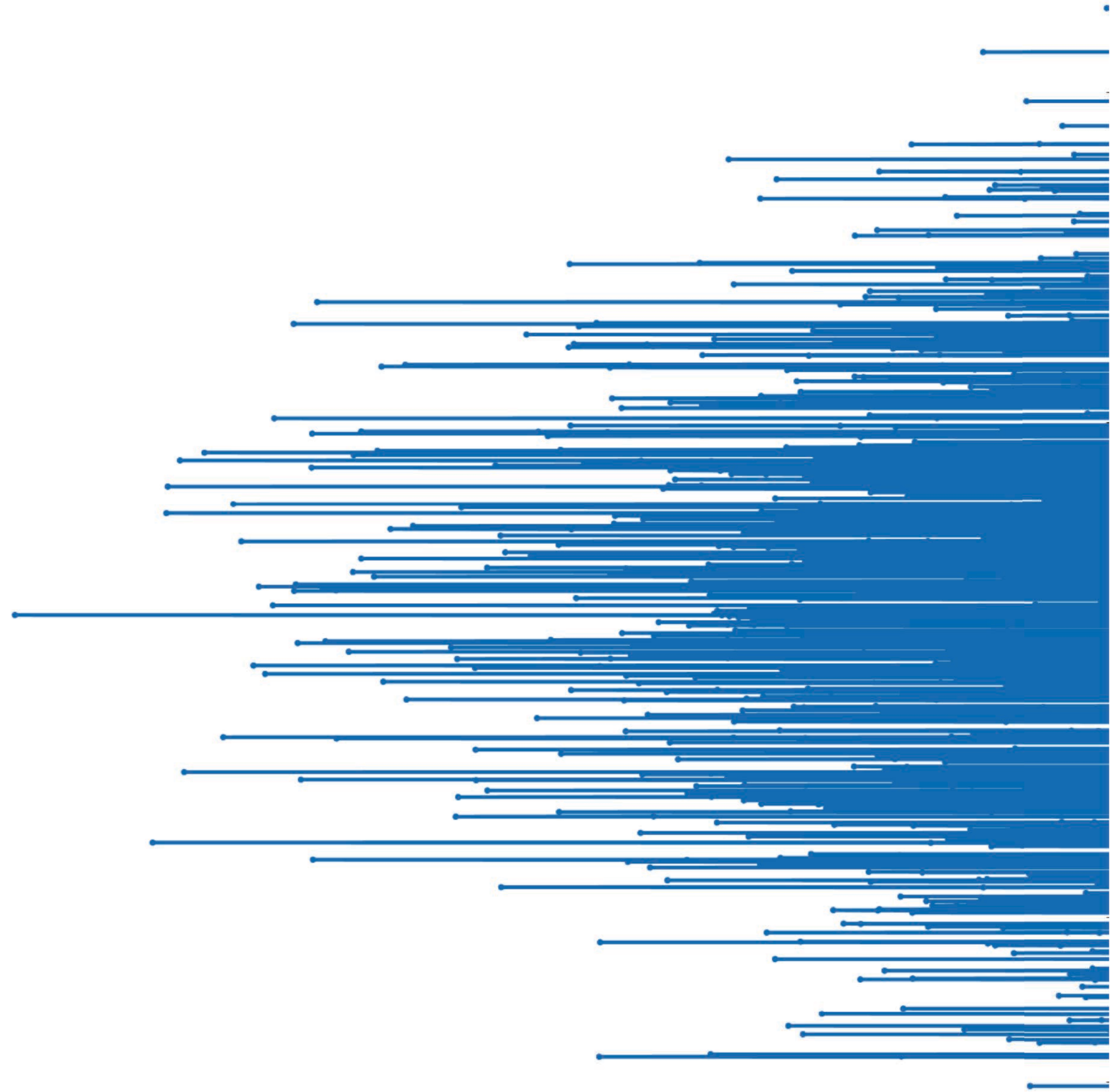
PHYS 2010 (W20)
Classical Mechanics



HW1 SOLUTIONS

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Ref. (re images):
Knudsen & Hjorth (2000), Kesten & Tauck (2012)

Problem 1 (SOL)

Problem 1. A first-order, linear differential equation with constant coefficients and a constant inhomogeneous (drive or input) term has an exponential solution. Therefore, the solution can be written in the form

$$n(t) = n_\infty + \left(n_0 - n_\infty\right) e^{-t/\tau},$$

where $n_0 = n(0)$ is the initial value of $n(t)$ and $n_\infty = \lim_{t\to\infty} n(t)$ is the final value of $n(t)$. The form of this solution can be verified by evaluating $n(t)$ at $t = 0$ and $t \to \infty$. Substitution into the differential equation shows that this solution satisfies the differential equation. The solutions for cases i-vi are shown in Figure 1. The solutions for part a (i and ii) have the same initial and final values but different time constants (by $t = 10$ s, curve ii is just above 6 and has not yet reached its final value of 10). The solutions for part b (iii and iv) have the same initial values and different final values. Although curve iv was calculated with the same time constant as in iii, it doesn't make sense to compare the time constants of the curves, since curve iv isn't changing. The solutions for part c (v and vi) have different initial and final values and the same time constants.

Note: The solution is essentially the same too, just written in a more general way

$$T(t) = T_0 + Ce^{-\alpha t}$$

Problem 1 (SOL cont)

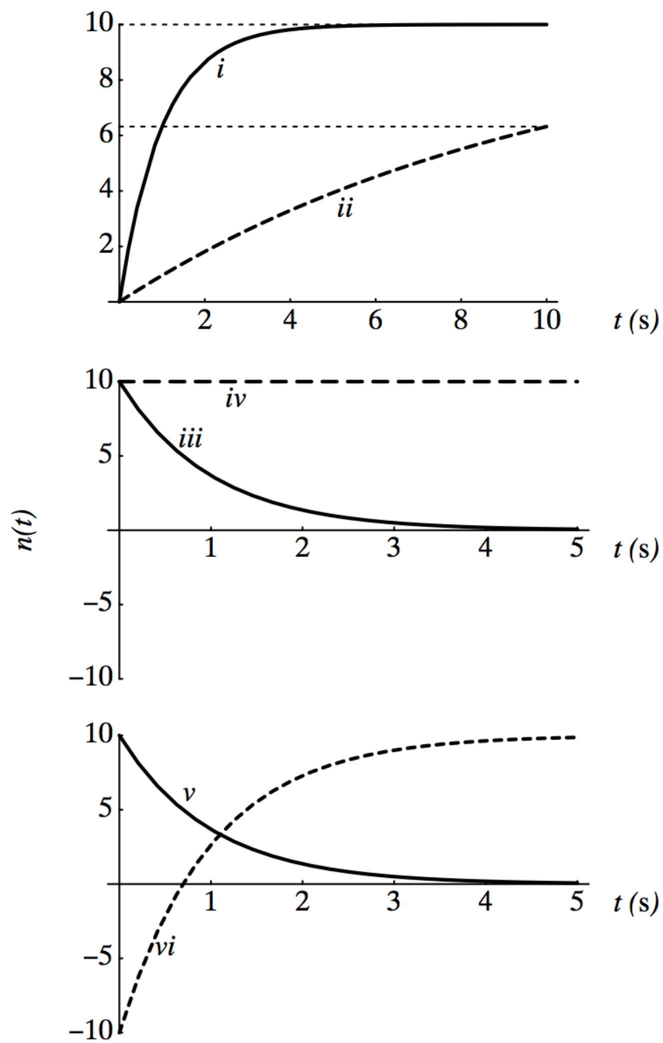

Figure 1. Solutions to parts i-vi. In the upper panel, horizontal dotted lines are shown at the final value of 10 and for the value of $n(t)$ at $t = \tau$, i.e., the line is at $10(1 - e^{-1})$.

Problem 2 (SOL)

Part a)

$$\Delta y = v_{0,y}t + \frac{1}{2}a_y t^2$$

$$\frac{1}{2}a_y t^2 + v_{0,y}t - \Delta y = 0$$

$$t = \frac{-v_{0,y} \pm \sqrt{v_{0,y}^2 - 4\left(\frac{1}{2}a_y\right)(-\Delta y)}}{2\left(\frac{1}{2}a_y\right)} = \frac{-v_0 \sin(30°) \pm \sqrt{v_0^2 \sin^2(30°) + 2a_y(\Delta y)}}{a_y}$$

$$= \frac{-\left(35.3\frac{\text{m}}{\text{s}}\right)\sin(30°) \pm \sqrt{\left(35.3\frac{\text{m}}{\text{s}}\right)\sin^2(30°) + 2\left(-9.8\frac{\text{m}}{\text{s2}}\right)(-255\text{ m})}}{-9.81\frac{\text{m}}{\text{s}^2}} = \frac{-17.65 \pm 72.87}{-9.81}\text{ s}$$

Taking the negative root, $\boxed{t = 9.24\text{ s}}$.

Part b)

$$v^2 - v_0^2 = 2a_y(\Delta y)$$

$$\Delta y = \frac{v^2 - v_0^2}{2a_y} = \frac{0 - \left(35.3\frac{\text{m}}{\text{s}}\right)^2 \sin^2(30°)}{2\left(-9.8\frac{\text{m}}{\text{s}^2}\right)} = 16\text{ m}$$

The ball's maximum height is 16 m above its initial position, or 255 m + 16 m = $\boxed{271\text{ m}}$.

Part c)

$$\Delta x = v_{0,x}t = \left(35.3\frac{\text{m}}{\text{s}}\right)\cos(30°)(9.24\text{ s}) = \boxed{282.5\text{ m}}$$

Problem 3 (SOL)

SET UP

Two cars, initially separated by 24 m, are traveling in a straight line. The blue car, which is in the lead, is traveling at a speed of 28 m/s, while the red car is traveling at a speed of 34 m/s. We can calculate the time it takes for the red car to catch up with the blue car by realizing the red car is traveling at a speed of 6 m/s relative to the blue car. The time is equal to the separation distance divided by this relative speed. Once we calculate this time, we can multiply it by the red car's actual speed of 34 m/s to determine the distance it covered. If the red car accelerated from an initial relative speed of 6 m/s at a rate of $a = (4/3)\ \text{m/s}^2$ instead, we can calculate the time it would take the car to cover a distance of 24 m using the constant acceleration equations.

SOLVE

Part a)

$$t = \frac{24\ \text{m}}{6\frac{\text{m}}{\text{s}}} = \boxed{4\ \text{s}}$$

Part b)

$$\Delta x_{\text{red}} = (4\ \text{s})\left(34\frac{\text{m}}{\text{s}}\right) = \boxed{136\ \text{m}}$$

Part c)

$$\Delta x = v_{0,x}t + \frac{1}{2}a_x t^2$$

$$\frac{1}{2}a_x t^2 + v_{0,x}t - \Delta x = 0$$

$$t = \frac{-v_{0,x} \pm \sqrt{v_{0,x}^2 - 4\left(\frac{1}{2}a_x\right)(-\Delta x)}}{a_x} = \frac{-v_{0,x} \pm \sqrt{v_{0,x}^2 + 2a_x(\Delta x)}}{a_x}$$

$$= \frac{-\left(6\frac{\text{m}}{\text{s}}\right) \pm \sqrt{\left(6\frac{\text{m}}{\text{s}}\right)^2 + 2\left(\frac{4\ \text{m}}{3\ \text{s}^2}\right)(24\ \text{m})}}{\left(\frac{4\ \text{m}}{3\ \text{s}^2}\right)} = \frac{-\left(6\frac{\text{m}}{\text{s}}\right) \pm \left(10\frac{\text{m}}{\text{s}}\right)}{\left(\frac{4\ \text{m}}{3\ \text{s}^2}\right)}$$

Taking the positive root, $t = 3$ s.

REFLECT

Because the car is accelerating in part (c), it makes sense that the time should be less than in part (a). We can double-check our answers by calculating the distance the blue car travels in 4 s: $\Delta x_{\text{blue}} = (4\ \text{s})\left(28\frac{\text{m}}{\text{s}}\right) = 112\ \text{m}$. The blue car travels 24 m less than the red car, which is exactly the original distance separating them.

Problem 4 (SOL)

11. A naval destroyer is testing five clocks. Exactly at noon, as determined by the WWV time signal, on the successive days of a week the clocks read as follows:

Clock	Sun.	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
A	12:36:40	12:36:56	12:37:12	12:37:27	12:37:44	12:37:59	12:38:14
B	11:59:59	12:00:02	11:59:57	12:00:07	12:00:02	11:59:56	12:00:03
C	15:50:45	15:51:43	15:52:41	15:53:39	15:54:37	15:55:35	15:56:33
D	12:03:59	12:02:52	12:01:45	12:00:38	11:59:31	11:58:24	11:57:17
E	12:03:59	12:02:49	12:01:54	12:01:52	12:01:32	12:01:22	12:01:12

How would you arrange these five clocks in the order of their relative value as good timekeepers? Justify your choice.

Best to worse: C, D, A, B, E
(main criteria is consistency, followed by the size of the daily variation)

Problem 5 (SOL)

Example 5. Calculate the speed of an artificial earth satellite, assuming that it is traveling at an altitude h of 140 miles above the surface of the earth where $g = 30 \text{ ft/sec}^2$. The radius of the earth R is 3960 miles.

Like any free object near the earth's surface the satellite has an acceleration g toward the earth's center. It is this acceleration that causes it to follow the circular path. Hence the centripetal acceleration is g, and from Eq. 4–9, $a = v^2/r$, we have

$$g = v^2/(R + h),$$

or

$$v = \sqrt{(R + h)g} = \sqrt{(3960 \text{ miles} + 140 \text{ miles})(5280 \text{ ft/mile})(30 \text{ ft/sec}^2)}$$

$$= 2.55 \times 10^4 \text{ ft/sec} = 17{,}400 \text{ miles/hr.}$$ ◀

Problem 6 (SOL)

Given the two vectors $\mathbf{A} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{B} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, find $\mathbf{A} \times \mathbf{B}$.

Find a unit vector normal to the plane containing the two vectors **A** and **B** above.

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix} = \mathbf{i}(2-1) + \mathbf{j}(-1-4) + \mathbf{k}(-2-1)$$

$$= \mathbf{i} - 5\mathbf{j} - 3\mathbf{k}$$

$$\mathbf{n} = \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|} = \frac{\mathbf{i} - 5\mathbf{j} - 3\mathbf{k}}{[1^2 + 5^2 + 3^2]^{1/2}}$$

$$= \frac{\mathbf{i}}{\sqrt{35}} - \frac{5\mathbf{j}}{\sqrt{35}} - \frac{3\mathbf{k}}{\sqrt{35}}$$

Problem 7 (SOL)

Given the three vectors $\mathbf{A} = \mathbf{i}$, $\mathbf{B} = \mathbf{i} - \mathbf{j}$, and $\mathbf{C} = \mathbf{k}$, find $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$.

Find $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ above.

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1(-1+0) = -1$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{i} - \mathbf{j})0 - \mathbf{k}(1 - 0) = -\mathbf{k}$$

Problem 8 (SOL)

The observer must be on the equator of the earth. The orbit of the space station is a large circle in the equatorial plane with center at the center of the earth. The radius of the orbit can be figured out using the orbiting period of 24 hours* as follows. Let the radius of the orbit be R and that of the earth be R_0.

*For a more accurate calculation, the orbiting period should be taken as 23 hours 56 minutes and 4 seconds.

We have

$$\frac{mv^2}{R} = \frac{GMm}{R^2},$$

where v is the speed of the space station, G is the universal constant of gravitation, m and M are the masses of the space station and the earth respectively, giving

$$v^2 = \frac{GM}{R}.$$

As

$$mg = \frac{GMm}{R_0^2},$$

we have

$$GM = R_0^2 g.$$

Hence

$$v^2 = \frac{R_0^2 g}{R}.$$

For circular motion with constant speed v, the orbiting period is

$$T = \frac{2\pi R}{v}.$$

Hence

$$\frac{4\pi^2 R^2}{T^2} = \frac{R_0^2 g}{R}$$

and

$$R = \left(\frac{R_0^2 T^2 g}{4\pi^2}\right)^{\frac{1}{3}} = 4.2 \times 10^4 \text{ km}.$$

Problem 9 (SOL)

10. The speed of filling (i.e. the amount of liquid which falls into the bucket during unit time) will not change, for although

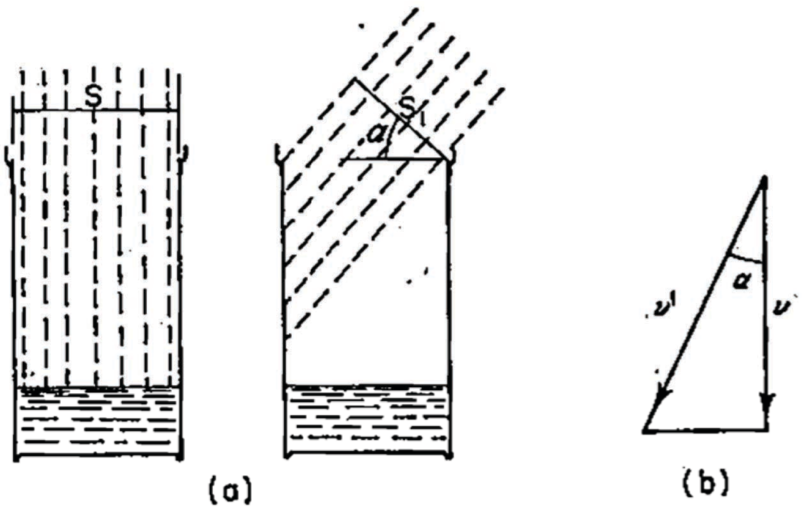

Fig. 157

the area of the cross-section of rain falling into the bucket decreases ($S_1 = S \cos \alpha$, Fig. 157a), the velocity of the drops not only changes direction but also increases in magnitude ($v' = v/\cos \alpha$, Fig. 157b). In other words, the speed at which the bucket fills up depends only on the vertical velocity of the drops, which is not altered by the wind.

Bonus (SOL)

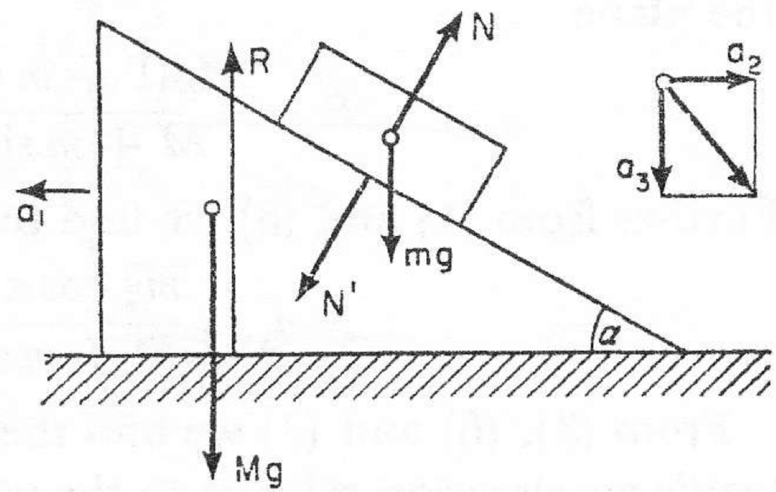

FIG. 166

Let us consider the forces acting on the load m and on the wedge M (Fig. 166). The load m is subject to: (1) its weight mg and (2) the reaction of the wedge N. The wedge is subject to (1) its own weight Mg, (2) the pressure exerted by the load N' and (3) the reaction of the plane, R. As a result of the horizontal component of the pressure exerted by the load, the wedge moves to the left relative to the plane with a horizontal acceleration a_1, which can be found from the equation

$$Ma_1 = N' \sin \alpha. \quad (1)$$

In the vertical direction the wedge has no acceleration, therefore

$$Mg - R + N' \cos \alpha = 0 \quad (2)$$

Bonus (SOL cont)

Let us call the horizontal component of the load's acceleration *relative to the wedge* a_2, and the vertical component a_3. Then the horizontal component of the load's acceleration *relative to the plane* will be $a_2 - a_1$, and the vertical component will be a_3. These accelerations may be found from the equations

$$m(a_2 - a_1) = N \sin\alpha \qquad (3)$$

and

$$ma_3 = mg - N\cos\alpha \qquad (4)$$

Plainly $N' = N$ and

$$a_3 = a_2 \tan\alpha. \qquad (5)$$

From equations (4), (5), (3) and (1) we find that the pressure of the load on the wedge is

$$N = \frac{mMg\cos\alpha}{M + m\sin^2\alpha} \qquad (6)$$

Now from equation (2) we can find the pressure of the wedge on the plane

$$R = \frac{Mg(1 + m\cos^2\alpha)}{M + m\sin^2\alpha}.$$

Further from (1) and (6) we find the wedge's acceleration

$$a_1 = \frac{mg\cos\alpha\sin\alpha}{M + m\sin^2\alpha}. \qquad (7)$$

Bonus (SOL cont)

From (3), (6) and (7) we find the horizontal component of the load's acceleration relative to the wedge

$$a_2 = \frac{(M+m)\, g \cos\alpha \sin\alpha}{M + m \sin^2\alpha}, \qquad (8)$$

and the horizontal component of the load's acceleration relative to the plane

$$a_2 - a_1 = \frac{Mg \cos\alpha \sin\alpha}{M + m \sin^2\alpha}.$$

From (8) and (5) we find that the vertical component of the acceleration of the load relative to the plane

$$a_3 = \frac{(M+m)\, g \sin^2\alpha}{M + m \sin^2\alpha}.$$