$\boxed{\text{HW4 Prob. 1}}$

$m = 15\,g \ , \quad f_0 = 4.0\ Hz$

a) $\omega_0 = \sqrt{k/m} = 2\pi f_0 \longrightarrow k = 4\pi^2 f_0^2 m$

$k = 4\pi^2 (4.0)^2 (0.015) \simeq \boxed{9.5\ N/m = k}$ $\left(\text{or } \left[\frac{N}{m}\right] = \left[\frac{kg}{s^2}\right]\right)$

b) Need to be a bit careful here about the coefficients. Assume we start w/

$$m\ddot{x} + b\dot{x} + kx = 0 \longrightarrow \ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

$[\text{where } \gamma \equiv b/m]$

- From our notes we know one solution is $x(t) = A_0 e^{-\gamma t/2}\cos(\omega t + \alpha)$ where $\omega = \sqrt{\frac{k}{m} - b^2/4m^2}$ and $A_0 = |x(0)|$ (assuming $\dot{x}(0) = 0$)

- Thus $A(t) = A_0 e^{-t\gamma/2}$. Note that this is no longer SHM (as the problem originally prescribed) and we do not know A_0 (it's not given!)

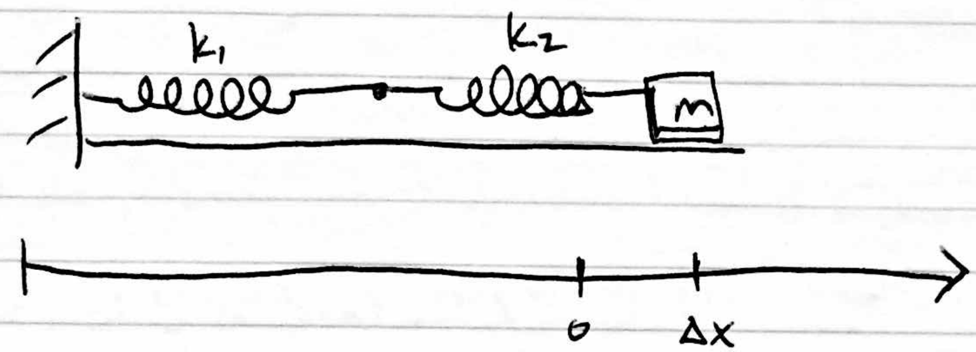
$0.5 = A_0 e^{-4.0\gamma/2}$

Best we can say: $\boxed{\gamma = -\frac{1}{2}\ln\left(\frac{0.5}{A_0}\right)}$ $\Longrightarrow$ Can't solve part b) as stated better than that

NOTE: If instead problem had "decreases by $\frac{1}{2}$ over 4.0 s", the unknown would cancel (by virtue of becoming a ratio)

$\therefore$

Prob. 2



- Assume the mass is pulled some distance ΔX from equilibrium, stretching the springs (let spring 1 be extended ΔX_1 and spring 2 by ΔX_2 such that $\Delta X = \Delta X_1 + \Delta X_2$)

- Let k_T be the total effective stiffness of the combined springs. Then the total force acting on the mass is

$$F_M = -k_T \Delta X$$

- <u>BUT</u> the force acting on the ~~spring~~ mass is also acting on spring 2. Which in turn is acting on spring 1. That is, the force extending spring 1 is the same as that extending spring 2 (and also that acting on the mass). Thus Newton's 3rd law tells us that

$$F_m = -k_1 \Delta X_1 = -k_2 \Delta X_2$$

- So then: $\Delta X = \Delta X_1 + \Delta X_2 = -\frac{F_m}{k_1} + \frac{F_m}{k_2} = -\frac{k_1+k_2}{k_1 k_2} F_m$

- But $F_M = -k_T \Delta X$

$$\Longrightarrow \boxed{k_T = \frac{k_1 k_2}{k_1 + k_2}}$$

<u>NOTE</u> springs in "series" is directly equivalent to capacitors in series!

cont

Prob. 2 (cont)

- Since $\omega_0 = \sqrt{\frac{k_T}{m}}$, we have $\omega_0 = \sqrt{\frac{k_1 k_2}{m(k_1+k_2)}} = 2\pi f_0$
- But we need f_0 in terms of f_1 and f_2

$$\omega_1^2 = k_1/m \longrightarrow k_1 = m\omega_1^2 = m4\pi^2 f_1^2$$
$$\omega_2^2 = k_2/m \longrightarrow k_2 = m\omega_2^2 = m4\pi^2 f_2^2$$

$$\Longrightarrow f_0 = \frac{1}{2\pi}\sqrt{\frac{(m4\pi^2)^2 f_1^2 f_2^2}{m(m4\pi^2)(f_1^2+f_2^2)}} = \frac{1}{2\pi}\sqrt{4\pi^2 \frac{f_1^2 f_2^2}{f_1^2+f_2^2}}$$

$$\Longrightarrow \boxed{f_0 = \sqrt{\frac{f_1^2 f_2^2}{f_1^2+f_2^2}}}$$

$\boxed{\text{HW 4 Prob. 3}}$

- SHO $\longrightarrow$ energy is conserved
- Known: x_1, $\dot{x}_1$, x_2 and $\dot{x}_2$ $\Longrightarrow$ Determine ω_0 and A

- Conserv. of energy reqs. $\cancel{\tfrac{1}{2}} m\dot{x}_1^2 + \cancel{\tfrac{1}{2}} k x_1^2 = \cancel{\tfrac{1}{2}} m\dot{x}_2^2 + \cancel{\tfrac{1}{2}} k x_2^2$

$$k(x_1^2 - x_2^2) = m(\dot{x}_2^2 - \dot{x}_1^2)$$

$$\rightarrow \omega_0 = \sqrt{\frac{k}{m}} = \boxed{\left[\frac{\dot{x}_2^2 - \dot{x}_1^2}{x_1^2 - x_2^2}\right]^{\frac{1}{2}} = \omega_0}$$

- Now the total energy of the system (call it E) is equivalent to the condition when the mass is at rest and the spring fully extended/compressed (i.e. $x = \pm A$). Thus

$$\frac{1}{2}kA^2 = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}kx_1^2 = \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}kx_2^2$$

So
$$A^2 = \frac{m}{k}\dot{x}_1^2 + x_1^2 = \dot{x}_1^2\left(\frac{x_1^2 - x_2^2}{\dot{x}_2^2 - \dot{x}_1^2}\right) + x_1^2$$

$$\Longrightarrow A = \left[\frac{\dot{x}_1^2 x_1^2 - \dot{x}_1^2 x_2^2 + x_1^2(\dot{x}_2^2 - \dot{x}_1^2)}{\dot{x}_2^2 - \dot{x}_1^2}\right]^{\frac{1}{2}}$$

$$\boxed{= \left[\frac{x_1^2\dot{x}_2^2 - \dot{x}_1^2 x_2^2}{\dot{x}_2^2 - \dot{x}_1^2}\right]}$$

$\therefore$

HW4 Prob 4

- Assume solution of the form $x(t) = Ae^{-\gamma t/2}\cos(\omega t + \alpha)$ where $\omega = \sqrt{\frac{k}{m} - \gamma^2/4}$

- Now $\frac{dx}{dt} = -\frac{\gamma}{2}Ae^{-\gamma t/2}\cos(\omega t+\alpha) - \omega Ae^{-\gamma t/2}\sin(\omega t+\alpha) = 0$ at extremas. Thus

$$\frac{\gamma}{2}\cos(\omega t+\alpha) + \omega\sin(\omega t+\alpha) = 0 \longrightarrow \tan(\omega t+\alpha) = -\frac{\gamma}{2\omega}$$

which repeats after the interval $T = 2\pi/\omega$

- Let x_i and x_{i+1} be the positions of two successive maxima. Then

$$x_i = Ae^{-\gamma t/2}\cos(\omega t+\alpha) \quad \text{and} \quad x_{i+1} = Ae^{-\gamma(t+T)/2}\cos[\omega(t+T)+\alpha]$$

Note that $\cos(\omega t+\alpha) = \cos[\omega(t+T)+\alpha]$

- Thus we have

$$\frac{x_i}{x_{i+1}} = \frac{e^{-\gamma t/2}}{e^{-\gamma(t+T)/2}} = \frac{e^{-\gamma t/2}}{e^{-\gamma t/2}\cdot e^{-\gamma T/2}} = e^{\gamma T/2}$$

$$\Longrightarrow \boxed{\frac{x_i}{x_{i+1}} = e^{\gamma\pi/\omega}}$$

(which is a const.)

NOTE: Another common assumed form of sol. is $x(t) = Ae^{-\gamma t}\cos(\omega t+\alpha)$, which leads to $x_i/x_{i+1} = e^{\gamma T} = e^{\gamma 2\pi/\omega}$. Really just hinges on assumed form of ODE (i.e. $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$ vs $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$)

HW 4 Prob. 5

- $\omega_d = 2\pi f_d = 200\pi$ Hz (since $f_d = 100$ Hz)

we are told this

- Now from the last problem, we know that $\frac{x_i}{x_{i+1}} = e^{-\gamma T/2} = \frac{1}{2}$ (i.e. we assume the ODE of the form $\ddot{x} + \gamma\dot{x} + \omega_0^2 = 0$ such that $\gamma = b/m$ from $m\ddot{x} + b\dot{x} + kx = 0$, not $\gamma = \frac{b}{2m}$)

Thus $\ln\left(e^{-\gamma T/2}\right) = -\frac{\gamma T}{2} = \ln\left(\frac{1}{2}\right) = -\ln 2$

$$\rightsquigarrow \gamma = \frac{1}{T_d} \cdot 2\ln 2 = f_d \cdot 2\ln 2 = 138.6$$

- Now $\omega_d^2 = \omega_0^2 - \frac{\gamma^2}{4}$

so $\omega_0^2 = (2\pi f_0)^2 = \omega_d^2 + \frac{\gamma^2}{4} = (2\pi f_d)^2 + \frac{\gamma^2}{4}$

$$\Rightarrow f_0 = \frac{1}{2\pi}\sqrt{4\pi^2(100)^2 + \frac{(138.6)^2}{4}} \simeq \boxed{100.6 \text{ Hz} = f_0}$$

- For ω_R (the resonant freq.), we have

$$\omega_R^2 = \omega_0^2 - \frac{\gamma^2}{2} \rightsquigarrow f_R = \frac{1}{2\pi}\sqrt{(2\pi f_0)^2 - \frac{\gamma^2}{2}}$$

$$= \frac{1}{2\pi}\sqrt{4\pi^2 \cdot 100.6^2 - \frac{138.6^2}{2}}$$

$$\boxed{\simeq 99.4 \text{ Hz} = \omega_R}$$

∴

HW4 Prob. 6

- Eqn. of motion is $m\ddot{x} + b\dot{x} + kx = F_0 e^{-\alpha t} \cos \omega t = F_{ext}$
- Note that from the hint: $\beta \equiv -\alpha + i\omega \rightarrow Re(e^{\beta t}) = \frac{F_{ext}}{F_0}$
- Let us assume a solution of the form: $z(t) = Ae^{\beta t - i\phi}$ (such that $x = Re(z)$). Plugging into our ODE, we obtain:

$$z = Ae^{\beta t - i\phi} = Ae^{\beta t}e^{-i\phi}$$
$$\dot{z} = \beta A e^{\beta t - i\phi} = \beta z \qquad \rightsquigarrow z(\beta^2 m + b\beta + k) = F_0 e^{\beta t}$$
$$\ddot{z} = \beta^2 z$$

$$\text{or} \quad A\cancel{e^{\beta t}} e^{-i\phi}(m\beta^2 + b\beta + k) = F_0 \cancel{e^{\beta t}} \longrightarrow m\beta^2 + b\beta + k = \frac{F_0}{A} e^{i\phi}$$

$\rightsquigarrow$ Goal now is to solve for A and ϕ (in terms of β, m, b and k and ω)
Couple approaches to do this, below just being one

$$\beta^2 = (-\alpha + i\omega)(-\alpha + i\omega) = \alpha^2 - \omega^2 - i2\alpha\omega$$

$$\text{so} \quad m\beta^2 + b\beta + k = m(\alpha^2 - \omega^2) - i2m\alpha\omega - b\alpha + ib\omega + k$$
$$= \frac{F_0}{A}\cos\phi + i\frac{F_0}{A}\sin\phi$$
$$= [m(\alpha^2 - \omega^2) - b\alpha + k] + i[b\omega - 2m\alpha\omega]$$

$\rightsquigarrow$ Can equate real and imaginary parts:

$$m(\alpha^2 - \omega^2) - b\alpha + k = \frac{F_0}{A}\cos\phi$$

$$\omega[b - 2m\alpha] = \frac{F_0}{A}\sin\phi$$

$\boxed{\text{Cont}}$

HW4 Prob. 6 (cont)

- Can divide two equations to obtain

$$\frac{\sin\phi}{\cos\phi} = \tan\phi = \frac{\omega[b-2m\alpha]}{m(\alpha^2-\omega^2)-b\alpha+k} \longrightarrow \boxed{\phi = \tan^{-1}\left[\frac{\omega(b-2m\alpha)}{m(\alpha^2-\omega^2)-b\alpha+k}\right]}$$

- To find A, we note that we must have $\sin^2\phi + \cos^2\phi = 1$. Thus

$$\left(\frac{A}{F_0}\right)^2\left[\omega(b-2m\alpha)\right]^2 + \left(\frac{A}{F_0}\right)^2\left[m(\alpha^2-\omega^2)-b\alpha+k\right]^2 = 1$$

⇝ Rearranging:

$$\boxed{A = \frac{F_0}{\left\{\left[\omega(b-2\alpha m)\right]^2+\left[m(\alpha^2-\omega^2)-b\alpha+k\right]^2\right\}^{\frac{1}{2}}}}$$

- Thus our solution is $\boxed{x(t) = Ae^{-\alpha t}\cos(\omega t-\phi)}$ where A and ϕ are noted above

<u>NOTE</u>: This last bit stems from $x = \mathrm{Re}(z) = \mathrm{Re}\left(Ae^{-i\phi}e^{-\alpha t}e^{i\omega t}\right)$

$$= \mathrm{Re}\left[Ae^{-\alpha t}\cdot e^{i(\omega t-\phi)}\right] = Ae^{-\alpha t}\cos(\omega t-\phi)$$

$\therefore$

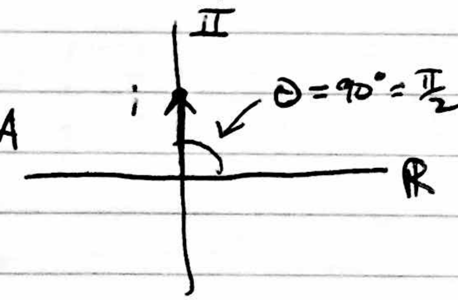
HW4 Prob. 7

<u>NOTE</u>: $i = j = \sqrt{-1}$

- The key to this is to re-write i as a complex exponential

$$i = Ae^{i\theta} = 1 \cdot e^{i\pi/2} = e^{i\pi/2}$$

$$|i| = 1 = A$$



$$\text{Thus } i^i = \left(e^{i\pi/2}\right)^i$$

$$= e^{-\pi/2}$$

- Two ways to figure out what x is for $x = e^{-\pi/2}$

a) calculator $\longrightarrow$ $x \simeq 0.208$

b) $x \simeq e^{-1.57} \simeq \dfrac{1}{e^{1.57}} \simeq \dfrac{1}{2.72^{1.57}} \simeq \left[2.75 \cdot \sqrt{2.75}\right]^{-1}$

$\simeq \left[2.75 \cdot 1.65\right]^{-1} \simeq \dfrac{1}{4.53} \simeq \dfrac{1}{4.5} \simeq 0.22$

(good practice to do this back-of-the-envelope!)

$\longrightarrow$ Either way: Yes, buy the object from the mathematician. But only really for memorie's sake (as it is chiefly "worth" 20¢, so you get what you pay for)

$\therefore$

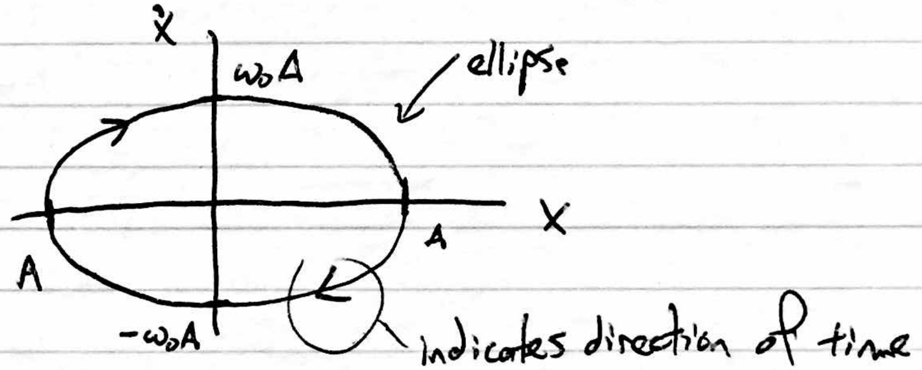
HW4 Prob. 8

- For SHO, energy is conserved such that

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = E = \frac{1}{2}kA^2 \longrightarrow \begin{array}{l} x \in [-A, A] \\ \dot{x} \in [-\sqrt{\frac{k}{m}}A, \sqrt{\frac{k}{m}}A] \end{array}$$

- Another way to see this is that if we assume $x(t) = A\cos(\omega_0 t + \theta)$, then $\dot{x}(t) = -\omega_0 A \sin(\omega_0 t + \theta)$, where $\omega_0 = \sqrt{k/m}$. Thus, x takes on values between $-A$ and A while $\dot{x}$ takes on values between $-\omega_0 A$ and $\omega_0 A$

- Either way:



- If there is damping, the system will lose energy, meaning that the max/min values for x and $\dot{x}$ slowly decrease. Since the system is autonomous, Existance + Uniqueness theorem guarantees that orbits in the phase plane do not cross. Putting all that together, we have a inward-going spiral