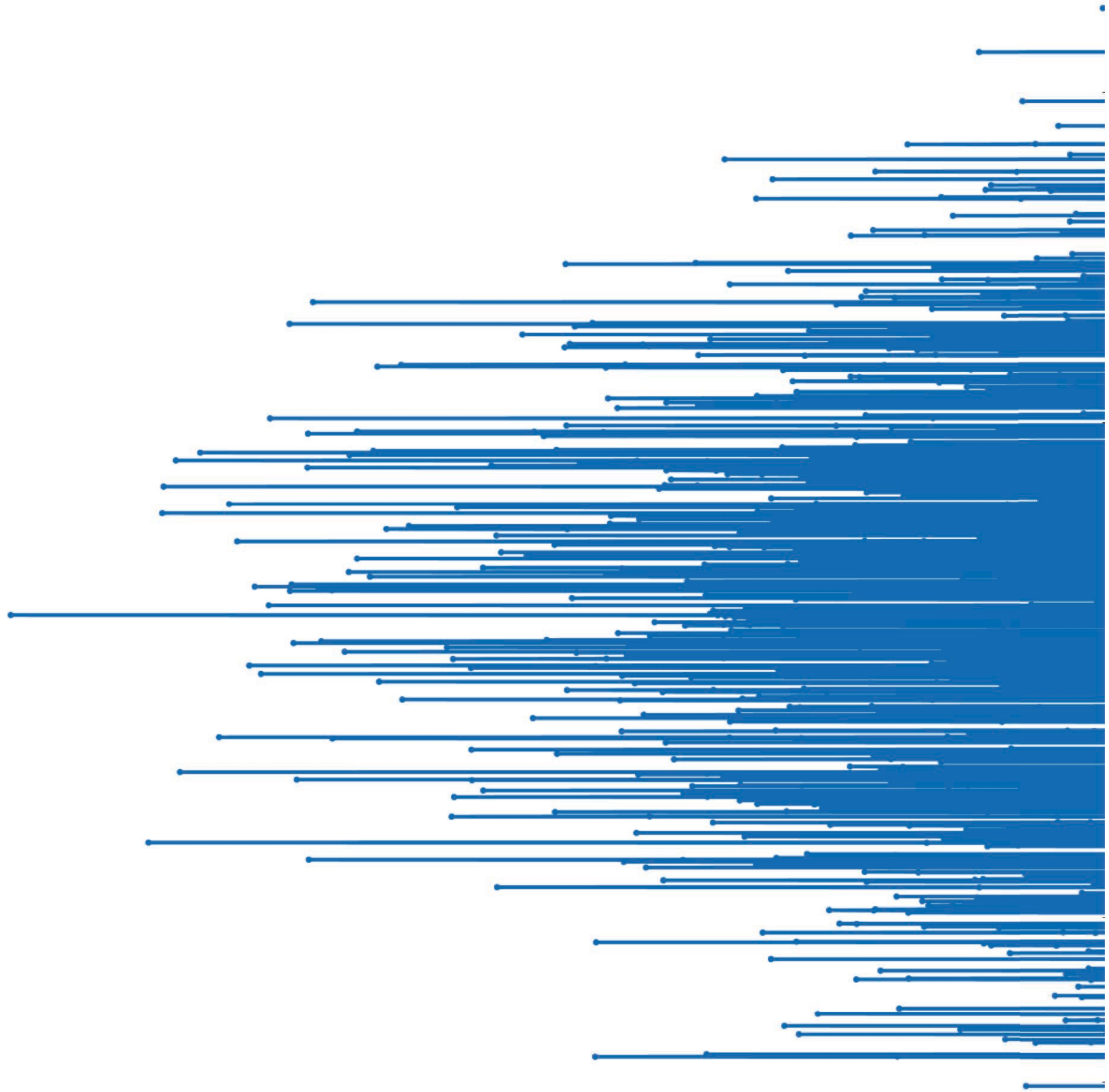
PHYS 2010 (W20)
Classical Mechanics



2020.01.16

Relevant reading:

Knudsen & Hjorth: 8.1-8.2

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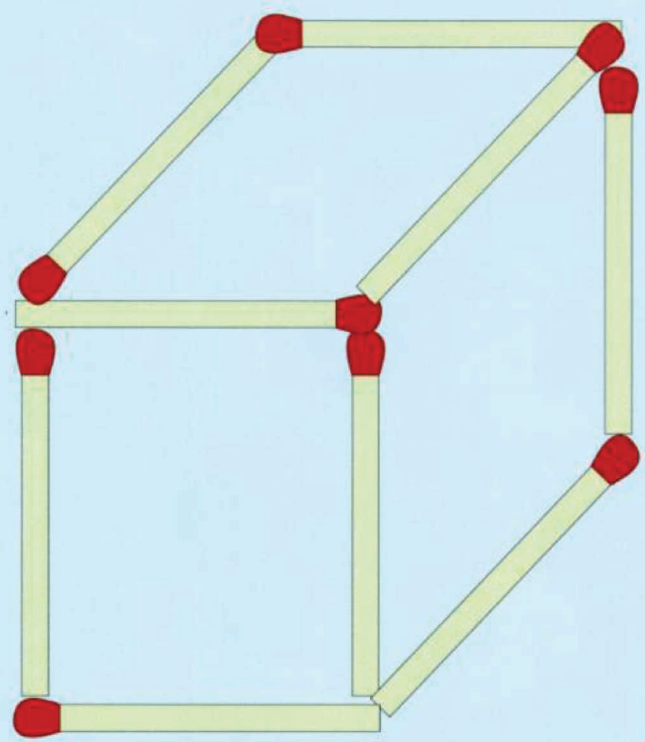
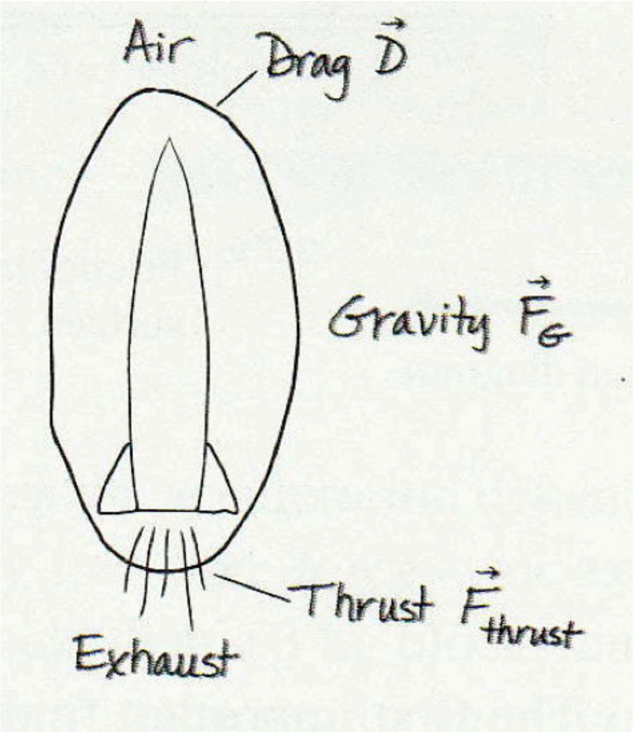
Ref.s:
Knudsen & Hjorth (2000), Fowles & Cassidy (2005)

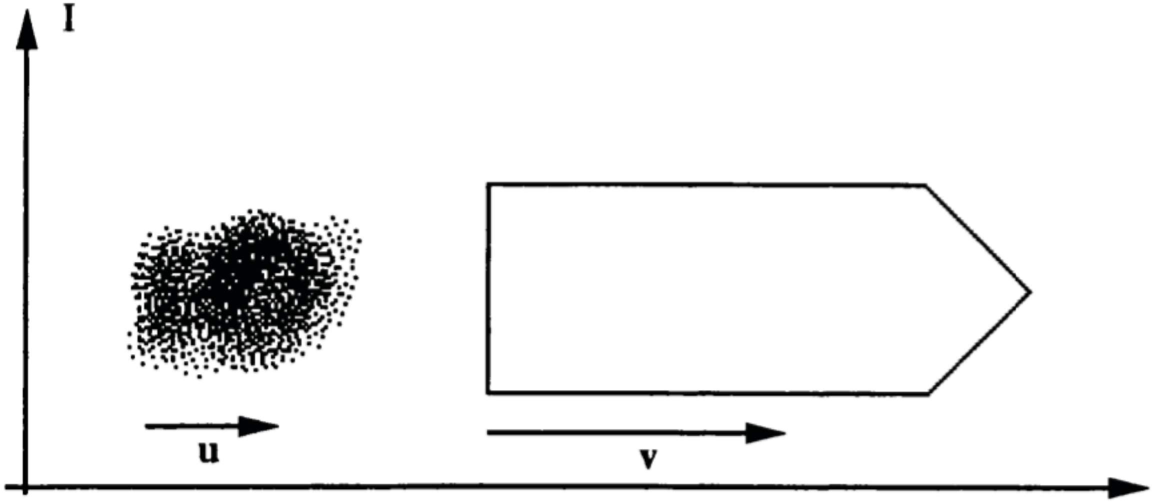
Here are nine matches, which have been arranged on a table to form a figure which looks like a cube.

Suppose two of the matches were removed. How could you rearrange the matches that remained so that they still formed the figure of a cube?



→ HW2 and rockets!

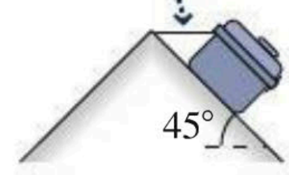




Ex.

GOT IT? 5.1 A roofer's toolbox rests on an essentially frictionless metal roof with a 45° slope, secured by a horizontal rope as shown. Is the rope tension (a) greater than, (b) less than, or (c) equal to the box's weight?

How does the rope tension compare with the toolbox weight?



Ex. (SOL)

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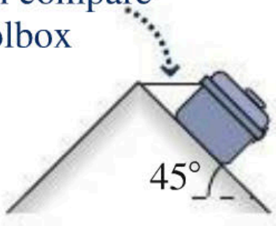


c

→ This one is not immediately intuitive per se. It's generally a good idea to draw a free-body diagram and set up the appropriate equations

A 73-kg climber finds himself dangling over the edge of an ice cliff, as shown in Fig. 5.7. Fortunately, he's roped to a 940-kg rock located 51 m from the edge of the cliff. Unfortunately, the ice is frictionless, and the climber accelerates downward. What's his acceleration, and how much time does he have before the rock goes over the edge? Neglect the rope's mass.

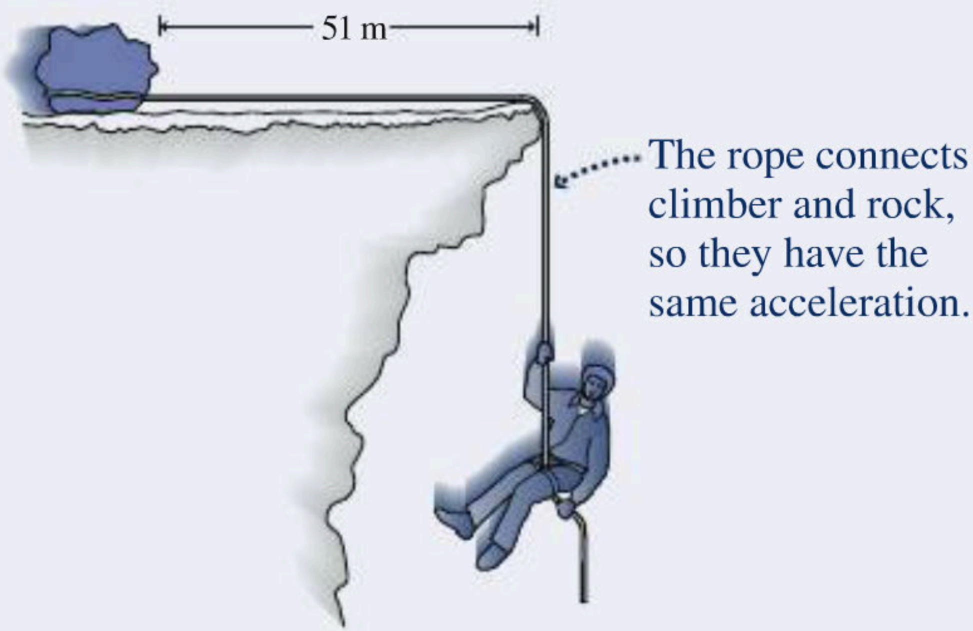
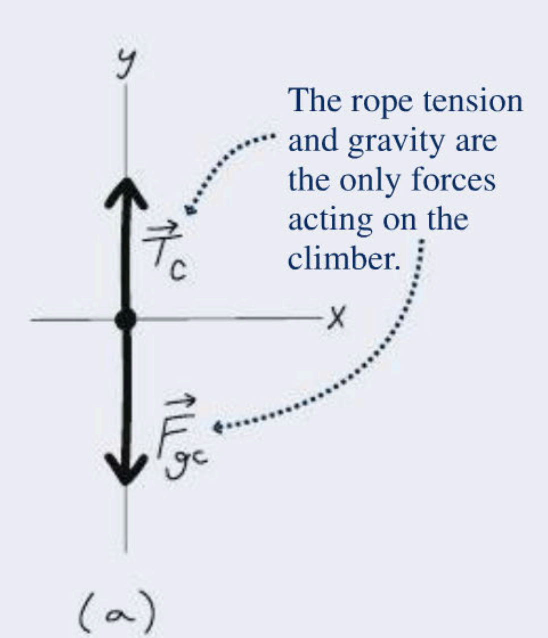
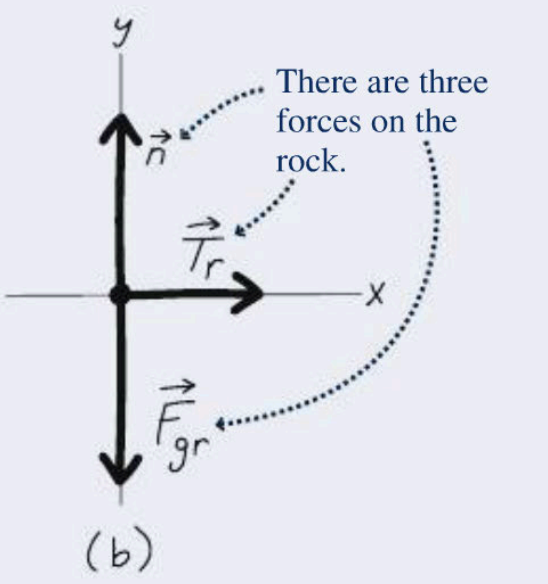


FIGURE 5.7 A climber in trouble.





$$\text{climber:} \quad \vec{T}_c + \vec{F}_{gc} = m_c\vec{a}_c$$
$$\text{rock:} \quad \vec{T}_r + \vec{F}_{gr} + \vec{n} = m_r\vec{a}_r$$

Note: These "vector equations" are essentially 2-D

$$\text{climber, } y\text{:} \quad T - m_c g = -m_c a$$
$$\text{rock, } x\text{:} \quad T = m_r a$$
$$\text{rock, } y\text{:} \quad n - m_r g = 0$$

→ Now broken up into a set(s) of 1-D eqns.

$$a = \frac{m_c g}{m_c + m_r}$$

→ This is a good problem to ensure that the "answer" makes sense (e.g., what if $m_r=0$?)

$$t = \sqrt{\frac{2x}{a}}$$

Looking further: What if we hadn't ignored the rope's mass?

Wolfson

Looking further: What if we hadn't ignored the rope's mass?

A related problem...

A chain of length x and mass m is hanging over the edge of a tall building and does not touch the ground. How much work is required to lift the chain to the top of the building?

To answer this, we'll need some more pieces:

- Definition of *work*
- Integration

Spatially-dependent forces → Energy

Let us assume the force depends only upon the "particle's" position (x), not on velocity or time

(e.g., gravitational and electrostatic forces are canonical examples of this)

$$F(x) = m\ddot{x}$$

Applying the chain rule (so to re-express a as):

$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{dx}{dt}\frac{d\dot{x}}{dx} = v\frac{dv}{dx}$$

$$F(x) = mv\frac{dv}{dx} = \frac{m}{2}\frac{d(v^2)}{dx} = \frac{dT}{dx}$$

Out pops ***kinetic energy***!

$$T = \frac{1}{2}mv^2$$

Rewriting in "integral form":

$$W = \int_{x_0}^{x} F(x)\,dx = T - T_0$$

Out pops ***work***!!

→ *Work* done is equal to the change in *kinetic energy* of the particle

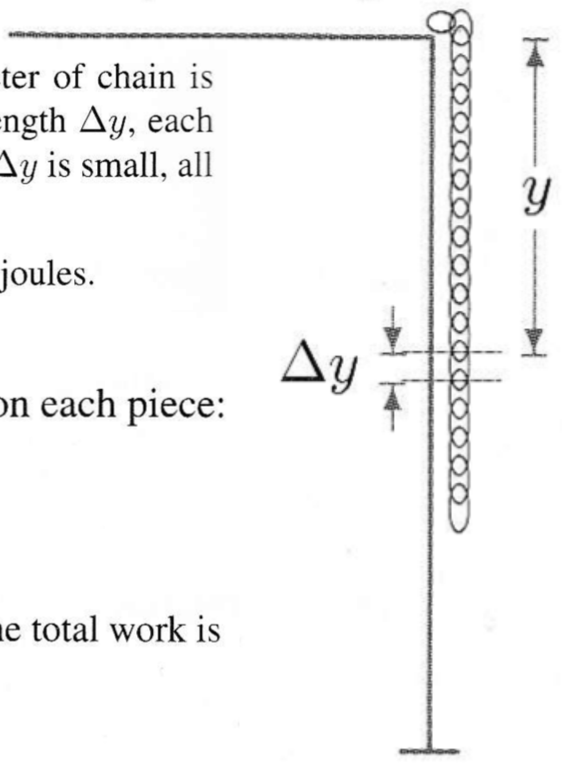
Fowles & Cassidy

Review: Work as an Integral

Note: Be careful re the specified units!
(solution below is sloppy!)

A 28-meter uniform chain with a mass 2 kilograms per meter is dangling from the roof of a building. How much work is needed to pull the chain up onto the top of the building?

→ Very useful starting point is to draw a diagram and set up the relevant variables!



Since 1 meter of the chain has mass density 2 kg, the gravitational force per meter of chain is $(2\,\text{kg})(9.8\,\text{m/sec}^2) = 19.6$ newtons. Let's divide the chain into small sections of length Δy, each requiring a force of $19.6\,\Delta y$ newtons to move it against gravity. See Figure 8.61. If Δy is small, all of this piece is hauled up approximately the same distance, namely y, so

$$\text{Work done on the small piece} \approx (19.6\,\Delta y\,\text{newtons})(y\,\text{meters}) = 19.6y\,\Delta y\,\text{joules}.$$

The work done on the entire chain is given by the total of the work done on each piece:

$$\text{Work done} \approx \sum 19.6y\,\Delta y\,\text{joules}.$$

As $\Delta y \to 0$, we obtain a definite integral. Since y varies from 0 to 28 meters, the total work is

$$\text{Work done} = \int_0^{28} (19.6y)\,dy = 9.8y^2\Big|_0^{28} = 7683.2\,\text{joules}.$$

→ *Work* done is also equal to a change in *potential energy*....

Hughes-Hallet et al (2005)

Spatially-dependent forces → Energy

$$W = \int_{x_0}^{x} F(x)\,dx = T - T_0$$

Let us define a function $V(x)$ such that:

$$-\frac{dV(x)}{dx} = F(x)$$

Weaving in ***potential energy*** here....

Rewriting work in terms of $V(x)$:

$$W = \int_{x_0}^{x} F(x)\,dx = -\int_{x_0}^{x} dV = -V(x) + V(x_0) = T - T_0$$

$$-[V(x) + C] + [V(x_0) + C] = -V(x) + V(x_0)$$

Note: An arbitrary constant (of integration) will cancel out

$$\boxed{T_0 + V(x_0) = \text{constant} = T + V(x) \equiv E}$$

Out pops ***conservation of energy***!

Fowles & Cassidy

Spatially-dependent forces → Energy

$$T_0 + V(x_0) = \text{constant} = T + V(x) \equiv E$$

Caution: There is a **MAJOR** catch here though!

$$-\frac{dV(x)}{dx} = F(x)$$

Here we assumed the force only depends upon x
[or put another way, the energy of the "particle" can be deduced from some function $V(x)$ that describes the "potential"]

$$F(x) = m\ddot{x}$$

→ Such is what is called a **"conservative" force**

However, many forces are non-conservative (e.g., friction, drag) and application of this "model" is not, umm, valid per se....

→ Let us briefly distinguish between conservative and non-conservative forces to crystallize things further....

Fowles & Cassidy

Conservative Forces

First consider 1-D case:

Recall: We now restrict ourselves to the special case where the force F depends on position only, $F = F(x)$. (In particular, F is not dependent on time).

> The change in the potential energy $U(x)$ over the segment $\mathrm{d}x$ is equal to minus the work done by the force F.

Equivalently: $$U(x) = -\int F(x)\mathrm{d}x$$

Leading back to: $$T + U(x) = E.$$

Here, E is just a constant of integration that depends upon the initial conditions!

> The total mechanical energy – i.e., the sum of kinetic and potential energy – for a particle moving in a conservative force field, is conserved (constant).

Knudsen & Hjorth

<u>Conservative Forces</u>

Generalizing (for a moment) to higher dimensions (e.g., notion of a *field*):

A force field is called *conservative* if the force vector $\mathbf{F}$ of the field depends only on the position $\mathbf{r}$ of the particle and the work integral $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path of integration, depending only on the initial point A and the final point B, of the path.

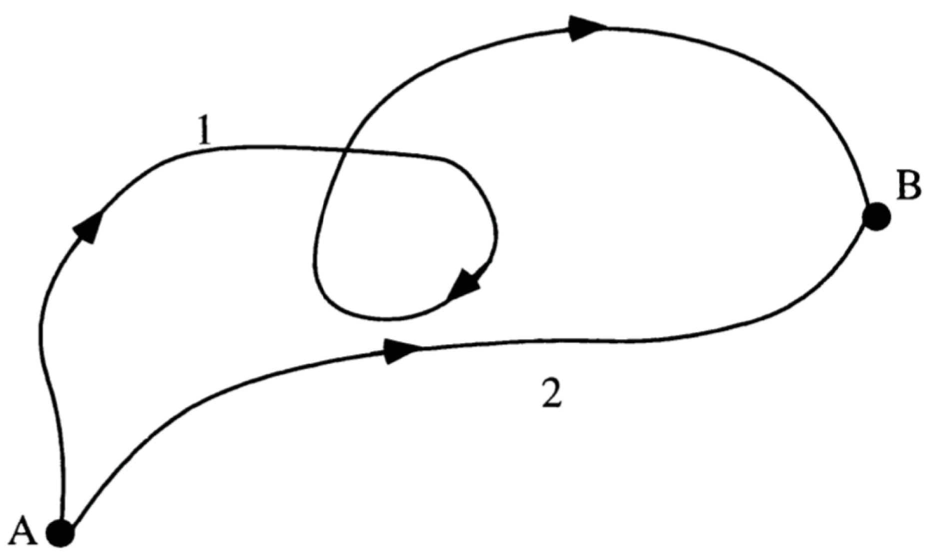


Fig. 8.2. Defining a conservative force field. The particle is moved from A to B along the path 1 or along the path 2

Knudsen & Hjorth

<u>Conservative Forces</u>

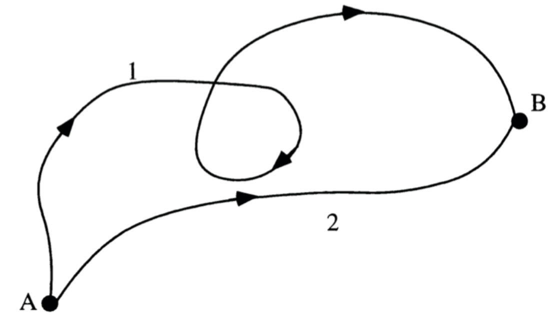


Fig. 8.2. Defining a conservative force field. The particle is moved from A to B along the path 1 or along the path 2

The condition that the force field is conservative is:

$$\int_{\substack{A \\ \text{along } 1}}^{B} \mathbf{F} \cdot d\mathbf{r} = \int_{\substack{A \\ \text{along } 2}}^{B} \mathbf{F} \cdot d\mathbf{r} \,,$$

Or more succinctly:

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0 \,.$$

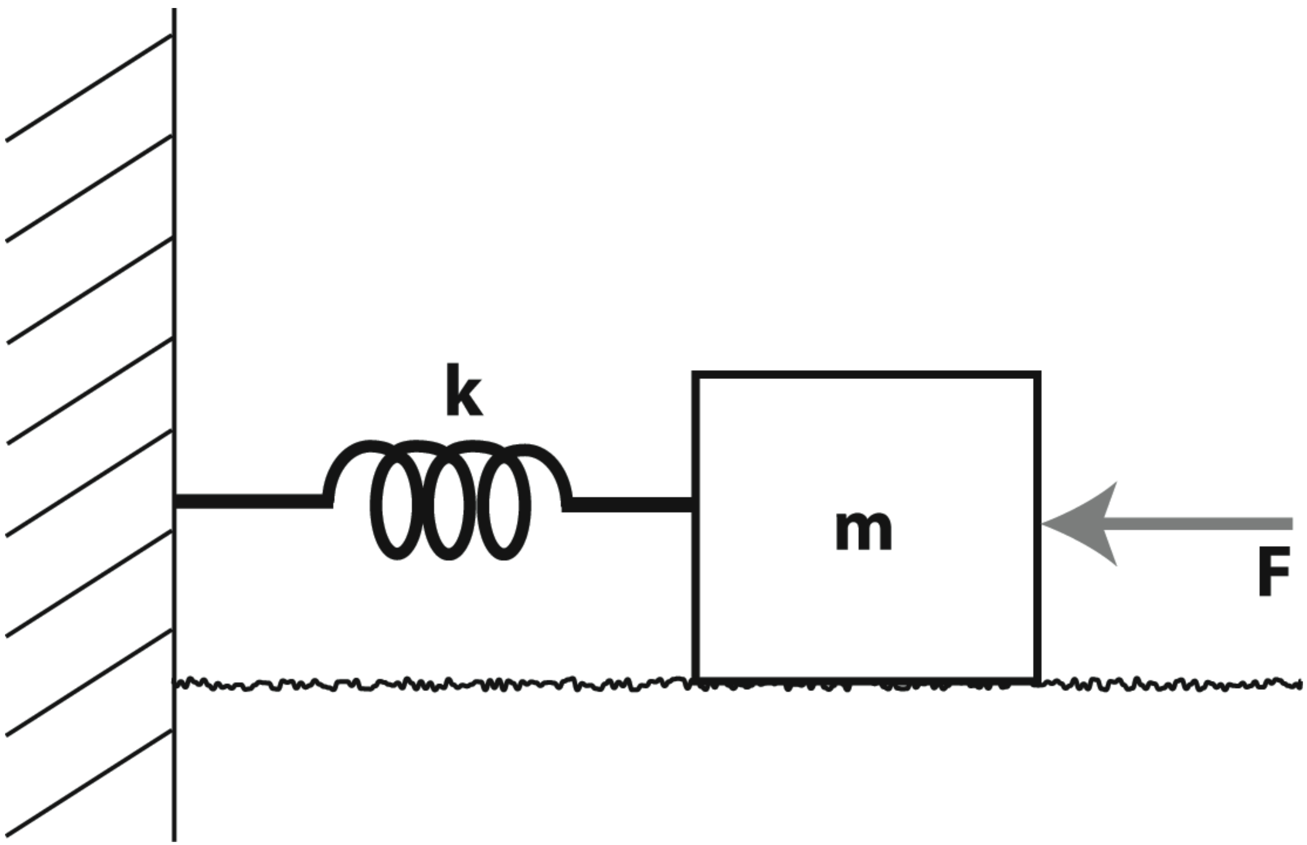
Equivalently:

$$\int_{\substack{A \\ \text{along } 1}}^{B} \mathbf{F} \cdot d\mathbf{r} - \int_{\substack{A \\ \text{along } 2}}^{B} \mathbf{F} \cdot d\mathbf{r} = 0 \,,$$

"This is the most common definition of a conservative force field: The work integral around any closed curve in the field, is zero."

Knudsen & Hjorth

(Non-)Conservative Forces: Intuitive example I



(Non-)Conservative Forces: Intuitive example II

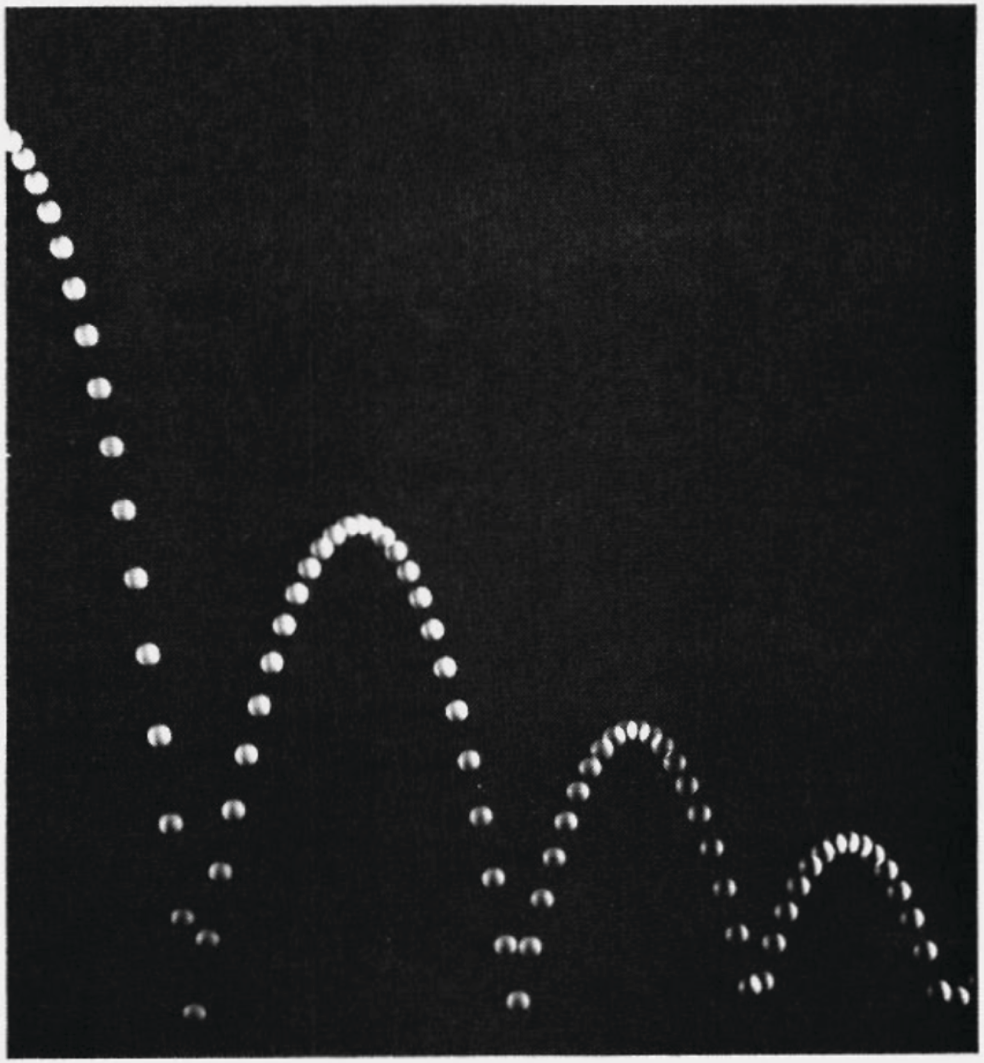


Fig. 2–1 Stroboscopic photograph of a motion. (From PSSC Physics, *D. C. Heath, Lexington, Massachusetts, 1965.)*

Situation here is a bit more nuanced...

French (1971)

Conservative Forces & Energy

Let's us return to conservative force for the moment:

$$F(x) = m\ddot{x} \qquad -\frac{dV(x)}{dx} = F(x)$$

[Why? Such will ultimately/directly lead into Lagrangian mechanics downstream...]

"Energy equation"

$$T_0 + V(x_0) = \text{constant} = T + V(x) \equiv E$$

Solving the energy equation for v:

$$v = \frac{dx}{dt} = \pm\sqrt{\frac{2}{m}[E - V(x)]}$$

Note that v is only real for $V <= E$

Equivalently:

$$\int_{x_0}^{x} \frac{dx}{\pm\sqrt{\frac{2}{m}[E - V(x)]}} = t - t_0$$

Fowles & Cassidy

Conservative Forces & Energy

$$v = \frac{dx}{dt} = \pm\sqrt{\frac{2}{m}[E - V(x)]}$$

Subsequent considerations:

- $V(x)$ defines a *potential well*
- The "particle" is confined to regions where $V <= E$
- v goes to zero when $V = E$ → "turning points" (particle comes to rest and reverses its motion)
- There will be regions that energetically are "not allowed"

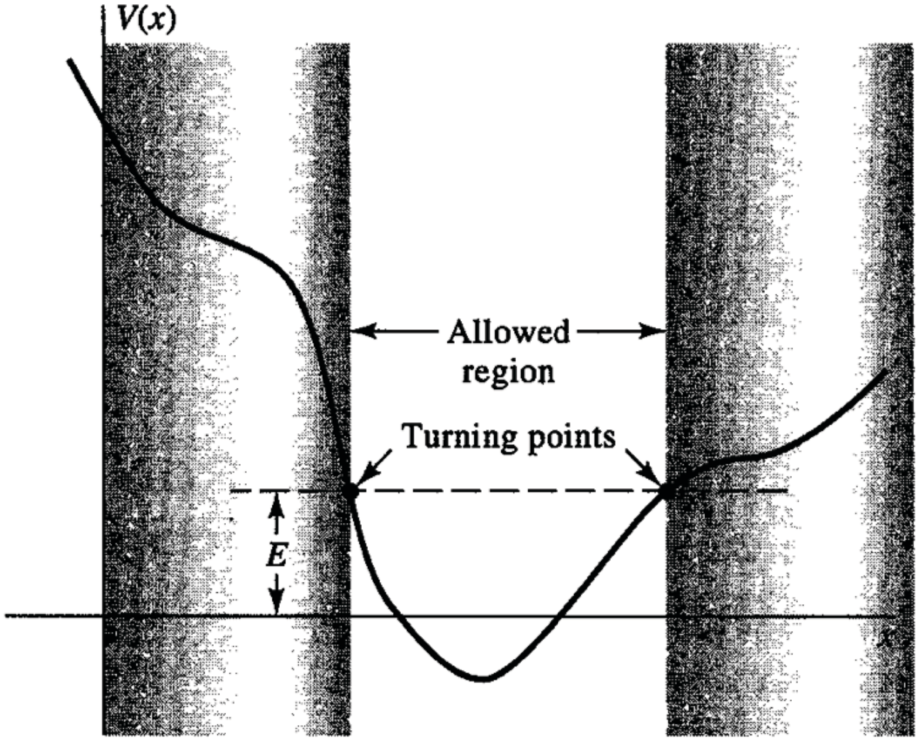


Figure 2.3.1 Graph of a one-dimensional potential energy function $V(x)$ showing the allowed region of motion and the turning points for a given value of the total energy E.

Fowles & Cassidy

Ex. Object in "free fall" due to gravity

Let x be vertical distance (positive when going upwards, away from Earth's surface)

Force due to gravity: $-mg$

Recall (by definition):

$$-\frac{dV(x)}{dx} = F(x) \longrightarrow -dV/dx = -mg,$$

Solving for V, one obtains: $V = mgx + C$

Note that C is an arbitrary const. of integration (and ties back to our *reference height*). Easiest to choose C =0 (i.e., V=0 when x=0)

Total energy (E) is then determined by the initial conditions. Consider the object shot upwards with initial velocity v_o, then:

$$E = mv_0^2/2 = mv^2/2 + mgx,$$

Rearranging, we obtain the familiar: $v^2 = v_0^2 - 2gx$

Leading to an expression for the "turning point": $h = x_{max} = \frac{v_0^2}{2g}$

→ A familiar 1st year problem recast through the lens of a "potential well"

Fowles & Cassidy

Ex. Vibrating diatomic molecule

The Morse function approximates the potential energy of a vibrating diatomic molecule as a function of x (the separation distance between the two atoms)

$$V(x) = V_0\left[1 - e^{-(x-x_0)/\delta}\right]^2 - V_0$$

Note: The three arbitrary constants can be tailored for a particular atomic pair

One constant in particular (x_o) is relevant here....

$$-\frac{dV(x)}{dx} = F(x)$$

... let us determine where F is at a minimum

$$F(x) = -\frac{dV(x)}{dx} = 0 = \quad 2\frac{V_0}{\delta}\left(1 - e^{-(x-x_0)/\delta}\right)\left(e^{-(x-x_0)/\delta}\right) = 0$$

$$1 - e^{-(x-x_0)/\delta} = 0 \quad \longrightarrow \quad \ln(1) = -(x - x_0)/\delta = 0 \qquad \therefore x = x_0$$

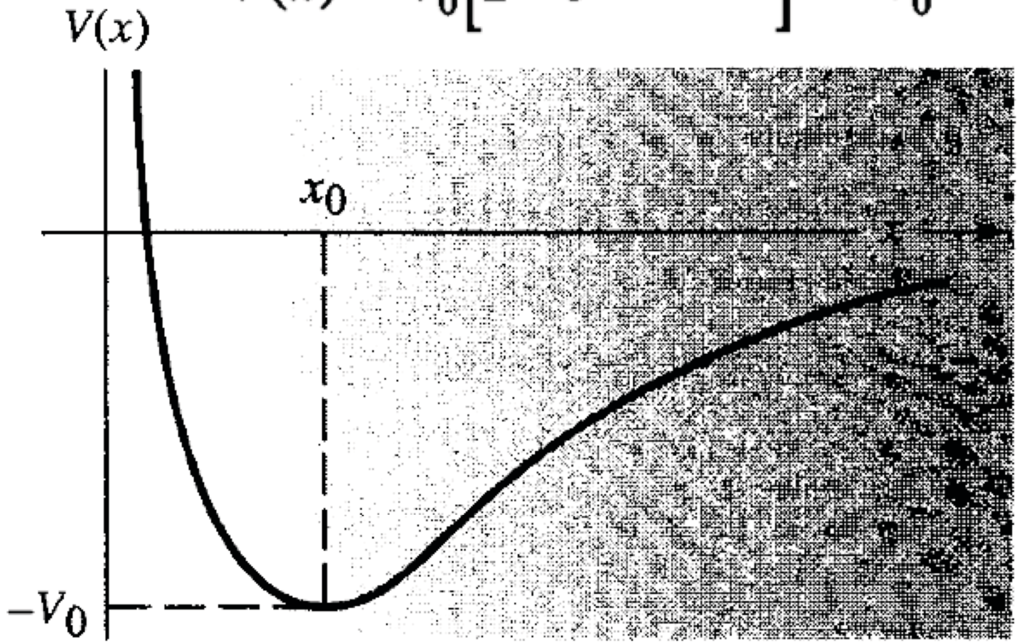
Leading to: $V(x_0) = -V_0.$

Fowles & Cassidy

Ex. Vibrating diatomic molecule

$$V(x) = V_0\left[1 - e^{-(x-x_0)/\delta}\right]^2 - V_0$$

$$V(x_0) = -V_0.$$



→ This is the *equilibrium* condition for the molecule....

.... but what is the associated *stability*?

That is, what happens if the system is perturbed and the separation moves slightly away from x_o (i.e., for small $|x - x_o|$)?

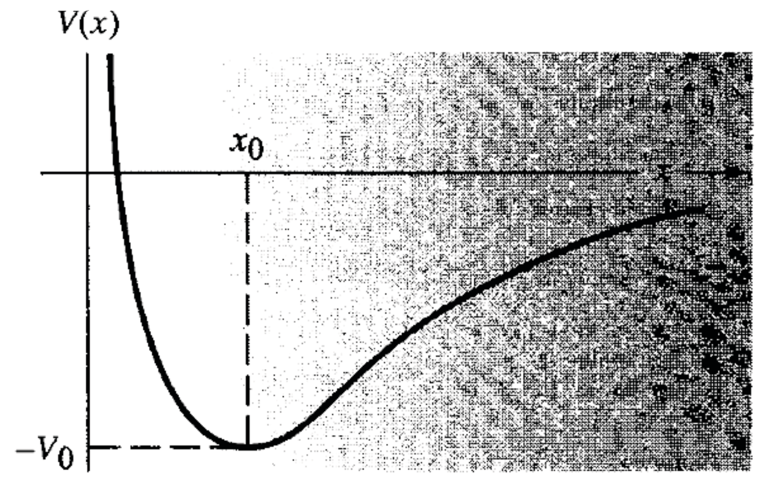
Given the exponential function, a simple Taylor series expansion should do the trick....

$$\exp x := \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

Fowles & Cassidy

Ex. Vibrating diatomic molecule

$$V(x) = V_0\left[1 - e^{-(x-x_0)/\delta}\right]^2 - V_0$$



$$\exp x := \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$V(x) \approx V_0\left[1 - \left(1 - \left(\frac{x - x_0}{\delta}\right)\right)\right]^2 - V_0$$

$$\approx \frac{V_0}{\delta^2}(x - x_0)^2 - V_0$$

A quadratic!

And the associated force:

$$F(x) = -\frac{dV(x)}{dx} = -\frac{2V_0}{\delta^2}(x - x_0)$$

A linear function (of x). And a **restorative** one too (i.e., F always points back towards x_o)

Note: We could readily carry our analysis out further (e.g., estimate the "***binding energy***")

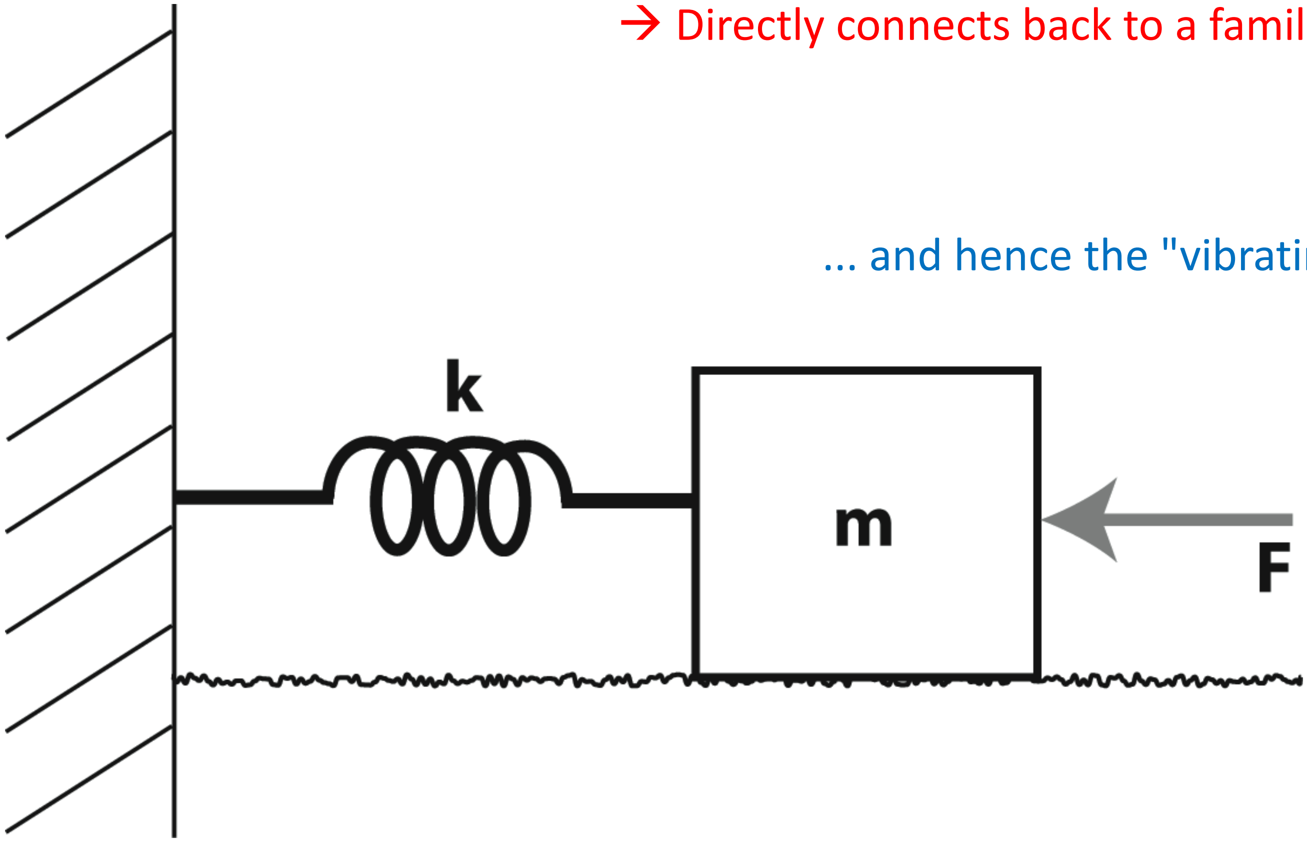
Fowles & Cassidy

Ex. Vibrating diatomic molecule

$$F(x) = -\frac{dV(x)}{dx} = -\frac{2V_0}{\delta^2}(x - x_0)$$

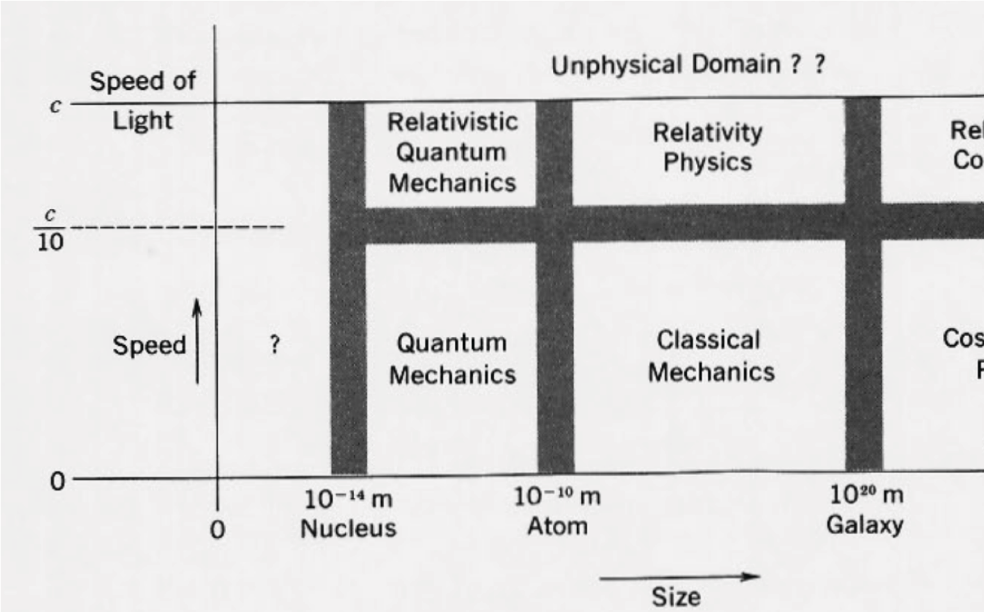
→ Directly connects back to a familiar problem!

... and hence the "vibrating" molecule



We will return to the HO in quite some detail soon....

<u>Caveat</u> (re the "Vibrating diatomic molecule")



French (1971)

→ An example of an application of classical mechanics applied to systems that aren't really "classical" per se...

(see also Lord Rayleigh and elastic scattering re "Why is the sky blue?")