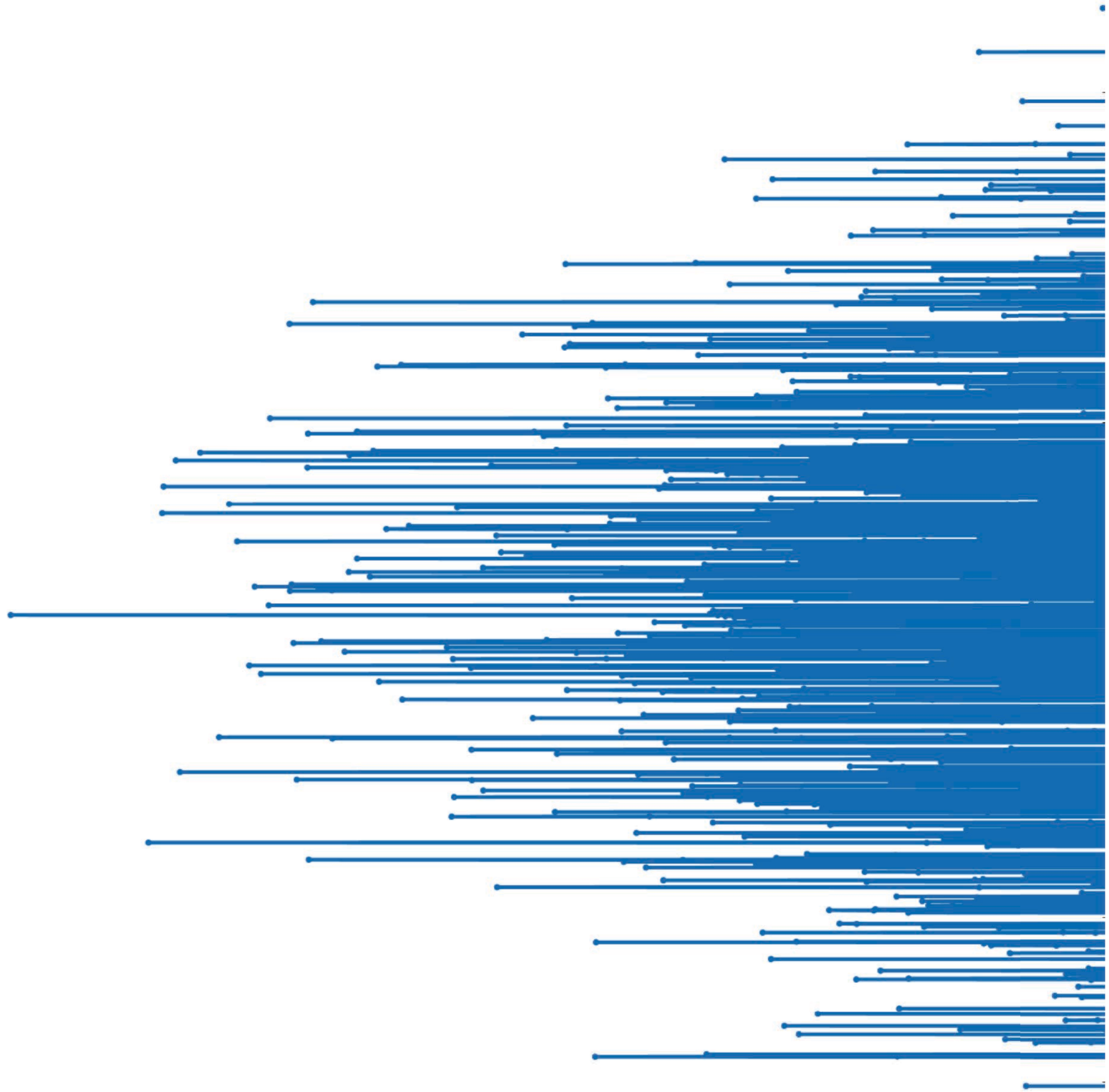
PHYS 2010 (W20)
Classical Mechanics



Due 2020.02.07
HW3

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Ref. (re images):
Knudsen & Hjorth (2000), Kesten & Tauck (2012)

Problem 1

Given the time-varying vector

$$\mathbf{A} = \mathbf{i}\alpha t + \mathbf{j}\beta t^2 + \mathbf{k}\gamma t^3$$

where α, β, and γ are constants, find the first and second time derivatives $d\mathbf{A}/dt$ and $d^2\mathbf{A}/dt^2$.

SOL

$$\frac{d\vec{A}}{dt} = \hat{i}\frac{d}{dt}(\alpha t) + \hat{j}\frac{d}{dt}(\beta t^2) + \hat{k}\frac{d}{dt}(\gamma t^3) = \hat{i}\alpha + \hat{j}2\beta t + \hat{k}3\gamma t^2$$

$$\frac{d^2\vec{A}}{dt^2} = \hat{j}2\beta + \hat{k}6\gamma t$$

Problem 2

A small ball is fastened to a long rubber band and twirled around in such a way that the ball moves in an elliptical path given by the equation

$$\mathbf{r}(t) = \mathbf{i}b\cos\omega t + \mathbf{j}2b\sin\omega t$$

where b and ω are constants. Find the speed of the ball as a function of t. In particular, find v at $t = 0$ and at $t = \pi/2\omega$, at which times the ball is, respectively, at its minimum and maximum distances from the origin.

SOL

$$\vec{v}(t) = -\hat{i}b\omega\sin(\omega t) + \hat{j}2b\omega\cos(\omega t)$$

$$|\vec{v}| = \left(b^2\omega^2\sin^2\omega t + 4b^2\omega^2\cos^2\omega t\right)^{\frac{1}{2}} = b\omega\left(1+3\cos^2\omega t\right)^{\frac{1}{2}}$$

$$\vec{a}(t) = -\hat{i}b\omega^2\cos\omega t - \hat{j}2b\omega^2\sin\omega t$$

$$|\vec{a}| = b\omega^2\left(1+3\sin^2\omega t\right)^{\frac{1}{2}}$$

$$\text{at} \quad t = 0, \quad |\vec{v}| = 2b\omega; \quad \text{at} \quad t = \frac{\pi}{2\omega}, \quad |\vec{v}| = b\omega$$

Fowles & Cassidy

Problem 3

A particle of mass m is released from rest a distance b from a fixed origin of force that attracts the particle according to the inverse square law:

$$F(x) = -kx^{-2}$$

Show that the time required for the particle to reach the origin is

$$\pi\left(\frac{mb^3}{8k}\right)^{1/2}$$

<u>Hint</u>: Treat this as a 1-D problem. Then just integrate. And integrate. And integrate some more!

SOL

$$a = v\frac{dv}{dx} = -\frac{k}{m}x^{-2}$$

$$\int_0^v v\,dv = \int_b^x -\frac{k\,dx}{mx^2}$$

$$\frac{1}{2}v^2 = \frac{k}{m}\left(\frac{1}{x} - \frac{1}{b}\right)$$

$$v = \frac{dx}{dt} = \left[\frac{2k}{m}\left(\frac{1}{x} - \frac{1}{b}\right)\right]^{\frac{1}{2}} = \left[\frac{2k}{mb}\left(\frac{b-x}{x}\right)\right]^{\frac{1}{2}}$$

$$\int_0^t dt = \int_b^0 \left[\frac{mb}{2k}\left(\frac{x}{b-x}\right)\right]^{\frac{1}{2}} dx = \left(\frac{mb^3}{2k}\right)^{\frac{1}{2}} \int_1^0 \left(\frac{\frac{x}{b}}{1-\frac{x}{b}}\right)^{\frac{1}{2}} d\left(\frac{x}{b}\right)$$

Since $x \le b$, say $\frac{x}{b} = \sin^2\theta$

$$t = \left(\frac{mb^3}{2k}\right)^{\frac{1}{2}} \int_{-\frac{\pi}{2}}^0 \frac{\sin\theta\left(2\sin\theta\cos\theta\,d\theta\right)}{\cos\theta} = \left(\frac{2mb^3}{k}\right)^{\frac{1}{2}} \int_{-\frac{\pi}{2}}^0 \sin^2\theta\,d\theta$$

$$t = \left(\frac{mb^3}{8k}\right)^{\frac{1}{2}}\pi$$

Problem 4

A surface-going projectile is launched horizontally on the ocean from a stationary warship, with initial speed v_0. Assume that its propulsion system has failed and it is slowed by a retarding force given by $F(v) = -Ae^{\alpha v}$. **(a)** Find its speed as a function of time, $v(t)$. Find **(b)** the time elapsed and **(c)** the distance traveled when the projectile finally comes to rest. A and α are positive constants.

<u>Hint</u>: There are a handful of ways to solve the ODE that arises in part a (e.g., make a substitution $u = e^{\alpha v}$. You should end up with something not too messy that has an $\ln$....

SOL

$$F(x) = -Ae^{\alpha\dot{x}} = m\ddot{x} \quad \text{or} \quad F(v) = -Ae^{\alpha v} = m\dot{v} \qquad \frac{dv}{e^{\alpha v}} = -\frac{A}{m}dt$$

$$\text{Let } u = e^{\alpha v} \qquad du = \alpha e^{\alpha v} dv \qquad dv = \frac{du}{\alpha e^{\alpha v}} = \frac{du}{\alpha u} \qquad \therefore \frac{du}{u^2} = -\frac{\alpha A}{m}dt$$

Integrating

$$\frac{1}{u} - \frac{1}{u_\circ} = \frac{A}{m}\alpha t \qquad \text{and substituting } e^{\alpha v} = u$$

(a) $v = v_\circ - \dfrac{1}{\alpha}\ln\left[1 + \dfrac{A}{m}e^{\alpha v_\circ}\alpha t\right]$

(b) $t = T @ v = 0$

$$\alpha v_\circ = \ln\left[1 + \frac{A}{m}e^{\alpha v_\circ}\alpha T\right]$$

$$e^{\alpha v_\circ} = 1 + \frac{A}{m}e^{\alpha v_\circ}\alpha T \qquad T = \frac{m}{\alpha A}\left[1 - e^{-\alpha v_\circ}\right]$$

(c) $v\dfrac{dv}{dx} = v = -\dfrac{A}{m}e^{\alpha v} \qquad \dfrac{vdv}{e^{\alpha v}} = -\dfrac{A}{m}dx$

again, let $u = e^{\alpha v} \qquad du = \alpha u dv \quad$ or $\quad dv = \dfrac{du}{\alpha u} \qquad v = \dfrac{1}{\alpha}\ln u$

$$\frac{\left[\frac{1}{\alpha}\ln u\right]\frac{du}{\alpha u}}{u} = -\frac{A}{m}dx \qquad \text{Integrating and solving}$$

$$x = \frac{m}{\alpha^2 A}\left[1 - (1 + \alpha v_\circ)e^{-\alpha v_\circ}\right]$$

Problem 5

Consider the two force functions

(a) $\mathbf{F} = \mathbf{i}x + \mathbf{j}y$

(b) $\mathbf{F} = \mathbf{i}y - \mathbf{j}x$

Verify that (a) is conservative and that (b) is nonconservative by showing that the integral $\int \mathbf{F} \cdot d\mathbf{r}$ is independent of the path of integration for (a), but not for (b), by taking two paths in which the starting point is the origin (0, 0), and the endpoint is (1, 1). For one path take the line $x = y$. For the other path take the x-axis out to the point (1, 0) and then the line $x = 1$ up to the point (1, 1).

SOL

(a) $\vec{F} = \hat{i}x + \hat{j}y$

on the path $x = y$: $d\vec{r} = \hat{i}dx + \hat{j}dy$

$$\int_{(0,0)}^{(1,1)} \vec{F}\cdot d\vec{r} = \int_0^1 F_x dx + \int_0^1 F_y dy = \int_0^1 x dx + \int_0^1 y dy = 1$$

on the path along the x-axis: $d\vec{r} = \hat{i}dx$

and on the line $x = 1$: $d\vec{r} = \hat{j}dy$

$$\int_{(0,0)}^{(1,1)} \vec{F}\cdot d\vec{r} = \int_0^1 F_x dx + \int_0^1 F_y dy = 1$$

$\vec{F}$ is conservative.

(b) $\vec{F} = \hat{i}y - \hat{j}x$

on the path $x = y$:

$$\int_{(0,0)}^{(1,1)} \vec{F}\cdot d\vec{r} = \int_0^1 F_x dx + \int_0^1 F_y dy = \int_0^1 y dx - \int_0^1 x dy$$

and, with $x = y$ $\quad \int_{(0,0)}^{(1,1)} \vec{F}\cdot d\vec{r} = \int_0^1 x dx - \int_0^1 y dy = 0$

on the x-axis: $\quad \int_{(0,0)}^{(1,0)} \vec{F}\cdot d\vec{r} = \int_0^1 F_x dx = \int_0^1 y dx$

and, with $y = 0$ on the x-axis $\quad \int_{(0,0)}^{(1,0)} \vec{F}\cdot d\vec{r} = 0$

on the line $x = 1$: $\quad \int_{(1,0)}^{(1,1)} \vec{F}\cdot d\vec{r} = \int_0^1 F_y dy = \int_0^1 x dy$

and, with $x = 1$ $\quad \int_{(1,0)}^{(1,1)} \vec{F}\cdot d\vec{r} = \int_0^1 dy = 1$

on this path $\quad \int_{(0,0)}^{(1,1)} \vec{F}\cdot d\vec{r} = 0 + 1 = 1$

$\vec{F}$ is *not* conservative.

Problem 6

Show that the vector field $\vec{F}(x, y) = y \cos x \vec{i} + (\sin x + y)\vec{j}$ is path-independent.

Hint: Suppose there is a potential function. What assumptions can you make then about that function?

SOL

Let's suppose $\vec{F}$ does have a potential function f, so that $\vec{F} = \operatorname{grad} f$. This means

$$\frac{\partial f}{\partial x} = y\cos x \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \sin x + y.$$

Integrating the expression for $\partial f/\partial x$ with respect to x shows that

$$f(x, y) = y\sin x + C(y) \qquad \text{where } C(y) \text{ is a function of } y \text{ only.}$$

The constant of integration here is an arbitrary function $C(y)$ of y, since $\partial(C(y))/\partial x = 0$. Differentiating this expression for $f(x, y)$ with respect to y and using $\partial f/\partial y = \sin x + y$ gives

$$\frac{\partial f}{\partial y} = \sin x + C'(y) = \sin x + y.$$

Thus, we must have $C'(y) = y$, so $g(y) = y^2/2 + A$, where A is some constant. Thus,

$$f(x, y) = y\sin x + \frac{y^2}{2} + A$$

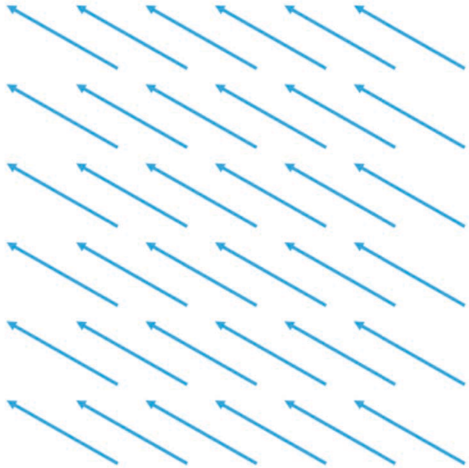
is a potential function for $\vec{F}$. Therefore, $\vec{F}$ is path-independent.

Hughes-Hallet et al

Problem 7

Indicate which vector fields are *conservative* and briefly justify.

A



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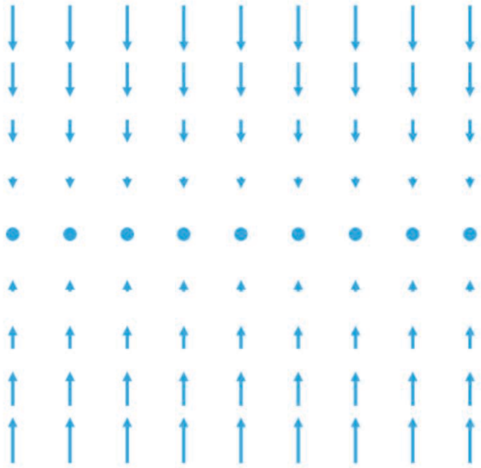

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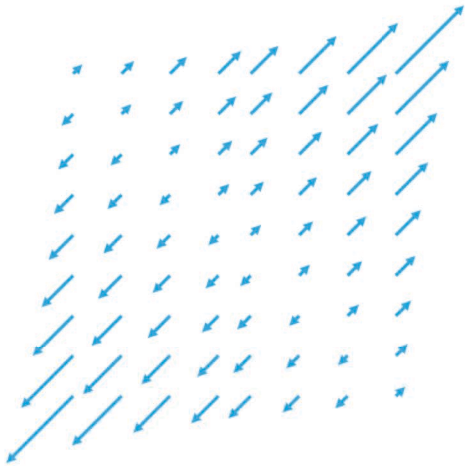

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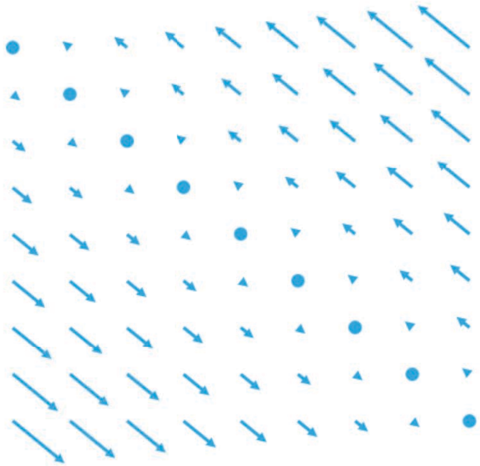


SOL

Problem 8

The sun is about 25,000 light years from the center of the galaxy and travels approximately in a circle with a period of 170,000,000 years. The earth is 8 light minutes from the sun. From these data alone, find the approximate gravitational mass of the galaxy in units of the sun's mass. You may assume that the gravitational force on the sun may be approximated by assuming that all the mass of the galaxy is at its center.

Hint: Connections between centripetal accelerations and Newtons' Law of Gravitation?

Yung-Kuo (1994)

SOL

For the motion of the earth around the sun,

$$\frac{mv^2}{r} = \frac{Gmm_s}{r^2},$$

where r is the distance from the earth to the sun, v is the velocity of the earth, m and m_s are the masses of the earth and the sun respectively.

For the motion of the sun around the center of the galaxy,

$$\frac{m_s V^2}{R} = \frac{Gm_s M}{R^2},$$

where R is the distance from the sun to the center of the galaxy, V is the velocity of the sun and M is the mass of the galaxy.

Hence

$$M = \frac{RV^2}{G} = \frac{R}{r}\left(\frac{V}{v}\right)^2 m_s .$$

Using $V = 2\pi R/T$, $v = 2\pi r/t$, where T and t are the periods of revolution of the sun and the earth respectively, we have

$$M = \left(\frac{R}{r}\right)^3 \left(\frac{t}{T}\right)^2 m_s .$$

With the data given, we obtain

$$M = 1.53 \times 10^{11} m_s .$$

Problem 9

An Olympic diver of mass m begins his descent from a 10 meter high diving board with zero initial velocity.

(a) Calculate the velocity V_0 on impact with the water and the approximate elapsed time from dive until impact (use any method you choose).

Assume that the buoyant force of the water balances the gravitational force on the diver and that the viscous force on the diver is bv^2.

(b) Set up the equation of motion for vertical descent of the diver through the water. Solve for the velocity V as a function of the depth x under water and impose the boundary condition $V = V_0$ at $x = 0$.

(c) If $b/m = 0.4\ \mathrm{m}^{-1}$, estimate the depth at which $V = V_0/10$.

(d) Solve for the vertical depth $x(t)$ of the diver under water in terms of the time under water.

Hint: Once the diver is in the water, the force due to gravity will be counterbalanced by buoyancy (i.e., only drag will create a non-zero net force). As for (d), you've seen something like this before....

SOL

(a)

$$V_0 = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 10} = 14 \text{ m/s} .$$

The time elapsed from dive to impact is

$$t = \frac{V_0}{g} = \frac{14}{9.8} = 1.43 \text{ s} .$$

(b) As the gravitational force on the diver is balanced by the buoyancy, the equation of motion of the diver through the water is

$$m\ddot{x} = -b\dot{x}^2 ,$$

or, using $\ddot{x} = \dot{x}d\dot{x}/dx$,

$$\frac{d\dot{x}}{\dot{x}} = -\frac{b}{m}dx .$$

Integrating, with $\dot{x} = V_0$ at $x = 0$, we obtain

$$V \equiv \dot{x} = V_0 e^{-\frac{b}{m}x} .$$

(c) When $V = V_0/10$,

$$x = \frac{m}{b}\ln 10 = \frac{\ln 10}{0.4} = 5.76 \text{ m} .$$

(d) As $dx/dt = V_0 e^{-\frac{b}{m}x}$,

$$e^{\frac{b}{m}x}dx = V_0 dt .$$

Integrating, with $x = 0$ at $t = 0$, we obtain

$$\frac{m}{b}(e^{\frac{b}{m}x} - 1) = V_0 t ,$$

or

$$x = \frac{m}{b}\ln\left(1 + bV_0\frac{t}{m}\right) .$$

Yung-Kuo (1994)

Bonus

A gun is located at the bottom of a hill of constant slope ϕ. Show that the range of the gun measured up the slope of the hill is

$$\frac{2v_0^2 \cos\alpha \sin(\alpha-\phi)}{g \cos^2\phi}$$

where α is the angle of elevation of the gun, and that the maximum value of the slope range is

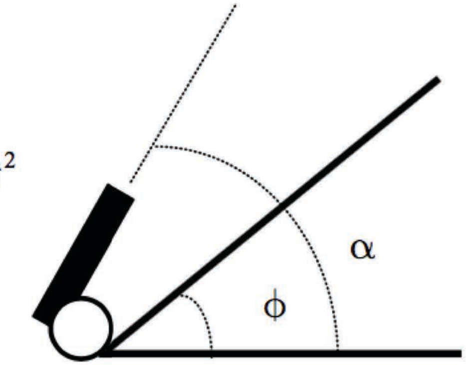
$$\frac{v_0^2}{g(1+\sin\phi)}$$

Hint: Note that α is relative to flat ground (not the hill). Drawing a diagram helps. Also, trig identities such as $\sin(\alpha+\theta)$ or the like might be useful

SOL

4.8 $x = R\cos\phi$ and $x = v_{\circ x}t = (v_\circ \cos\alpha)t$

so $t = \dfrac{R\cos\phi}{v_\circ \cos\alpha}$



$y = R\sin\phi$ and $y = v_{\circ y}t - \frac{1}{2}gt^2 = (v_\circ \sin\alpha)t - \frac{1}{2}gt^2$

$$R\sin\phi = (v_\circ \sin\alpha)\frac{R\cos\phi}{v_\circ\cos\alpha} - \frac{1}{2}g\left(\frac{R\cos\phi}{v_\circ\cos\alpha}\right)^2$$

$$\sin\phi = \tan\alpha\cos\phi - \frac{gR\cos^2\phi}{2v_\circ^2\cos^2\alpha}$$

$$R = \frac{2v_\circ^2\cos^2\alpha}{g\cos^2\phi}(\tan\alpha\cos\phi - \sin\phi) = \frac{2v_\circ^2\cos\alpha}{g\cos^2\phi}(\sin\alpha\cos\phi - \cos\alpha\sin\phi)$$

From Appendix B, $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$

$$R = \frac{2v_\circ^2\cos\alpha}{g\cos^2\phi}\sin(\alpha-\phi)$$

R is a maximum for $\dfrac{dR}{d\alpha} = 0 = \dfrac{2v_\circ^2}{g\cos^2\phi}\left[-\sin\alpha\sin(\alpha-\phi) + \cos\alpha\cos(\alpha-\phi)\right]$

Implies that $\cos\alpha\cos(\alpha-\phi) - \sin\alpha\sin(\alpha-\phi) = 0$

From appendix B, $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

so $\cos(2\alpha-\phi) = 0$

$$2\alpha - \phi = \frac{\pi}{2} \qquad\qquad \alpha = \frac{\pi}{4} + \frac{\phi}{2}$$

$$R_{\max} = \frac{2v_\circ^2}{g\cos^2\phi}\cos\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\sin\left(\frac{\pi}{4}-\frac{\phi}{2}\right)$$

Now $\sin\left(\frac{\pi}{4}-\frac{\phi}{2}\right)=\cos\left[\frac{\pi}{2}-\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\right]=\cos\left(\frac{\pi}{4}+\frac{\phi}{2}\right)$

$$R_{\max}=\frac{2v_\circ^2}{g\cos^2\phi}\cos^2\left(\frac{\pi}{4}+\frac{\phi}{2}\right)$$

Again using Appendix B, $\cos 2\theta=\cos^2\theta-\sin^2\theta=2\cos^2\theta-1$

$$R_{\max}=\frac{2v_\circ^2}{g\cos^2\phi}\left[\frac{1}{2}\cos\left(\frac{\pi}{2}+\phi\right)+\frac{1}{2}\right]=\frac{v_\circ^2}{g\cos^2\phi}\left[\cos\left(\frac{\pi}{2}+\phi\right)+1\right]$$

Using $\cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta$,

$$R_{\max}=\frac{v_\circ^2}{g\left(1-\sin^2\phi\right)}\left(1-\sin\phi\right)$$

$$R_{\max}=\frac{v_\circ{}^2}{g\left(1+\sin\phi\right)}$$