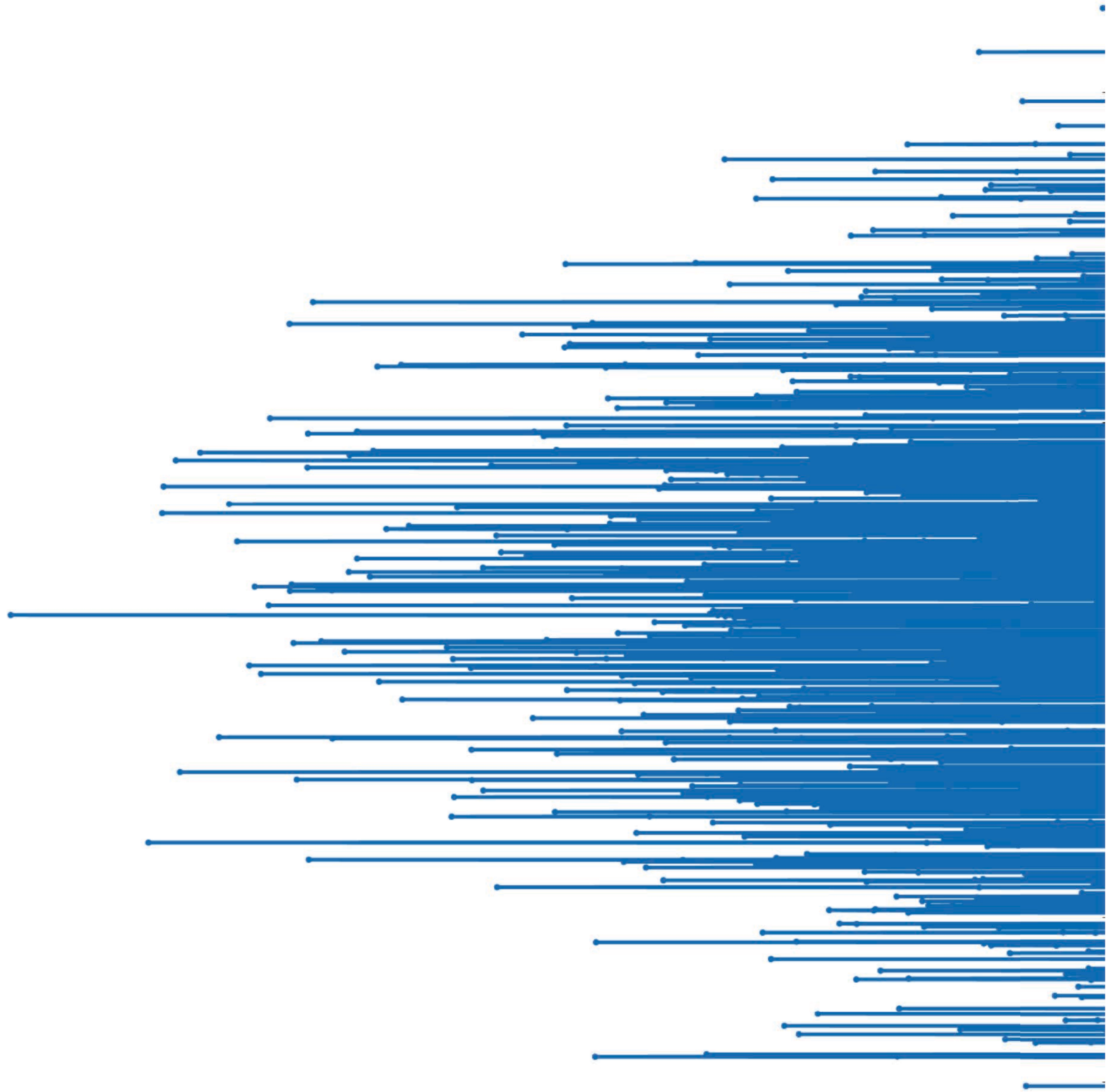
PHYS 2010 (W20)
Classical Mechanics



2020.01.09

Relevant reading:
Knudsen & Hjorth: Appendix
(pgs.433ff)

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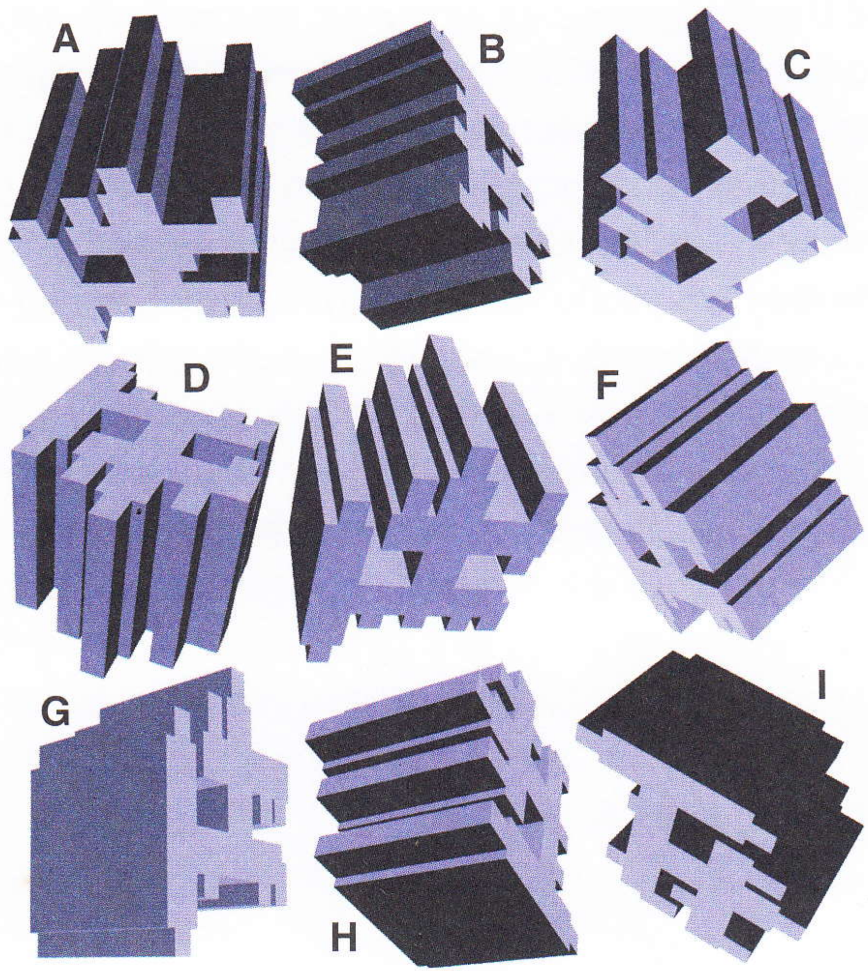
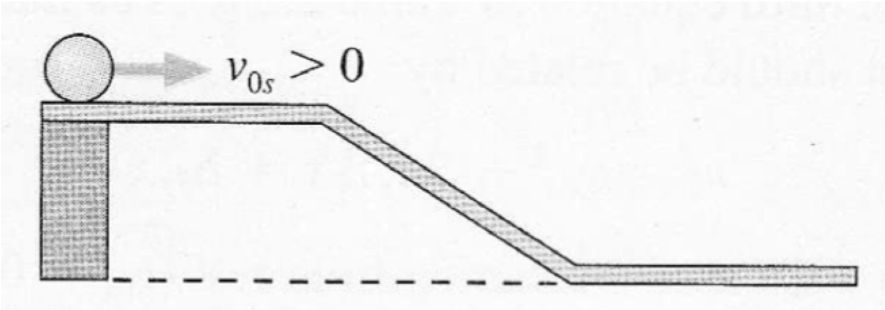
Ref. (re images):
Fowles & Cassidy (2007)

Where's the Pair?

Only two of the shapes below are exactly the same – can you find the matching pair?

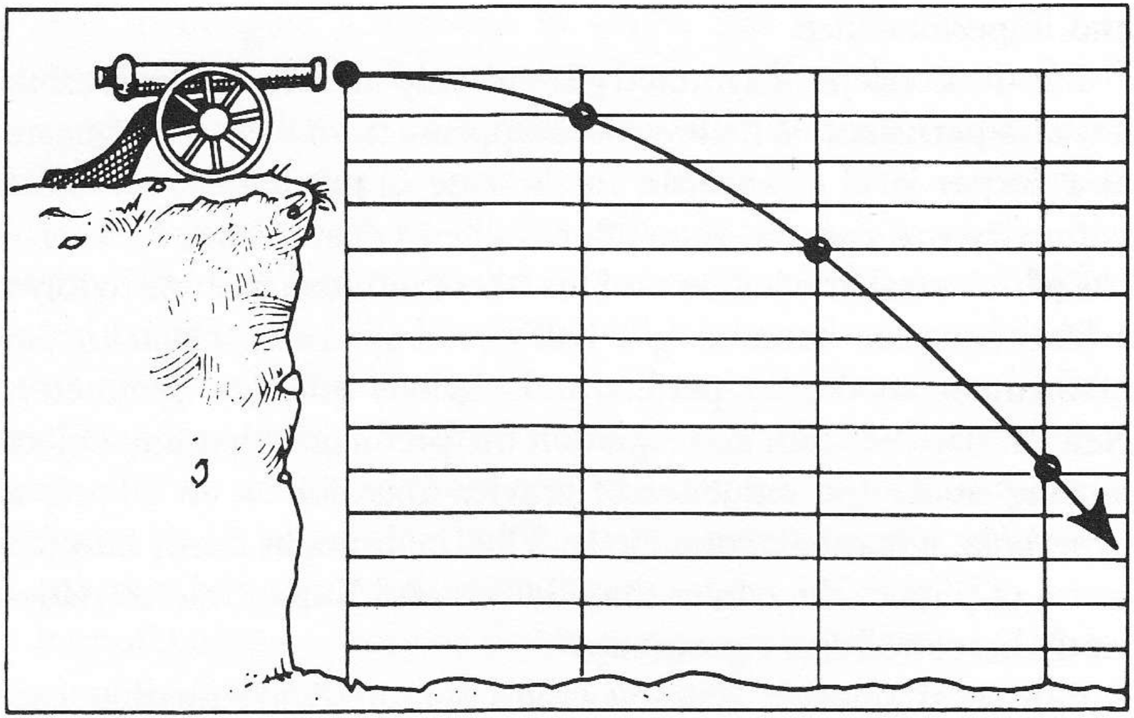


<u>Mechanics:</u> Shifting to higher dimensions....



- Perhaps for a ball *<u>confined</u>* to a track, a one-spatial-dimensional (1-D) description is sufficient....

- ... but for the cannonball, a higher dimensionality is needed (e.g., both horizontal and vertical position matters)



von Baeyer

Vectors....

What does this equation represent?

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -\mathbf{k}gt + \mathbf{v}_0$$

Note: Look carefully! Notice the little things, like the **bold font**

→ Projectile motion

$$m\frac{d^2\mathbf{r}}{dt^2} = -\mathbf{k}mg$$

We may have started off from a different place... (i.e., Newton's 2nd Law and a handful of assumptions)

$$\mathbf{r} = \mathbf{i}x + \mathbf{k}z$$

And we need to ensure we defined the relevant quantities of interest!

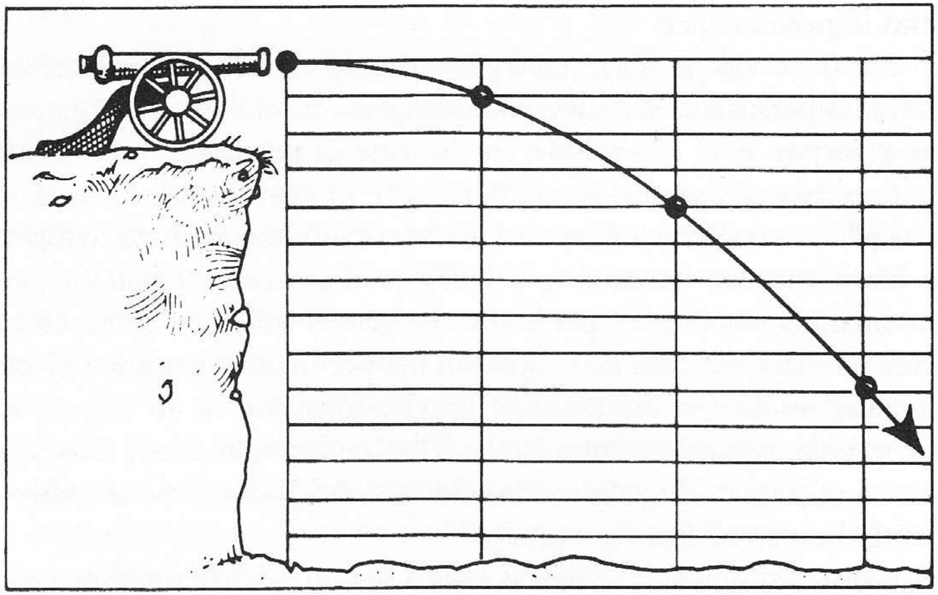
(e.g., r describes what precisely?)

Niccolò Tartaglia (1499-1557)

Question: What angle of elevation would a cannon achieve its greatest range?

"Tartaglia's correct theoretical answer of 45^o surprised the experts; they thought it would be smaller [...] but he refrained from publication. The reason for his diffidence is highly creditable: He felt it would be immoral to use science to help [soliders] slaughter [soliders] more efficiently"

→ As we will see later on, 45^o is not technically correct for "real" cannonballs....



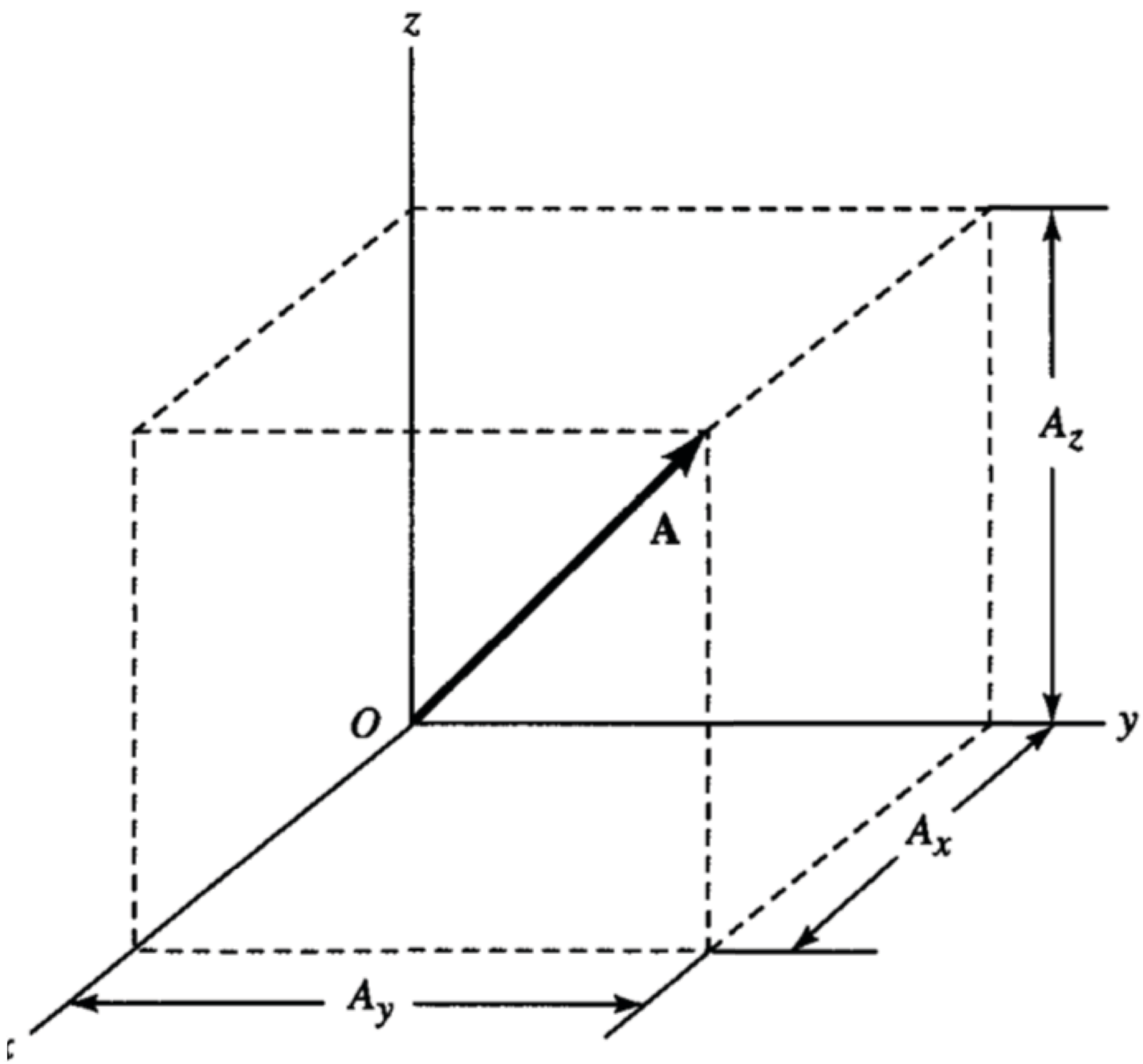
von Baeyer

Mechanics: Shifting to higher dimensions....

- Standard coordinate system: 3-D Cartesian

- Vectors will be a key means to represent various physical quantities

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -\mathbf{k}gt + \mathbf{v}_0$$



Fowles & Cassidy

<u>Review:</u> Vector Representations

- Vectors are very useful for representing various physical quantities....

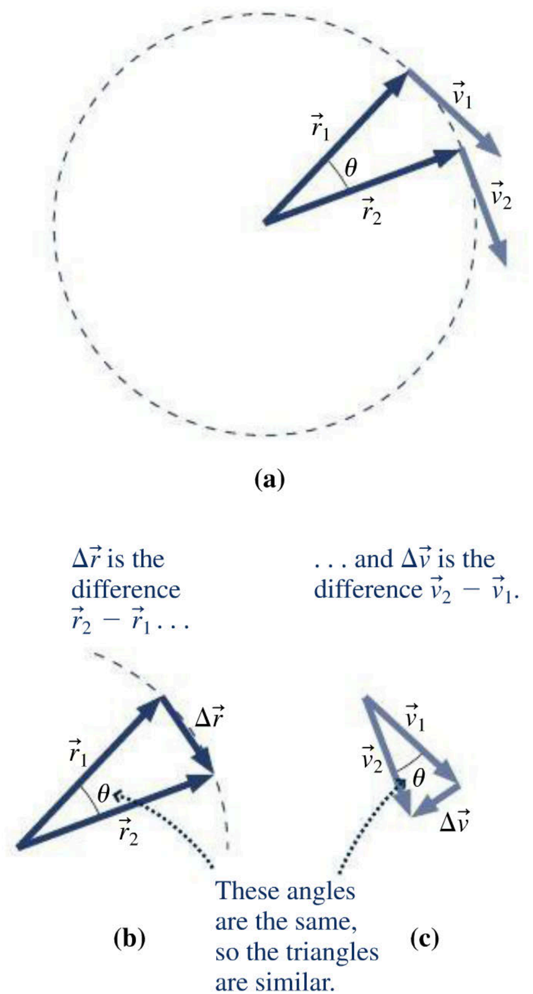


FIGURE 3.22 Position and velocity vectors for two nearby points on the circular path.

Position, velocity, acceleration

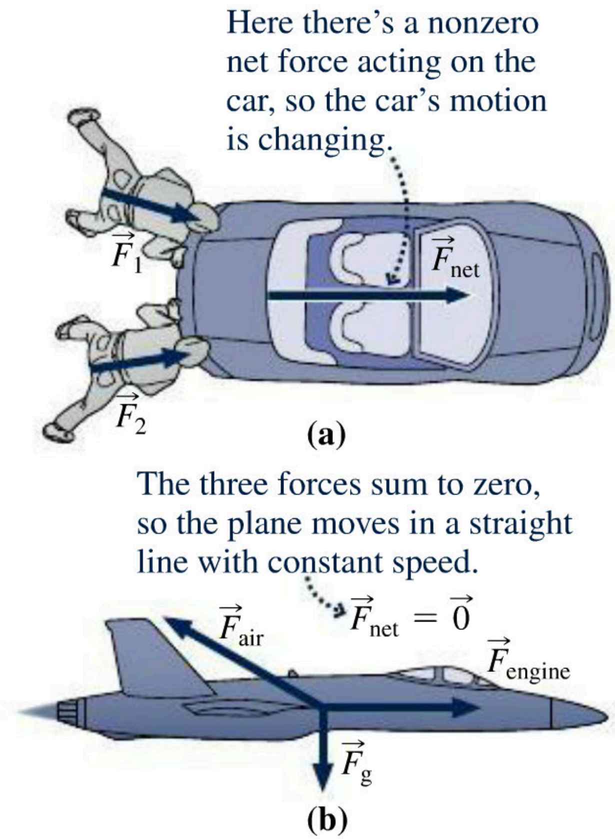


FIGURE 4.2 The net force determines the change in an object's motion.

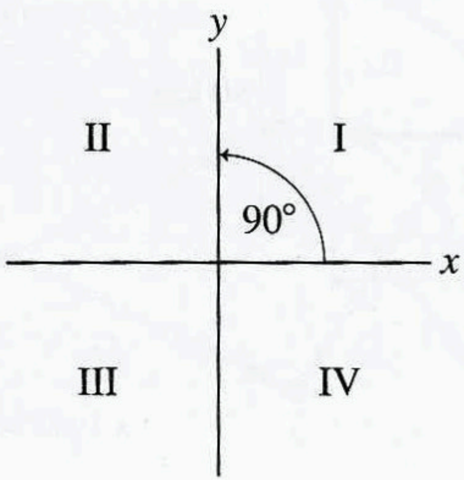
Forces

Wolfson

<u>Review:</u> Vector Components & Unit Vectors

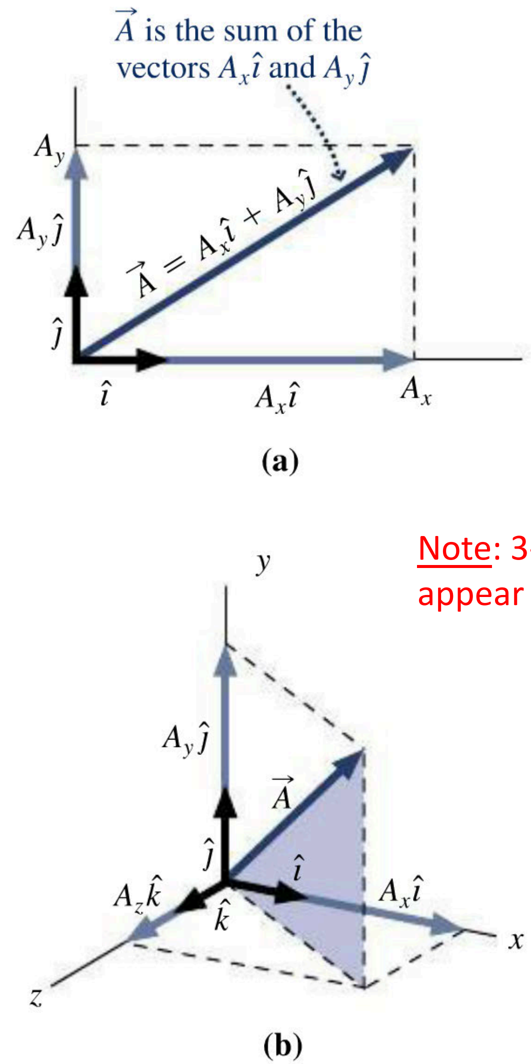
- Choose a coordinate system (such provides a key *frame of reference*)

FIGURE 3.9 A conventional *xy*-coordinate system and the quadrants of the *xy*-plane.



Knight

→ Cartesian system is a good starting choice for 2-D problems



<u>Note</u>: 3-D pictures can appear a bit crowded

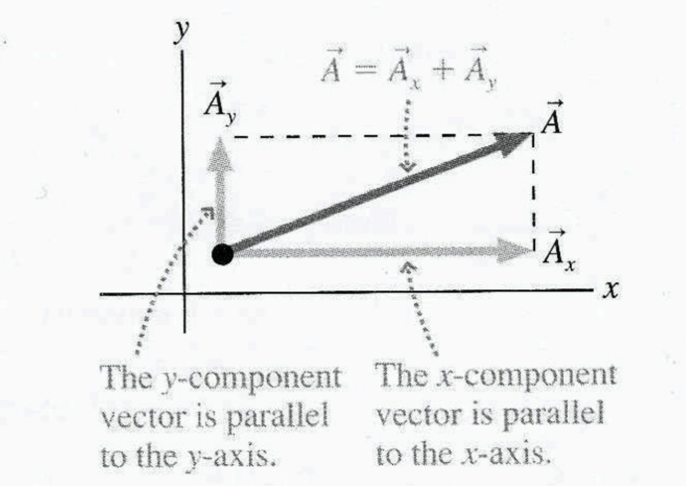
FIGURE 3.5 Vectors in (a) a plane and (b) space, expressed using unit vectors.

Wolfson

Review: Vector Components & Unit Vectors

- Unit vectors stem directly from the chosen frame and allow a compact way to express vectors via *components*

FIGURE 3.10 Component vectors $\vec{A}_x$ and $\vec{A}_y$ are drawn parallel to the coordinate axes such that $\vec{A} = \vec{A}_x + \vec{A}_y$.



Knight

- "Components" can be vectors or scalars combined w/ the unit vectors

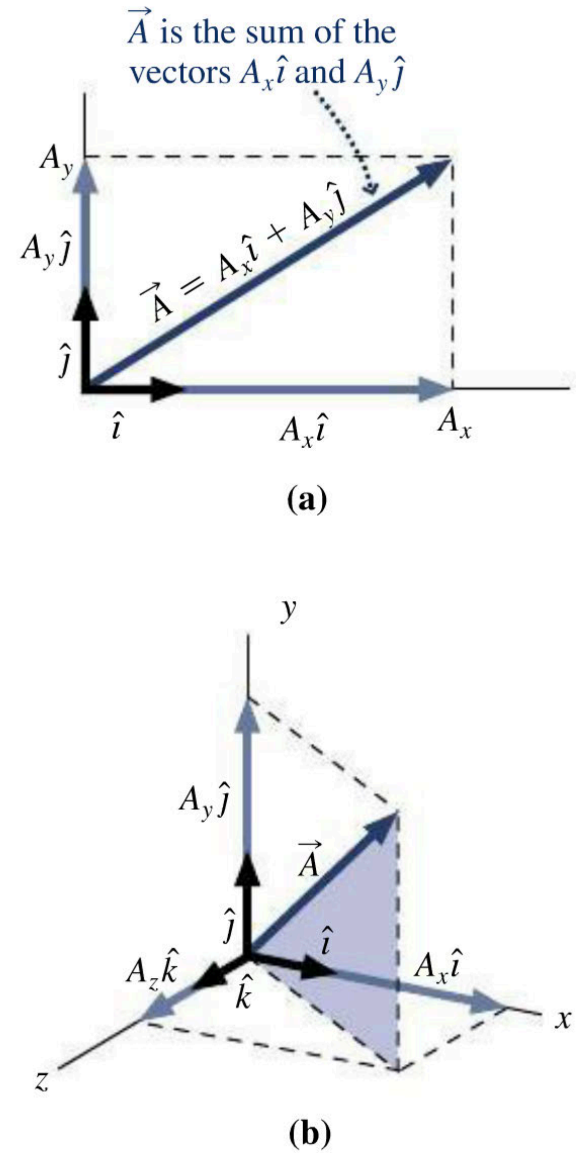


FIGURE 3.5 Vectors in (a) a plane and (b) space, expressed using unit vectors.

Review: Vector Components & Unit Vectors

To summarize:
[2-D] Two pieces of information can be expressed in different ways:

- x—y coordinates
- magnitude & direction (i.e., phase)
- component vectors
- components tied to unit vectors

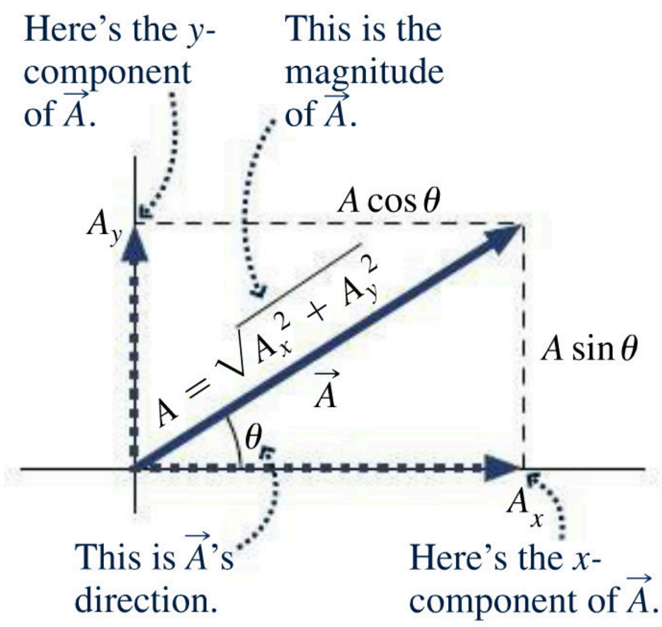


FIGURE 3.4 Magnitude/direction and component representations of vector $\vec{A}$.

A dummy's guide to "component vectors":

TACTICS BOX 3.1 **Determining the components of a vector**

1. The absolute value $|A_x|$ of the x-component A_x is the magnitude of the component vector $\vec{A}_x$.
2. The *sign* of A_x is positive if $\vec{A}_x$ points in the positive x-direction, negative if $\vec{A}_x$ points in the negative x-direction.
3. The y-component A_y is determined similarly.

Knight

<u>Review</u>: Vector "Math"

I. *Equality of Vectors*
The equation

$$\mathbf{A} = \mathbf{B}$$

or

$$(A_x, A_y, A_z) = (B_x, B_y, B_z)$$

is equivalent to the three equations

$$A_x = B_x \qquad A_y = B_y \qquad A_z = B_z$$

II. *Vector Addition*
The addition of two vectors is defined by the equation

$$\mathbf{A} + \mathbf{B} = (A_x, A_y, A_z) + (B_x, B_y, B_z) = (A_x + B_x, A_y + B_y, A_z + B_z)$$

III. *Multiplication by a Scalar*
If c is a scalar and $\mathbf{A}$ is a vector,

$$c\mathbf{A} = c(A_x, A_y, A_z) = (cA_x, cA_y, cA_z) = \mathbf{A}c$$

Fowles & Cassidy

<u>Review</u>: Vector "Math"

IV. *Vector Subtraction*
Subtraction is defined as follows:

$$\mathbf{A}-\mathbf{B}=\mathbf{A}+(-1)\mathbf{B}=(A_x-B_x,\,A_y-B_y,\,A_z-B_z) \tag{1.3.5}$$

That is, subtraction of a given vector **B** from the vector **A** is equivalent to adding −**B** to **A.**

V. *The Null Vector*
The vector $\mathbf{O}=(0,0,0)$ is called the *null* vector. The direction of the null vector is undefined. From (IV) it follows that $\mathbf{A}-\mathbf{A}=\mathbf{O}$. Because there can be no confusion when the null vector is denoted by a zero, we shall hereafter use the notation $\mathbf{O}=0$.

VI. *The Commutative Law of Addition*
This law holds for vectors; that is,

$$\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A} \tag{1.3.6}$$

because $A_x+B_x=B_x+A_x$, and similarly for the y and z components.

VII. *The Associative Law*
The associative law is also true, because

$$\begin{aligned}\mathbf{A}+(\mathbf{B}+\mathbf{C}) &= (A_x+(B_x+C_x),\,A_y+(B_y+C_y),\,A_z+(B_z+C_z))\\ &= ((A_x+B_x)+C_x,\,(A_y+B_y)+C_y,\,(A_z+B_z)+C_z)\\ &= (\mathbf{A}+\mathbf{B})+\mathbf{C}\end{aligned} \tag{1.3.7}$$

VIII. *The Distributive Law*
Under multiplication by a scalar, the distributive law is valid because, from (II) and (III),

$$\begin{aligned}c(\mathbf{A}+\mathbf{B}) &= c(A_x+B_x,\,A_y+B_y,\,A_z+B_z)\\ &= (c(A_x+B_x),\,c(A_y+B_y),\,c(A_z+B_z))\\ &= (cA_z+cB_x,\,cA_y+cB_y,\,cA_z+cB_z)\\ &= c\mathbf{A}_x+c\mathbf{B}\end{aligned} \tag{1.3.8}$$

Review: Vector "Math"

IX. *Magnitude of a Vector*

The magnitude of a vector **A**, denoted by $|\mathbf{A}|$ or by A, is defined as the square root of the sum of the squares of the components, namely,

$$A = |\mathbf{A}| = \left(A_x^2 + A_y^2 + A_z^2\right)^{1/2} \tag{1.3.9}$$

where the positive root is understood. Geometrically, the magnitude of a vector is its length, that is, the length of the diagonal of the rectangular parallelepiped whose sides are A_x, A_y, and A_z, expressed in appropriate units. See Figure 1.3.5.

X. *Unit Coordinate Vectors*

A *unit vector* is a vector whose magnitude is unity. Unit vectors are often designated by the symbol **e**, from the German word *Einheit*. The three unit vectors

$$\mathbf{e}_x = (1,0,0) \qquad \mathbf{e}_y = (0,1,0) \qquad \mathbf{e}_z = (0,0,1) \tag{1.3.10}$$

are called *unit coordinate vectors* or *basis vectors*. In terms of basis vectors, any vector can be expressed as a vector sum of components as follows:

$$\begin{aligned} \mathbf{A} &= (A_x, A_y, A_z) = (A_x, 0, 0) + (0, A_y, 0) + (0, 0, A_z) \\ &= A_x(1,0,0) + A_y(0,1,0) + A_z(0,0,1) \\ &= \mathbf{e}_x A_x + \mathbf{e}_y A_y + \mathbf{e}_z A_z \end{aligned} \tag{1.3.11}$$

A widely used notation for Cartesian unit vectors uses the letters **i**, **j**, and **k**, namely,

Common convention

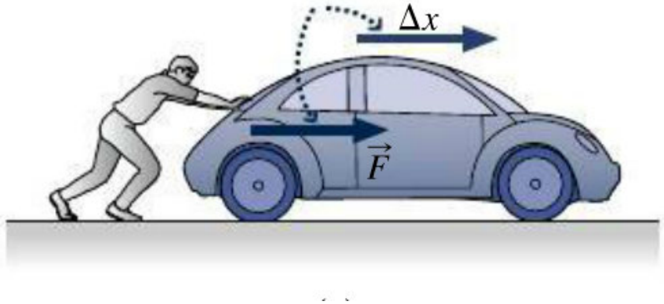
$$\mathbf{i} = \mathbf{e}_x \qquad \mathbf{j} = \mathbf{e}_y \qquad \mathbf{k} = \mathbf{e}_z \tag{1.3.12}$$

Fowles & Cassidy

Review: Vector "Math" & Work

→ What about multiplying vectors?

- Work is the energy transferred between systems via an applied force



Common definition of work

$$W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}$$

Common definition of the "dot product"

$$\vec{A} \cdot \vec{B} = AB\cos\theta$$

"Scalar Product"

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

Distributive Rule:

$$\begin{aligned}\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) &= A_x(B_x + C_x) + A_y(B_y + C_y) + A_z(B_z + C_z) \\ &= A_x B_x + A_y B_y + A_z B_z + A_x C_x + A_y C_y + A_z C_z \\ &= \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}\end{aligned}$$

$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{\mathbf{A} \cdot \mathbf{B}}{AB}$$

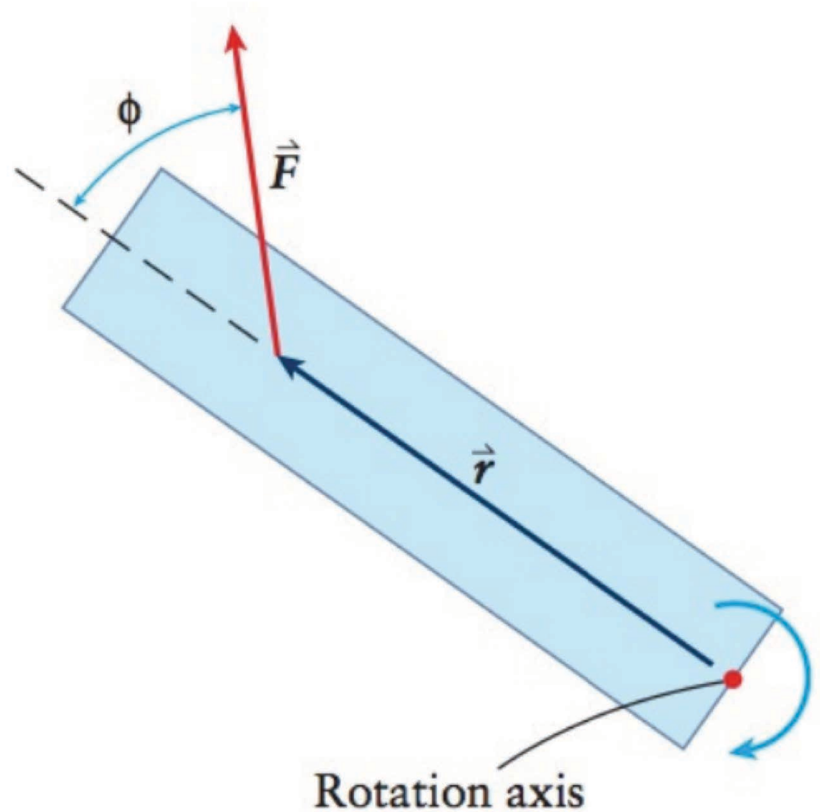
$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$$

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{k} = 0$$

Review: Vector "Math" & Torque

→ What about multiplying vectors?

Figure 8-24 Torque τ is the rotational analog of force and takes into account the distance r between where a force F is applied and the rotation axis. Torque also takes into account the angle φ between the force vector $\vec{F}$ and the $\vec{r}$ vector that points from the rotation axis to the point at which the force is applied.



$$\vec{\tau} = \vec{r} \times \vec{F}$$

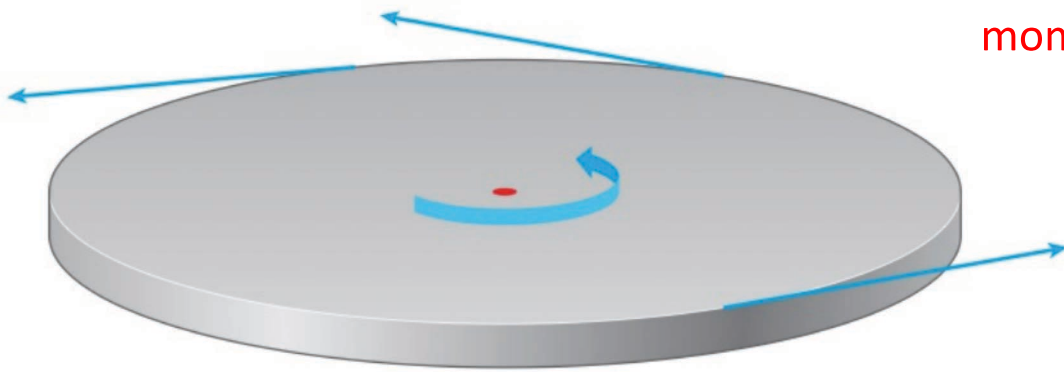
$$\tau = rF\sin\varphi$$

Scalar version (figure above motivates where this comes from...)

<u>Aside/Looking Ahead:</u> Vector nature of angular quantities

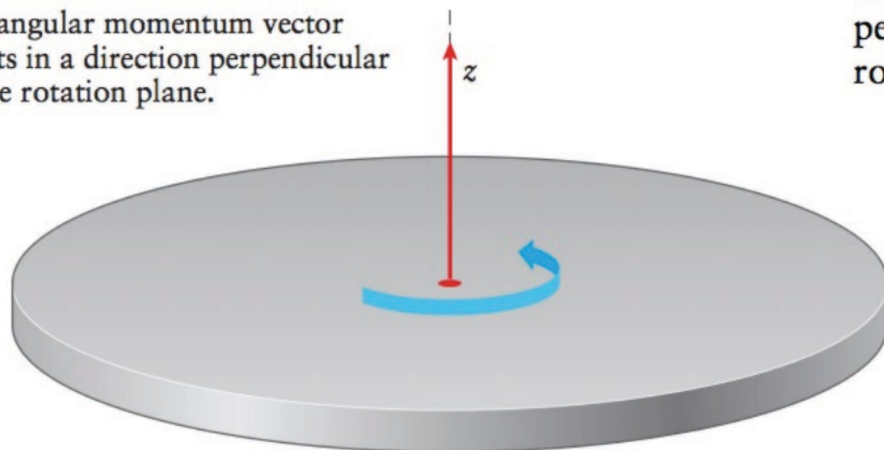
We'll come back to angular momentum a bit downstream...

(a)



Which one vector represents the rotational motion? (None of them)

(b) The angular momentum vector points in a direction perpendicular to the rotation plane.

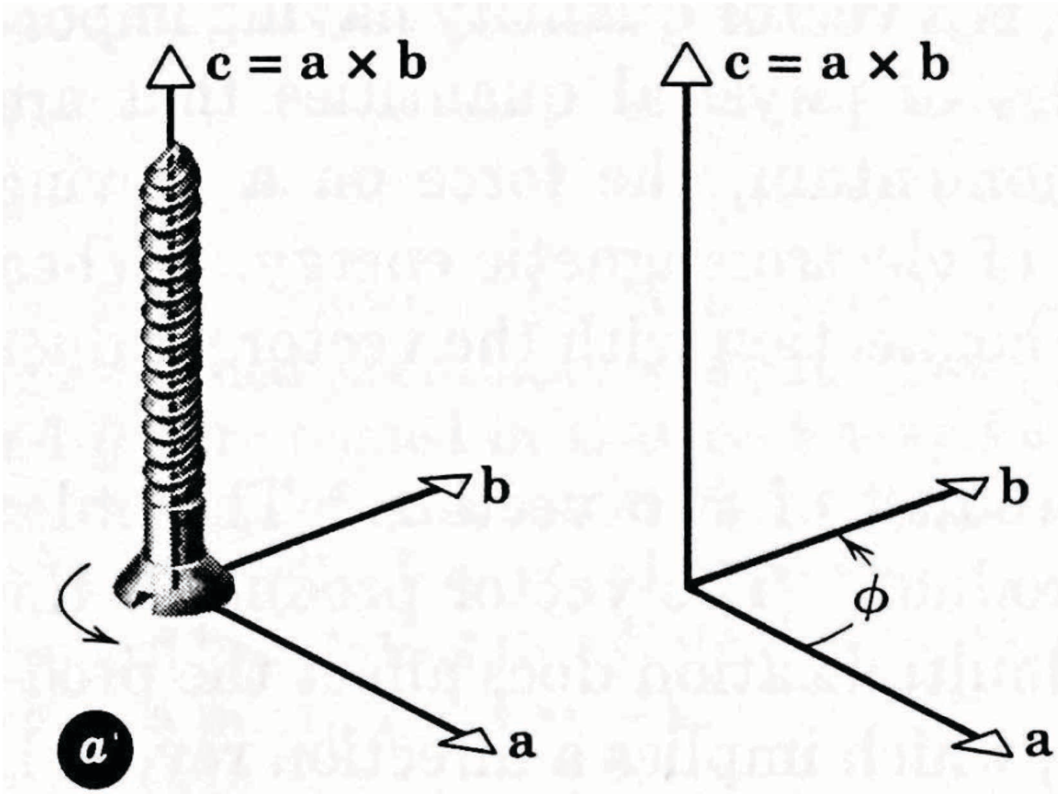


Is angular momentum up or down? The direction is up in a right-handed sense. Curl the fingers on your right hand in the direction of motion and stick your thumb straight out; your thumb points in the direction of the angular momentum vector.

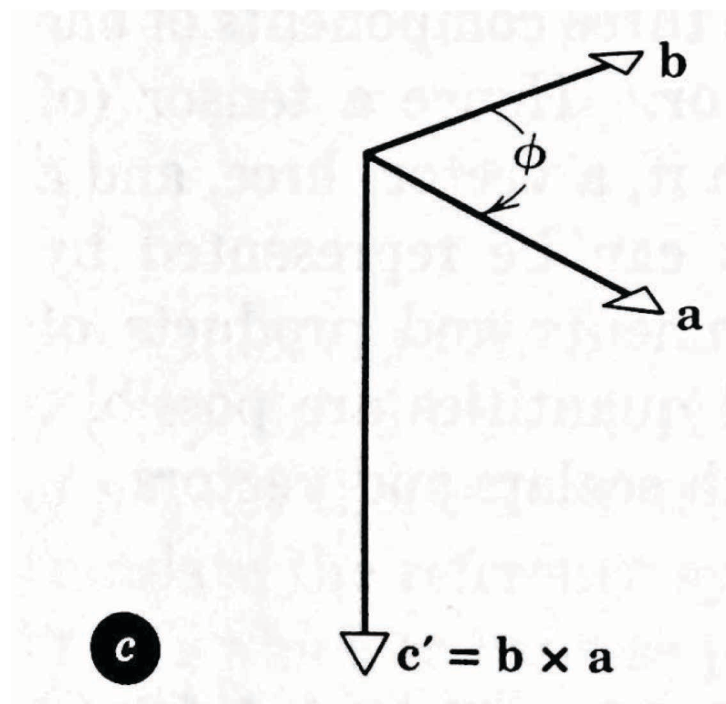
Figure 8-35 (a) No single vector that lies in the plane of rotation indicates the direction of rotation. (b) The angular momentum vector $\vec{L}$ points in a direction perpendicular to the plane of rotation.

Kesten & Tauck

Review: Cross Product



Order matters!

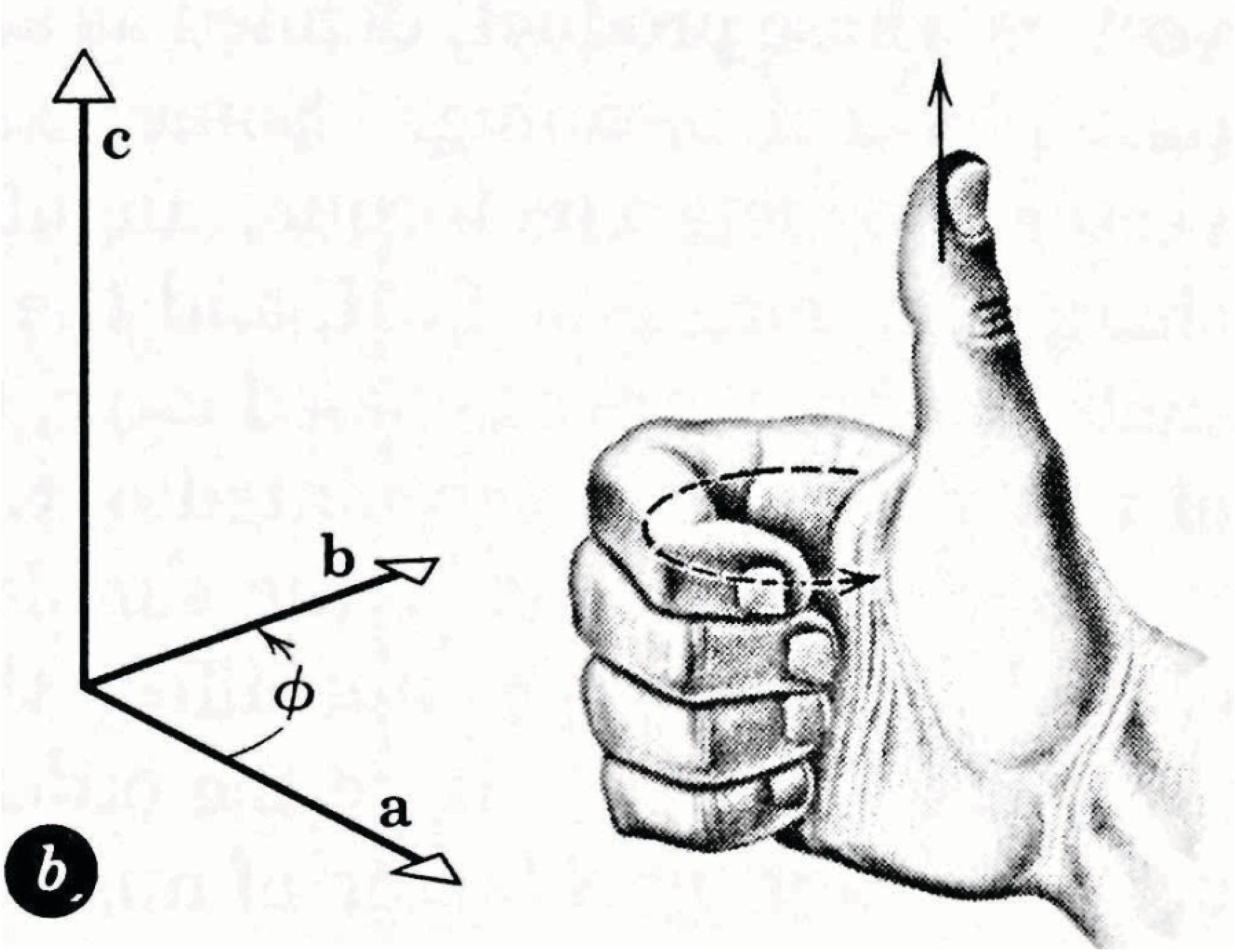


Ultimately, this is a convention....

Resnick & Halliday

Right-hand Rule (RHR)

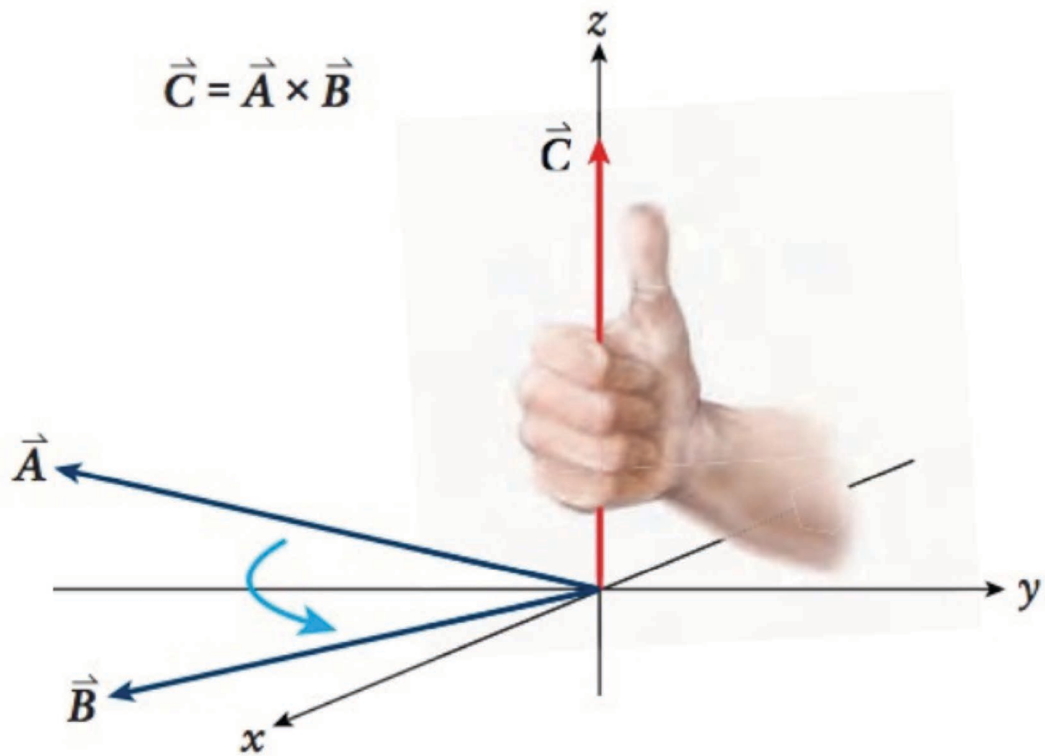
Key aspect here to get correct is that your fingers turn **a** towards **b** through the *smaller angler*



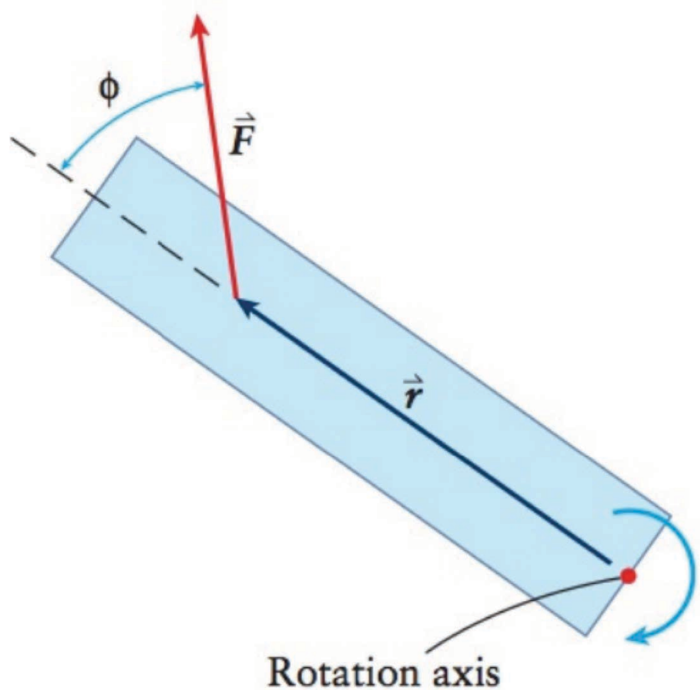
Resnick & Halliday

Right-hand Rule (RHR)

Right hand rule: Curl the fingers on your right hand from $\vec{A}$ to $\vec{B}$ along the closest path. Stick out your thumb; it points in the direction of $\vec{C}$, the result of the cross product $\vec{A} \times \vec{B}$.



$\vec{A}$ and $\vec{B}$ lie in the *xy* plane in this example. $\vec{C}$ points in the positive *z* direction.



Vector Product

"Vector Product"

$$\mathbf{A}\times\mathbf{B}=(A_yB_z-A_zB_y,\ A_zB_x-A_xB_z,\ A_xB_y-A_yB_x)$$

$$\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=0$$
$$\mathbf{j}\times\mathbf{k}=\mathbf{i}=-\mathbf{k}\times\mathbf{j}$$
$$\mathbf{i}\times\mathbf{j}=\mathbf{k}=-\mathbf{j}\times\mathbf{i}$$
$$\mathbf{k}\times\mathbf{i}=\mathbf{j}=-\mathbf{i}\times\mathbf{k}$$

$$\mathbf{A}\times\mathbf{B}=\mathbf{i}(A_yB_z-A_zB_y)+\mathbf{j}(A_zB_x-A_xB_z)+\mathbf{k}(A_xB_y-A_yB_x)$$

Adding determinants to the fray...

$$\mathbf{A}\times\mathbf{B}=\mathbf{i}\begin{vmatrix}A_y & A_z\\ B_y & B_z\end{vmatrix}+\mathbf{j}\begin{vmatrix}A_z & A_x\\ B_z & B_x\end{vmatrix}+\mathbf{k}\begin{vmatrix}A_x & A_y\\ B_x & B_y\end{vmatrix}$$

$$\mathbf{A}\times\mathbf{B}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}$$

$$\mathbf{A}\times\mathbf{B}=-\mathbf{B}\times\mathbf{A}$$
$$\mathbf{A}\times(\mathbf{B}+\mathbf{C})=\mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C}$$
$$n(\mathbf{A}\times\mathbf{B})=(n\mathbf{A})\times\mathbf{B}=\mathbf{A}\times(n\mathbf{B})$$

Fowles & Cassidy

Vector Product

How does one go from the "vector" version of the cross product to the scalar version?

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad \tau = rF\sin\varphi$$

$$\mathbf{A}\times\mathbf{B} = \mathbf{i}(A_yB_z - A_zB_y) + \mathbf{j}(A_zB_x - A_xB_z) + \mathbf{k}(A_xB_y - A_yB_x)$$

$$|\mathbf{A}\times\mathbf{B}|^2 = (A_yB_z - A_zB_y)^2 + (A_zB_x - A_xB_z)^2 + (A_xB_y - A_yB_x)^2$$

$$|\mathbf{A}\times\mathbf{B}|^2 = \left(A_x^2 + A_y^2 + A_z^2\right)\left(B_x^2 + B_y^2 + B_z^2\right) - (A_xB_x + A_yB_y + A_zB_z)^2$$

$$|\mathbf{A}\times\mathbf{B}|^2 = A^2B^2 - (\mathbf{A}\cdot\mathbf{B})^2 \qquad |\mathbf{A}\times\mathbf{B}| = AB(1-\cos^2\theta)^{1/2} = AB\sin\theta$$

Fowles & Cassidy

Connecting Vectors to Mechanics

3.2 Velocity and Acceleration Vectors

We defined velocity in one dimension as the rate of change of position. In two or three dimensions it's the same thing, except now the change in position—displacement—is a vector. So we write

$$\bar{\vec{v}} = \frac{\Delta \vec{r}}{\Delta t} \quad \text{(average velocity vector)} \tag{3.3}$$

for the average velocity, in analogy with Equation 2.1. Here division by Δt simply means multiplying by $1/\Delta t$. As before, instantaneous velocity is given by a limiting process:

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} \quad \text{(instantaneous velocity vector)} \tag{3.4}$$

Again, that derivative $d\vec{r}/dt$ is shorthand for the result of the limiting process, taking ever smaller time intervals Δt and the corresponding displacements $\Delta \vec{r}$. Another way to look at Equation 3.4 is in terms of components. If $\vec{r} = x\hat{\imath} + y\hat{\jmath}$, then we can write

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} = v_x\hat{\imath} + v_y\hat{\jmath}$$

where the velocity components v_x and v_y are the derivatives of the position components.

Acceleration is the rate of change of velocity, so we write

$$\bar{\vec{a}} = \frac{\Delta \vec{v}}{\Delta t} \quad \text{(average acceleration vector)} \tag{3.5}$$

for the average acceleration and

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} \quad \text{(instantaneous acceleration vector)} \tag{3.6}$$

for the instantaneous acceleration. We can also express instantaneous acceleration in components, as we did for velocity:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{\imath} + \frac{dv_y}{dt}\hat{\jmath} = a_x\hat{\imath} + a_y\hat{\jmath}$$

$$\vec{v} = \vec{v}_0 + \vec{a}t \quad \text{(for constant acceleration only)} \tag{3.8}$$

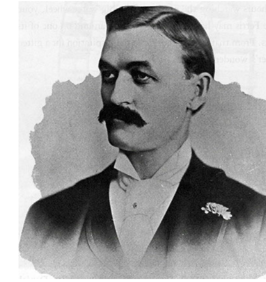
$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \tfrac{1}{2}\vec{a}t^2 \quad \text{(for constant acceleration only)} \tag{3.9}$$

→ Obvious implication here is that our "vector math" also needs to include *vector calculus*

Wolfson

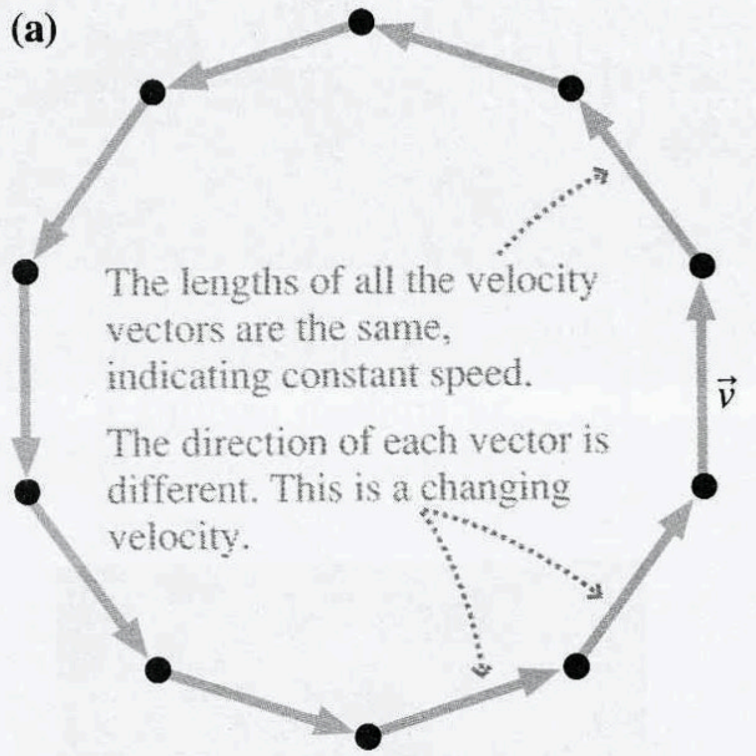
Ex.

George Washington Gale Ferris Jr.

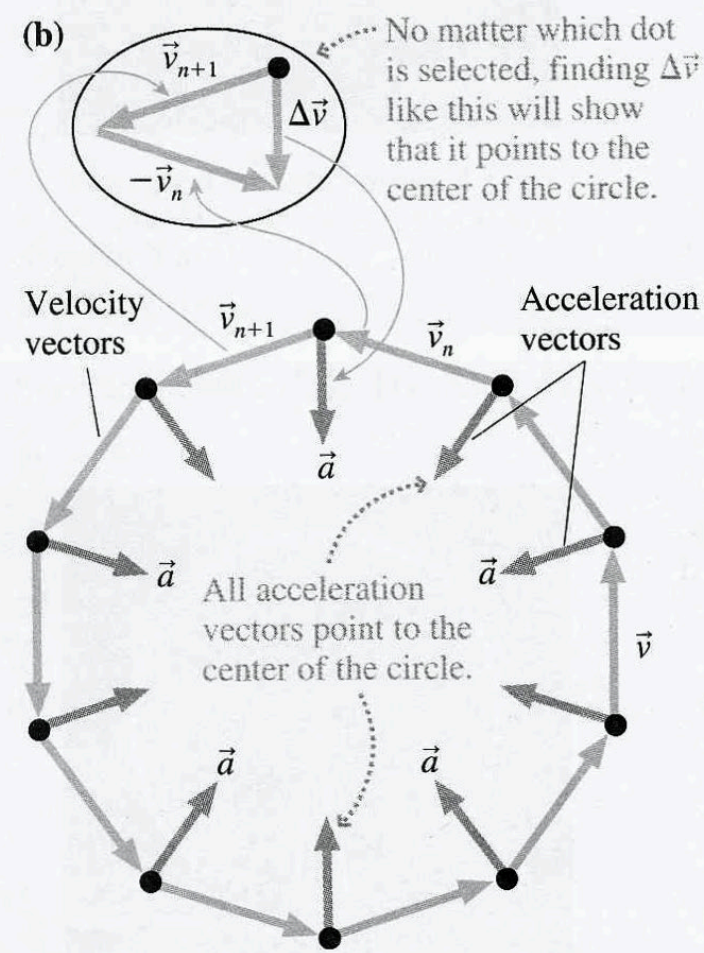


A classic example: "Maria" riding a Ferris wheel

FIGURE 4.1 Using Tactics Box 4.1 to find Maria's acceleration on the Ferris wheel.



Maria moves at constant speed but *not* at constant velocity. Thus she is accelerating.

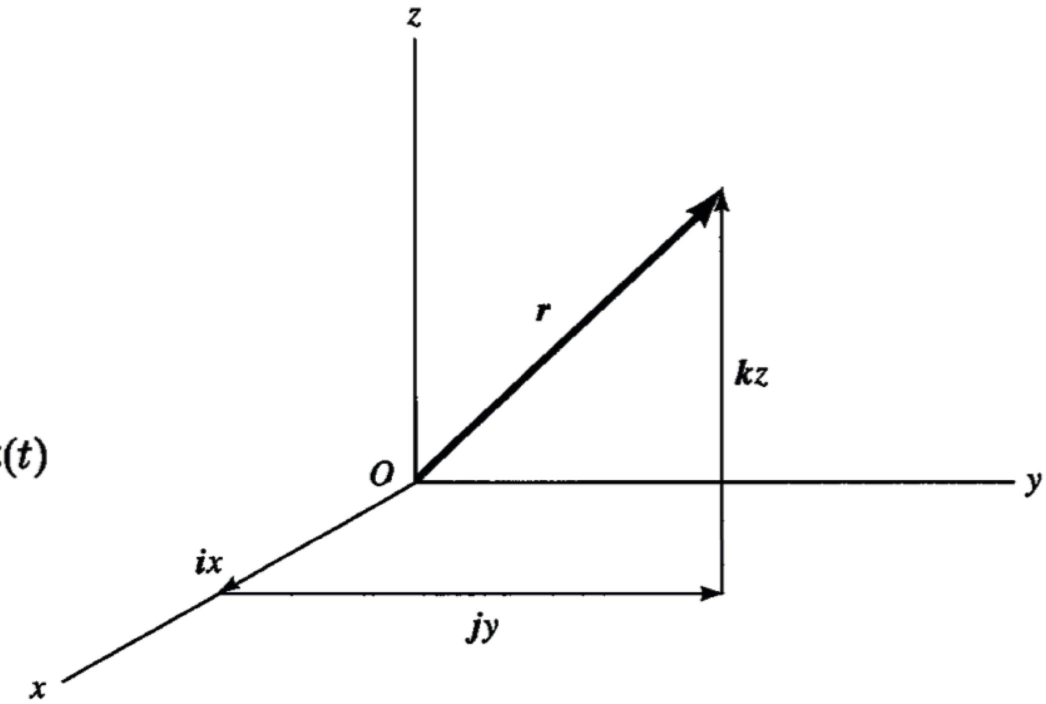


Maria's acceleration is an acceleration of changing direction, not of changing speed.

Knight

Vector "Calculus" & Mechanics

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$$



$$x = x(t) \qquad y = y(t) \qquad z = z(t)$$

$$v = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2]^{1/2}}{\Delta t}$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i}\dot{x} + \mathbf{j}\dot{y} + \mathbf{k}\dot{z}$$

Note: Diacritical dots (i.e., dots above a variable) is a common convention for a time derivative

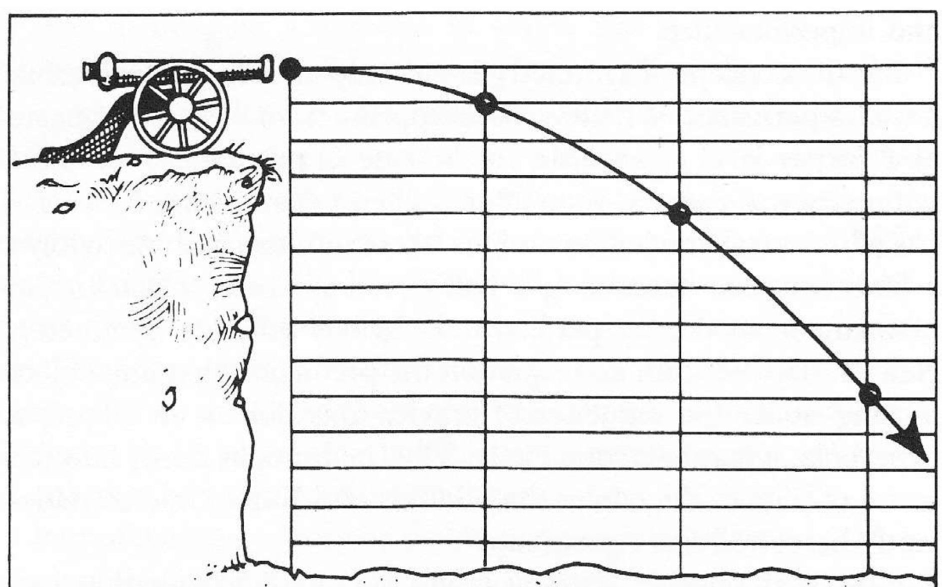
$$\mathbf{a} = \mathbf{i}\ddot{x} + \mathbf{j}\ddot{y} + \mathbf{k}\ddot{z}$$

Fowles & Cassidy

Ex./Review: Projectile Motion

Niccolò Tartaglia (1499-1557)

“Tartaglia’s correct theoretical answer of 45^o surprised the experts; they thought it would be smaller [...] but he refrained from publication. The reason for his diffidence is highly creditable: He felt it would be immoral to use science to help [soliders] slaughter [soliders] more efficiently”



Fowles & Cassidy

Ex./Review: Projectile Motion

Vector representation of the problem

FIGURE 4.14 A projectile launched with initial velocity $\vec{v}_0$.

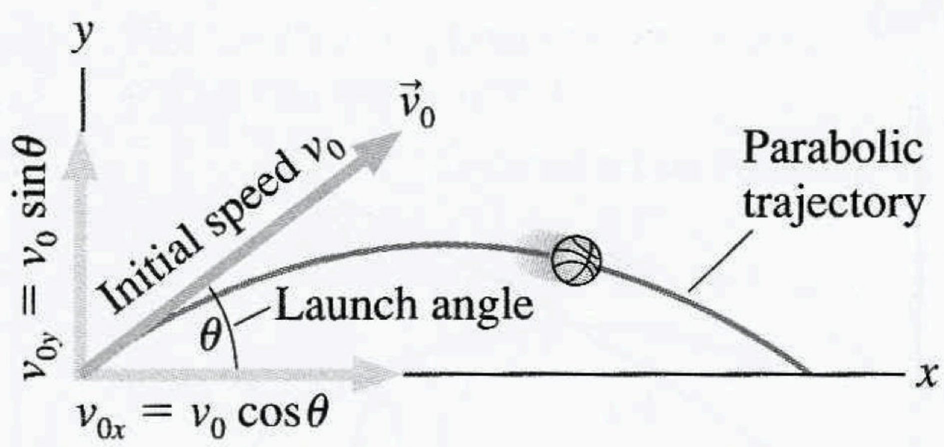
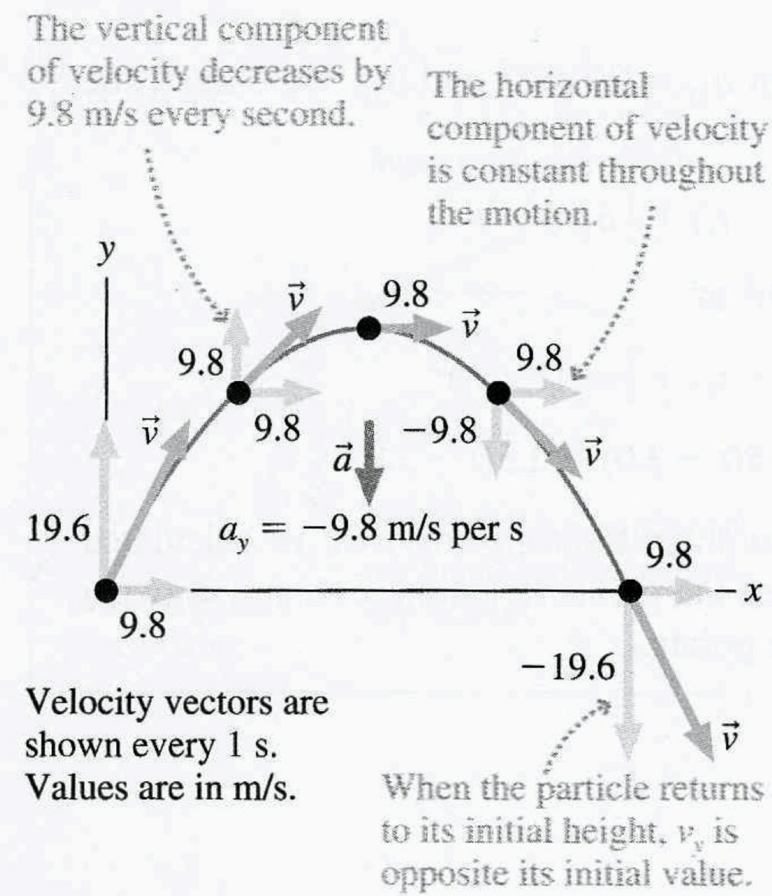


FIGURE 4.15 The velocity and acceleration vectors of a projectile moving along a parabolic trajectory.



Breaking it up into components

Two Approaches we could take:

1) Pair of 1-D calculations
2) Vector calculation

Knight

Ex./Review: Projectile Motion 1) Pair of 1-D calculations

Eqns. for linear motion under constant acceleration:

$$\vec{v} = \vec{v}_0 + \vec{a}t$$

$$\vec{x} - \vec{x}_0 = \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$$

Breaking up into components:

$$v_x = v_{0x} + a_x t$$

$$v_y = v_{0y} + a_y t$$

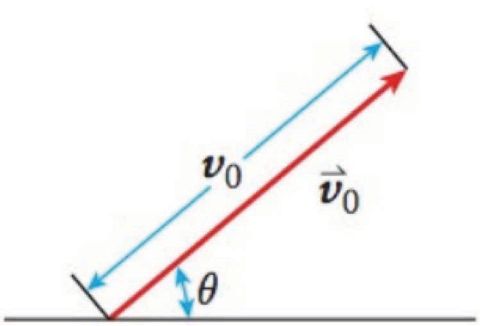
$$x - x_0 = v_{0x} t + \frac{1}{2} a_x t^2$$

$$y - y_0 = v_{0y} t + \frac{1}{2} a_y t^2$$

Initial condition:

$$v_{0x} = v_0 \cos\theta$$

$$v_{0y} = v_0 \sin\theta$$



But (assuming no air resistance) $a_x = 0\ \mathrm{m/s^2}$

$$v_x = v_{0x}$$

$$x - x_0 = v_{0x} t$$

Kesten & Tauck

Ex./Review: Projectile Motion 1) Pair of 1-D calculations

$$v_x = v_{0x}$$
$$x - x_0 = v_{0x} t$$

This allows us to effectively re-express time....

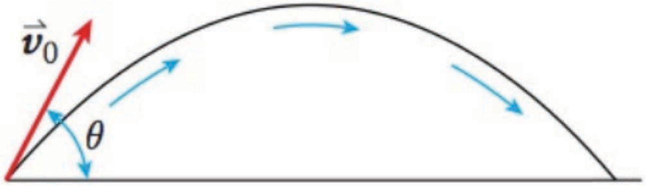
$$t = \frac{x - x_0}{v_{0x}}$$

$$x - x_0 = v_{0x} t + \frac{1}{2} a_x t^2$$
$$y - y_0 = v_{0y} t + \frac{1}{2} a_y t^2$$

$$y - y_0 = v_{0y}\left(\frac{x - x_0}{v_{0x}}\right) - \frac{1}{2} g \left(\frac{x - x_0}{v_{0x}}\right)^2$$

An object launched only under the influence of Earth's gravity follows a parabolic path.

$$y - y_0 = \left(\frac{v_{0y}}{v_{0x}}\right)(x - x_0) - \frac{1}{2}\left(\frac{g}{v_{0x}^2}\right)(x - x_0)^2$$



$$y = a\,x + b\,x^2$$

Both the velocity and angle of the path with respect to the horizontal are always changing during projectile motion.

Ex./Review: Projectile Motion 2) Vector calculation

$$\mathbf{r}(t) = \mathbf{i}bt + \mathbf{j}\left(ct - \frac{gt^2}{2}\right) + \mathbf{k}0$$

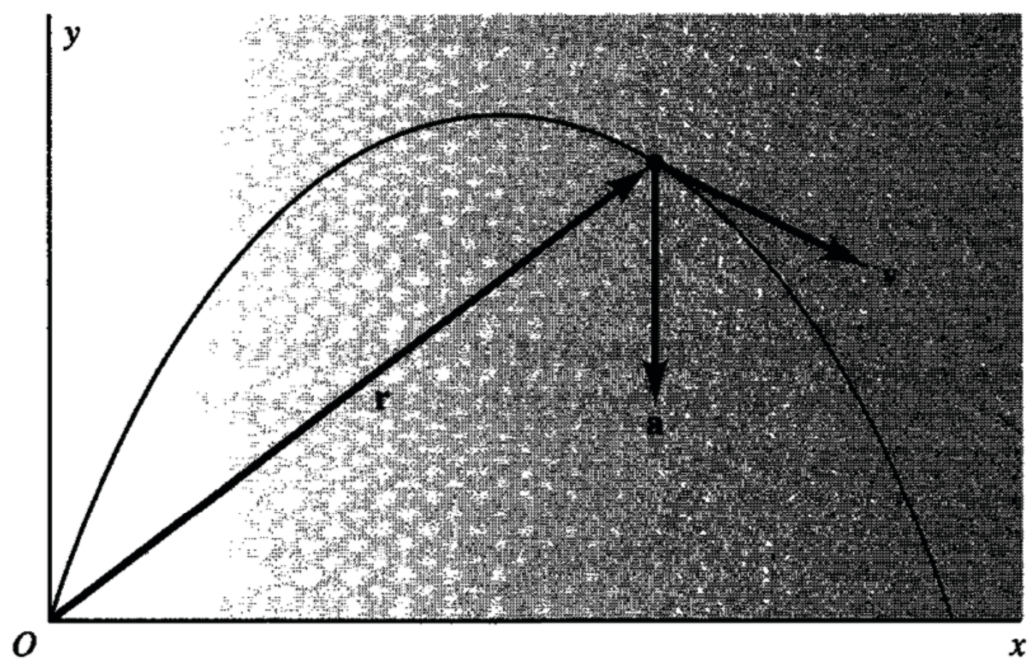
$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i}b + \mathbf{j}(c - gt)$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -\mathbf{j}g$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -\mathbf{k}gt + \mathbf{v}_0$$

Note the (slight) change in coord system as assumed at start of lecture!

$$v = [b^2 + (c - gt)^2]^{1/2}$$



Fowles & Cassidy

Ex./Review: Projectile Motion 1) Pair of 1-D calculations

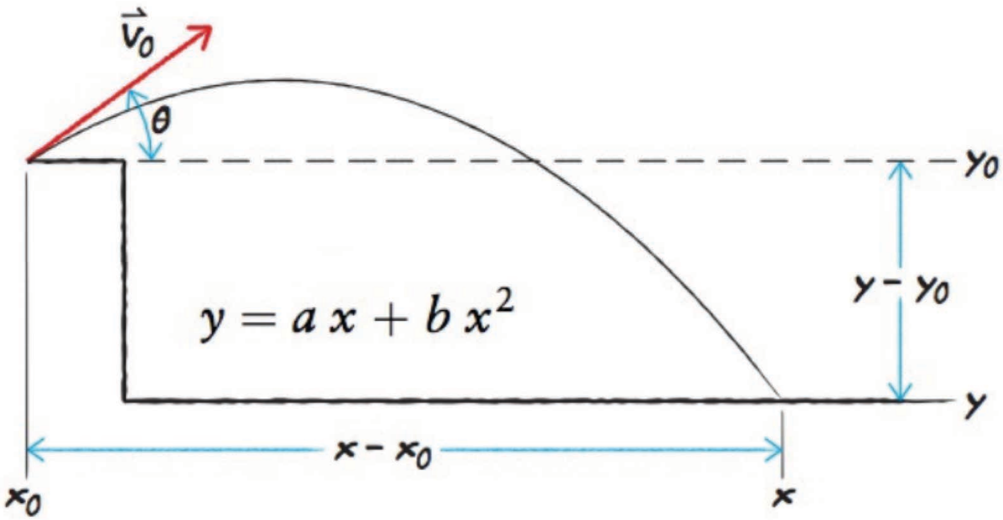
We can carry the analysis out a bit further...

$$y - y_0 = \left(\frac{v_{0y}}{v_{0x}}\right)(x - x_0) - \frac{1}{2}\left(\frac{g}{v_{0x}^2}\right)(x - x_0)^2$$

$$y - y_0 = \left(\frac{v_0 \sin\theta}{v_0 \cos\theta}\right)(x - x_0) - \frac{1}{2}\left(\frac{g}{(v_0 \cos\theta)^2}\right)(x - x_0)^2$$

$$y - y_0 = \tan\theta\,(x - x_0) - \frac{g(x - x_0)^2}{2v_0^2 \cos^2\theta}$$

"projectile trajectory"



"projectile range"

$$(x - x_0)_{\text{range}} = \frac{2v_{0y}v_{0x}}{g} = \frac{2v_0^2 \sin\theta \cos\theta}{g}$$

Kesten & Tauck

Ex./Review: Projectile Motion

$$x = \frac{v_0{}^2}{g} \sin 2\theta_0 \qquad \text{(horizontal range)}$$

$$y = x \tan \theta_0 - \frac{g}{2v_0{}^2 \cos^2 \theta_0} x^2 \qquad \text{(projectile trajectory)}$$

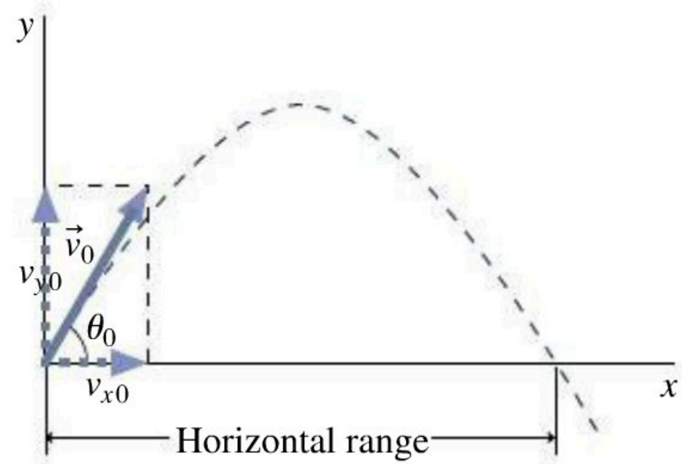
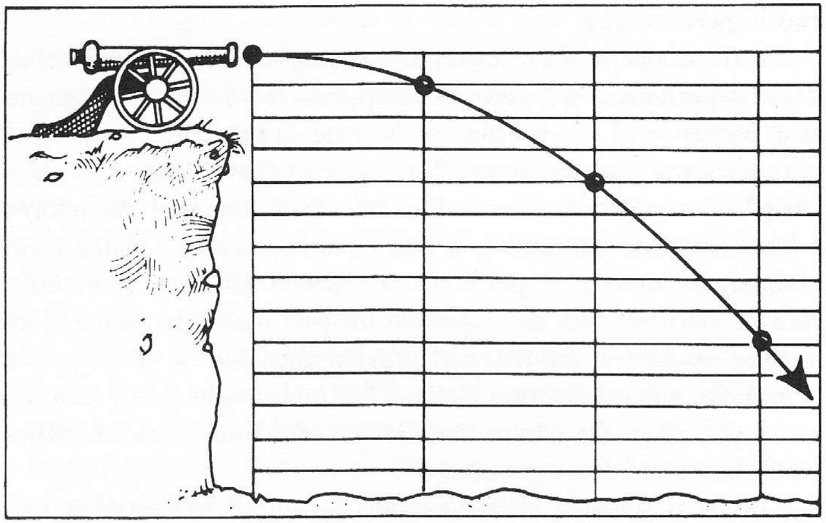


FIGURE 3.17 Parabolic trajectory of a projectile.

Wolfson

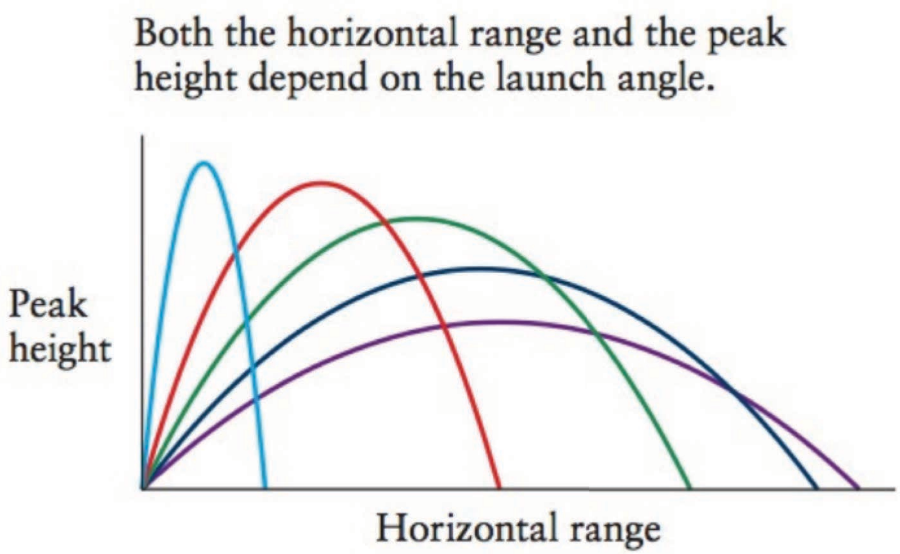
Note: $\sin 2A = 2 \sin A \cos A$

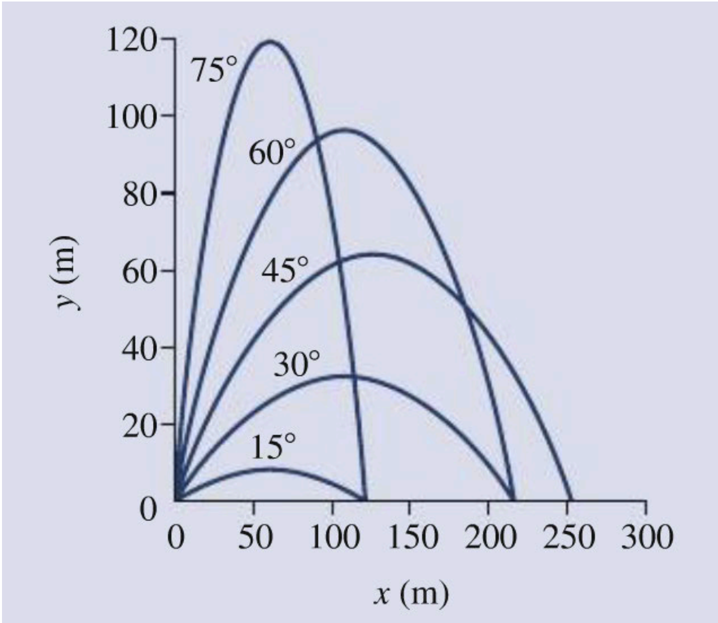


Tartaglia's discovery should now be readily apparent...

$$x = \frac{v_0{}^2}{g} \sin 2\theta_0 \qquad \text{(horizontal range)}$$

$$y = x \tan \theta_0 - \frac{g}{2 v_0{}^2 \cos^2 \theta_0} x^2 \qquad \text{(projectile trajectory)}$$





Kesten & Tauck

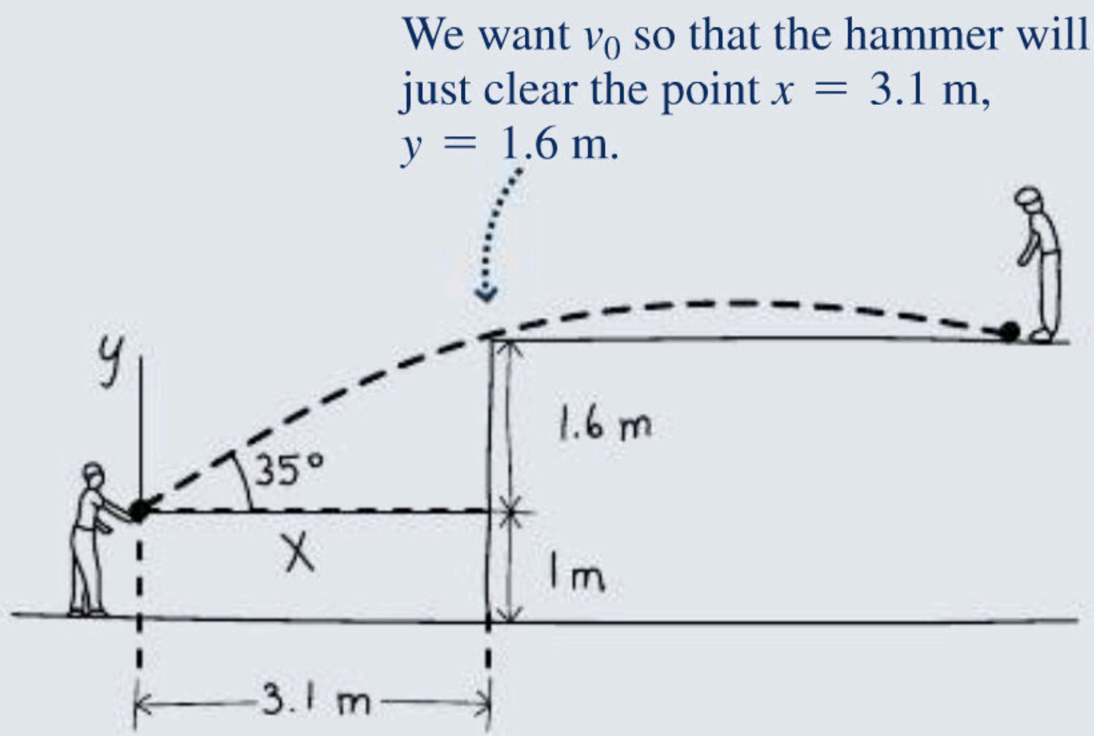
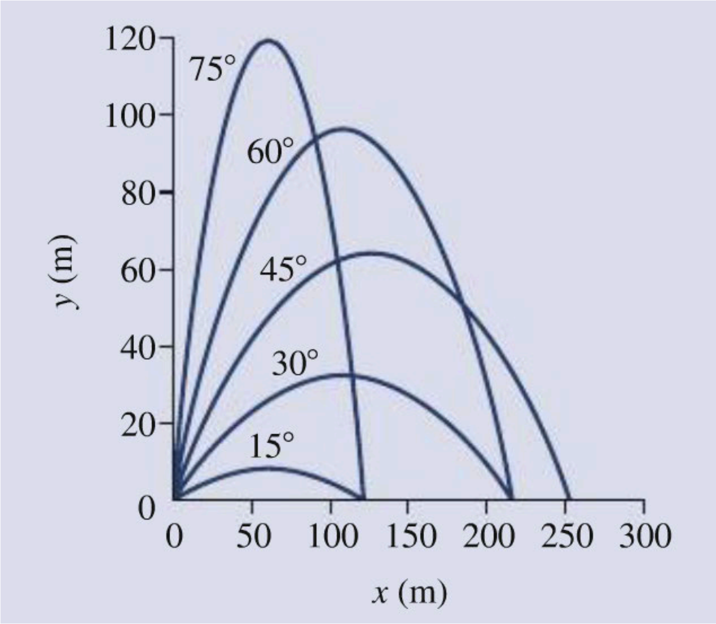


FIGURE 3.18 Our sketch for Example 3.5.

→ Practice these sorts of "projectile motion" problems, keeping careful track of what assumptions are stated (or need to be presumed!)

Projectile Motion

$$x = \frac{v_0^{\,2}}{g} \sin 2\theta_0 \qquad \text{(horizontal range)}$$



A harder problem: What happens if there is "drag" (i.e., air resistance)?

Recall

Falling body: Terminal velocity

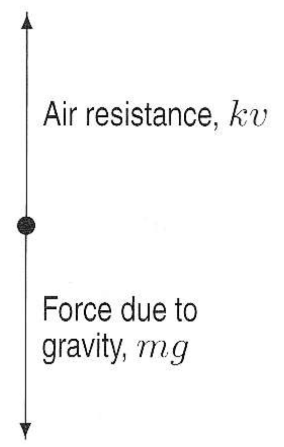


Figure 11.44: Forces acting on a falling object

Assume air resistance is proportional to velocity, the Newton's 2nd Law leads to:

$$m\frac{dv}{dt} = mg - kv$$

$$\frac{dv}{dt} = -\frac{k}{m}\left(v - \frac{mg}{k}\right)$$

Linear 1st order ODE

Solution $$v = \frac{mg}{k}\left(1 - e^{-kt/m}\right)$$

Back to our vector equation for a projectile

$$m\frac{d^2\mathbf{r}}{dt^2} = -\mathbf{k}mg$$

Adding in **linear** air resistance

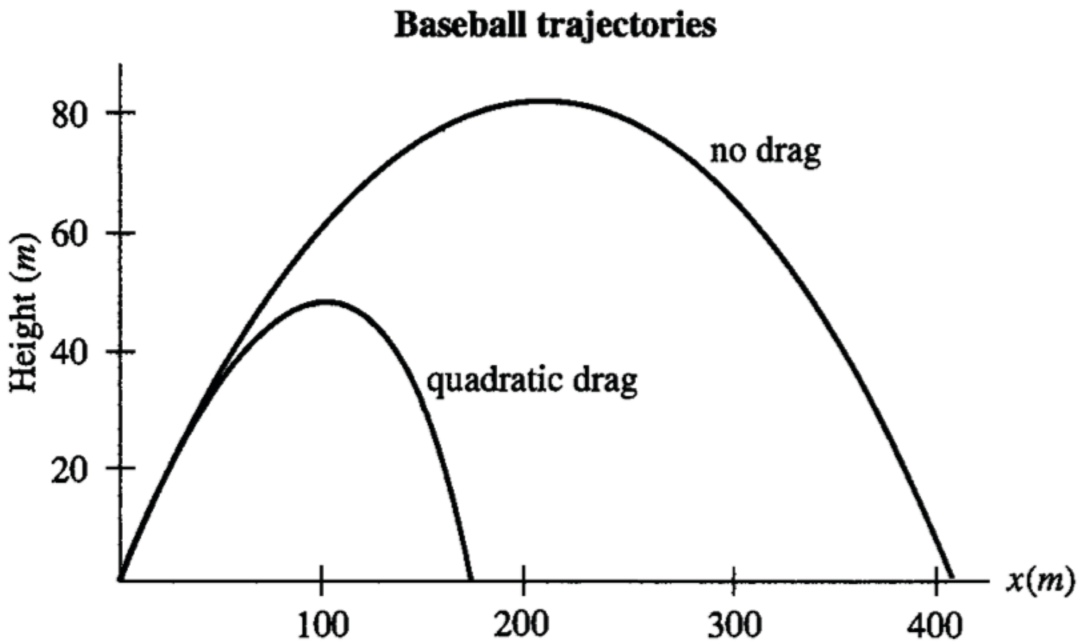
$$\frac{d^2\mathbf{r}}{dt^2} = -\gamma\mathbf{v} - \mathbf{k}g$$

Or in component form:

$$\ddot{x} = -\gamma\dot{x}$$
$$\ddot{y} = -\gamma\dot{y}$$
$$\ddot{z} = -\gamma\dot{z} - g$$

→ Let's come back a bit later on in the semester to this problem...

Hughes-Hallett et al. (2005)



Short version: In the "real world", Tartaglia was wrong (you want a launch angle a bit less than 45°)

“Tartaglia’s correct theoretical answer of 45° surprised the experts; they thought it would be smaller [...] but he refrained from publication. The reason for his diffidence is highly creditable: He felt it would be immoral to use science to help [soliders] slaughter [soliders] more efficiently”

→ So the "experts" were right!

Ex.

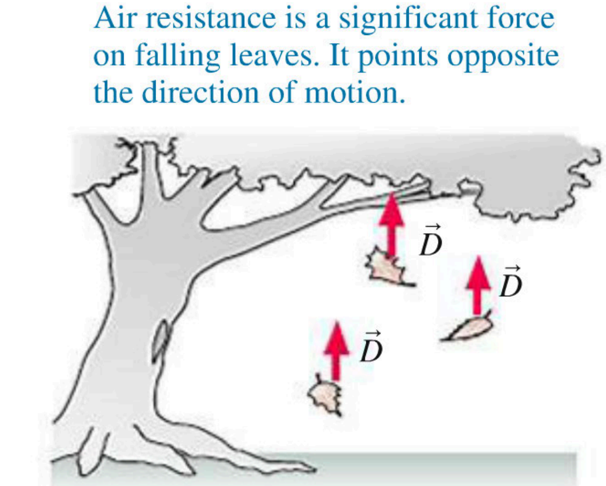
A bomb is dropped from an aeroplane flying horizontally at a constant speed. Where will the aeroplane be when the bomb hits the ground?

Shaskol'Skaya & El'Tsin (1963)

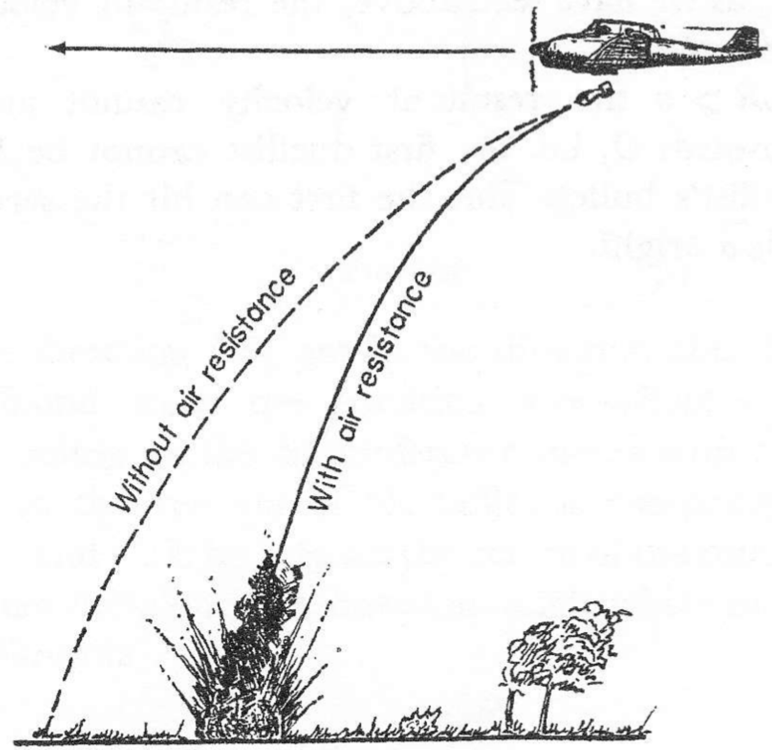
Ex. (SOL)

The plane flies horizontally with constant speed v. The bomb follows the path of a parabola, since its motion is compounded of horizontal motion with initial velocity v and uniformly accelerated vertical fall. If there were no air-resistance, the bomb's horizontal velocity would be no different from that of the plane, and the plane would be directly above the bomb the whole time—in particular, when the bomb hits the ground. But in fact, as a result of air-resistance, the bomb's horizontal velocity is decreasing all the time, and so it falls behind the plane (Fig. 163). Therefore the fall to earth and explosion of the bomb take place not underneath the plane, but considerably behind it.

FIGURE 5.11 Air resistance is an example of drag.



Knight (2013)



Shaskol'Skaya & El'Tsin (1963)