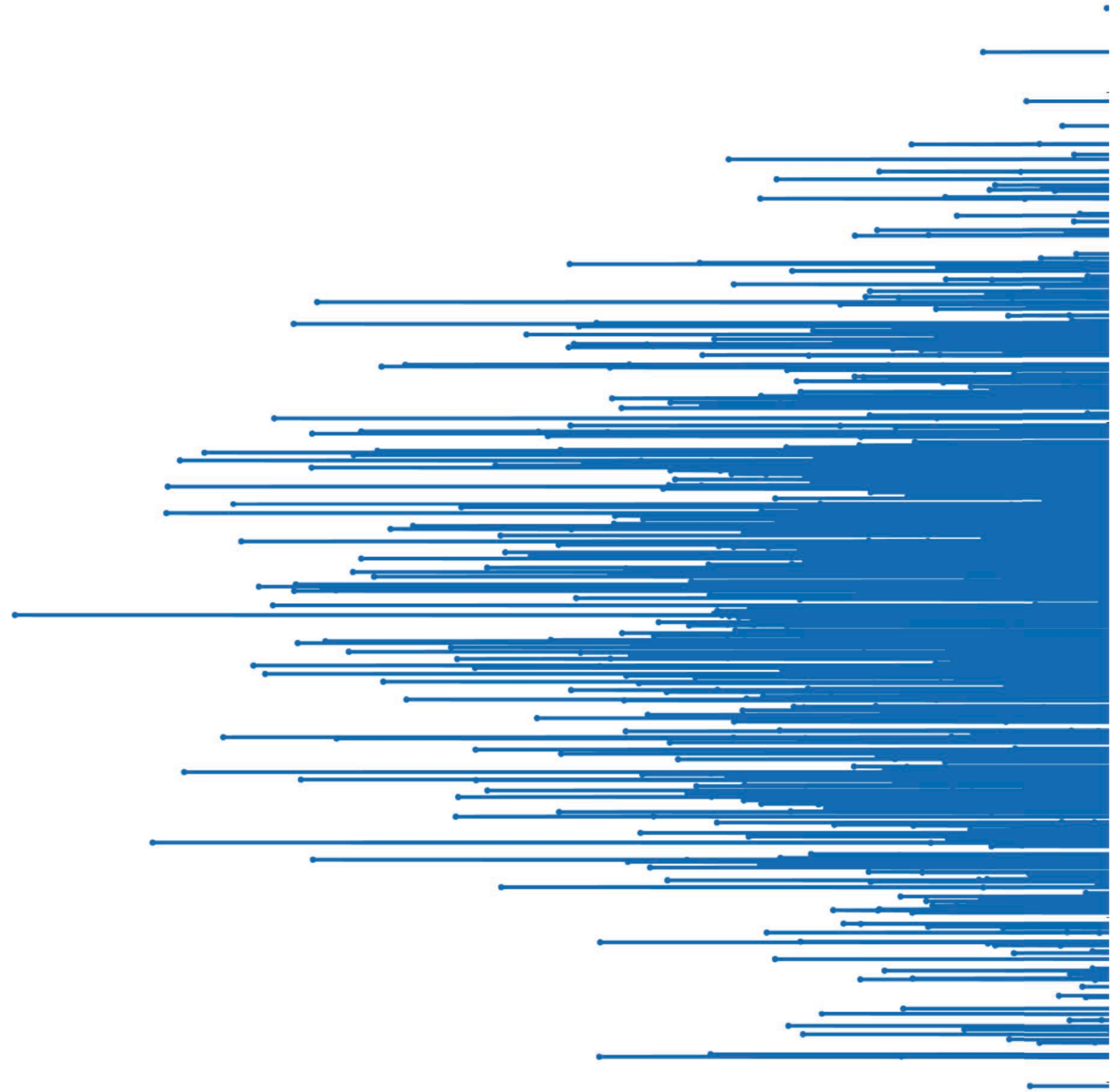
PHYS 2010 (W20)
Classical Mechanics



HW5

Christopher Bergevin
York University, Dept. of Physics & Astronomy
Office: Petrie 240   Lab: Farq 103
cberge@yorku.ca

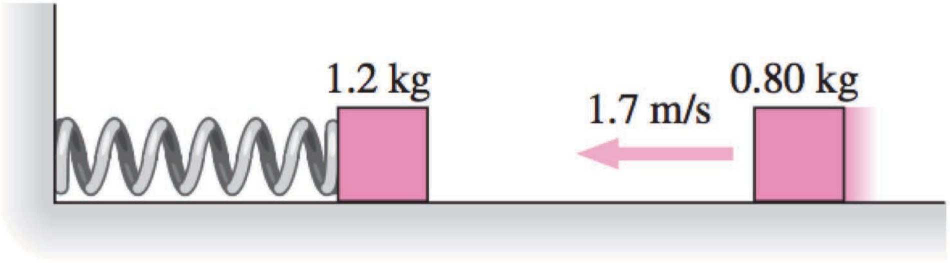
Ref. (re images):
Knudsen & Hjorth (2000), Kesten & Tauck (2012)

Problem 1

A mass-spring system has $b/m = \omega_0/5$, where b is the damping constant and ω_0 the natural frequency. How does its amplitude at ω_0 compare with its amplitude when driven at frequencies 10% above and below ω_0?

Problem 2

A 1.2-kg block rests on a frictionless surface and is attached to a horizontal spring of constant $k = 23$ N/m (see below) The block oscillates with amplitude 10 cm and phase constant $\phi = -\pi/2$. A block of mass 0.80 kg moves from the right at 1.7 m/s and strikes the first block when the latter is at the rightmost point in its oscillation. The two blocks stick together. Determine the frequency, amplitude, and phase constant (relative to the *original* $t = 0$) of the resulting motion.



Problem 3

A mass m is free to slide on a frictionless track whose height y as a function of horizontal position x is $y = ax^2$, where a is a constant with units of inverse length. The mass is given an initial displacement from the bottom of the track and then released. Find an expression for the period of the resulting motion.

Problem 4

A damped harmonic oscillator is driven by an external force of the form

$$F_{ext} = F_0 \sin \omega t$$

Show that the steady-state solution is given by

$$x(t) = A(\omega) \sin(\omega t - \phi)$$

where $A(\omega)$ and ϕ are identical to the expressions given by

$$\tan \phi = \frac{2\gamma\omega}{\omega_0^2 - \omega^2}$$

$$A(\omega) = \frac{F_0/m}{\left[\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2\right]^{1/2}}$$

Problem 5

Show that the Fourier series for a periodic square wave is

$$f(t) = \frac{4}{\pi}\left[\sin(\omega t) + \tfrac{1}{3}\sin(3\omega t) + \tfrac{1}{5}\sin(5\omega t) + \cdots\right]$$

where

$$f(t) = +1 \qquad \text{for } 0 < \omega t < \pi,\ 2\pi < \omega t < 3\pi, \text{ and so on}$$

$$f(t) = -1 \qquad \text{for } \pi < \omega t < 2\pi,\ 3\pi < \omega t < 4\pi, \text{ and so on}$$

Hint: There are lots of approaches one can take here. If one were to use complex exponentials, since f(t) has a relatively simple form (i.e., either -1 or 1), the integral becomes relatively straightforward and can be reduced to a fairly simple expression....

$$f(t) = \sum_n c_n e^{in\omega t} \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-in\omega t}\, dt$$

Problem 6

A popular undergrad classical mechanics text (JR Taylor, 2005) expresses the following eqn. for the DDHO:

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = f_o \cos(\omega t).$$

Further, they express a general (steady-state) solution as:

$$x(t) = A\cos(\omega t - \delta)$$

a) Show that the amplitude (squared) is given by

$$A^2 = \frac{f_o^{\,2}}{(\omega_o^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

b) Verify that the following is in fact true:

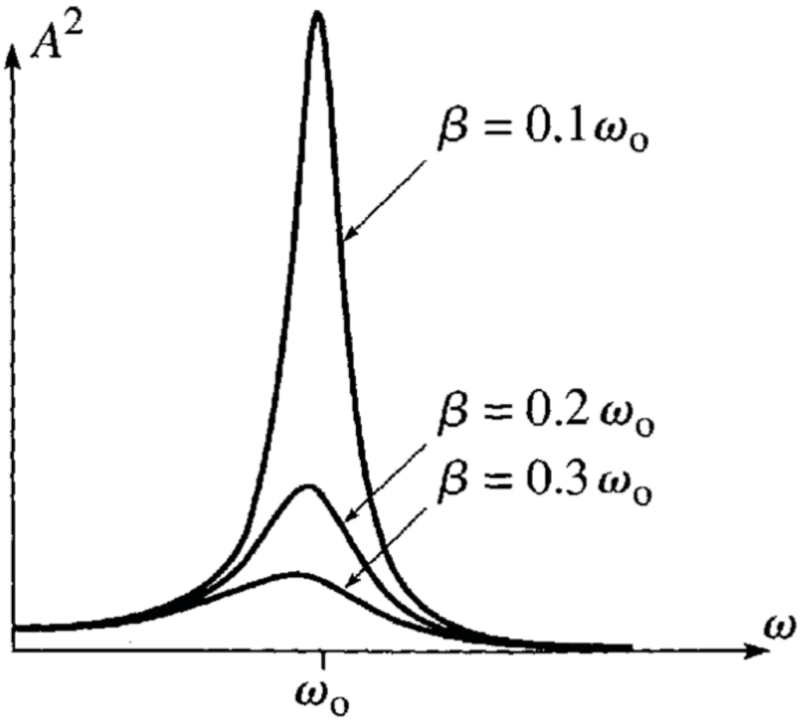
$$\omega_2 = \sqrt{\omega_o^2 - 2\beta^2} = \text{value of } \omega \text{ at which response is maximum.}$$

Problem 6 (cont)

c) Show that the peak of the response takes on the following approximate value

$$A_{\max} \approx \frac{f_0}{2\beta\omega_0}.$$

d) Using a computer, reproduce this plot (i.e., trace the three curves). Briefly explain how this relates to changes in damping as well as the associated "Q-value"



Problem 6 (cont)

e) For a curve of this form, one can characterize the effective "bandwidth" by virtue of the "Full Width at Half Maximum" (FWHM). Show that this is given simply by:

$$\text{FWHM} \approx 2\beta$$

Problem 7

Another undergrad classical mechanics text (Knudsen & Hjorth, 200) expresses the following eqn. for the DDHO:

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t$$

Further, they express the amplitude of the (steady-state) solution as:

$$A = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}} \, .$$

They note that the power absorbed by the system in response to the driving force is the mean value of the product of the driving force and response velocity, being given by:

$$P = \frac{1}{2} \frac{F_0^2}{\gamma m} \frac{\gamma^2 \omega^2}{[(\gamma \omega)^2 + (\omega_0^2 - \omega^2)^2]}$$

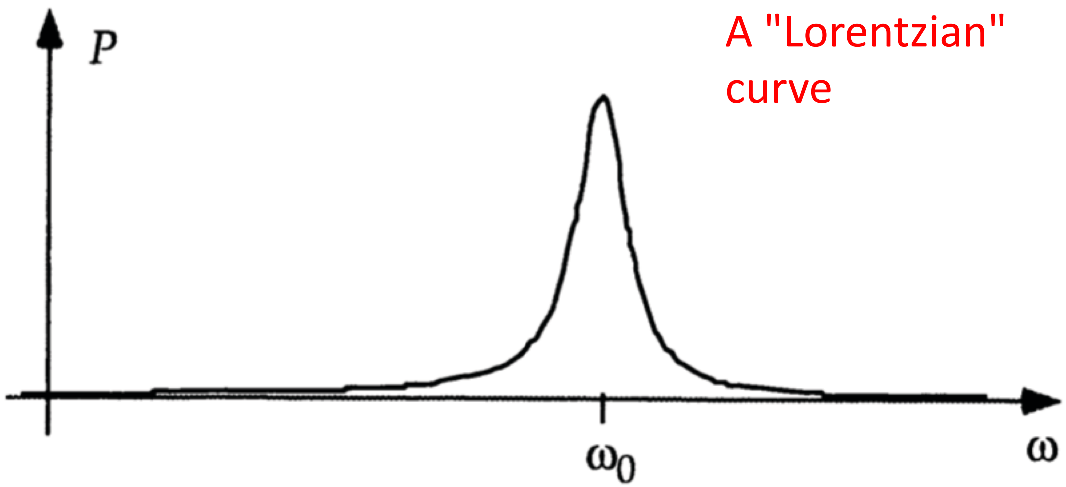
a) Using a computer, make a plot of absorbed power as a function of driving frequency. Make sure to clearly note the frequency where the peak is

Problem 7 (cont)

b) Make the assumption that the driving frequency ω is very close to ω_o (e.g., $\omega_o + \omega \sim 2\omega_o$). Show then that this results in a general expression for the so-called Lorentzian curve (call it P_L), as given below. Further, briefly mention places where this sort of function is commonly used (Hint: Google "spectral line shape")

$$P_L = \frac{1}{2}\frac{F_0^2}{\gamma m}\frac{\gamma^2/4}{(\omega_0 - \omega)^2 + \gamma^2/4}$$

c) Determine the FWHM for P



A "Lorentzian" curve