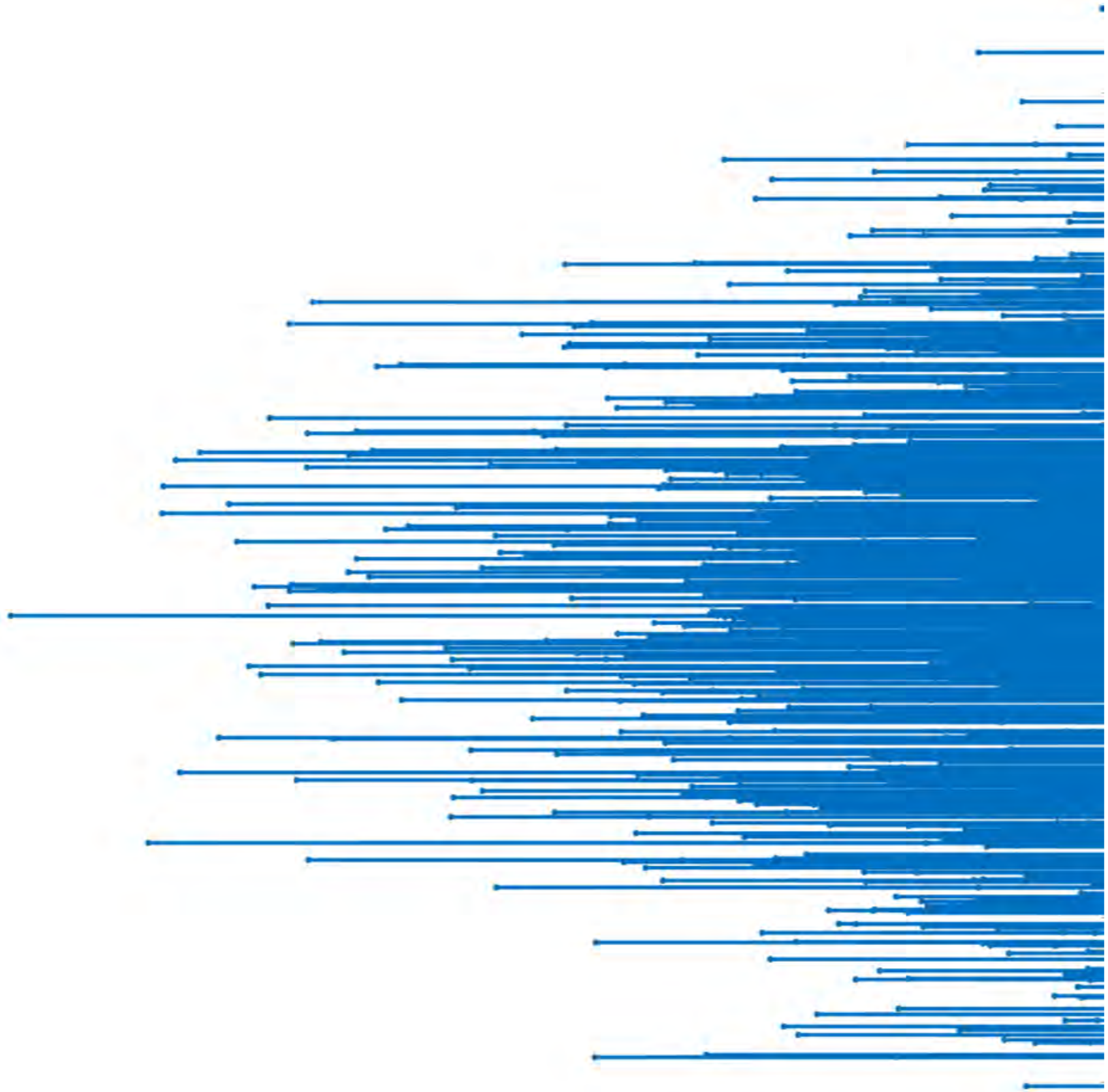
PHYS 2010 (W20)
Classical Mechanics



2020.01.07

Relevant reading:

Knudsen & Hjorth: 1.1-1.5

Christopher Bergevin
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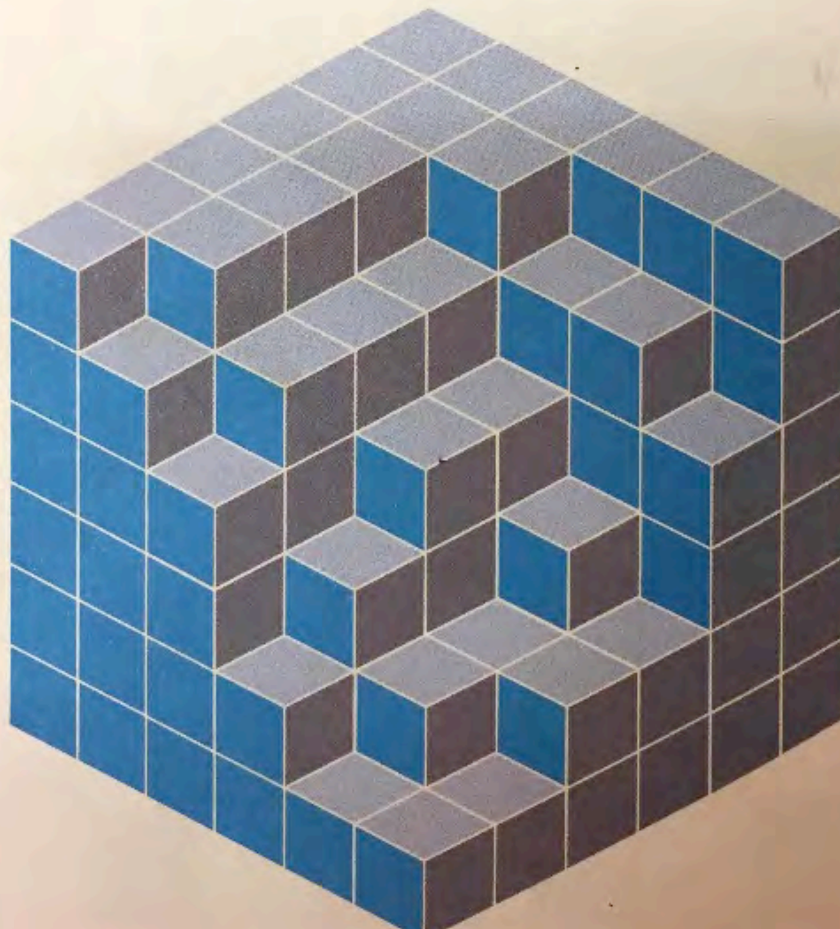
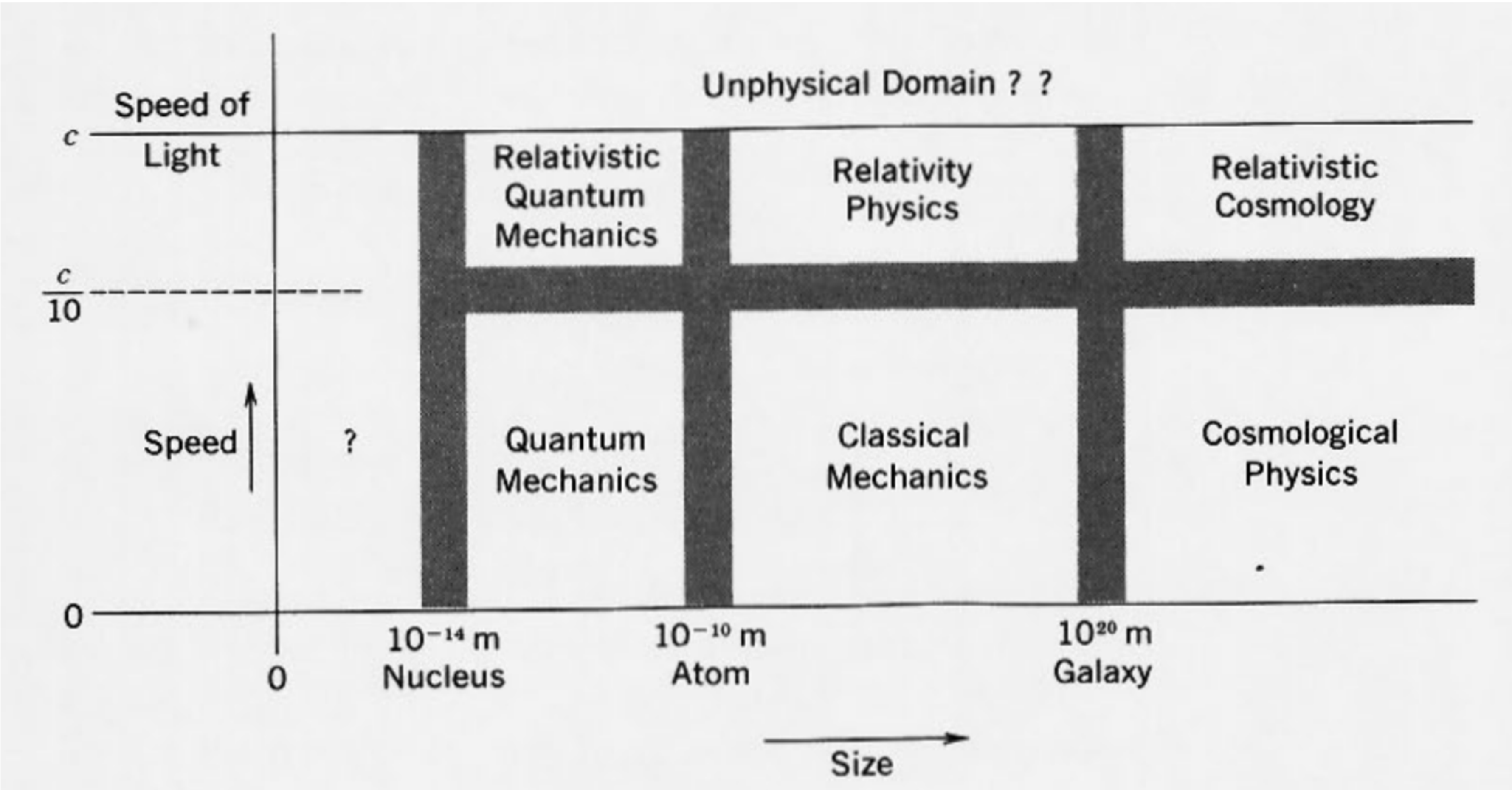
Ref. (re images):
Knudsen & Hjorth (2000), Kesten & Tauck (2012)

Cube Volume

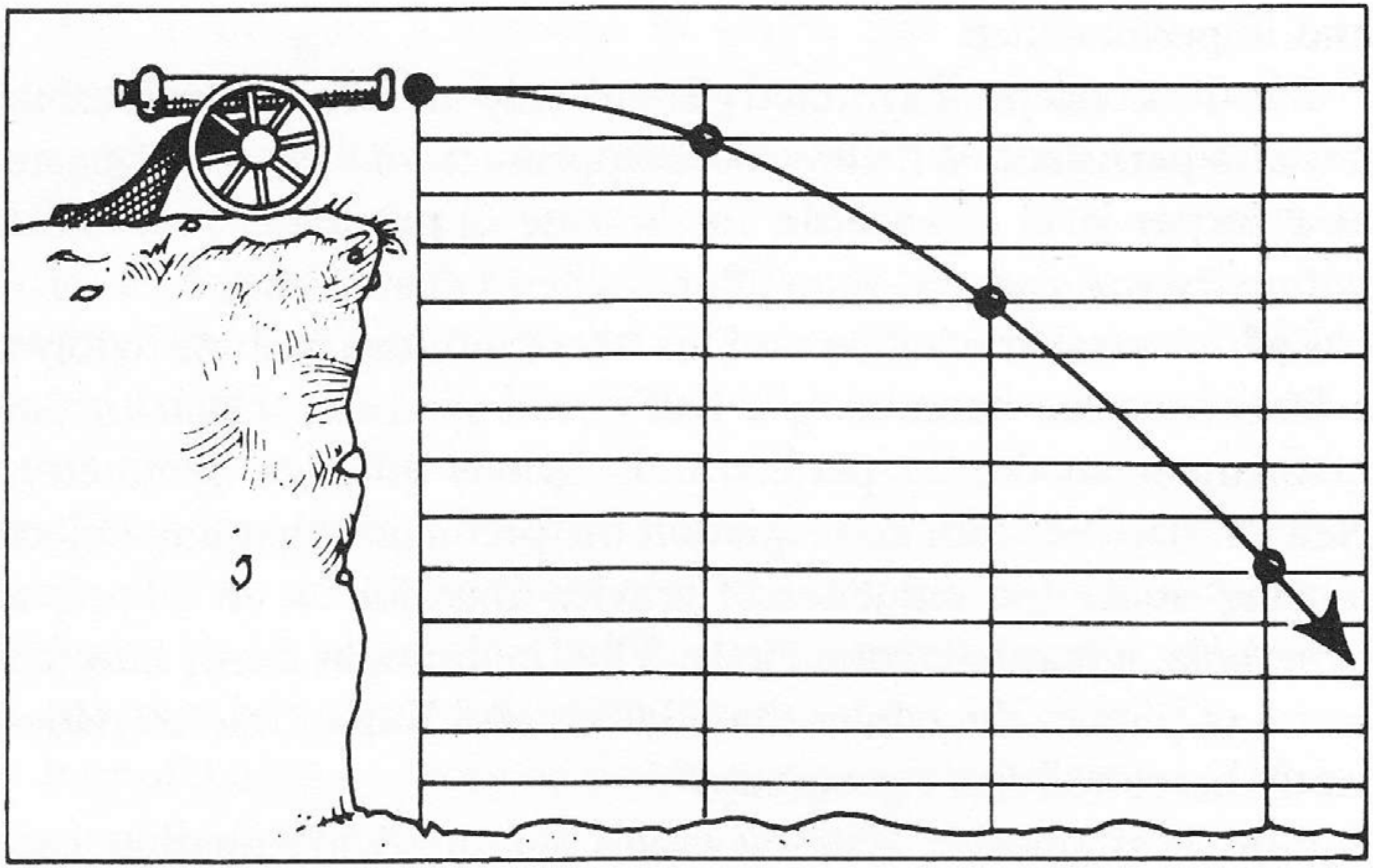
These little cubes originally made a big cube measuring 18 cm x 18 cm x 18 cm. Now some of the little cubes have been removed, can you work out what volume the remaining cubes have? Assume all invisible cubes are present.





Unphysical Domain ? ?
c
Speed of
Light
Relativistic
Quantum
Mechanics
Relativity
Physics
Relativistic
Cosmology
c
10
Speed
?
Quantum
Mechanics
Classical
Mechanics
Cosmological
Physics
0
0
10⁻¹⁴ m
Nucleus
10⁻¹⁰ m
Atom
10²⁰ m
Galaxy
Size

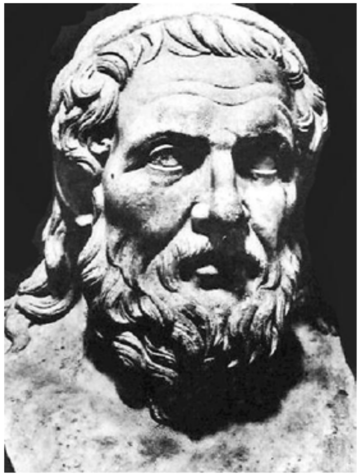
French (1971)



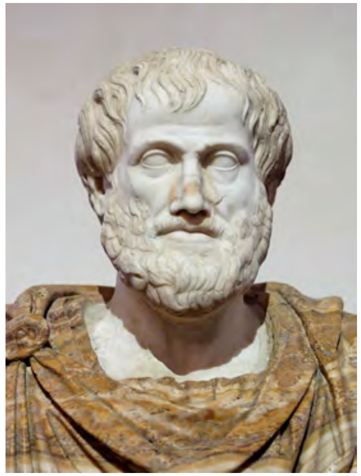
von Baeyer

Mechanics

Apollonius of Perga

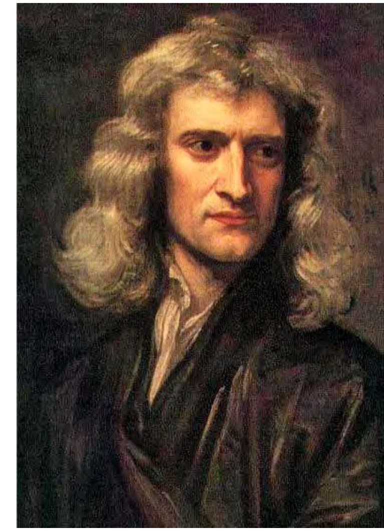


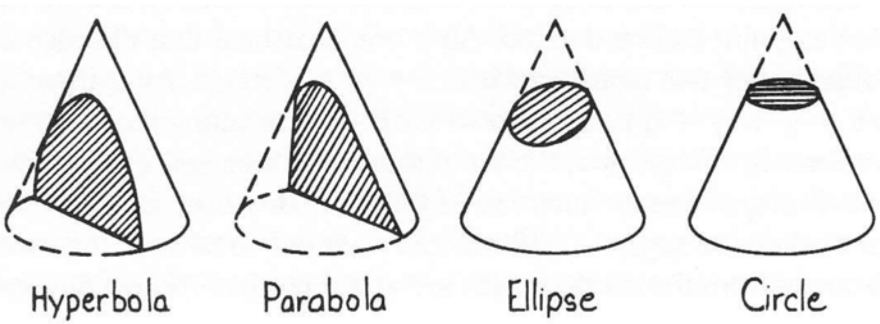
Aristotle

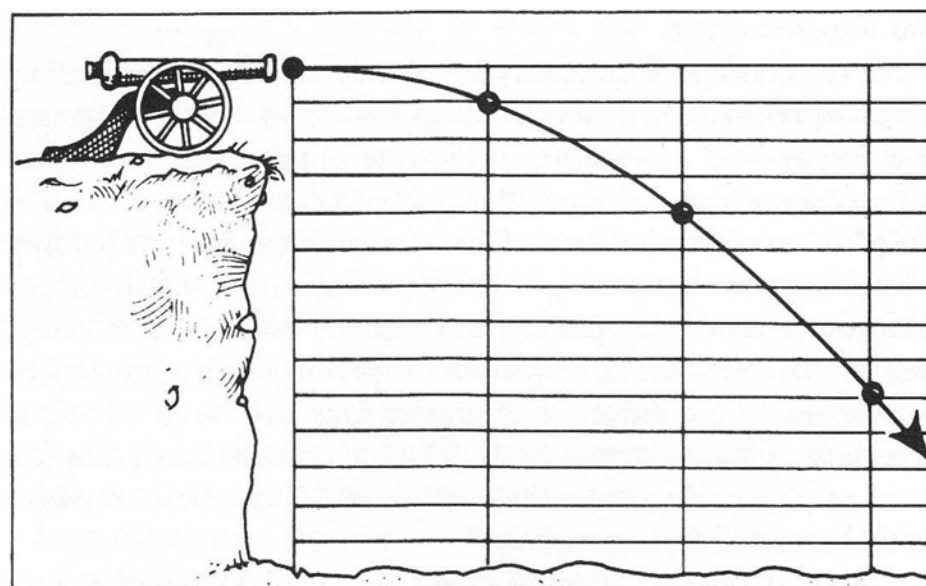


G. Galileo

I. Newton







von Baeyer

More parabolas....

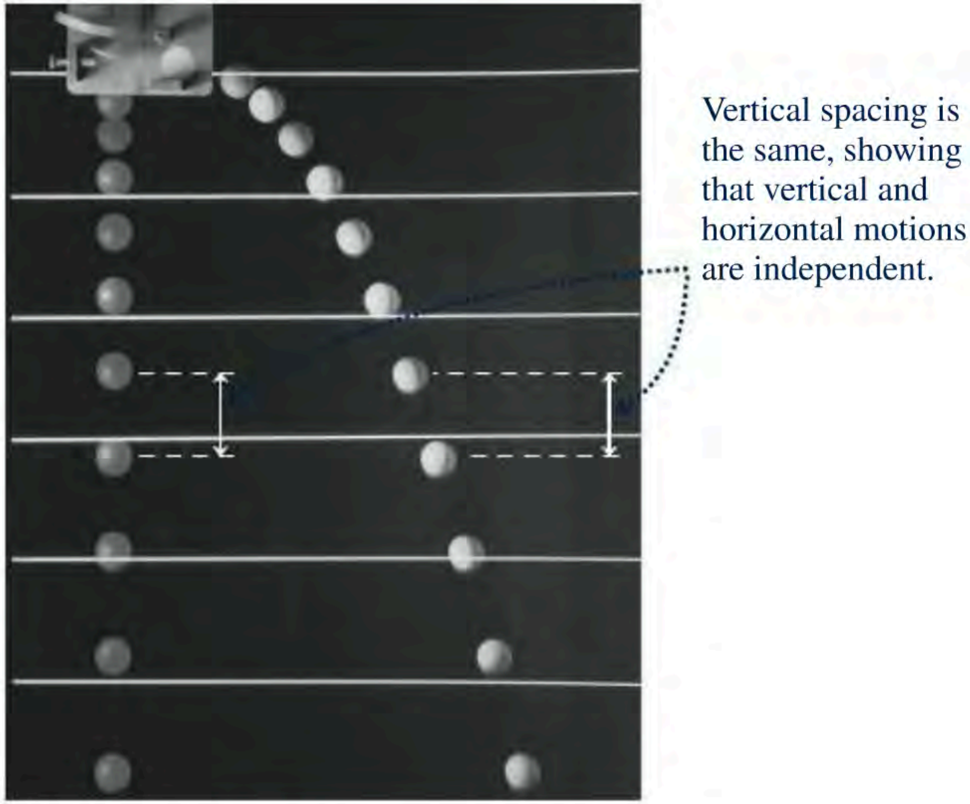


FIGURE 3.11 Two marbles, one dropped and the other projected horizontally.

Much easier to study a jet of water than a falling ball (they behave the same!)

FIGURE 3.14 Water droplets–each an individual projectile–combine to form graceful parabolic arcs in this fountain.

Wolfson

York University
PHYS 2010: Classical Mechanics (3 credits)
Winter 2020

Time & Location
Lecture: TTh 10:00–11:30 (DB 0014)
Recitation: F 9:30-10:30 (DB 0014)

Instructor: Christopher Bergevin
Office: Petrie 240
Email: cberge@yorku.ca
Office Hours: Check course website (or email for appt.)

Course Website:
`http://www.yorku.ca/cberge/2010W2020.html`

Textbook:
Elements of Newtonian Mechanics, (2000, 3rd Ed.) JM Knudsen & PG Hjorth (*Springer)*

Prerequisites: SC/PHYS 1010 6.00, or SC/PHYS 1800 3.00 and SC/PHYS 1801 3.00, or SC/ISCI 1310 6.0 or a minimum grade of C in SC/PHYS 1410 6.00 or SC/PHYS 1420 6.00; SC/MATH 1014 3.00 or equivalent; SC/MATH 1025 3.00 or equivalent; SC/MATH 2015 3.00 or equivalent. Corequisite: SC/MATH 2271 3.00. PRIOR TO FALL 2010: Prerequisites: SC/PHYS 1010 6.00, or a minimum grade of C in SC/PHYS 1410 6.00 or SC/PHYS 1420 6.00; SC/MATH 1014 3.00 or equivalent; SC/MATH 1025 3.00 or equivalent; Corequisite: SC/MATH 2015 3.00.

Course Theme/Topics: Newtonian mechanics of mass points and rigid bodies. Accelerated reference frames and rotational motion, centrifugal and Coriolis forces. Central force motion in celestial mechanics. Euler's equations: precession and nutation in the gyroscope. Topics will include (but are not limited to):

- 1-D to 3-D motion (via differential equations)
- Newton's Laws
- Various coordinate systems
- Oscillations (incl. complex numbers)
- Harmonic oscillator
- Nonlinear dynamics
- Gravitation
- Collisions

http://www.yorku.ca/cberge/2010W2020.html

→ This is the course website and will be the main "go to" place for all course-related info (e.g., syllabus, slides, chapters for reading, exam info, etc…)

J.M. Knudsen
P.G. Hjorth

Elements of Newtonian Mechanics

Including Nonlinear Dynamics

Third, Revised and Enlarged Edition

There will be 100 total possible points in the course. Point breakdowns are as follows:

- Homework – **25 points**
- Exams – **60 points**
- Project – **15 points**

Homework: Assignments will be given on a regular basis (there will be ~7–8 assignments). Each student is expected to turn in his or her own assignment. Points may be deducted for lack of explanation/clarity/completeness. It is crucial for students to spend considerable effort on these problem sets in order to be successful in the class.

Exams: Two exams will be given, the first being an in-class midterm on **Feb. 13** and the second a 3 hour final during the assigned final exam time (TBD). Note, as specified in the *lateness policy* below, there are no makeups.

Project: There will be a computational-centric project that will have students working in (randomized) groups to create a testable hypothesis and program/run a numeric simulation relevant to a topic covered in (or relevant to) class. Details will be provided later in the semester. In general, Matlab will be utilized in class, but students are free to use other programming languages as they see fit (e.g., Python).

Lateness: Unfortunately, some deadlines in the *real world* are quite harsh and allow no room for lateness. Given such, this course will implement two policies:

1. **There will be no makeup exams.** It is very important that you are present in class for the exams and the project presentation (as these determine more than 75% of your final grade!). Exceptions in extreme cases may be granted, but only upon prior approval or for an (excused) emergency.

2. All other due dates (i.e., for HW, lab reports, and project deadlines) will be subject to a severe lateness penalty. The grade for a particular assignment will be multiplied by a lateness factor

$$L = 0.3e^{-t/4} + 0.7e^{-t/72}$$

where t is the number of hours late. See figure for the lateness factor plotted as a function of time. Notice that the maximum grade for a report that is more than ONE DAY LATE is less than 50%.

Why introduce computing in 2010?

Slate

Quora THE BEST ANSWER TO ANY QUESTION. DEC. 28 2015 7:06 AM

Will Coding Still Be Relevant in a Decade?

By Quora Contributor

100 19

Students learn how to code at an Apple Store through Apple's Hour of Code workshop program on Dec. 9, 2015, in New York City.

Photo by Andrew Burton/Getty Images

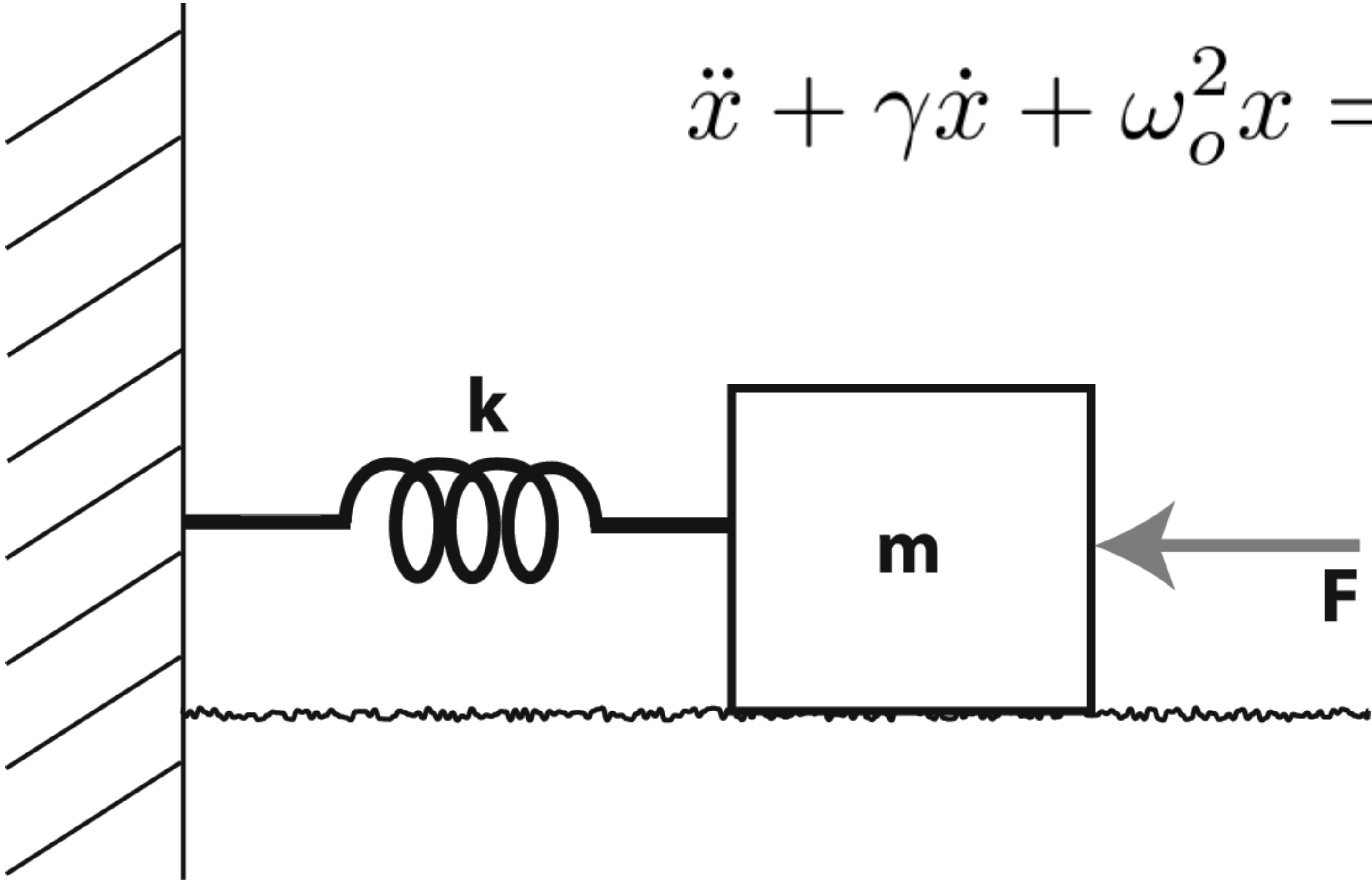
"Absolutely. Not only will coding be relevant in 10 years, it will be *more* relevant than it is today. However, the syntax of coding languages will continue becoming easier. When it started, coding was about holes in pieces of cardboard. Then it looked like this: 00101010101. It now looks a lot more like English. As coding languages become more English-like, they will be easier to learn, less arcane, and thus more popular. And as computing systems permeate our lives, telling these devices what we want them to do, and inventing new uses for them, will continue to be more popular."

http://www.slate.com/blogs/quora/2015/12/28/will_coding_still_be_relevant_in_a_decade.html

Looking Ahead: Harmonic Oscillator

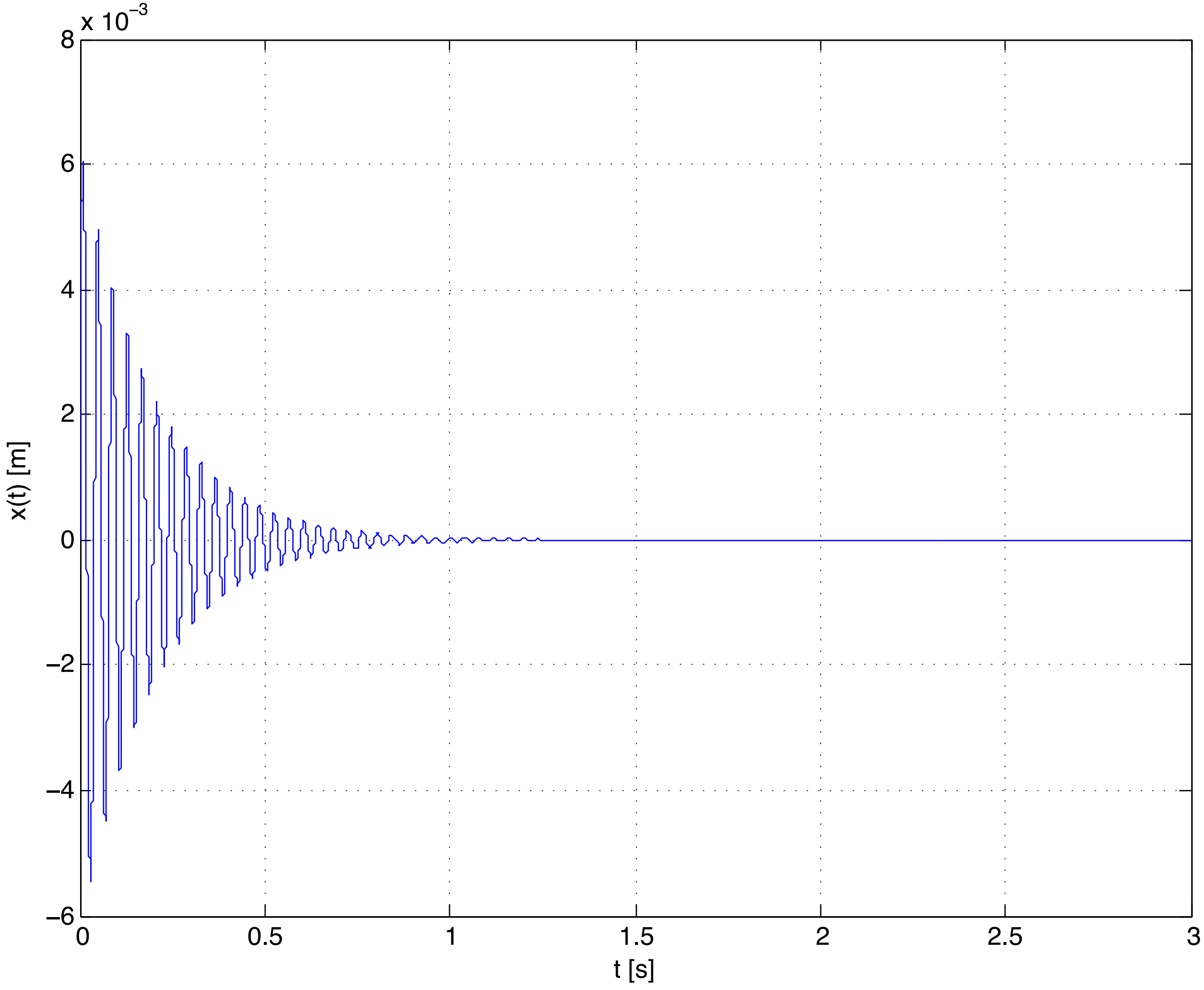
→ Harmonic oscillator will be a key topic in PHYS 2010

$$\ddot{x} + \gamma \dot{x} + \omega_o^2 x = 0$$

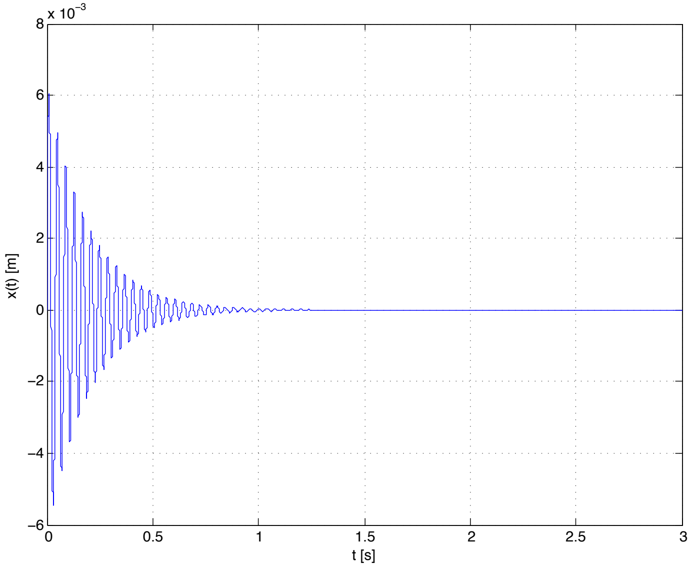


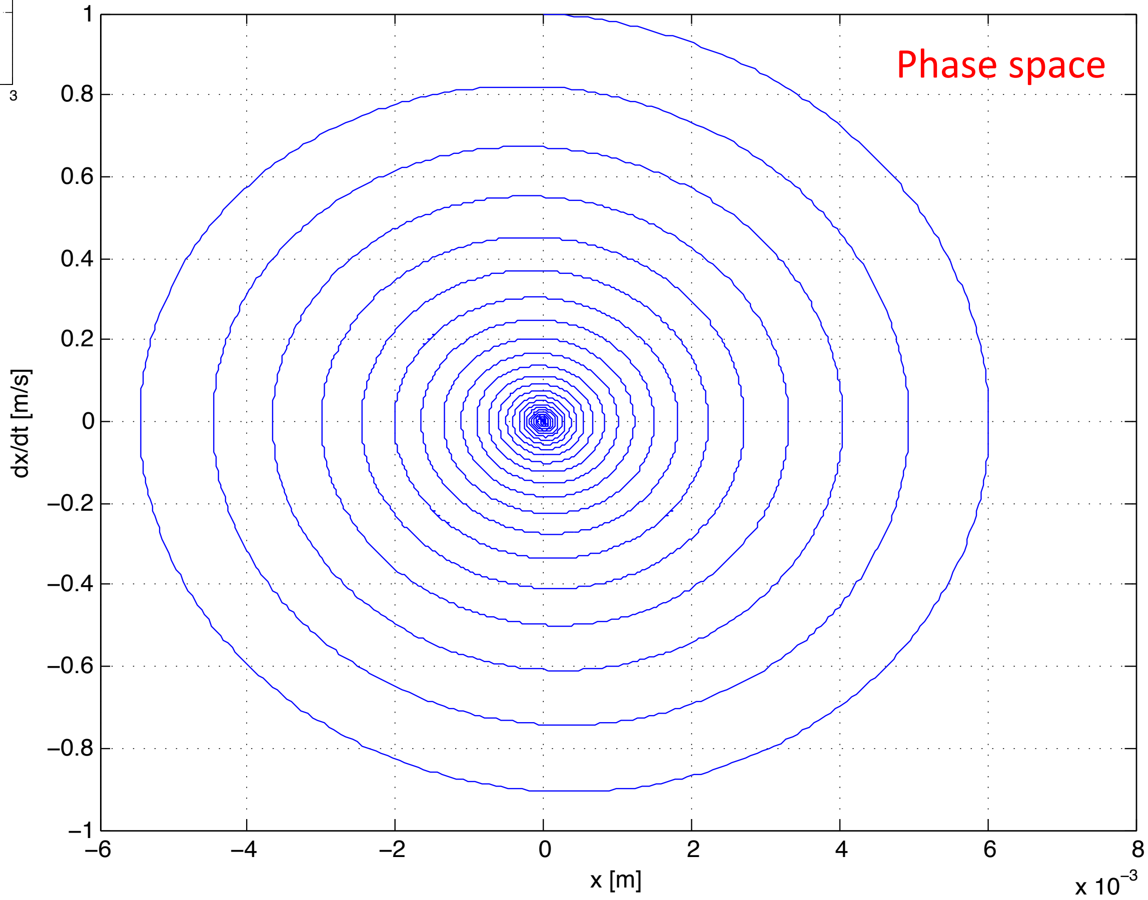
```matlab
% ### HOode45EX.m ###
% Numerically integrate the damped/driven harmonic oscillator
%   m*x''+ b*x' + k*x = A*sin(wt)
clear
% -----------------------------------------------------
% User input (Note: All paramters are stored in a structure)
P.y0(1) = 0.0;   % initial position [m]
P.y0(2) = 1.0;   % initial velocity [m/s]
P.b= 0.1;  % damping coefficient [kg/s]
P.k= 250.0;   % stiffness [N m]
P.m= 0.01;    % mass [kg]

% sinusoidal driving term
P.A= 0.0;   % amplitude [N] (set to zero to turn off)
fD= 1.05*sqrt(P.k/P.m)/(2*pi);  % freq. (Hz) [expressed as fraction of resonant freq.]

% Integration limits
P.t0 = 0.0;   % Start value
P.tf = 3.0;   % Finish value
P.dt = 0.0001;  % time step
% ---------------------------------------------------------------------
% +++
% spit back out some basic derived quantities
P.wr= 2*pi*fD;  % convert to angular freq.
disp(sprintf('Resonant frequency ~%g [Hz]', sqrt(P.k/P.m)/(2*pi)));
Q = (sqrt(P.k/P.m))/(P.b/P.m);  % quality factor
disp(sprintf('Q-value = %g', Q));
% +++
% use built-in ode45 to solve
[t y] = ode45('HOfunction', [P.t0:P.dt:P.tf],P.y0,[],P);

% ---------------------------------------------------------
% visualize
figure(1); clf;
plot(t,y(:,1)); hold on; grid on;
xlabel('t [s]');    ylabel('x(t) [m]')
% Phase plane
figure(2); clf;
plot(y(:,1), y(:,2)); hold on; grid on;
xlabel('x [m]');    ylabel('dx/dt [m/s]’)
```

```matlab
function [out1] = HOfunction(t,y,flag,P)
% -----------
%   y(1) ... position x
%   y(2) ... velocity dx/dt
out1(1)= y(2);
out1(2)= -1*(P.b/P.m)*y(2) - (P.k/P.m)*y(1)
           + (P.A/P.m)*sin(P.wr*t);
out1= out1’;
```


x 10^-3
8
6
4
2
0
−2
−4
−6
x(t) [m]
0
0.5
1
1.5
2
2.5
3
t [s]

HOode45EX.m





Things to try:

- Change the damping
- Change the stiffness or mass
- Change the initial conditions
- Turn on the (non-autonomous) sinusoidal driving term
- Other types of driving terms? (e.g., an impulse)
- Other changes?

Resonance

Consider the sinusoidally "driven" case:

$$\ddot{x} + \gamma \dot{x} + \omega_o^2 x = \frac{F_o}{m} e^{i\omega t}$$

Steady-state frequency response

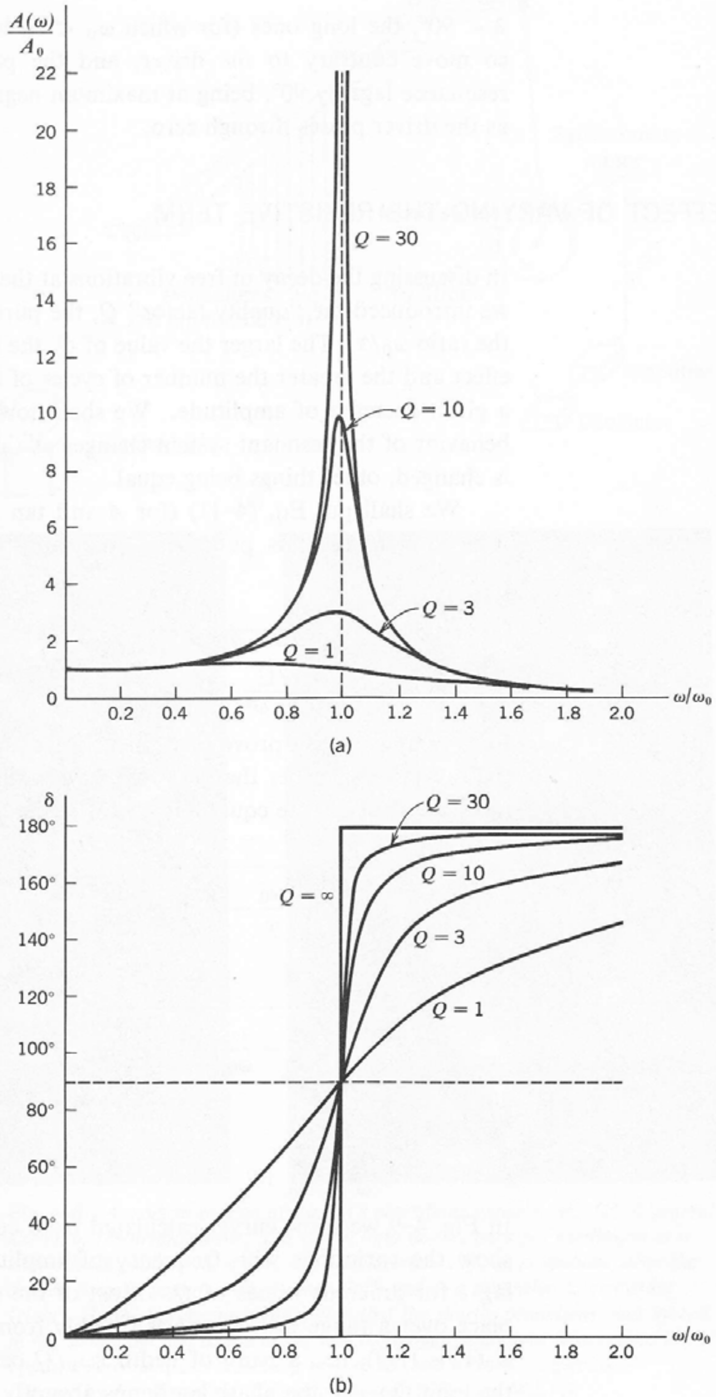


Fig. 4–9 (a) Amplitude as function of driving frequency for different values of Q, assuming driving force of constant magnitude but variable frequency. (b) Phase difference δ as function of driving frequency for different values of Q.

$$Q = \omega_o / \gamma$$

Q is the 'quality factor'

→ Think about how you would go about making these plots…

French (1971)

Some Useful #s

Physical Magnitudes

Gravitational acceleration (g)	$10\ m/sec^2$
Densities of solids and liquids	10^3–$10^4\ kg/m^3$
Density of air at sea level	$1\ kg/m^3$ (approx.)
Length of day	10^5 sec (approx.)
Length of year	3.16×10^7 sec $\approx 10^{7.5}$ sec
Earth's radius	6400 km
Angle subtended by finger thickness at arm's length	1° (approx.)
Thickness of paper	0.1 mm (approx.)
Mass of a paperclip	0.5 g (approx.)
Highest mountains, deepest oceans	10 km (approx.)
Earth–moon separation	3.8×10^5 km
Earth–sun separation	1.5×10^8 km
Atmospheric pressure	Equivalent to weight of $1\ kg/cm^2$ or a 10-m column of water
Avogadro's number	6.0×10^{23}
Atomic masses	1.6×10^{-27} kg to 4×10^{-25} kg
Linear dimensions of atoms	10^{-10} m (approx.)
Molecules/cm^3 in gas at STP	2.7×10^{19}
Atoms/cm^3 in solids	10^{23} (approx.)
Elementary charge (e)	1.6×10^{-19} C
Electron mass	10^{-30} kg (approx.)
Speed of light	3×10^8 m/sec
Wavelength of light	6×10^{-7} m (approx.)

$\pi^2 \approx 10$	$\log_{10} 4 \approx 0.60$
$e \approx 2.7$	$\log_{10} e \approx 0.43$
$\log_{10} 2 \approx 0.30$	$\log_{10} \pi \approx 0.50$
$\log_{10} 3 \approx 0.48$	$\log_e 10 \approx 2.3$

Angle (radians) = arc length/radius. Full circle = 2π rad.
1 rad $\approx 0.16 \times$ full circle $\approx 57°$.
Solid angle (steradians) = area/(radius)^2. Full sphere = 4π sr.
1 sr $\approx 0.08 \times$ full sphere.

Approximations

Binomial theorem:

For $x \ll 1$, $(1+x)^n \approx 1 + nx$

e.g., $(1+x)^3 \approx 1 + 3x$

$(1-x)^{1/2} \approx 1 - \frac{1}{2}x \approx (1+x)^{-1/2}$

For $b \ll a$, $(a+b)^n = a^n\left(1 + \frac{b}{a}\right)^n \approx a^n\left(1 + n\frac{b}{a}\right)$

Other expansions:

For $\theta \ll 1$ rad, $\sin\theta \approx \theta - \frac{\theta^3}{6} \to \theta$

$\cos\theta \approx 1 - \frac{\theta^2}{2} \to 1$

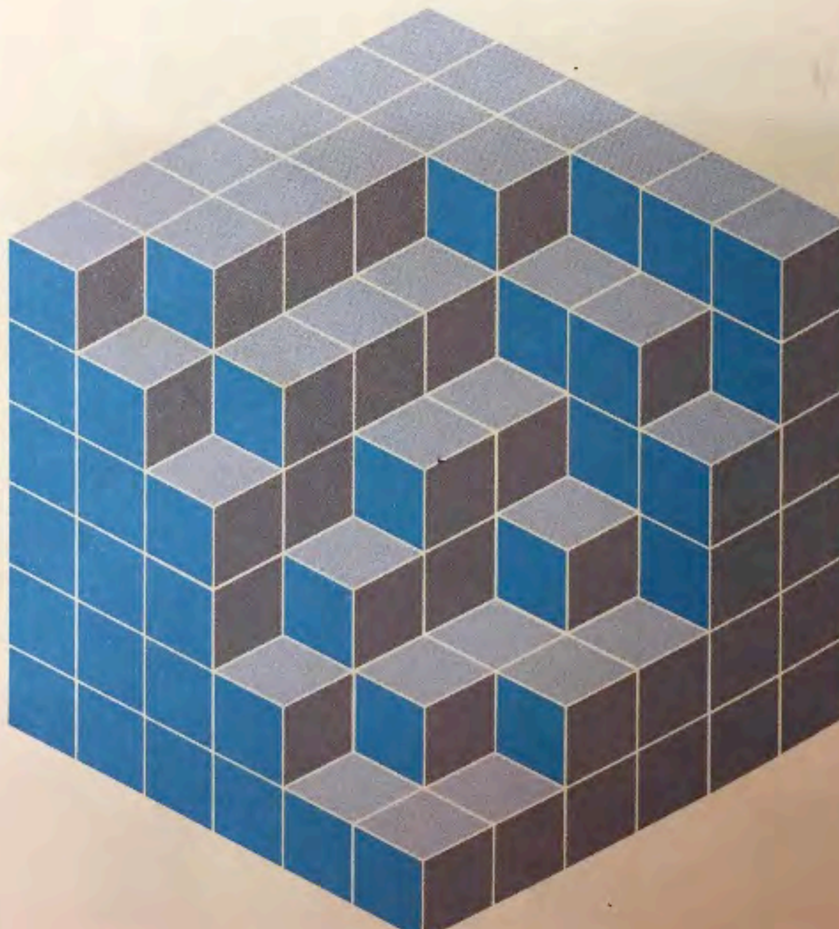
For $x \ll 1$, $\log_e(1+x) \approx x$

$\log_{10}(1+x) \approx 0.43x$

French (1971)

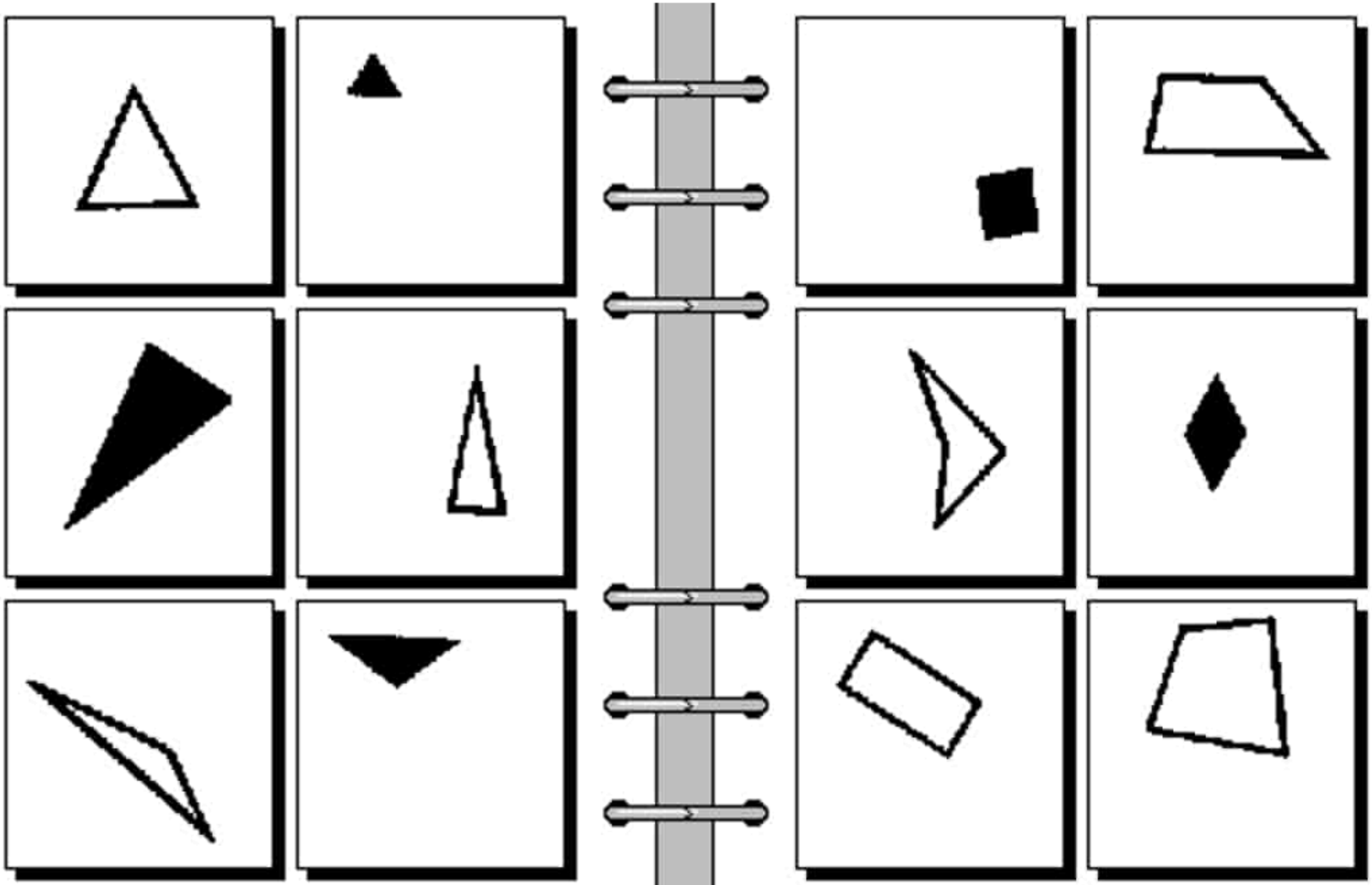
Cube Volume

These little cubes originally made a big cube measuring 18 cm x 18 cm x 18 cm. Now some of the little cubes have been removed, can you work out what volume the remaining cubes have? Assume all invisible cubes are present.



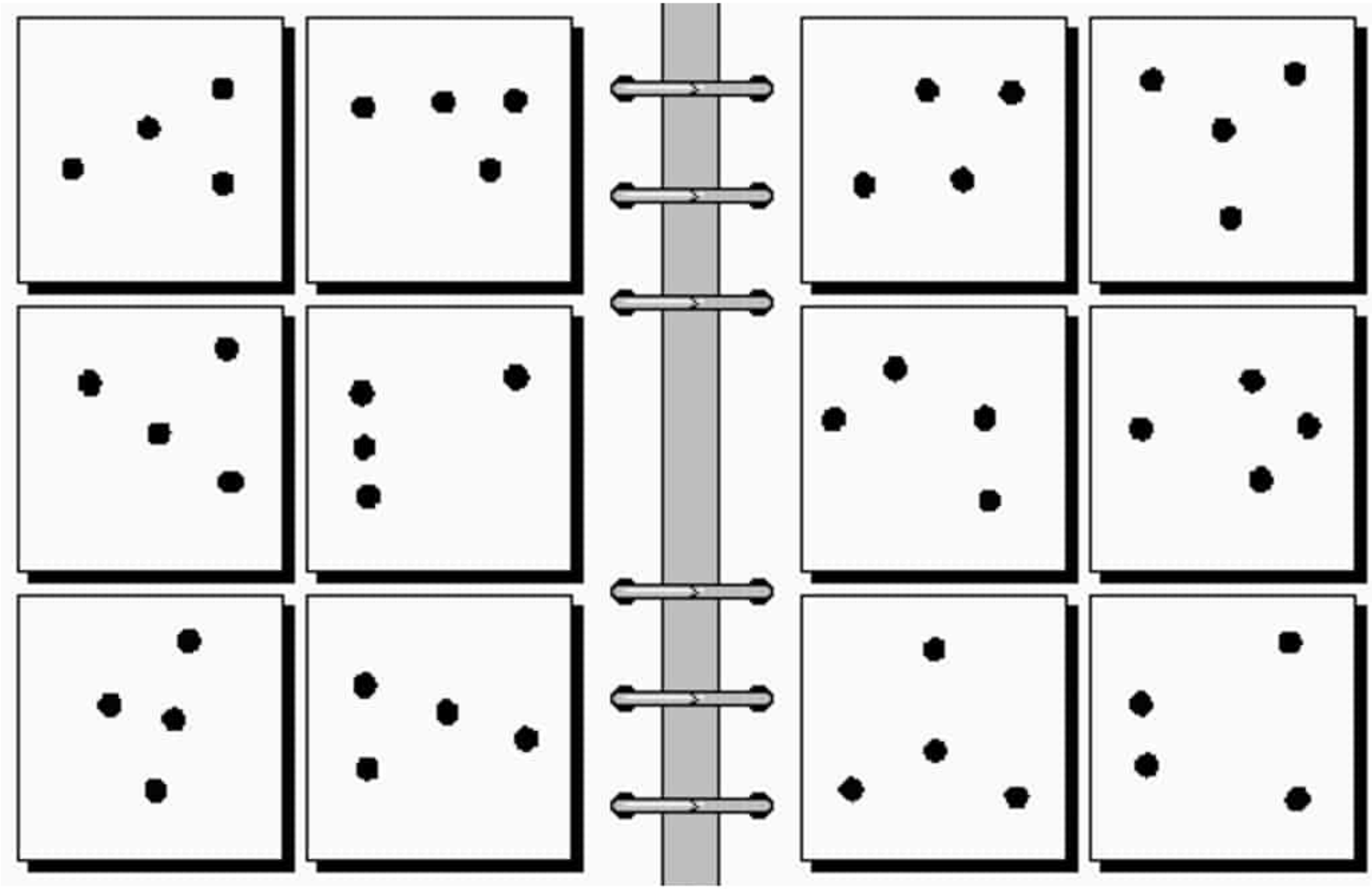
"Puzzles" – Bongard problems

Determine the "rule" that is different between the two sides (each of six)

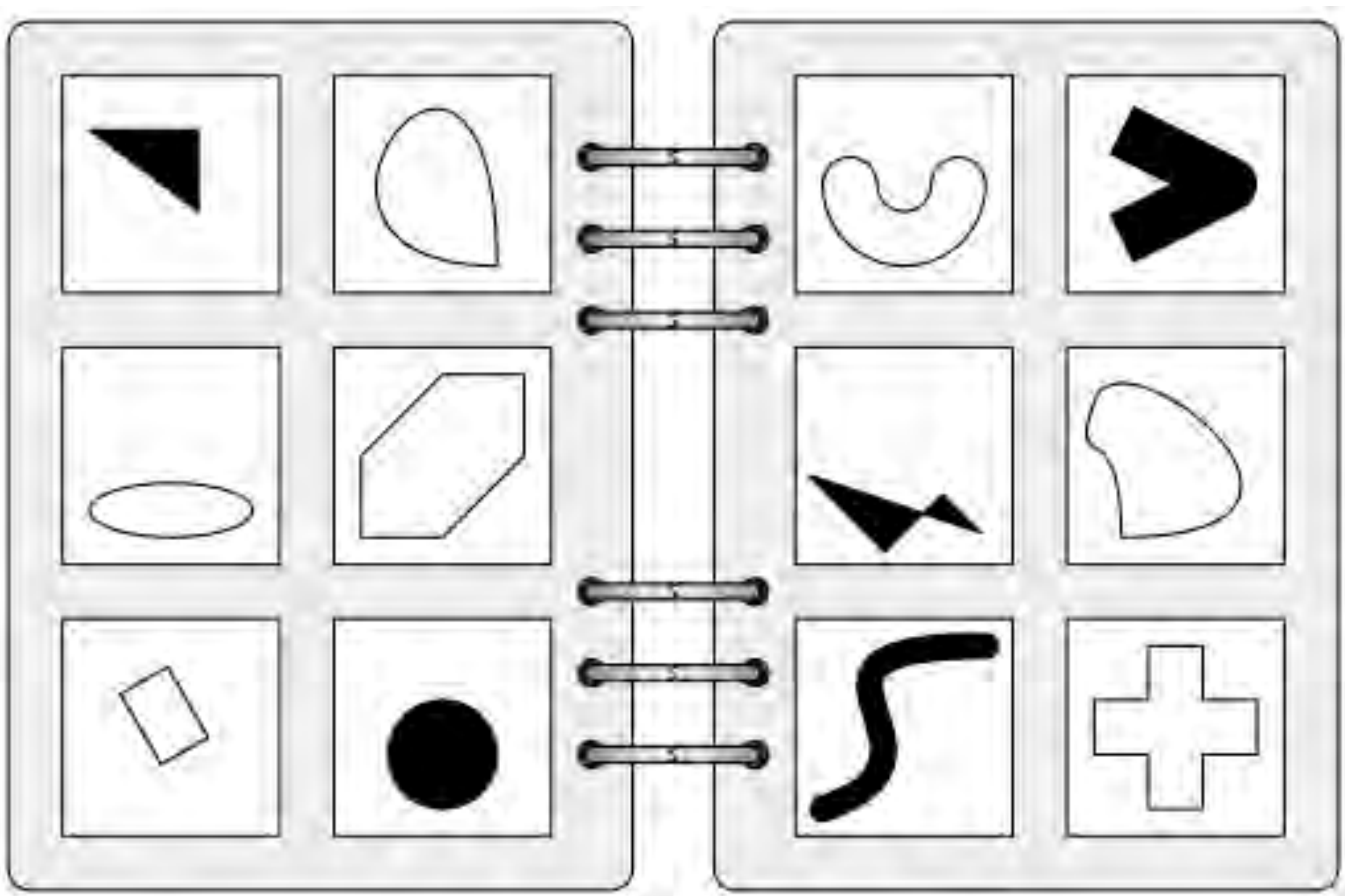


e.g., http://www.foundalis.com/res/bps/bpidx.htm

<u>"Puzzles"</u> – Bongard problems



"Puzzles" – Bongard problems



Problem Solving

1 What is the order of magnitude of the number of times that the earth has rotated on its axis since the solar system was formed?

2 During the average lifetime of a human being, how many heart-beats are there? How many breaths?

3 Make reasoned estimates of (a) the total number of ancestors you would have (ignoring inbreeding) since the beginning of the human race, and (b) the number of hairs on your head.

4 The present world population (human) is about 3×10^9.

(a) How many square kilometers of land are there per person? How many feet long is the side of a square of that area?

(b) If one assumes that the population has been doubling every 50 years throughout the existence of the human race, when did Adam and Eve start it all? If the doubling every 50 years were to continue, how long would it be before people were standing shoulder to shoulder over all the land area of the world?

5 Estimate the order of magnitude of the mass of (a) a speck of dust; (b) a grain of salt (or sugar, or sand); (c) a mouse; (d) an elephant; (e) the water corresponding to 1 in. of rainfall over 1 square mile; (f) a small hill, 500 ft high; and (g) Mount Everest.

6 Estimate the order of magnitude of the number of atoms in (a) a pin's head, (b) a human being, (c) the earth's atmosphere, and (d) the whole earth.

French (1971)

Problem Solving

7 Estimate the fraction of the total mass of the earth that is now in the form of living things.

8 Estimate (a) the total volume of ocean water on the earth, and (b) the total mass of salt in all the oceans.

9 It is estimated that there are about 10^{80} protons in the (known) universe. If all these were lumped into a sphere so that they were just touching, what would the radius of the sphere be? Ignore the spaces left when spherical objects are packed and take the radius of a proton to be about 10^{-15} m.

10 The sun is losing mass (in the form of radiant energy) at the rate of about 4 million tons per second. What fraction of its mass has it lost during the lifetime of the solar system?

11 Estimate the time in minutes that it would take for a theatre audience of about 1000 people to use up 10% of the available oxygen if the building were sealed. The average adult absorbs about one sixth of the oxygen that he or she inhales at each breath.

12 Solar energy falls on the earth at the rate of about 2 cal/cm^2/min. Estimate the amount of power, in megawatts or horsepower, represented by the solar energy falling on an area of 100 square miles—about the area of a good-sized city. How would this compare with the total power requirements of such a city? (1 cal = 4.2 J; 1 W = 1 J/sec; 1 hp = 746 W.)

French (1971)

Problem Solving

15 The astronomer Tycho Brahe made observations on the angular positions of stars and planets by using a quadrant, with one peephole at its center of curvature and another peephole mounted on the arc. One such quadrant had a radius of about 2 m, and Tycho's measurements could usually be trusted to 1 minute of arc ($\frac{1}{60}^\circ$). What diameter of peepholes would have been needed for him to attain this accuracy?

24 Two students want to measure the speed of sound by the following procedure. One of them, positioned some distance away from the other, sets off a firecracker. The second student starts a stopwatch when he sees the flash and stops it when he hears the bang. The speed of sound in air is roughly 300 m/sec, and the students must admit the possibility of an error (of undetermined sign) of perhaps 0.3 sec in the elapsed time recorded. If they wish to keep the error in the measured speed of sound to within 5%, what is the minimum distance over which they can perform the experiment?

Problem Solving

26 The radius of a sphere is measured with an uncertainty of 1%. What is the percentage uncertainty in the volume?

29 The universe appears to be undergoing a general expansion in which the galaxies are receding from us at speeds proportional to their distances. This is described by Hubble's law, $v = \alpha r$, where the constant α corresponds to v becoming equal to the speed of light, c ($= 3 \times 10^8$ m/sec), at $r \approx 10^{26}$ m. This would imply that the mean mass per unit volume in the universe is decreasing with time.

(a) Suppose that the universe is represented by a sphere of volume V at any instant. Show that the fractional increase of volume per unit time is given by

$$\frac{1}{V}\frac{dV}{dt} = 3\alpha$$

(b) Calculate the fractional decrease of mean density per second and per century.

4.1 The Wrong Question

Actually, "What keeps things moving?" is the wrong question. In the early 1600s, Galileo Galilei did experiments that convinced him that a moving object has an intrinsic "quantity of motion" and needs no push to keep it moving (Fig. 4.1). Instead of answering "What keeps things moving?," Galileo declared that the question needs no answer. In so doing, he set the stage for centuries of progress in physics, beginning with the achievements of Issac Newton and culminating in the work of Albert Einstein.

The Right Question

Our first question—about why the spacecraft keeps moving—is the wrong question. So what's the right question? It's the second one, about why the baseball's motion *changed*. Dynamics isn't about what causes motion itself; it's about what causes *changes* in motion. Changes include starting and stopping, speeding up and slowing down, and changing direction. Any *change* in motion begs an explanation, but motion itself does not. Get used to this important idea and you'll have a much easier time with physics. But if you remain a "closet Aristotelian," secretly looking for causes of motion itself, you'll find it difficult to understand and apply the simple laws that actually govern motion.

→ The notion of *change* is a lynchpin of physics....

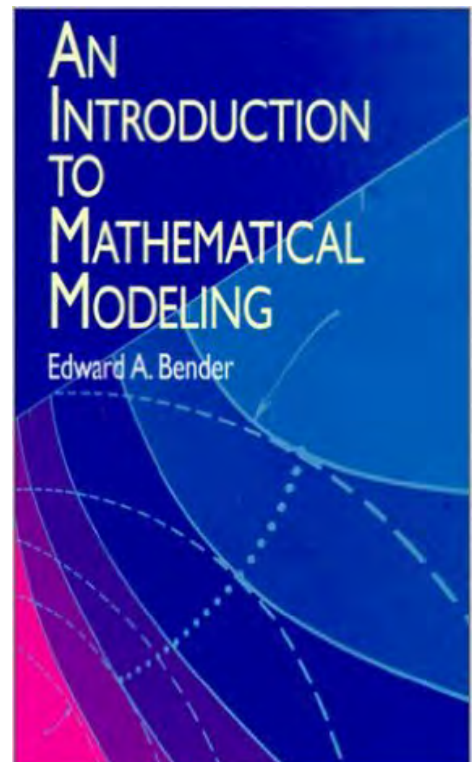
“Modeling”

- To help put some context in place for the physics ahead, let’s take a slight detour....
- Calculus provides wonderful tools to help study *change*
- In particular, a very useful extension of calculus is known as *differential equations*

Table 2.1 Equations of Motion for Constant Acceleration

Equation	Contains	Number
$v = v_0 + at$	v, a, t; no x	2.7
$x = x_0 + \frac{1}{2}(v_0 + v)t$	x, v, t; no a	2.9
$x = x_0 + v_0 t + \frac{1}{2}at^2$	x, a, t; no v	2.10
$v^2 = v_0^2 + 2a(x - x_0)$	x, v, a; no t	2.11

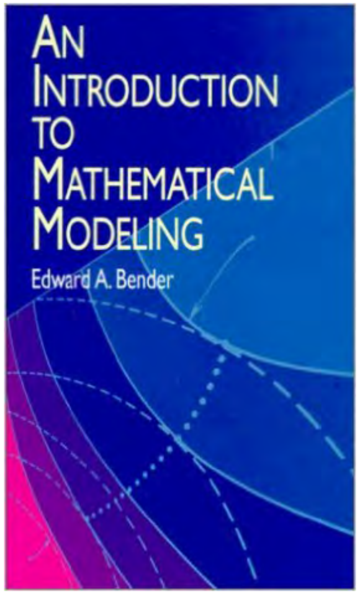
Whether you realize it or not, you have already been dealing with DEs in some fashion....



- Here comes the fun part: Many problems fall under the purview of *mathematical modeling*

“Mathematical Modeling”

From the preface (1978)



“This book is designed to teach students how to apply mathematics by formulating, analyzing, and criticizing models.”

“The first part of the book requires only elementary calculus and, in one chapter, basic probability theory.”

“Although the level of mathematics required is not high, this is not an easy text: Setting up and manipulating models requires thought, effort, and usually discussion.”

“Often problems have no single best answer, because different models can illuminate different facets of a problem. Discussion of homework in class by the students is an integral part of the learning process; in fact, my classes have spent about half the time discussing homework..”

Bender

"The theoretical and scientific study of a situation centers around a model, that is, something that mimics relevant features of the situation being studied. For example, a road map, a geological map, and a plant collection are all models that mimic different aspects of a portion of the earth's surface."

"The ultimate test of a model is how well it performs when it is applied to the problems it was designed to handle. (You cannot reasonably criticize a geological map if a major highway is not marked on it; however, this would be a serious deficiency in a road map.) When a model is used, it may lead to incorrect predictions. The model is often modified, frequently discarded, and sometimes used anyway because it is better than nothing. *This is the way science develops*."

"Here we are concerned exclusively with mathematical models, that is, models that mimic reality by using the language of mathematics. [...] What makes mathematical models useful? If we "speak in mathematics, then:

1. We must formulate our ideas precisely and so are less likely to let implicit assumptions slip by.
2. We have a concise "language" which encourages manipulation.
3. We have a large number of potentially useful theorems available.
4. We have high speed computers available for carrying out calculations.

Bender

"Modeling" & Differential equations (DEs)

→ A very common/useful tool in our toolbox....

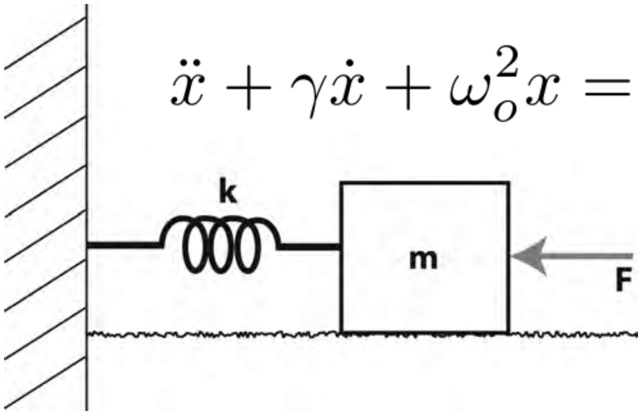
Wave equation

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}$$

Note: Though DEs pervade much of 2010 material, you are not expected to become super-adept at solving them for 2010

Harmonic oscillator

$$\ddot{x} + \gamma \dot{x} + \omega_o^2 x = 0$$



Note: This just a specific case of Newton's 2nd law (F=ma)!

Laplace's equation

$$\Delta f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

Maxwell's equations

$$\nabla \cdot \mathbf{E} = \frac{\rho_v}{\varepsilon} \quad \text{(Gauss' Law)}$$

$$\nabla \cdot \mathbf{H} = 0 \quad \text{(Gauss' Law for Magnetism)}$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad \text{(Faraday's Law)}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \quad \text{(Ampere's Law)}$$

Several basic flavors apparent:

- Ordinary (ODE)
- Partial (PDE)
- Scalar vs. Vector

“Modeling” & Differential equations

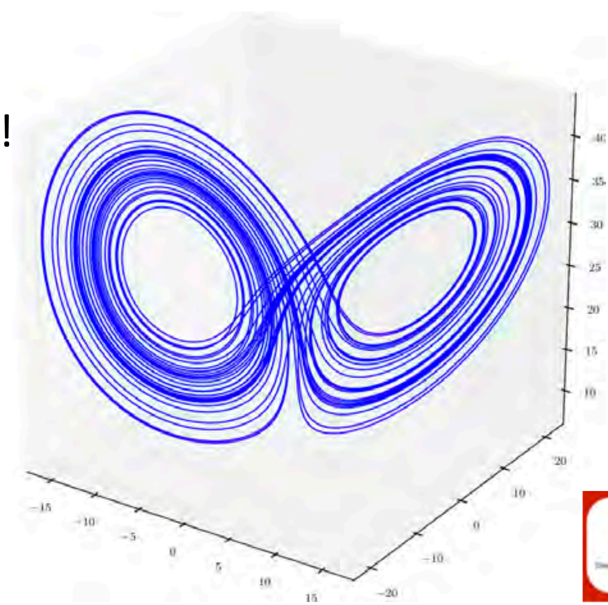
Lorenz equations

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = rx - y - xz$$

$$\frac{dz}{dt} = xy - bz$$

→ Chaos!



SIR model
(‘compartmental’ model in epidemiology)

S = the number of *susceptibles*, the people who are not yet sick but who could become sick

I = the number of *infecteds*, the people who are currently sick

R = the number of *recovered*, or *removed*, the people who have been sick and can no longer infect others or be reinfected.

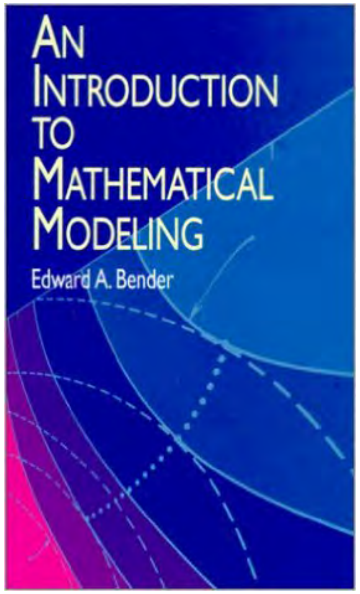
$$\frac{dS}{dt} = -\beta IS$$

$$\frac{dI}{dt} = \beta IS - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$

York University
Centre for Disease Modelling

<u>"Mathematical Modeling"</u>



"Mathematics and physical science each had important effects on the development of the other. Mathematics is starting to play a greater role in the development of the life and social sciences, and these sciences are starting to influence the development of mathematics."

"We begin with a definition based on the previous discussion: A mathematical model is an abstract, simplified, mathematical construct related to a part of reality and created for a particular purpose. [...] As far as a model is concerned the world can be divided into three parts:

1. Things whose effects are neglected.
2. Things that affect the model but whose behavior the model is not designed to study.
3. Things the model is designed to study the behavior of.

<u>Two key ingredients should be apparent here:</u>

- Figuring out what question you want to try to answer
- What assumptions you are willing to make

Bender

Ex.

Question: How fast does a person learn?

Ex.

Question: How fast does a person learn?

(very) Simple model: Rate a person learns = Percentage of task not yet learned

y is the percentage learned as a function of time t

$$\frac{dy}{dt} = 100 - y$$

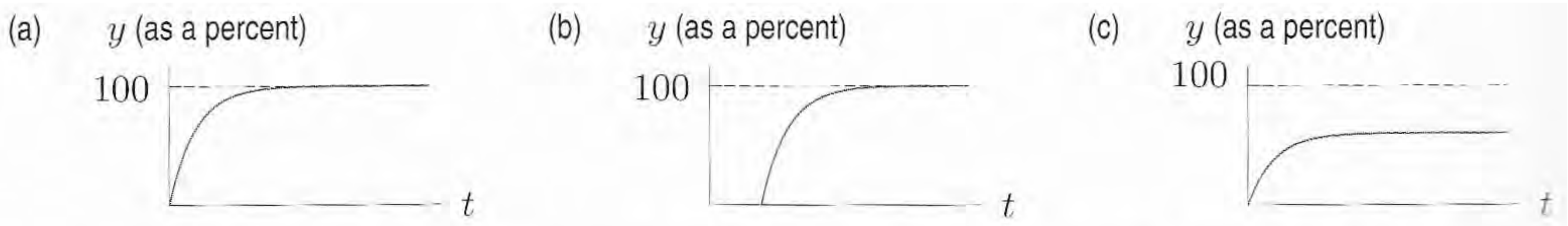


Figure 11.1: Possible graphs showing percentage of task learned, y, as a function of time, t

Solution
(e.g., via "separation of variables")

$$y(t) = 100 - Ce^{-t}$$

Hughes-Hallett et al. (2005)

<u>Ex.</u>

$$\frac{dy}{dt} = 100 - y$$

$$y(t) = 100 - Ce^{-t}$$

→ Note that our 'model' (redundantly) allows for y greater than 100

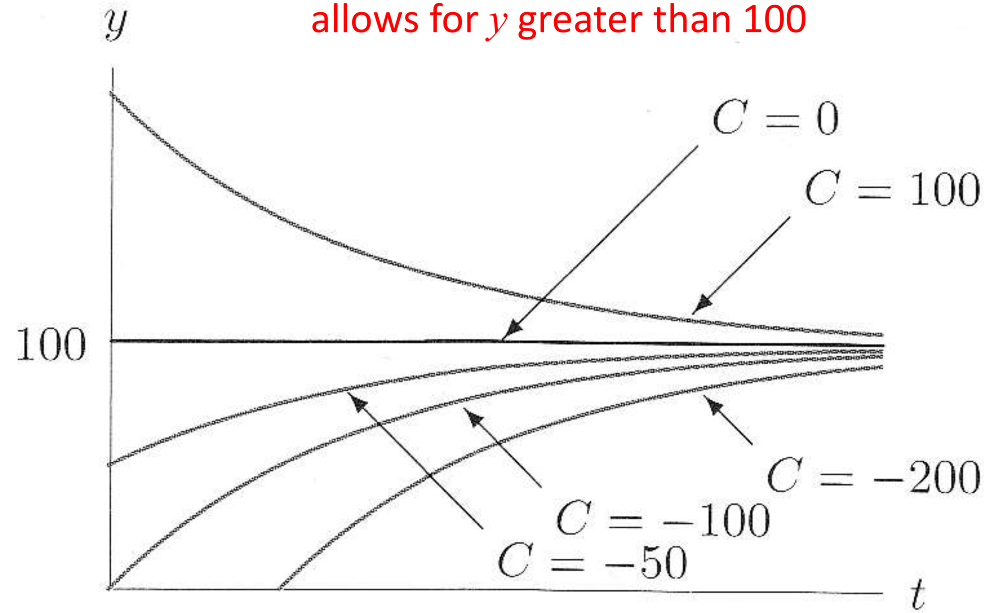


Figure 11.2: Solution curves for $dy/dt = 100 - y$: Members of the family $y = 100 + Ce^{-t}$

- Equilibrium points?

  Values of $y(t)$ where $dy/dt = 0$

  $y(t) = 100$

- Stability?

  Do solutions move towards or away from the equilibrium if starting nearby?

  stable (solution move towards $y(t) = 100$ with increasing t)

- What determines the value of C?

  initial conditions (→ E&U theorem!)

Hughes-Hallett et al. (2005)

Some further common examples

Solution

Exponential growth/decay

$$\frac{dP}{dt} = kP$$

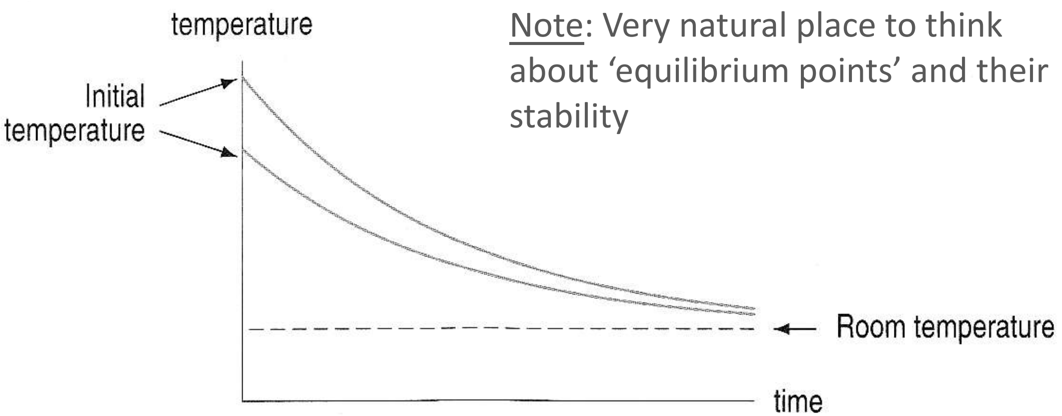
$$P = P_0 e^{kt}$$

e.g., Nuclear decay, 1st order chemical reaction, bacterial growth

Newton's law of heating/cooling

"Newton proposed that the temperature of a hot object decreases at a rate proportional to the difference between its temperature and that of its surroundings. Similarly, a cold object heats up at a rate proportional to the temperature difference between the object and its surroundings."

$$\frac{dT}{dt} = \alpha(T_o - T)$$



Note: Very natural place to think about 'equilibrium points' and their stability

Solution

$$T(t) = T_0 + Ce^{-\alpha t}$$

Hughes-Hallett et al. (2005)

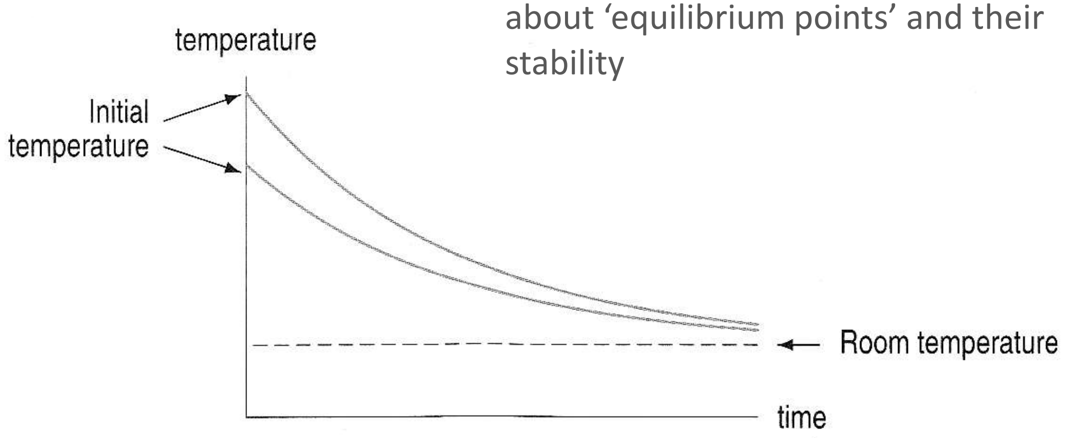
Stability

Newton's law of heating/cooling

$$\frac{dT}{dt} = \alpha(T_o - T)$$

Solution

$$T(t) = T_0 + Ce^{-\alpha t}$$



Note: Very natural place to think about 'equilibrium points' and their stability

- An **equilibrium solution** is constant for all values of the independent variable. The graph is a horizontal line.
- An equilibrium is **stable** if a small change in the initial conditions gives a solution which tends toward the equilibrium as the independent variable tends to positive infinity.
- An equilibrium is **unstable** if a small change in the initial conditions gives a solution curve which veers away from the equilibrium as the independent variable tends to positive infinity.

Hughes-Hallett et al. (2005)

Some further common examples

Note: We will come back to this in more detail next lecture

Falling body: Terminal velocity

Assume air resistance is proportional to velocity, the Newton's 2nd Law leads to:

$$m\frac{dv}{dt} = mg - kv$$

$$\frac{dv}{dt} = -\frac{k}{m}\left(v - \frac{mg}{k}\right)$$

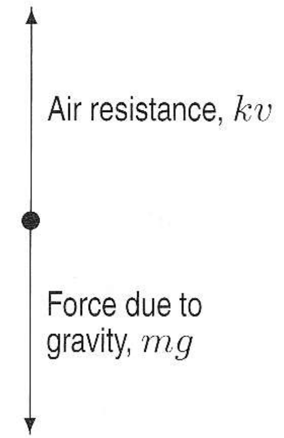


Figure 11.44: Forces acting on a falling object

Solution $$v = \frac{mg}{k}\left(1 - e^{-kt/m}\right)$$

Hughes-Hallett et al. (2005)

Reference

Problem 1. A variable $n(t)$ is described by a first-order linear differential equation with constant coefficients

$$\tau \frac{dn(t)}{dt} + n(t) = n_\infty$$

where τ and n_∞ are constants. Let $n(0) = n_0$.

a) For $t \geq 0$, determine an expression for $n(t)$ in terms of τ, n_∞, and n_0.

b) Plot $n(t)$ versus t for the following two cases and explain the difference between the two plots:
   - i) $n_0 = 0, \quad n_\infty = 10, \quad \tau = 1,$
   - ii) $n_0 = 0, \quad n_\infty = 10, \quad \tau = 10,$

c) Plot $n(t)$ versus t for the following two cases and explain the difference between the two plots:
   - iii) $n_0 = 10, \quad n_\infty = 0, \quad \tau = 1,$
   - iv) $n_0 = 10, \quad n_\infty = 10, \quad \tau = 1,$

d) Plot $n(t)$ versus t for the following two cases and explain the difference between the two plots:
   - v) $n_0 = 10, \quad n_\infty = 0, \quad \tau = 1,$
   - vi) $n_0 = -10, \quad n_\infty = 10, \quad \tau = 1.$

Note: This is essentially the same form of eqn. as others we saw earlier (e.g., Newton's Law of Cooling)

$$\frac{dT}{dt} = \alpha(T_o - T)$$

Reference (SOL)

Problem 1. A first-order, linear differential equation with constant coefficients and a constant inhomogeneous (drive or input) term has an exponential solution. Therefore, the solution can be written in the form

$$n(t) = n_\infty + \left(n_0 - n_\infty\right) e^{-t/\tau},$$

where $n_0 = n(0)$ is the initial value of $n(t)$ and $n_\infty = \lim_{t\to\infty} n(t)$ is the final value of $n(t)$. The form of this solution can be verified by evaluating $n(t)$ at $t = 0$ and $t \to \infty$. Substitution into the differential equation shows that this solution satisfies the differential equation. The solutions for cases i-vi are shown in Figure 1. The solutions for part a (i and ii) have the same initial and final values but different time constants (by $t = 10$ s, curve ii is just above 6 and has not yet reached its final value of 10). The solutions for part b (iii and iv) have the same initial values and different final values. Although curve iv was calculated with the same time constant as in iii, it doesn't make sense to compare the time constants of the curves, since curve iv isn't changing. The solutions for part c (v and vi) have different initial and final values and the same time constants.

Note: The solution is essentially the same too, just written in a more general way

$$T(t) = T_0 + Ce^{-\alpha t}$$

Reference (SOL)

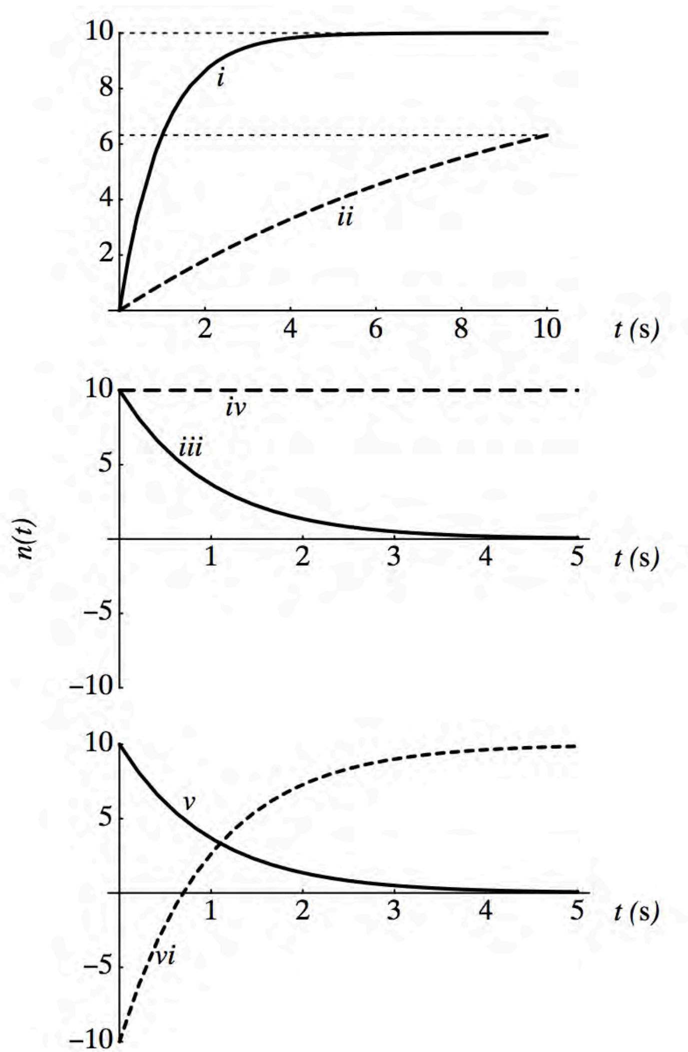
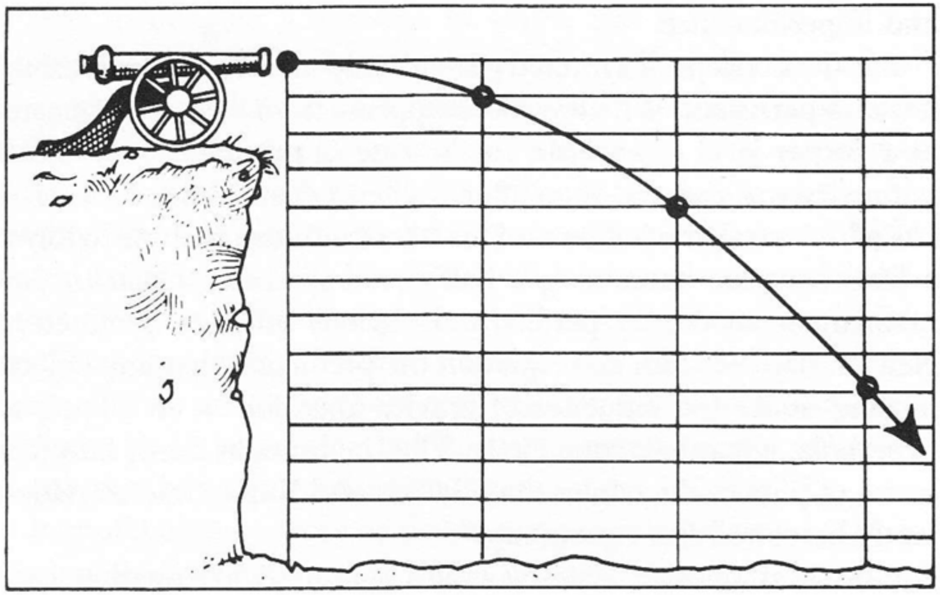


Figure 1. Solutions to parts i-vi. In the upper panel, horizontal dotted lines are shown at the final value of 10 and for the value of $n(t)$ at $t = \tau$, i.e., the line is at $10(1 - e^{-1})$.

Mechanics → "Change"



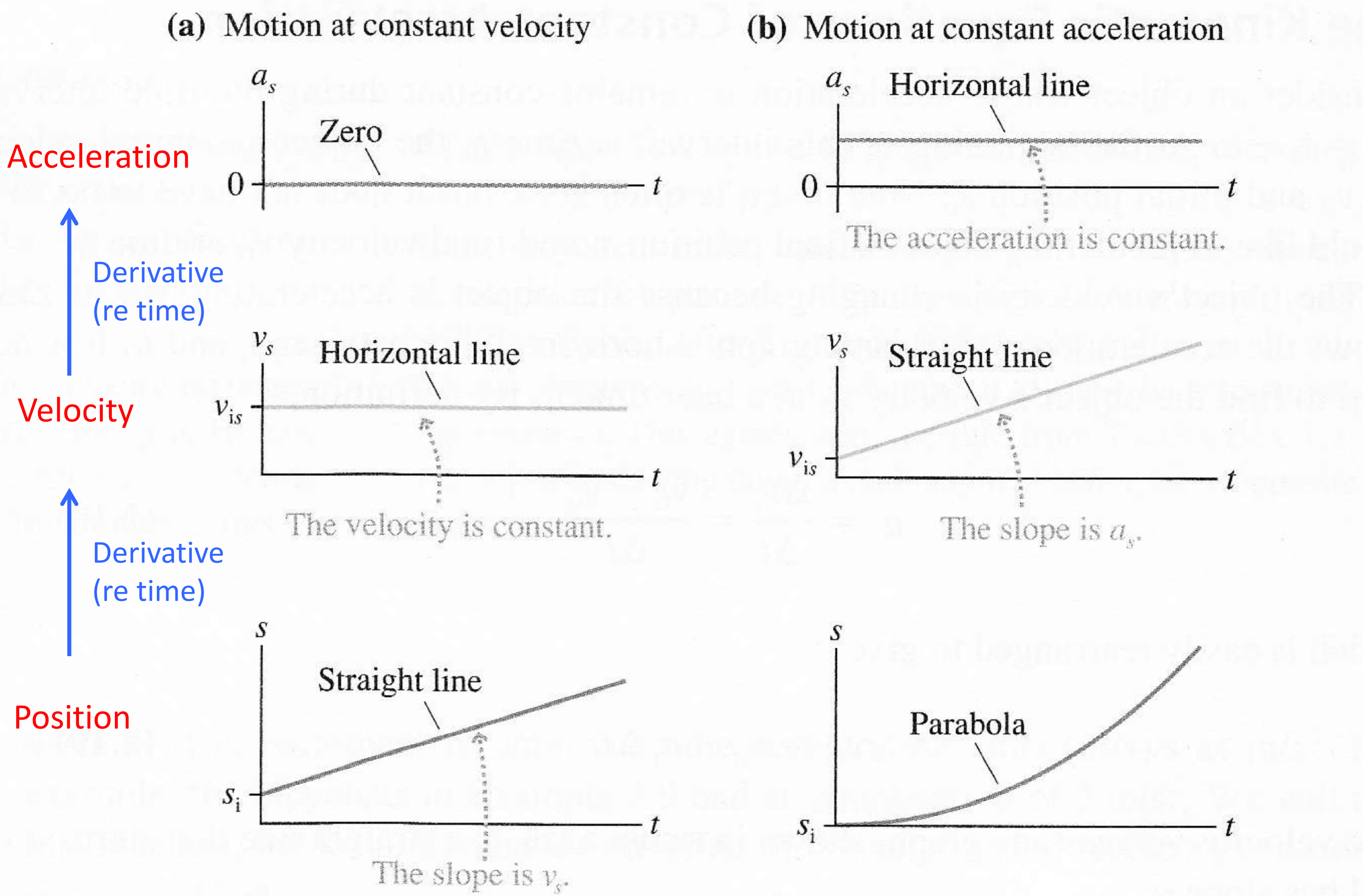
- Where is the cannonball?
  "When" matters too, right?

- Let's just consider 1-D for now (e.g., height of the cannonball; we'll come back to 2-D shortly)

- Consider three basic quantities:
  - Position [m]
  - Speed or velocity [m/s]
  - Acceleration [m/s^2]

- These are all inter-related via how things are ***changing with time***

FIGURE 2.24 Motion with constant velocity and constant acceleration. These graphs assume $s_i = 0$, $v_{is} > 0$, and (for constant acceleration) $a_s > 0$.



Knight

The door swings both ways.....

Position, Velocity, and Acceleration

Derivative Form

If $s = s(t)$ is the position function of an object at time t, then

Velocity $= v = \frac{ds}{dt}$ Acceleration $= a = \frac{dv}{dt}$

Integral Form

$$s(t) = \int v(t)dt \qquad v(t) = \int a(t)dt$$

Acceleration

↓ Integrate (re time) ↑ Derivative (re time)

Velocity

↓ Integrate (re time) ↑ Derivative (re time)

Position

→ Sometimes integration is called "anti-differentiation"

	Derivative Form
Position	$r(t)$
Velocity	$v(t) = \frac{dr}{dt}$
Acceleration	$a(t) = \frac{dv}{dt} = \frac{d^2r}{dt^2}$

NOTE: Numerically, integration is typically much easier than differentiation

Ex.

EXAMPLE 2.16 From track to graphs

Draw position, velocity, and acceleration graphs for the ball on the frictionless track of FIGURE 2.35.

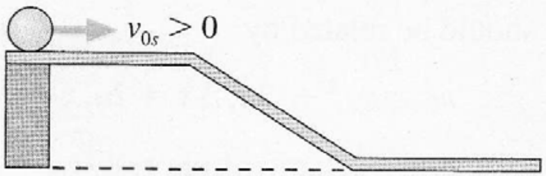


FIGURE 2.35 A ball rolling along a track.

Note: Implicitly buried in the "model" here is the notion that we treat the ball like a "particle" (or better yet, a *point*). That is, we don't worry about its rotation, the moment of intertia, etc… Further, note that we also make other (implicit) simplifications, such as neglecting friction, etc…

→ Generally helpful to consider what (stated & unstated) simplifying assumptions are being made….

Knight

Ex.

EXAMPLE 2.16 From track to graphs

Draw position, velocity, and acceleration graphs for the ball on the frictionless track of FIGURE 2.35.

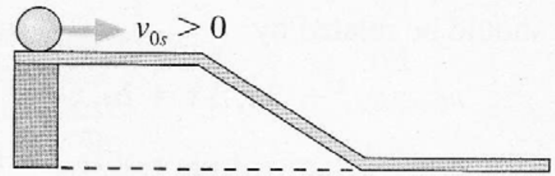
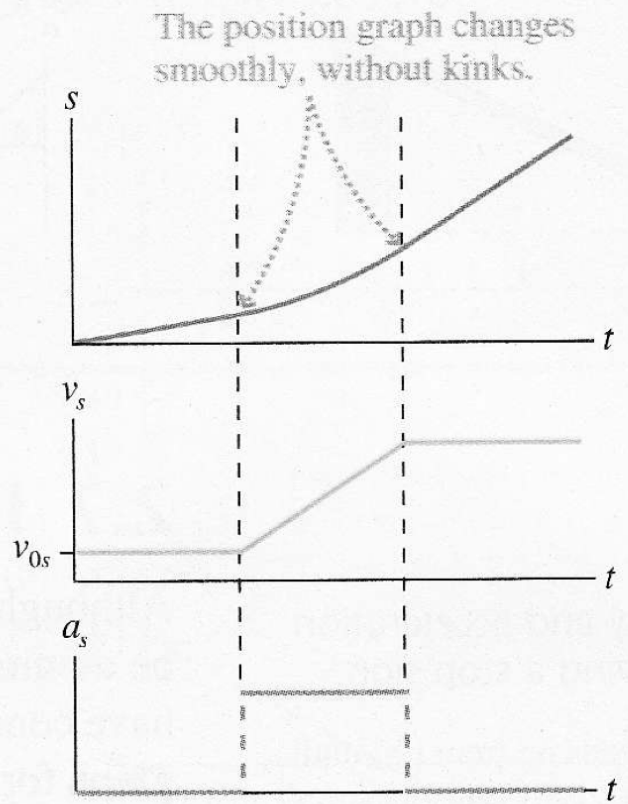


FIGURE 2.35 A ball rolling along a track.

ANS

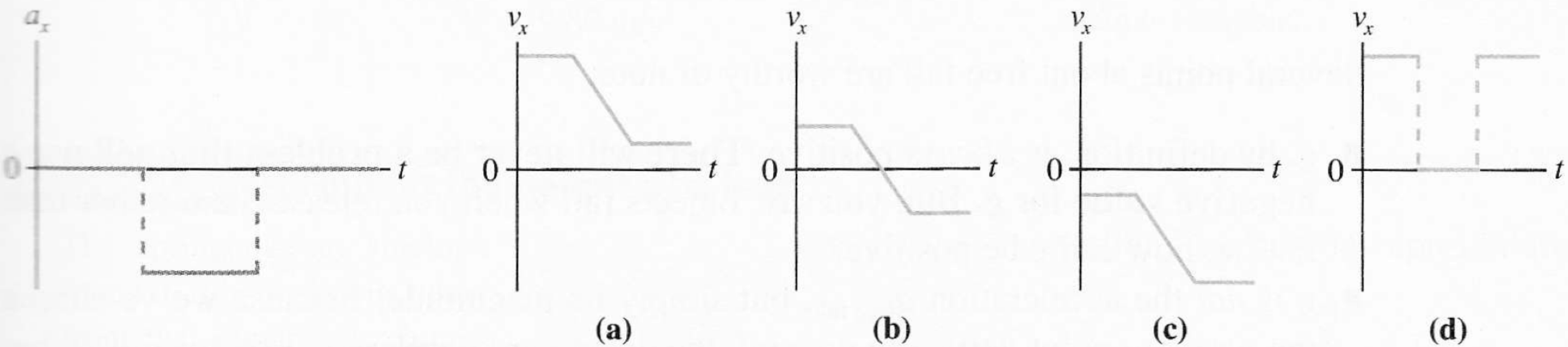
FIGURE 2.36 Motion graphs for the ball in Example 2.16.



Question: Why is acceleration only non-zero on the downward incline?

Ex. - Velocity vs Speed

STOP TO THINK 2.4 Which velocity-versus-time graph or graphs go with the acceleration-versus-time graph? The particle is initially moving to the right.



ANS
a & b only (why?)

→ Think carefully about what implicit assumptions are built-in to things...

Equations of motion

Table 2.1 Equations of Motion for Constant Acceleration

Equation	Contains	Number
$v = v_0 + at$	v, a, t; no x	2.7
$x = x_0 + \frac{1}{2}(v_0 + v)t$	x, v, t; no a	2.9
$x = x_0 + v_0 t + \frac{1}{2}at^2$	x, a, t; no v	2.10
$v^2 = v_0^2 + 2a(x - x_0)$	x, v, a; no t	2.11

Question: Where do these formulae (which are useful for solving problems!) come from?

→ You should feel comfortable deriving these equations

Position, Velocity, and Acceleration

Derivative Form

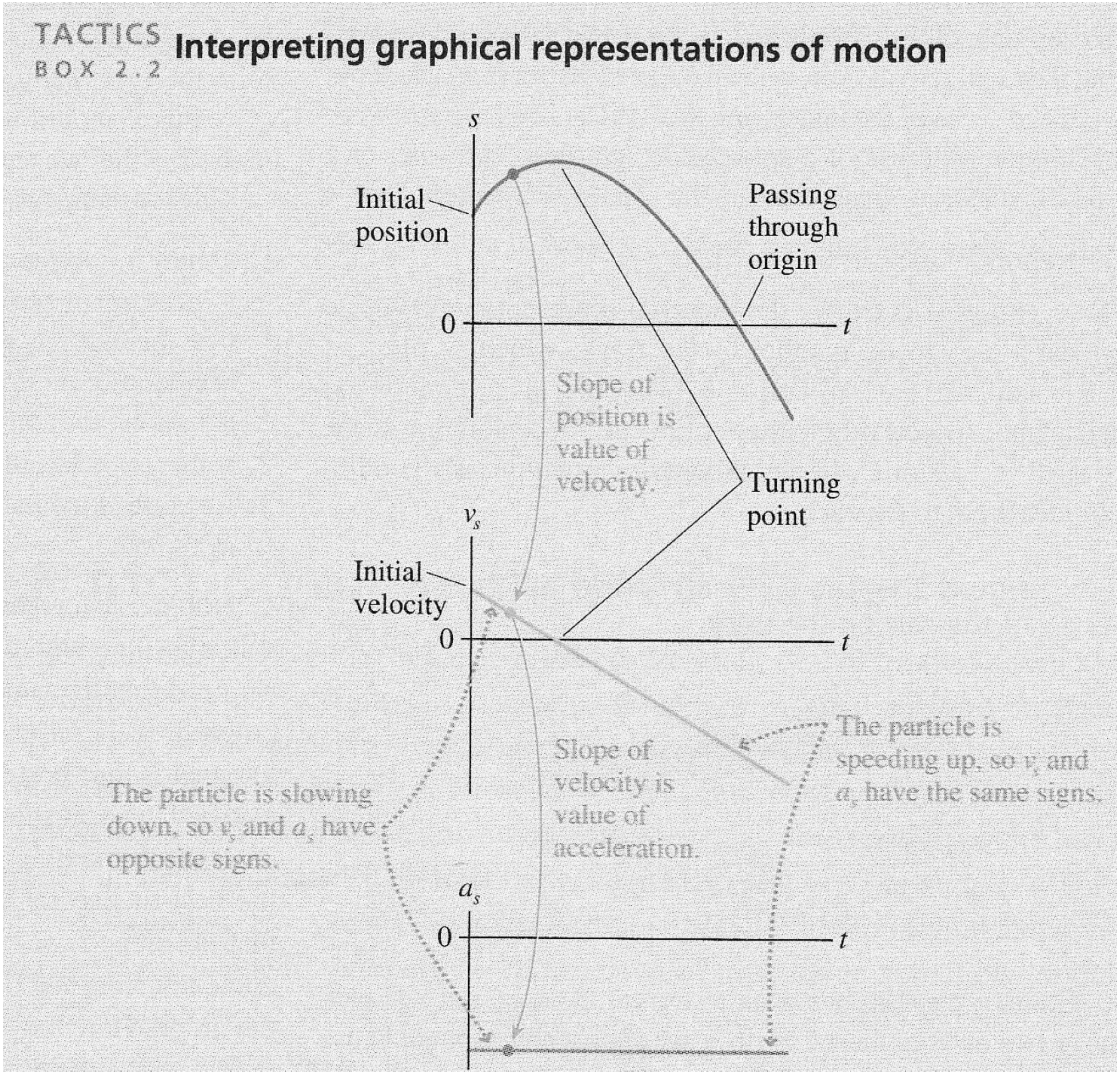
If $s = s(t)$ is the position function of an object at time t, then

$$\text{Velocity} = v = \frac{ds}{dt} \qquad \text{Acceleration} = a = \frac{dv}{dt}$$

Integral Form

$$s(t) = \int v(t)dt \qquad v(t) = \int a(t)dt$$

Wolfson



→ "Initial conditions" (ICs) matter

Knight