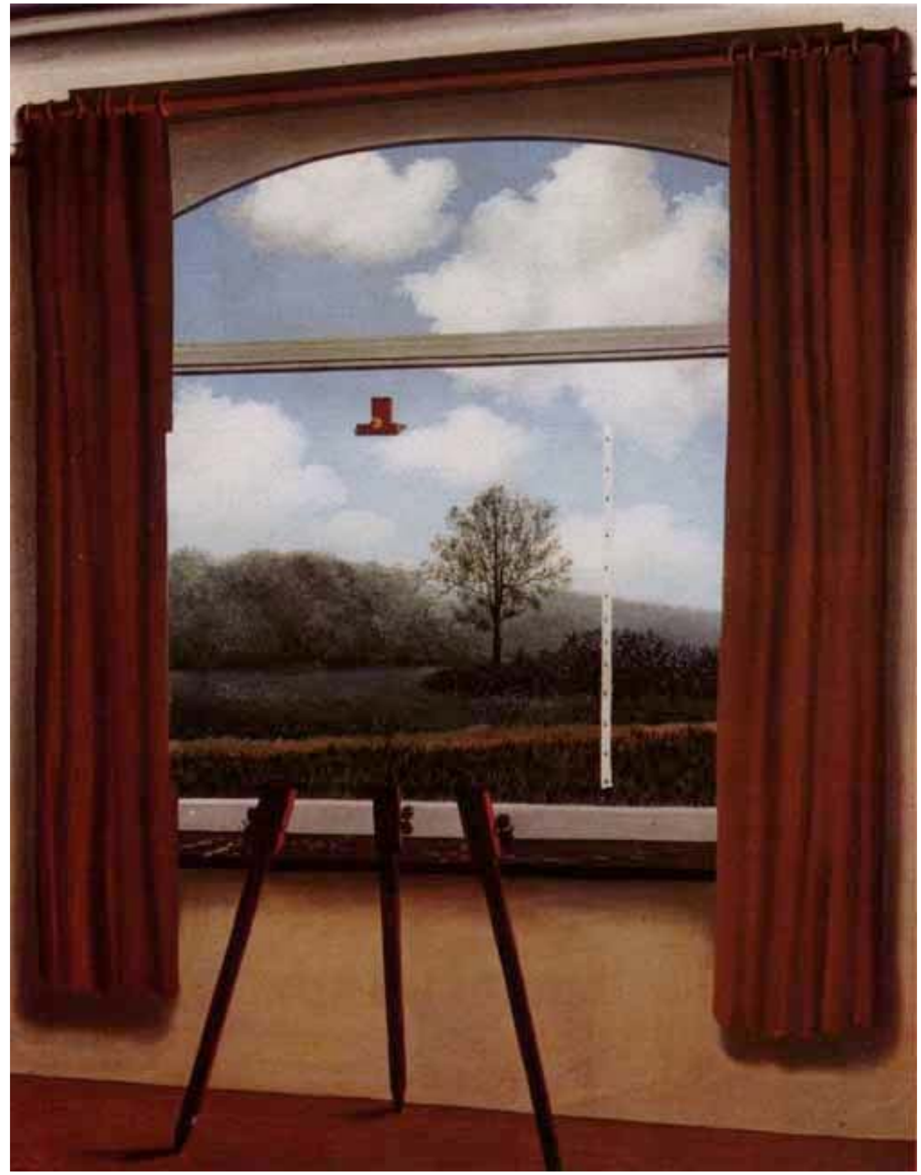
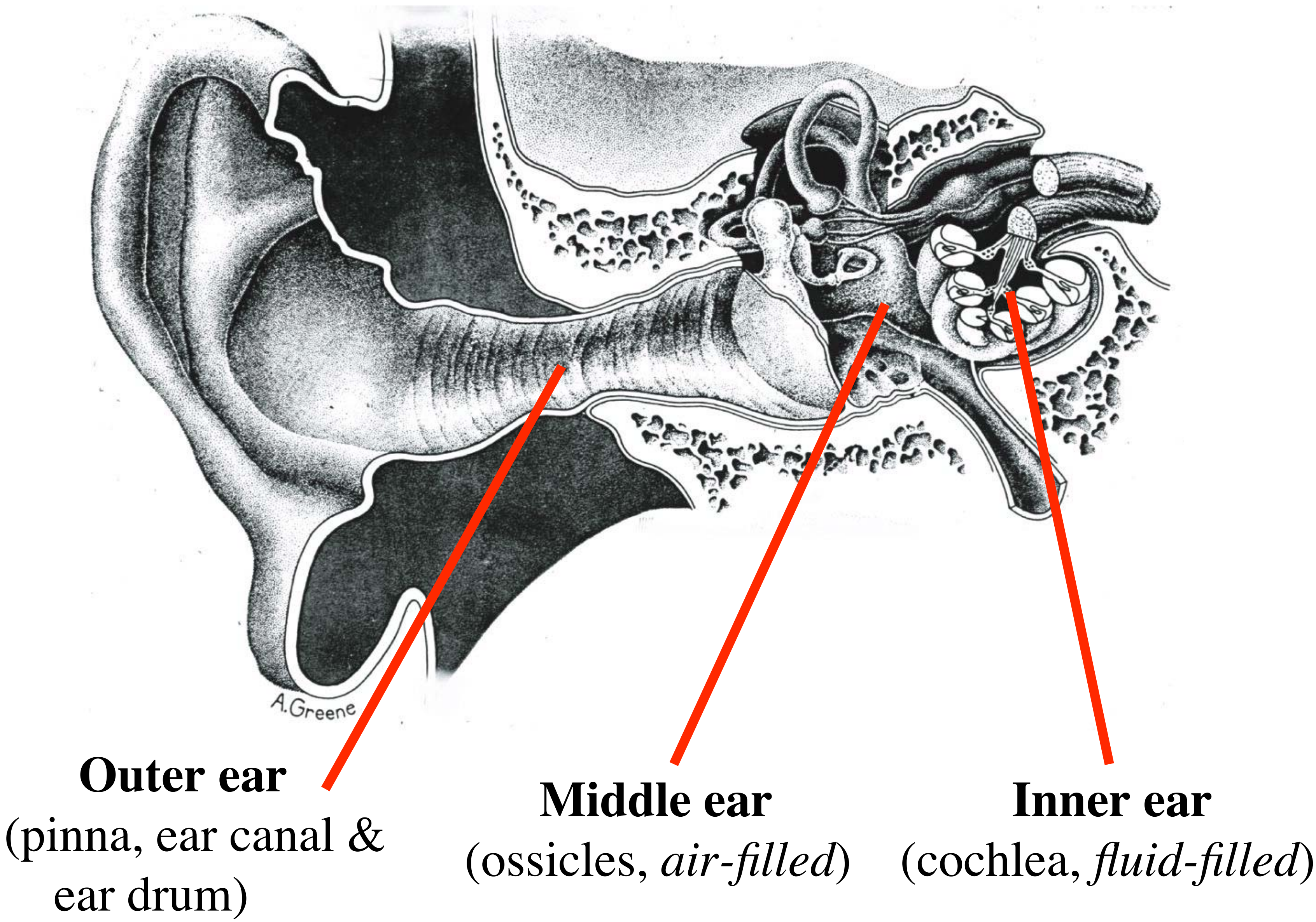
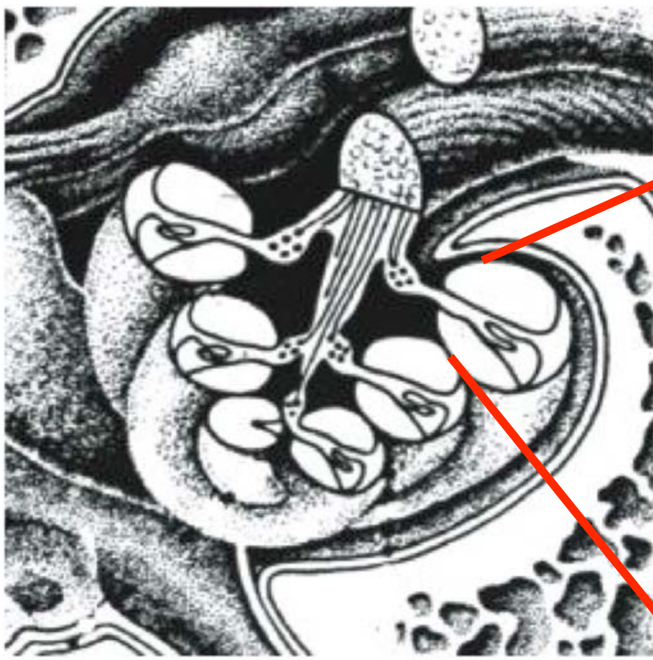


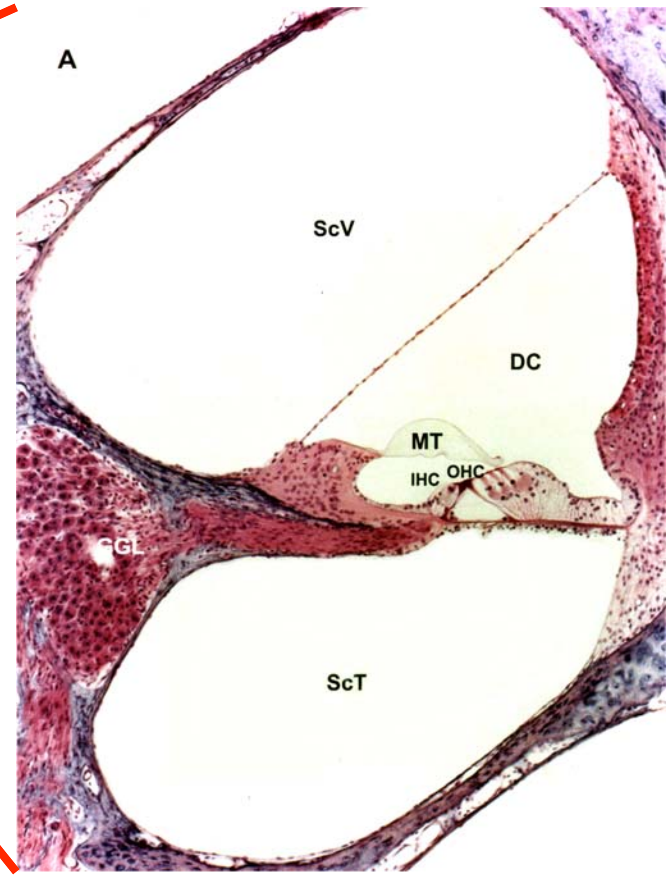
Rene Magritte

Disclaimer: $j \equiv \sqrt{-1} = i$


A.Greene
Outer ear
(pinna, ear canal &
ear drum)
Middle ear
(ossicles, air-filled)
Inner ear
(cochlea, fluid-filled)

Inner ear - Organ of Corti





- Three fluid filled compartments

- Sensory complex sits atop a flexible membrane (*basilar membrane*)

Georg von Bekesy (1899-1972)

(Nobel Prize in 1961)

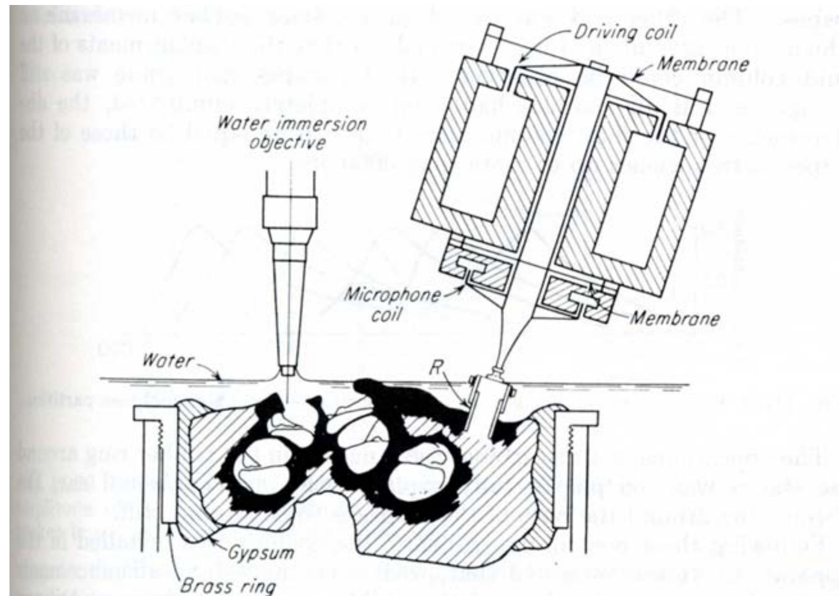


Fig. 11-48. Method of measuring the amplitude of vibration of the cochlear partition in response to volume displacements of the stapes.

Experiments in Hearing (1960)

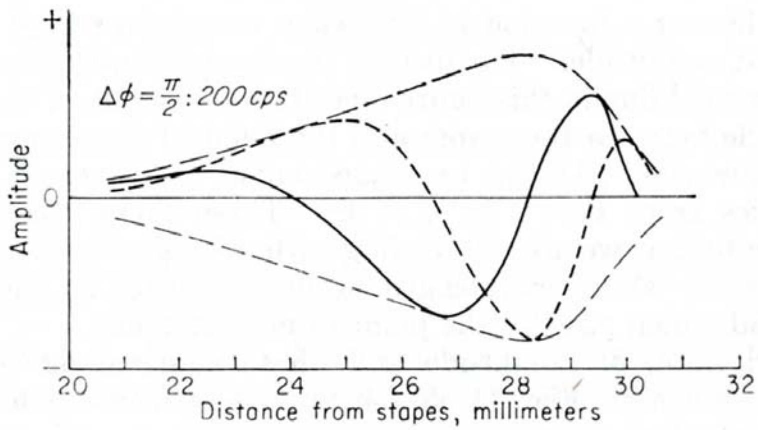


FIG. 11-59. Detail of the form of vibration of the cochlear partition for 200 cps at two instants within a cycle.

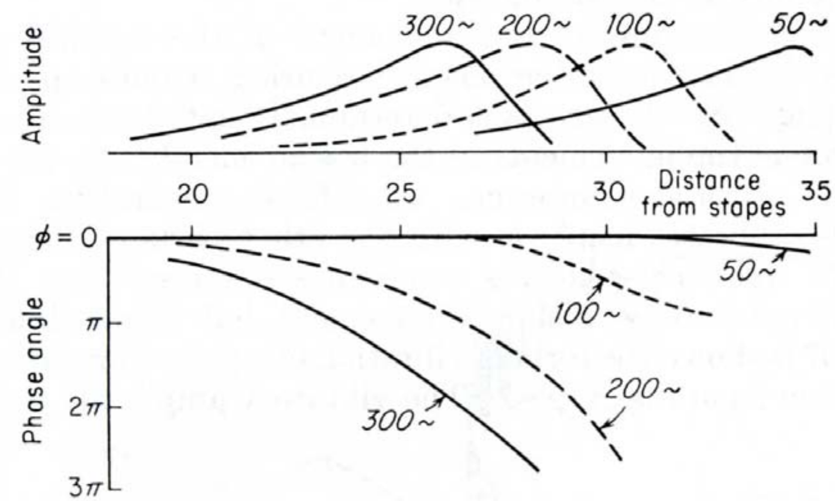


FIG. 11-58. Phase displacement and resonance curves for four low tones.

Question: Can we develop a model based upon the anatomy that captures the observed physiological features?

Goal: Model should serve as a foundation

⇒ 1-D transmission-line model solved using WKB approximation

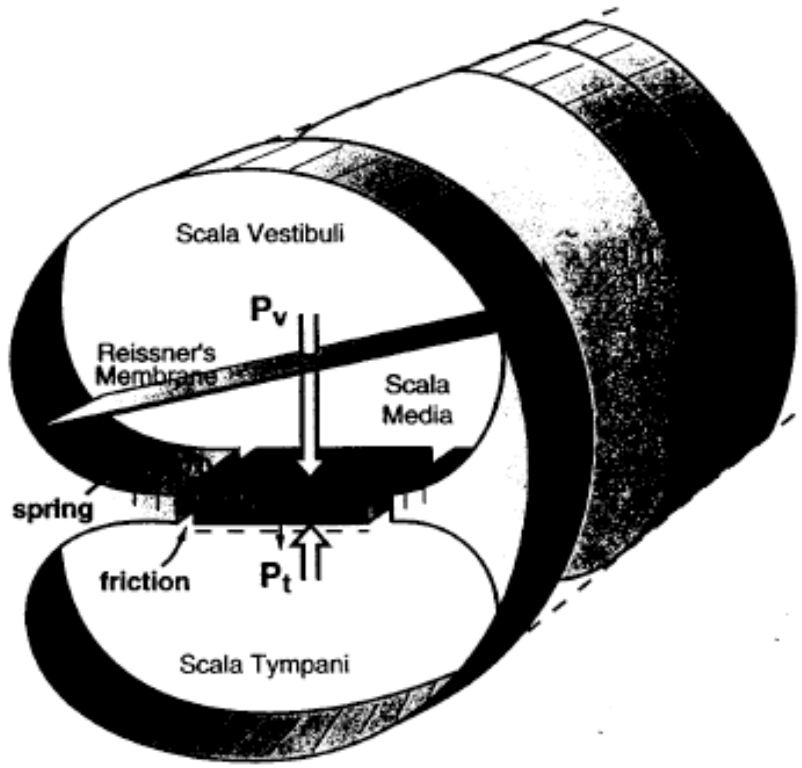
Assumptions

- effect of coiling is negligible
  (allows us to ‘unroll’ the cochlea)

- height of traveling wave is small relative to the height of the scalae
  (pressure is uniform in both cross-sections and depends only upon longitudinal distance)

- fluid is incompressible and viscosity negligible

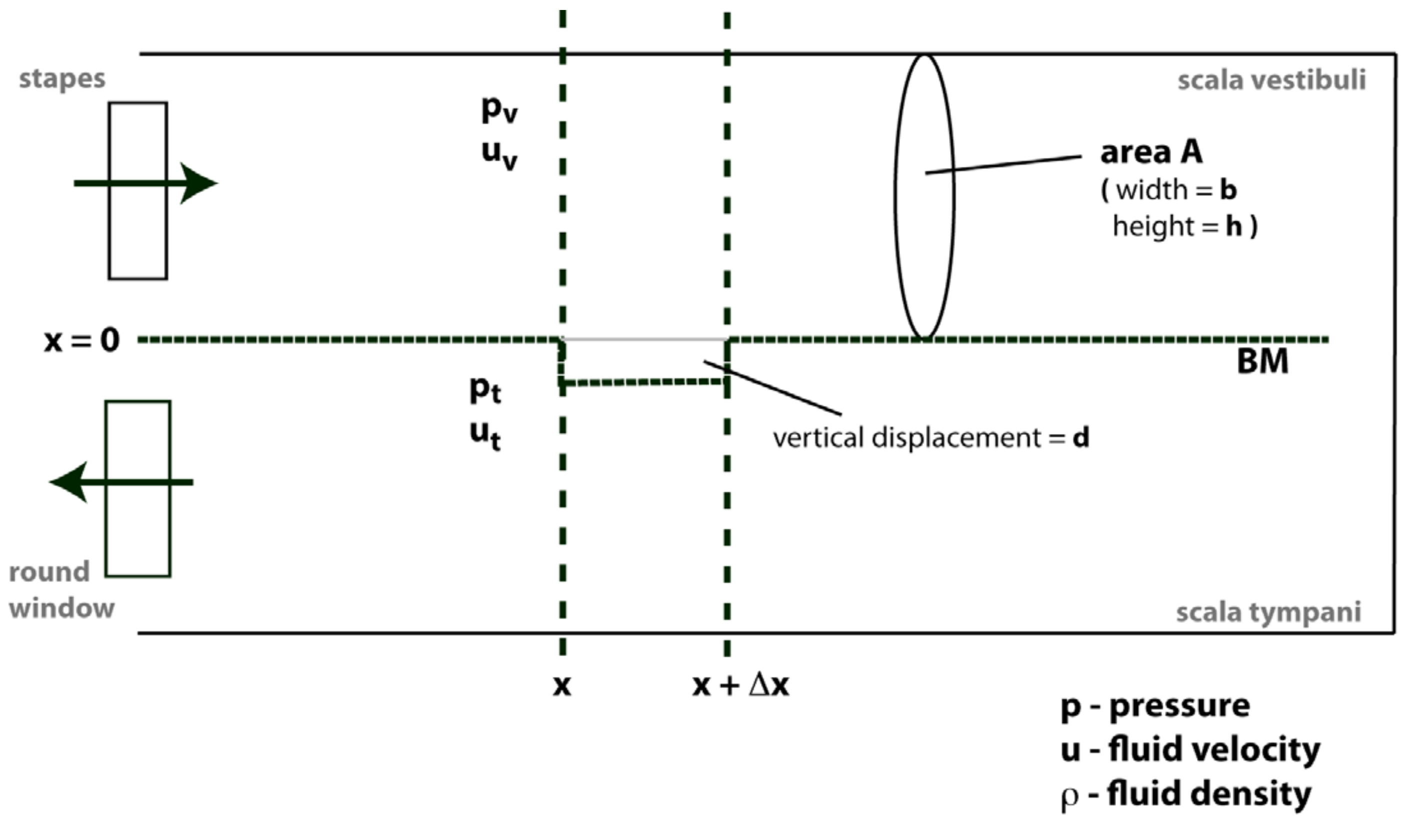
Geometry



x - longitudinal distance along cochlea

b and *h* - width and height of the scalae (assumed constant for now)

d - vertical (transverse) displacement of the BM


stapes
scala vestibuli
p_v
u_v
area A
(width = b
height = h)
x = 0
BM
p_t
u_t
vertical displacement = d
round
window
scala tympani
x
x + Δx
p - pressure
u - fluid velocity
ρ - fluid density

Fluid Flow Due to Pressure Difference

Consider element of scala vestibuli. Using Newton's 2nd law (no fluid viscosity):

$$\Delta t \cdot A_v \cdot [p_v(x,t) - p_v(x+\Delta x,t)] = \rho \cdot \Delta x \cdot A_v \cdot [u_v(x,t+\Delta t) - u_v(x,t)]$$

$$\frac{\partial p_v}{\partial x} = -\rho \frac{\partial u_v}{\partial t}$$

$$\frac{\partial p_t}{\partial x} = -\rho \frac{\partial u_t}{\partial t}$$

Fluid Velocity to Membrane Displacement

From the conservation of mass (incompressible fluid):

$$\Delta t \cdot A_v \cdot \rho \cdot [u(x,t) - u(x+\Delta x, t)] = \Delta x \cdot \rho \cdot b \cdot [d(x, t+\Delta t) - d(x,t)]$$

$$A_v \frac{\partial u_v}{\partial x} = -b \frac{\partial d}{\partial t}$$

$$A_t \frac{\partial u_t}{\partial x} = b \frac{\partial d}{\partial t}$$

BM Motion Due to Pressure Difference

Consider all forces acting on the BM:

$$F = F_{stiffness} + F_{drag} + F_{pressure} = \mu \cdot \Delta x \cdot \frac{\delta^2 d}{\delta t^2}$$

$$\mathbf{b} \cdot [\mathbf{p_v}(\mathbf{x}, \mathbf{t}) - \mathbf{p_t}(\mathbf{x}, \mathbf{t})] = \mu \cdot \frac{\delta^{\mathbf{2}} \mathbf{d}(\mathbf{x}, \mathbf{t})}{\delta \mathbf{t}^{\mathbf{2}}} + \alpha \cdot \frac{\delta \mathbf{d}(\mathbf{x}, \mathbf{t})}{\delta \mathbf{t}} + \kappa \cdot \mathbf{d}(\mathbf{x}, \mathbf{t})$$

Some consequences

$$u_v(x,t) = -u_t(x,t)$$

$$p_v(x,t) + p_t(x,t) = \alpha$$

Simplifications

Let: $p \equiv p_v - p_t$ $u \equiv A_{cs}(u_v - u_t)/2$

Assume both *u* and *p* are in sinusoidal steady-state (stimulus frequency ω) such that:

$$\mathbf{p(x,t)} = \Re[\mathbf{P}(\mathbf{x},\omega)\mathbf{e^{j\omega t}}] \qquad \mathbf{u(x,t)} = \Re[\mathbf{U}(\mathbf{x},\omega)\mathbf{e^{j\omega t}}]$$

$$\Rightarrow \frac{\partial \mathbf{P}}{\partial \mathbf{x}} = -\frac{\mathbf{2}\rho}{\mathbf{A_{cs}}}\mathbf{i}\omega\mathbf{U} = -\mathbf{ZU}$$

(eqn. I)

Simplifications II

$$\frac{\partial p}{\partial x} = \frac{A_{cs}}{2}\left[-\frac{b}{A_{cs}}\left(\frac{\partial d}{\partial t} + \frac{\partial d}{\partial t}\right)\right] = -b\frac{\partial d}{\partial t}$$

$$d(t) = \int -\frac{e^{j\omega t}}{b}\frac{\partial U}{\partial x}dt = -\frac{e^{j\omega t}}{j\omega b}\frac{\partial U}{\partial x}$$

Plugging back into the equation of motion (relating pressure and displacement):

$$\frac{\partial \mathbf{U}}{\partial \mathbf{x}} = -\frac{\mathbf{P}}{\mathbf{j}\omega\frac{\mu}{\mathbf{b^2}} + \frac{\alpha}{\mathbf{b^2}} + \frac{\mathbf{1}}{\mathbf{j}\omega\frac{\mathbf{b^2}}{\kappa}}} = -\mathbf{YP}$$

(eqn. II)

Wave Equation

$$\frac{\partial \mathbf{P}}{\partial \mathbf{x}} = -\frac{\mathbf{2}\rho}{\mathbf{A_{cs}}}\mathbf{i}\omega\mathbf{U} = -\mathbf{ZU}$$

$$\frac{\partial \mathbf{U}}{\partial \mathbf{x}} = -\frac{\mathbf{P}}{\mathbf{j}\omega\frac{\mu}{\mathbf{b^2}} + \frac{\alpha}{\mathbf{b^2}} + \frac{\mathbf{1}}{\mathbf{j}\omega\frac{\mathbf{b^2}}{\kappa}}} = -\mathbf{YP}$$

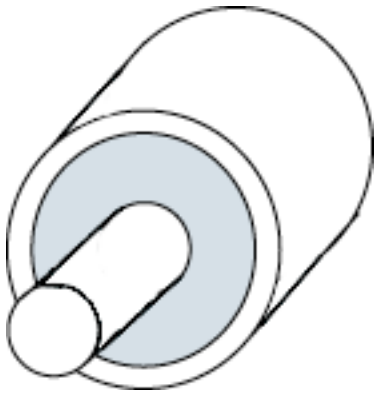
$$\frac{\partial^2 P}{\partial x^2} + \frac{1}{\ell^2}P = 0$$

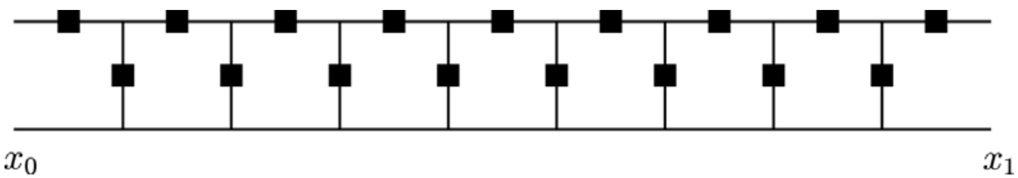
$$\ell \equiv j\sqrt{1/ZY}$$

$$\frac{\partial^2 U}{\partial x^2} + \frac{1}{\ell^2}U = 0$$

$$p(x,t) = P_1 e^{i(x/\ell+\omega t)} + P_2 e^{i(-x/\ell+\omega t)}$$

Analogy to Electrical Transmission Line





Electrical Case (loss-less)

$$-\frac{\partial v}{\partial x} = L\frac{\partial i}{\partial t}$$

$$-\frac{\partial i}{\partial x} = C\frac{\partial v}{\partial t}$$

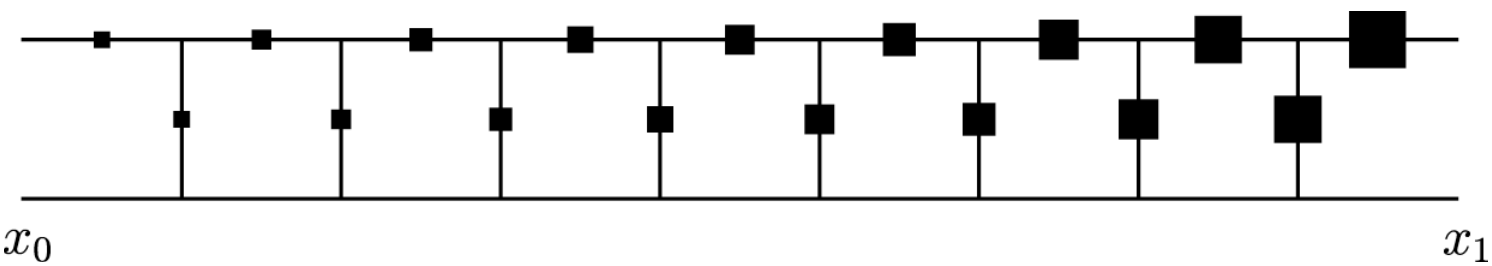
Cochlear Case

$$\frac{\partial \mathbf{P}}{\partial \mathbf{x}} = -\mathbf{ZU}$$

$$\frac{\partial \mathbf{U}}{\partial \mathbf{x}} = -\mathbf{YP}$$

Assumptions Revisited

b, A not constant ⇒ $Z \longrightarrow Z(x)$, $Y \longrightarrow Y(x)$



⇒ non-uniform transmission line

WKB Approximation

Wentzel, Kramers and Brillouin (1926)

Particle with energy *E* moving in constant potential field *V*:

$$\psi(x) = Ae^{\pm jkx} \qquad k \equiv \sqrt{2m(E-V)}/\hbar$$

⇒ What if *V* is not constant, but varies gradually?

$$\psi(x) \cong \frac{C}{\sqrt{k(x)}} e^{\frac{j}{\hbar}\int k(x)dx} \qquad k(x) \equiv \sqrt{2m[E-V(x)]}/\hbar$$

WKB Applied to Cochlea

Assumption: Cochlear parameters (e.g. BM stiffness, scalae area) vary *gradually* such that cochlea behaves like a uniform transmission line *locally*

$$P(x,\omega) = A(x)e^{-j\int_0^x dx'/\ell(x',\omega)}$$

$$\ell(x,\omega) = j\left[\frac{j\omega L + R + 1/j\omega C}{j\omega M}\right]^{1/2} \quad \Longrightarrow \quad \ell(x,\omega) = \frac{l}{4N}\frac{(1-\beta^2+j\delta\beta)^{1/2}}{\beta}$$

$$\omega_r(x) \equiv 1/\sqrt{LC} \qquad \beta(x,\omega) \equiv \omega/\omega_r(x) \qquad \delta \equiv \omega_r(x)RC \qquad N \equiv (l/4)\sqrt{M/L}$$

Transfer Function

$$T(x,\omega) = \frac{\text{BM velocity}}{\text{stapes velocity}}$$

$$\text{BM velocity} = \frac{e^{j\omega t}}{b}\ell^2(x,\omega) Y e^{-j\int_0^x dx'/\ell(x',\omega)}$$

$$\omega_r(x) = \omega_{max} e^{-x/l}$$

WKB Applied to Cochlea (cont.)

$$\beta = \omega/\omega_r \qquad\qquad dx = \ell \frac{d\beta}{\beta}$$

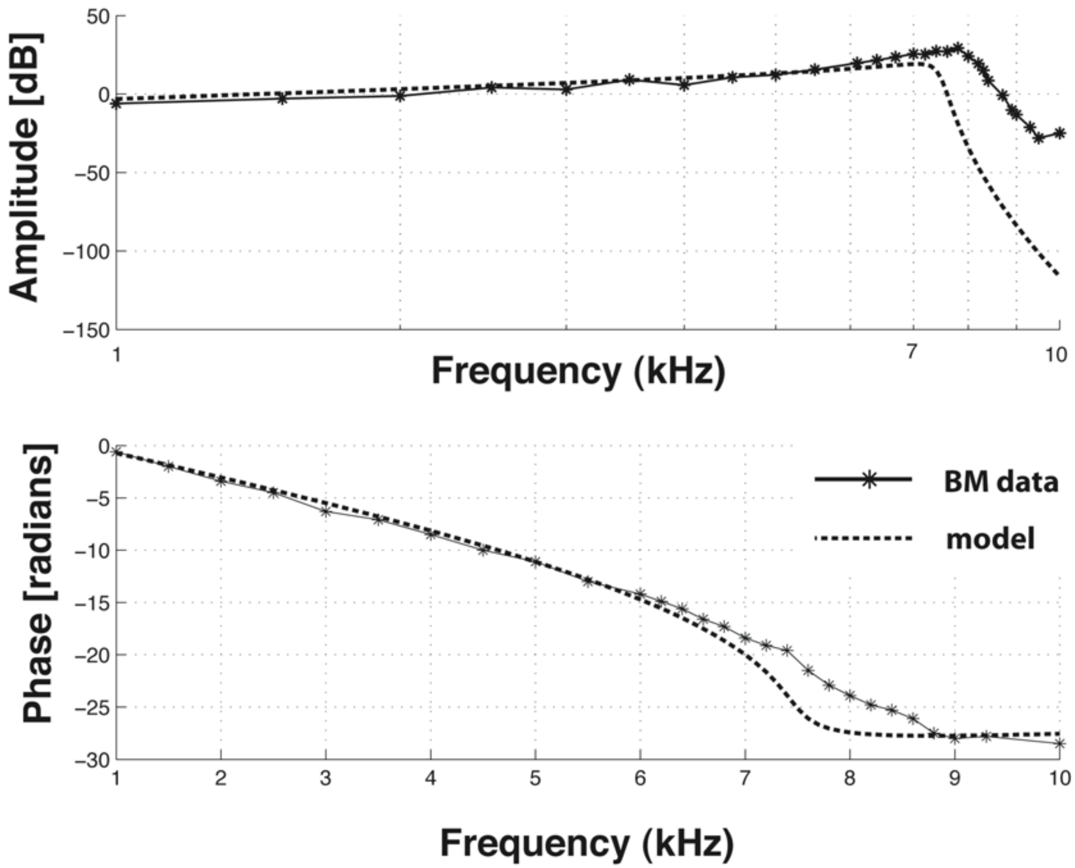
$$\Rightarrow \quad 4N \int_0^x \frac{d\beta}{\left[1 - \beta^2 + j\delta\beta\right]^{1/2}}$$

$$\mathbf{T}(\mathbf{x},\omega) \approx \mathbf{T_0 j}\beta(\mathbf{x},\omega)\left[\frac{\omega_{\mathbf{max}}}{\omega_{\mathbf{r}}(\mathbf{x})}\right]\frac{\mathbf{e}^{\mathbf{j4N}\{\mathbf{sin}^{-1}[\beta(\mathbf{x},\omega)-\mathbf{j}\delta/\mathbf{2}]-\mathbf{sin}^{-1}[\beta(\mathbf{0},\omega)-\mathbf{j}\delta/\mathbf{2}]\}}}{[\mathbf{1}-\beta^{\mathbf{2}}(\mathbf{x},\omega)+\mathbf{j}\delta\beta(\mathbf{x},\omega)]^{\mathbf{3/4}}}$$

(Zweig, 1991)

Comparison to Data

Physiological measurements of BM motion relative to stapes displacement



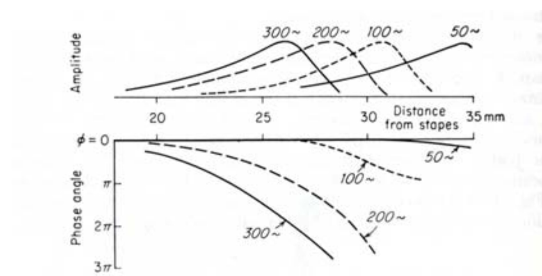
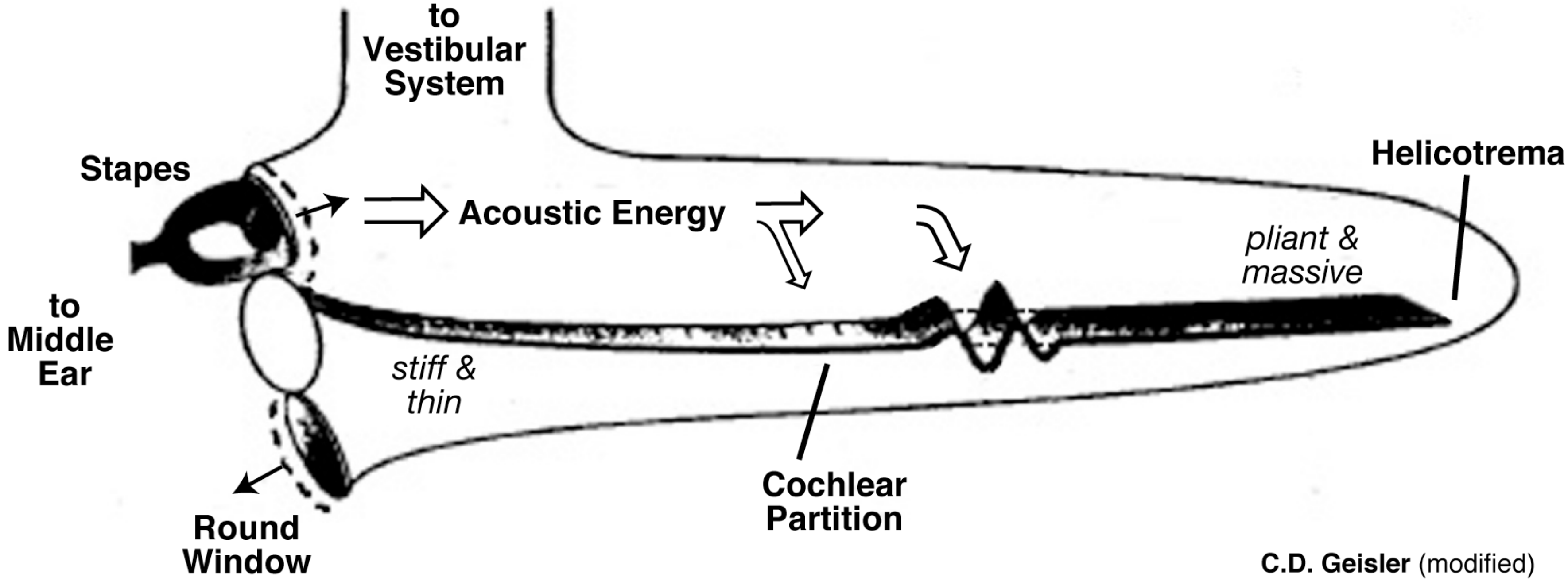


FIG. 11-58. Phase displacement and resonance curves for four low tones.

[Rhode, 1971]

Mammalian Cochlea Uncoiled

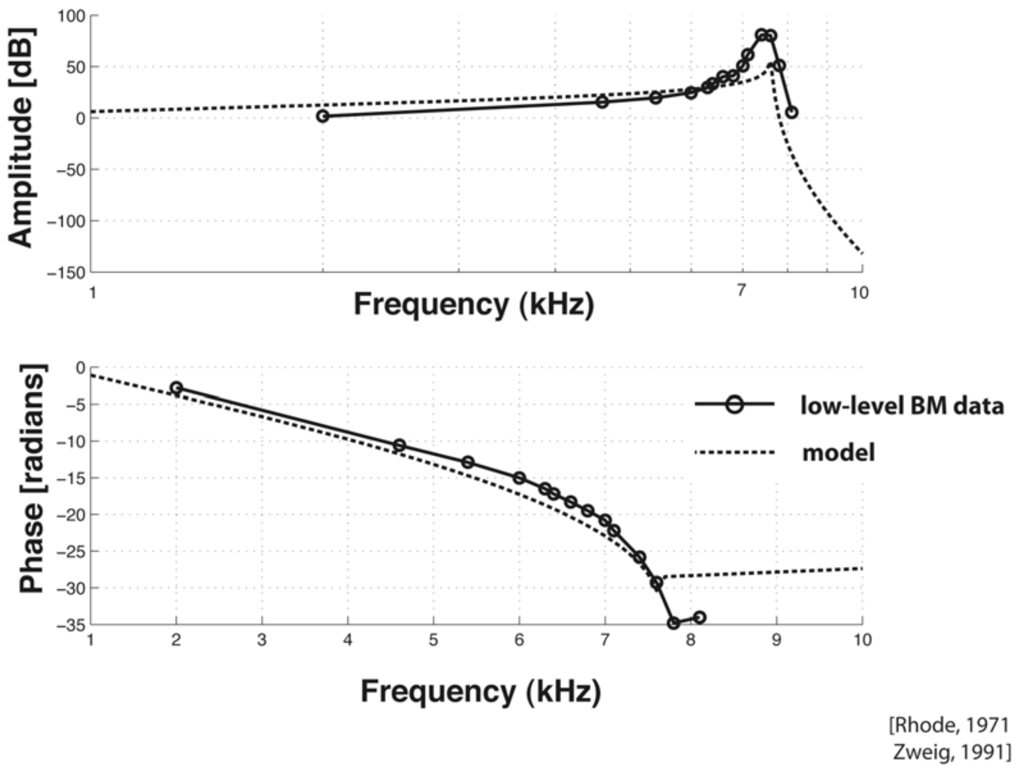


C.D. Geisler (modified)

Model provides a starting point for thinking about cochlear dynamics

⇒ What features are present in a real ear that we would like the model to capture?

I. Near Threshold in Viable Ears



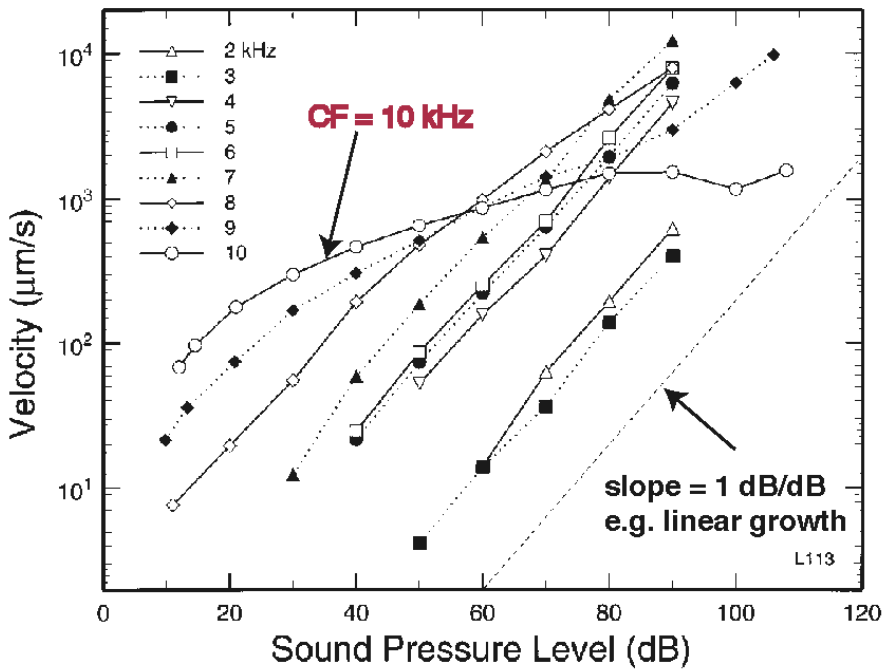
Model can not capture sharp tuning/large group delay at CF

⇒ **Need for _active mechanisms_?** [Neely and Kim, 1983]

II. Nonlinearity

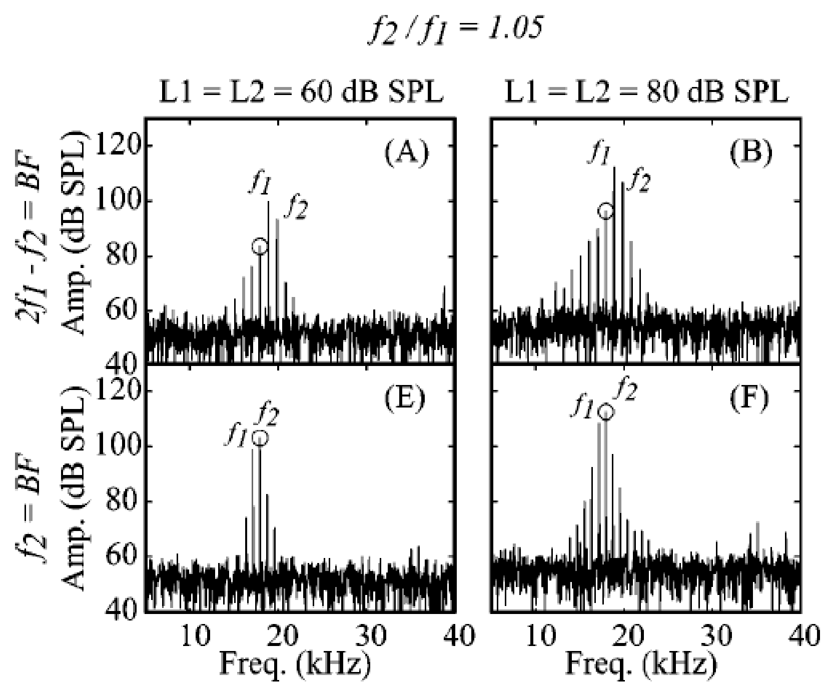
Compressive BM Growth (on CF)

BM response at a fixed spot to different frequency tones



from Ruggero et al. (JASA, 1997)

Two-tone Distortion in Pressure

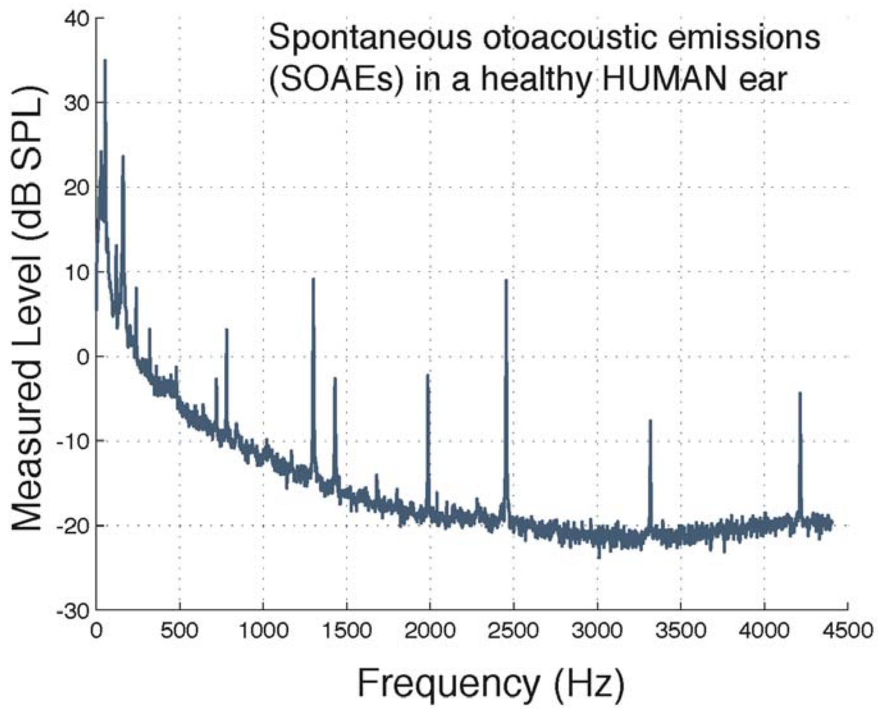


Dong and Olson (JASA, 2005)

⇒ Source of nonlinearity?

Otoacoustic Emissions (OAEs)

sounds emitted by a normal, healthy ear

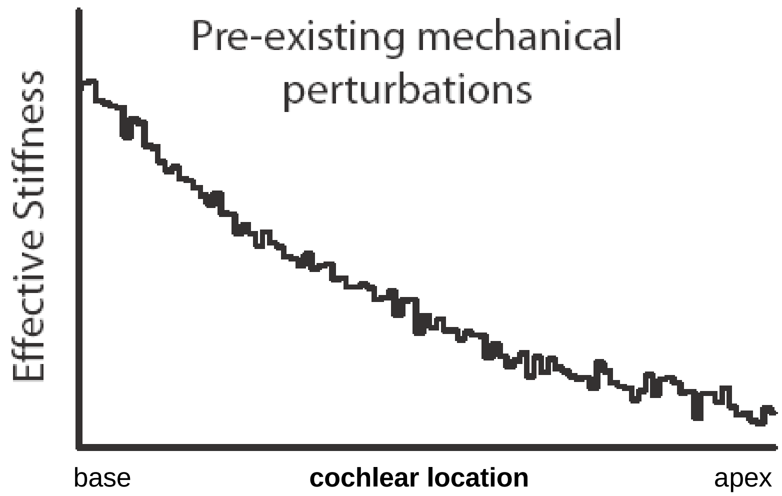


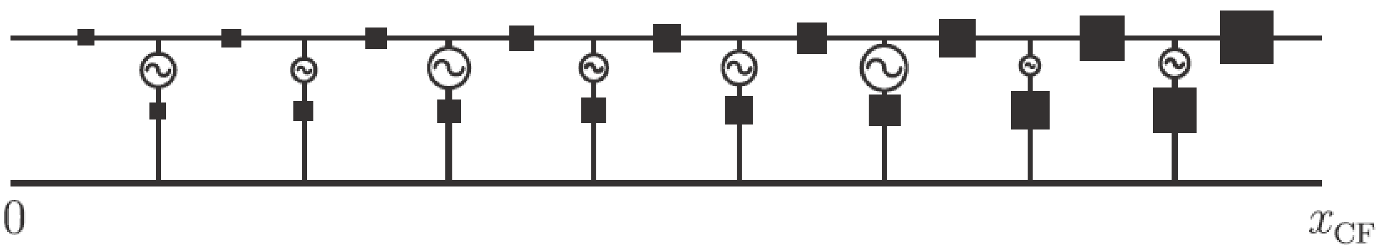
spontaneous or evoked

⇒ Model could serve to elucidate OAE generation

III. OAEs (cont.)

Reflection of energy?





Transmission line with random irregular 'sources'

Shera and Guinan (2007)

Summary

Developed a passive, linear 1-D transmission line model for the cochlea

Simple model captures essential features of the cochlea and can serve as a foundation for more realistic iterations

Ultimate goal is to use cochlear models to better understand auditory function/physiology and potential clinical applications

Georg von Bekesy (1899-1972)

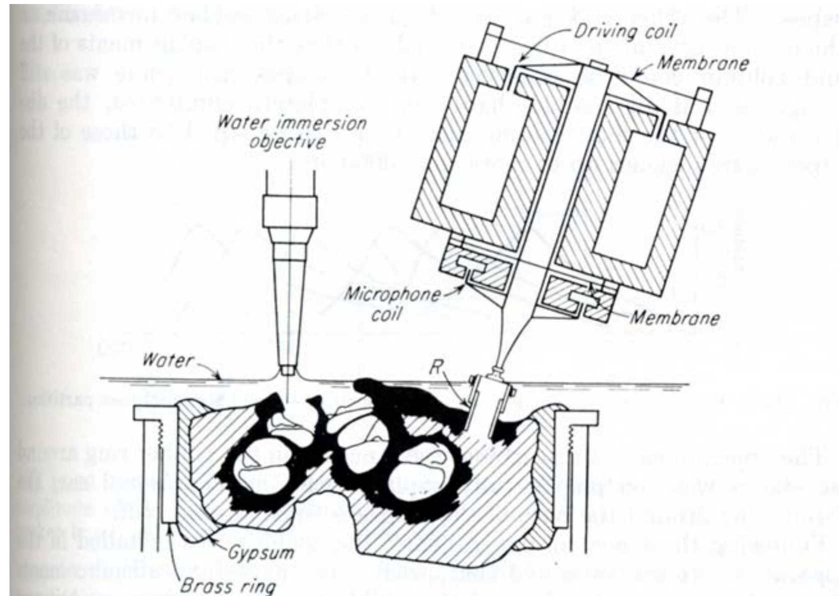


FIG. 11-48. Method of measuring the amplitude of vibration of the cochlear partition in response to volume displacements of the stapes.

Experiments in Hearing (1960)