$\boxed{\text{HW5 Prob. 1}}$

- Assume the eqn. of motion $m\ddot{x} + b\dot{x} + kx = F_0 \cos(\omega t)$.
  We are told that $\frac{b}{m} (\equiv \gamma) = \omega_0 / 5$
- As derived in class, the steady-state amplitude is given by

$$A(\omega) = \frac{F_0/m}{\left[(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2\right]^{\frac{1}{2}}}$$

- When driven at ω_0 (i.e. $\omega = \omega_0$), we have $A(\omega_0) = \dfrac{F_0/m}{\gamma\omega_0} = \dfrac{F_0}{b\omega_0}$

  Now we want an expression for the ratio $\dfrac{A(\omega)}{A(\omega_0)}$
  (as we need that for what the problem asks)

$$\frac{A(\omega)}{A(\omega_0)} = \frac{b\omega_0}{F_0} \cdot \frac{F_0/m}{\left[(\omega_0^2-\omega^2)^2 + (\gamma\omega)^2\right]^{\frac{1}{2}}} = \frac{\omega_0 b/m}{\left[(\omega_0^2-\omega^2)^2 + \left(\frac{b}{m}\right)^2 \omega^2\right]^{\frac{1}{2}}}$$

$$= \frac{1}{\cancel{\frac{m}{\omega_0 b}} \cdot \cancel{\frac{b\omega_0}{m}} \cdot \left[\left(\frac{m}{b\omega_0}\right)^2 (\omega_0^2 - \omega^2)^2 + \left(\frac{\omega}{\omega_0}\right)^2\right]^{\frac{1}{2}}}$$

NOTE
$(\omega_0^2-\omega^2)^2 = \left[\omega_0^2\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)\right]^2$
$= \omega_0^4 \left[1 - \left(\frac{\omega}{\omega_0}\right)^2\right]^2$

$$= \left[\left(\frac{m\omega_0}{b}\right)^2 \left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \left(\frac{\omega}{\omega_0}\right)^2\right]^{-\frac{1}{2}}$$

- ok, now we can compute the necessary ratios!

NOTE
$\dfrac{m\omega_0}{b} = \dfrac{5\omega_0}{\omega_0} = 5$

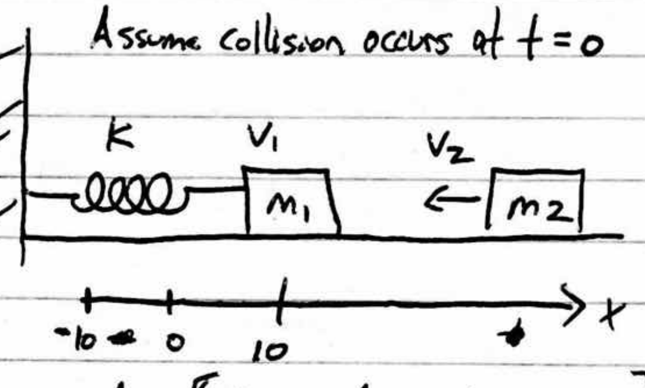
$$\frac{A(1.1\omega_0)}{A(\omega_0)} = \left[5^2\left[1 - \frac{(1.1\omega_0)^2}{\omega_0^2}\right]^2 + (1.1)^2\right]^{-\frac{1}{2}} = 0.658$$

$$\frac{A(0.9\omega_0)}{A(\omega_0)} = \left[5^2(1-0.9^2)^2 + (0.9)^2\right]^{-\frac{1}{2}} \simeq 0.764$$

$$\Longrightarrow \boxed{\frac{A(1.1\omega_0)}{A(\omega_0)} = 65.8\% \text{ and } \frac{A(0.9\omega_0)}{A(\omega_0)} = 76.4\%} \quad \therefore$$

HW5 Prob.2

Assume collision occurs at $t=0$



see below

- $V_1(t=0) = 0$, $m_1 = 1.2\,kg$
  $k = 23\ N/m$
  $V_2(t=0) = -1.7\ m/s$, $m_2 = 0.80\,kg$

- At $t=0$, m_2 collides w/ m_1 at point $x = A_1$ [where $A_1 = 10$ cm re $x=0$; amplitude of mass 1 SHO] when m_1 is (instantaneously) at rest under its SHO motion

- We can immediately derive a few relevant quantities of interest:

  - $\omega_{o1} = \sqrt{k/m_1} = \sqrt{23/1.2} = 4.38\ \frac{rads}{s}$ or $f_{o1} = 0.70\,Hz$] freq. of block 1 before collision
    (since $f_o = \omega_o/2\pi$)

  - $\omega_{oc} = \sqrt{\frac{k}{m_1+m_2}} = \sqrt{\frac{23}{1.2+0.8}} = 3.39\ \frac{rads}{s}$ or $\boxed{f_{oc} = 0.54\,Hz}$

    $\leadsto$ freq. of oscillation AFTER collision (since k presumably doesn't change)

  - During an inelastic collision (which occurs at $t=0$), energy is not conserved, but momentum is, thus:

    velocity of combined masses right after collision

    $$m_2 V_2(0) = (m_1+m_2) V_c(0)$$

    $$\leadsto V_c(0) = \frac{m_2 V_2(0)}{m_1+m_2} = \frac{0.80}{1.2+0.8}(-1.7) = -0.68\ m/s$$

- Now after the collision, the combined masses will still exhibit SHO and thus we can express such as

$$x_c(t) = A_c \cos(\omega_{oc} t + \phi_c) \quad \text{and} \quad V_c(t) = -\omega_{oc} A_c \sin(\omega_{oc} t + \phi_c)$$

(Cont)

$\boxed{\text{HW5 Prob. 2 (cont)}}$

We know ω_{oc}, but still need to determine A_c and ϕ_c.

- We do know $x(t=0)$ and $v(t=0)$, leading to

$$x(t=0) = 10 = A_c \cos(\omega_{oc} \cdot 0 + \phi_c) \quad \leadsto \quad 10 = \cos \phi_c$$

$$v(t=0) = -68 \left[\tfrac{cm}{s}\right] = -\omega_{oc} A_c \sin(\omega_{oc} \cdot 0 + \phi_c)$$

$$\leadsto \quad 20.1 = A_c \sin \phi_c$$

$\Rightarrow$ two equations, two unknowns

- $$\frac{20.1}{10} = \frac{A_c \sin \phi_c}{A_c \cos \phi_c} = \tan \phi_c \quad \leadsto \quad \boxed{\phi_c = \tan^{-1}(2.01) = 1.11 \text{ rads}}$$

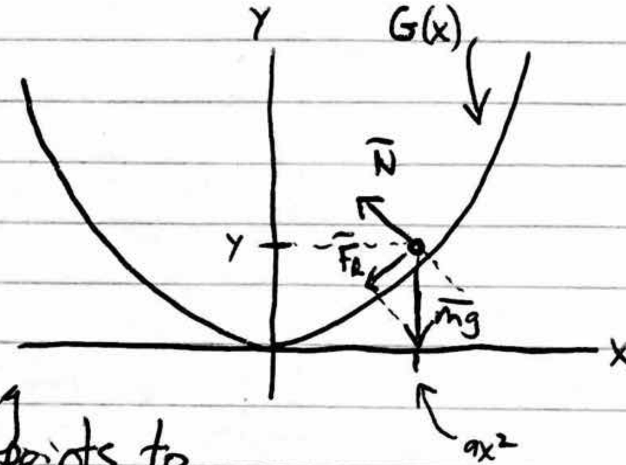
And plugging back into the first equation:

$$10 = A_c \cos(1.11) \longrightarrow \boxed{A_c = 22.5 \text{ cm}}$$

Thus after the collision, the combined masses move according to

$$\boxed{x_c(t) = 22.5 \cos[3.39t + 1.11]}$$

$\therefore$



HW5 Prob. 3

- Here we have an object that'll undergo periodic motion (since there is no friction). This is due to gravity and the normal force of the track creating a "restoring force" ($\bar{F}_R$) that always points to $x=0$ (except at $x=0$, where $F_R = 0$)

- Two approaches to this problem

1) Let $G(x) = ax^2$ act as the potential function governing the motion. Then

$$\bar{F}_R \propto -\frac{dG}{dx} = -2ax$$

the constant of proportionality being mg. $\bar{F}_R$ is a 2-D vector, but considering just the x-component ($\equiv F_{Rx}$), we have

$$F_{Rx} = -2mgax = m\frac{d^2x}{dt^2} = -kx$$

$\rightsquigarrow$ Thus the effective spring constant is $k = 2gam$. This leads to $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{1}{m}\cdot 2gam} = \sqrt{2ga} = 2\pi f_0 = \frac{2\pi}{T}$

$$\longrightarrow \boxed{T = \frac{2\pi}{\sqrt{2ga}}}$$

2) The potential energy of the mass relative to the bottom of the track is

$U(x) = mgy = mgax^2$. For SHO, $U(x) = \frac{1}{2}kx^2 \rightarrow k = 2mga$ (which leads to the same result)

$\therefore$

HW5 Prob. 4

- Best place to start is w/ the presumed equation of motion. Take this to be:

$$m\ddot{x} + c\dot{x} + kx = F_{ext} = F_0 \sin \omega t$$

- What we care about here is the steady-state solution, which has the form $x(t) = A(\omega) \sin(\omega t - \varphi(\omega))$ our job is to derive $A(\omega)$ and $\varphi(\omega)$

- Let us use complex exponentials as follows:

$$z(t) = A e^{i(\omega t - \varphi)} \quad \text{where} \quad x(t) = Re(z)$$

- Now we have: $\dot{z} = i\omega z$ and $\ddot{z} = -\omega^2 z$ leading to

$$m\ddot{z} + c\dot{z} + kz = F_0 e^{i\omega t} \longrightarrow z\left[-\omega^2 m + i\omega c + k\right] = F_0 e^{i\omega t}$$

$$\text{but } z = Ae^{i(\omega t - \varphi)} = Ae^{-i\varphi} e^{i\omega t} \longrightarrow Ae^{-i\varphi}\left[k - \omega^2 m + i\omega c\right] = F_0$$

- While there are two unknowns, (A and φ), because the equation is complex, there are effectively two eqns. (e.g. real and imaginary parts). Breaking such up, we have

$$A\left[k - \omega^2 m + i\omega c\right] = F_0 e^{i\varphi} = F_0(\cos\varphi + i\sin\varphi)$$

now just equate real and imaginary parts

$$\longrightarrow \begin{aligned} A(k - \omega^2 m) &= F_0 \cos\varphi \\ A\omega c &= F_0 \sin\varphi \end{aligned}$$

- Right off the bat, we can divide both equations to obtain

$$\tan\varphi = \frac{\omega c}{k - \omega^2 m} = \frac{\omega \cdot c}{m\left(\frac{k}{m} - \omega^2\right)} = \frac{\omega \cdot c/m}{\omega_0^2 - \omega^2} = \frac{\omega \cdot 2\gamma}{\omega_0^2 - \omega^2}$$

cont

$\boxed{\text{HW5 Prob. 4 (cont)}}$

where $\gamma = \frac{c}{2m}$ and $\omega_0 = \sqrt{\frac{k}{m}}$. Thus $\boxed{\phi = \tan^{-1}\left(\frac{2\gamma m}{\omega_0^2 - \omega^2}\right)}$

☐ To determine A, we use the identity $\sin^2\theta + \cos^2\theta = 1$

$$(F_0 \sin\phi)^2 + (F_0 \cos\phi)^2 = F_0^2(\sin^2\phi + \cos^2\phi) = F_0^2$$

$$= (A\omega c)^2 + [A(k - \omega^2 m)]^2 = A^2[(\omega c)^2 + (k - \omega^2 m)^2]$$

$$\Rightarrow A = \frac{F_0}{[\omega^2 c^2 + k^2 + \omega^4 m^2 - 2k\omega^2 m]^{1/2}} = \frac{F_0}{[(\omega^4 m^2 - 2km\omega^2 + k^2) + \omega^2 c^2]^{1/2}}$$

$$= \frac{F_0}{[(\omega^2 m - k)^2 + \omega^2 c^2]^{1/2}} = \frac{F_0}{[\{m(\omega^2 - \frac{k}{m})\}^2 + m^2 4\gamma^2\omega^2]^{1/2}}$$

$$= \frac{F_0/m}{[(\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2]^{1/2}}$$

$$\Longrightarrow \boxed{A(\omega) = \frac{F_0/m}{[(\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2]^{1/2}}}$$

<u>NOTE</u>: $(\omega^2 - \omega_0^2)^2 = (\omega_0^2 - \omega^2)^2$

HW5 Prob. 5

□ Assume we can treat f(t) as an infinite series of sinusoids using complex exps.

$$f(t) = \sum_n c_n e^{i\omega n t} \quad \text{where } c_n \in \mathbb{C} \text{ and } c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-i\omega n t}\, dt$$

$n = 0, \pm 1, \pm 2, \pm 3, \dots$

$$T = \frac{2\pi}{\omega} \quad \left(\text{thus } \frac{1}{T} = \frac{\omega}{2\pi} \text{ and } \frac{T}{2} = \frac{\pi}{\omega}\right)$$

$$\int -e^{-in\omega t} dt = \frac{1}{in\omega} e^{-in\omega t}$$

$$\rightarrow c_n = \frac{\omega}{2\pi}\int_{-\pi/\omega}^{\pi/\omega} f(t) e^{-in\omega t} dt$$

$$= \frac{\omega}{2\pi}\left[\int_{-\pi/\omega}^{0} (-1) e^{-in\omega t} dt + \int_0^{\pi/\omega} (1) e^{-in\omega t} dt\right]$$

$$= \frac{\omega}{2\pi}\left[\frac{1}{in\omega} e^{-in\omega t}\Big|_{-\pi/\omega}^{0} - \frac{1}{in\omega} e^{-in\omega t}\Big|_{0}^{\pi/\omega}\right]$$

$$= \frac{\omega}{2\pi}\left[\frac{1}{in\omega}\left(1 - e^{+in\pi}\right) - \frac{1}{in\omega}\left(e^{-in\pi} - 1\right)\right]$$

$$= \frac{1}{2\pi i n}\left[1 - e^{+in\pi} - e^{-in\pi} + 1\right]$$

$e^{in\pi} = e^{-in\pi}$ for $n \in \mathbb{N}$

$$e^{in\pi} = \begin{cases} -1 & \text{odd } n \\ 1 & \text{even } n \end{cases}$$

$$= \frac{4}{2\pi i n} \quad \text{for } n = \pm 1, \pm 3, \pm 5, \dots$$

(i.e. $c_n = 0$ for even n)

$$\rightarrow f(t) = \sum_n \frac{4}{2\pi i n} e^{in\omega t} \quad (\text{for odd } n \text{ as noted above})$$

$$= \sum_n \frac{4}{\pi}\frac{1}{n}\frac{1}{2i}\left(e^{in\omega t} - e^{-in\omega t}\right) \quad \text{for } n = 1, 3, 5, \dots$$

$$= \frac{4}{\pi}\sum_n \frac{1}{n}\sin(n\omega t) \quad \left(\text{since } \sin x = \frac{e^{ix} - e^{-ix}}{2i}\right)$$

$$= \frac{4}{\pi}\left[\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \cdots\right]$$

∴

HW5 Prob. 6 $\qquad \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos(\omega t)$

a) Assuming a solution of the form $x(t) = A\cos(\omega t - \delta)$, we can connect back to our solution to prob. 4. Here, we have (comparing back to the eqn. of motion in prob. 4)

$c = m \cdot 2\beta \rightarrow \beta = \gamma$ (since $\gamma = \frac{2c}{m}$) and $f_0 = F_0/m$

$$\rightsquigarrow A(\omega) = \frac{F_0/m}{[(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2]^{1/2}} = \frac{f_0}{[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2]^{1/2}}$$

Now we simply have $\boxed{A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$

b) Now for A^2 to have a max, the denominator must be a minimum. To show this, let

$$f(\omega) = (\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2$$

We want ω_m such that $\left.\frac{df}{d\omega}\right|_{\omega = \omega_m} = 0$

$$\frac{df}{d\omega} = 2(\omega_0^2 - \omega^2)(-2\omega) + 8\beta^2\omega = 8\beta^2\omega - 4\omega(\omega_0^2 - \omega^2)$$

Now $8\beta^2\omega_m - 4\omega_m(\omega_0^2 - \omega_m^2) = 0 \rightsquigarrow 2\beta^2 = \omega_0^2 - \omega_m^2$

or $\boxed{\omega_m = \sqrt{\omega_0^2 - 2\beta^2}}$

NOTE: $\frac{d^2 f}{d\omega^2} = 8\beta^2 - 4(\omega_0^2 - \omega^2)(-2\omega)$
$= 8\beta^2 + 8\omega(\omega_0^2 - \omega^2) > 0$
(because $\omega_m < \omega_0$) guarenteeing we found a max. (remember that f is the denominator!)

CONT

HW5 Prob. 6 (cont)

c) Here, we interpret the "$\approx$" as $\omega_0 \approx \omega$. Thus

$$A_{max} \approx \left[\frac{f_0^2}{(\omega_0^2-\omega_0^2)^2 + 4\beta^2\omega_0^2} \right]^{\frac{1}{2}} = \frac{f_0}{2\beta\omega_0}$$

d) see EXhw5Prob6.m

e) Ignoring part c, we can determine the true max for A^2 ($= A_{max}^2$) by plugging in $\omega = \sqrt{\omega_0^2 - 2\beta^2}$

$$A_{max}^2 = \frac{f_0^2}{(\cancel{\omega_0^2}-\cancel{\omega_0^2}+2\beta^2)^2 + 4\beta^2(\omega_0^2-2\beta^2)} = \frac{f_0^2}{4\beta^2(\omega_0^2-\beta^2)}$$

Now we want ω_{FWHM} such that $A^2(\omega_{FWHM}) = \frac{1}{2} A_{max}^2$

$$\Rightarrow \frac{f_0^2}{(\omega_0^2-\omega_{FWHM}^2)^2 + 4\beta^2\omega_{FWHM}^2} = \frac{f_0^2}{8\beta^2(\omega_0^2-\beta^2)}$$

$$8\beta^2(\omega_0^2-\beta^2) = (\omega_0^2-\omega_{FWHM}^2)^2 + 4\beta^2\omega_{FWHM}^2 = \omega_0^4 + \omega_{FWHM}^4 - 2\omega_0^2\omega_{FWHM}^2 + 4\beta^2\omega_{FWHM}^2$$

$$\Rightarrow \omega_{FWHM}^4 + \omega_{FWHM}^2(4\beta^2 - 2\omega_0^2) + [\omega_0^4 - 8\beta^2(\omega_0^2-\beta^2)] = 0$$

→ Getting a tad messy (but still doable). But let's try another approach: Let's plug in the answer and verify the equality

$$FWHM = 2\beta \longrightarrow \omega = \omega_0 \pm \beta$$

HW5 Prob. 6 (cont. #)

which should equal $\frac{1}{2} A^2_{max}$

$$\rightsquigarrow A_w^2 = A^2(\omega = \omega_0 \pm \beta)$$

NOTE:
$(\omega_0 \pm \beta)^2$
$= \omega_0^2 + \beta^2 \pm 2\omega_0 \beta$

$$= \frac{f_0^2}{[\omega_0^2 - (\omega_0 \pm \beta)^2]^2 + 4\beta^2(\omega_0 \pm \beta)^2}$$

$$= \frac{f_0^2}{[\cancel{\omega_0^2} - \cancel{\omega_0^2} - \beta^2 \mp 2\omega_0\beta]^2 + 4\beta^2(\omega_0^2 + \beta^2 \pm 2\omega_0\beta)}$$

$$= \frac{f_0^2}{\beta^4 + 4\omega_0^2\beta^2 \pm 4\omega_0\beta^3 + 4\beta^2\omega_0^2 + 4\beta^4 \pm 8\beta^3\omega_0}$$

☐ I seem to be mentally stalling on this one. Too much coronavirus...

HW5 Prob. 7

a) See attached

b) Note that if ω is close to ω_0, then $\omega + \omega_0 \simeq 2\omega_0$

$$\rightsquigarrow \omega_0^2 - \omega^2 = (\omega_0 - \omega)(\omega_0 + \omega) \simeq 2\omega_0(\omega_0 - \omega)$$

$$\rightsquigarrow P_L = \frac{1}{2}\frac{F_0^2}{\gamma m}\frac{\gamma^2\omega^2}{(\omega_0^2-\omega^2)+\gamma^2\omega^2} \simeq \frac{1}{2}\frac{F_0^2}{\gamma m}\frac{\gamma^2\omega^2}{4\omega_0^2(\omega_0-\omega)^2+\gamma^2\omega^2}$$

$$= \frac{1}{2}\frac{F_0^2}{\gamma m}\frac{1}{4\omega^2}\frac{\gamma^2\omega^2}{\left(\frac{\omega_0}{\omega}\right)^2(\omega_0-\omega)^2+\gamma^2/4} \simeq \frac{1}{2}\frac{F_0^2}{\gamma m}\frac{\gamma^2/4}{(\omega_0-\omega)^2+\gamma^2/4}$$

(since we assume $\frac{\omega_0}{\omega} \simeq 1$)

<u>NOTE</u>: The general form for a Cauchy/Lorenz probability distribution function is

$$\frac{1}{\pi\gamma\left[1+\frac{(x-x_0)^2}{\gamma^2}\right]}$$

where γ affects the width and x_0 the peak location. If we manipulated our expression for P_L further, we would obtain

$P_L \simeq \frac{F_0^2}{2m}\frac{1}{\gamma}\cdot\frac{1}{\frac{4(\omega_0-\omega)^2}{\gamma^2}+1}$ which has the same basic form

HW5 Prob. 7 (cont)

c) To determine the FWHM, we note that P_L takes on its max. value when $\omega = \omega_0$ (since $\omega_0 - \omega_0 = 0$ and thus the denominator is the smallest). There

$$P_L(\omega_0) = F_0^2 / 2\gamma m \quad \text{so we want } \omega_n \text{ such that } P(\omega_n) = \frac{1}{2} \cdot \frac{1}{2} \frac{F_0^2}{\gamma m}$$

$$\frac{1}{2} \cdot \frac{1}{2} \frac{F_0^2}{\gamma m} = \frac{1}{2} \frac{F_0^2}{\gamma m} \cdot \frac{\gamma^2/4}{(\omega_0 - \omega_n)^2 + \gamma^2/4} \rightsquigarrow \frac{1}{2} = \frac{\gamma^2/4}{(\omega_0 - \omega_n)^2 + \gamma^2/4}$$

$$\text{So } (\omega_0 - \omega_n)^2 + \frac{\gamma^2}{4} = \frac{\gamma^2}{2} \rightsquigarrow (\omega_0 - \omega_n)^2 = \frac{\gamma^2}{4} = \omega_0^2 + \omega_n^2 - 2\omega_n\omega_0$$

$$\omega_n^2 - \omega_n 2\omega_0 + \omega_0^2 - \frac{\gamma^2}{4} = 0 \rightsquigarrow \text{quadratic for } \omega_n$$

$$\omega_n = \frac{2\omega_0 \pm \sqrt{4\omega_0^2 - 4(\omega_0^2 - \gamma^2/4)}}{2} = \omega_0 \pm \frac{\gamma}{2}$$

$$\text{Thus } \Delta\omega = \omega_{n+} - \omega_{n-} = \omega_0 + \frac{\gamma}{2} - \left(\omega_0 - \frac{\gamma}{2}\right) = \gamma$$

$$\Longrightarrow \boxed{\text{FWHM}(P_L) = \gamma}$$

∴