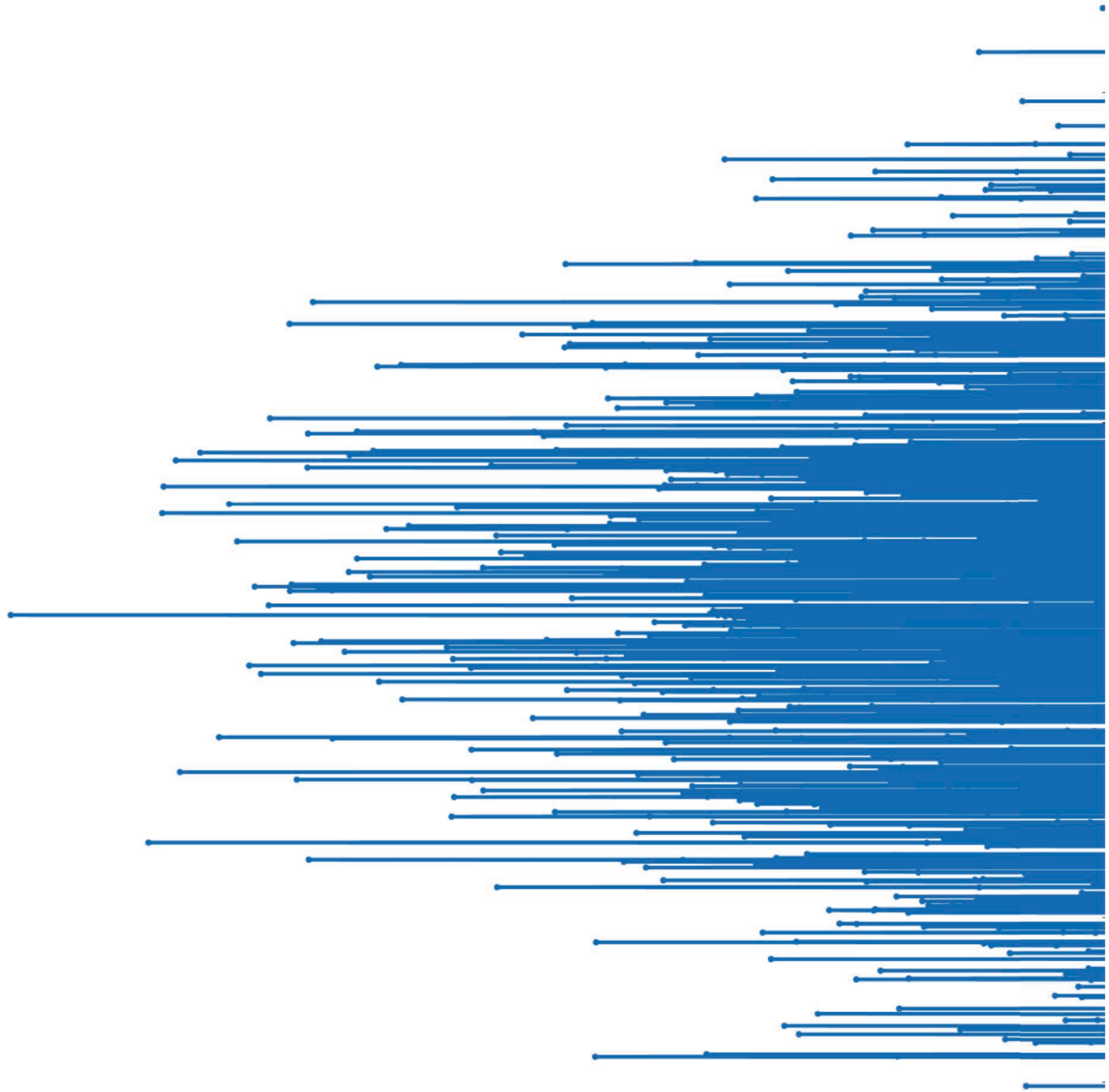
PHYS 2010 (W20)
Classical Mechanics



2020.01.21

<u>Relevant reading</u>:
Knudsen & Hjorth: 7.1, 8.3-8.5

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<u>Ref.s</u>:
Knudsen & Hjorth (2000), Fowles & Cassidy (2005), Hughes-Hallet et al. (2013)

Here's a more serious, *practical* math/physics question for you to ponder. If you are making a round-trip flight from A to B and then back to A, does a steady wind blowing from A to B increase, decrease, or leave unchanged, the total travel time compared with when no wind is blowing? *Don't guess*—make a mathematical analysis (it's just high school algebra).

Nahin (2016)

Projectiles are hurled at a horizontal distance R from the edge of a cliff of height h in such a way as to land a horizontal distance x from the bottom of the cliff. If you want x to be as small as possible, how would you adjust θ_0 and v_0, assuming that v_0 can be varied from zero to some maximum finite value and that θ_0 can be varied continuously? Only one collision with the ground is allowed (see Fig. 4–14).

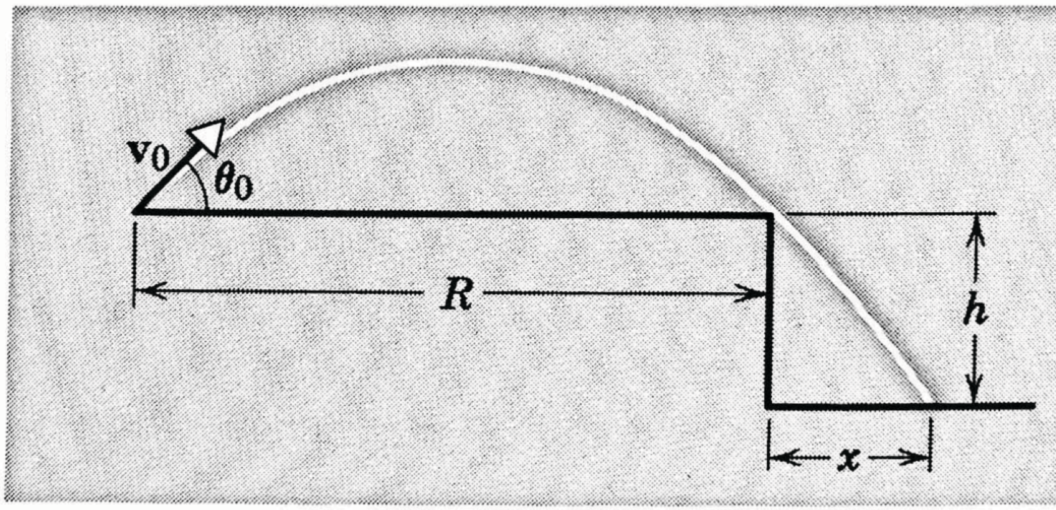


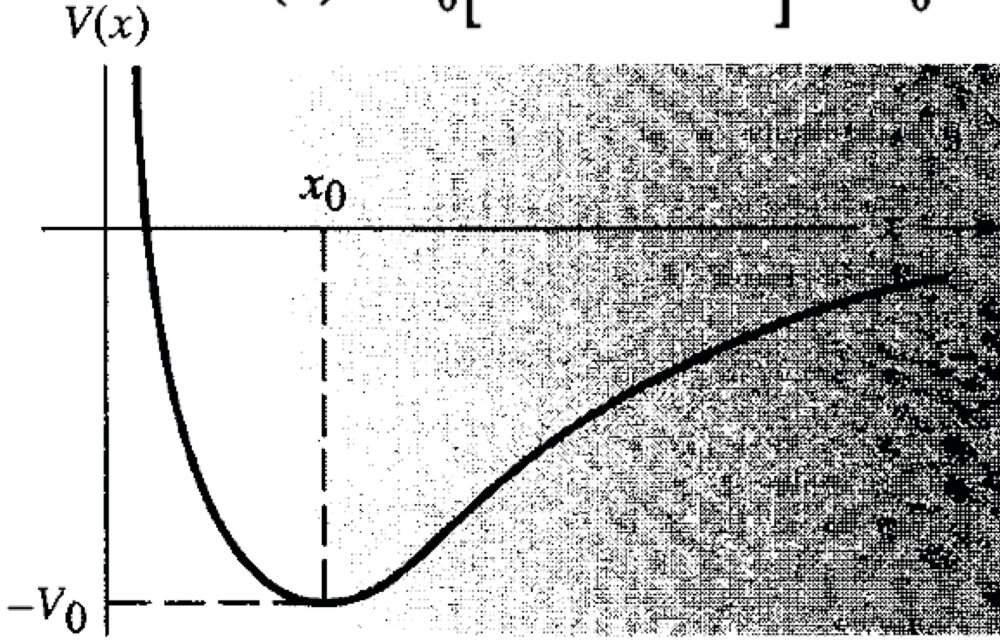
Fig. 4–14

Resnick & Halliday (1968)

Ex. Vibrating diatomic molecule

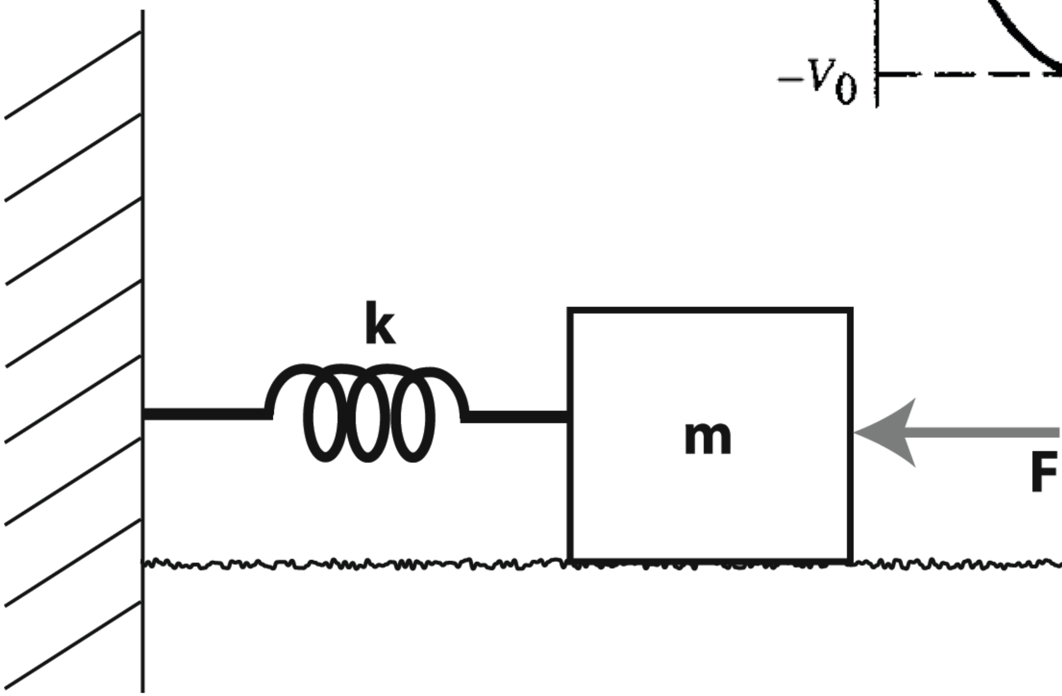
"Morse Function"

$$V(x) = V_0\left[1 - e^{-(x-x_0)/\delta}\right]^2 - V_0$$



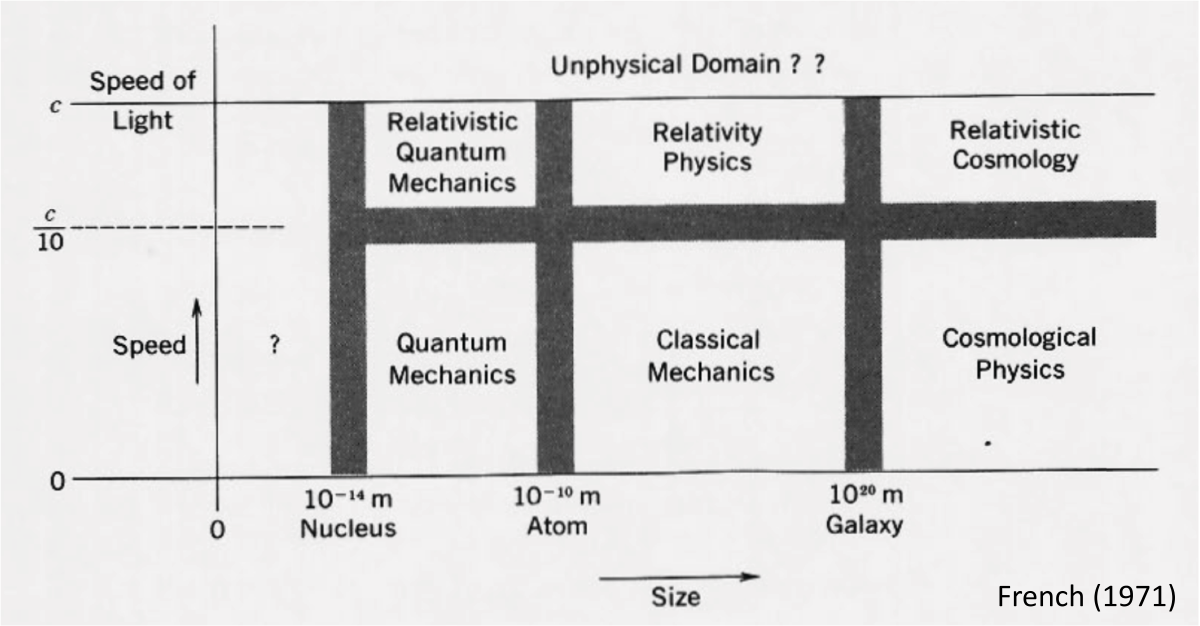
$$F(x) = -\frac{dV(x)}{dx} = -\frac{2V_0}{\delta^2}(x - x_0)$$

→ Directly connects back to a familiar problem....



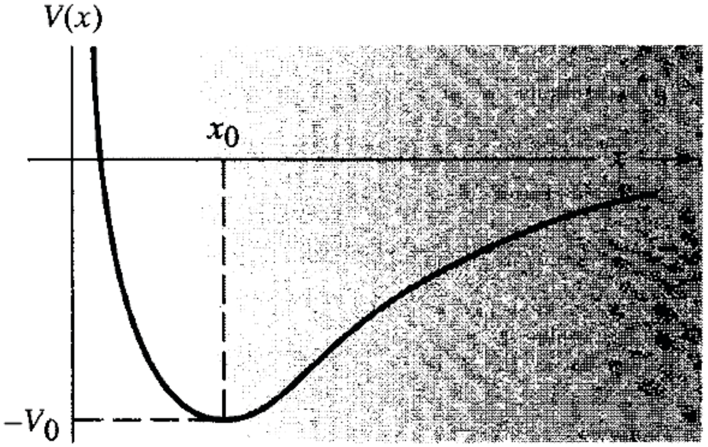
... and hence the "vibrating" molecule

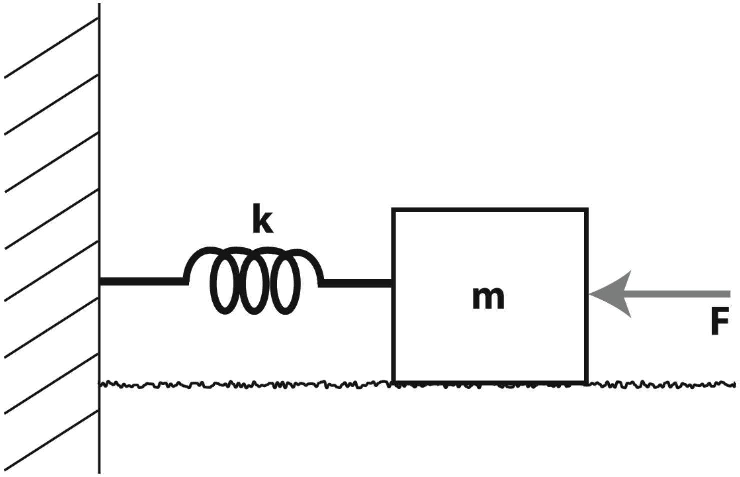
Caveat (re the "Vibrating diatomic molecule")



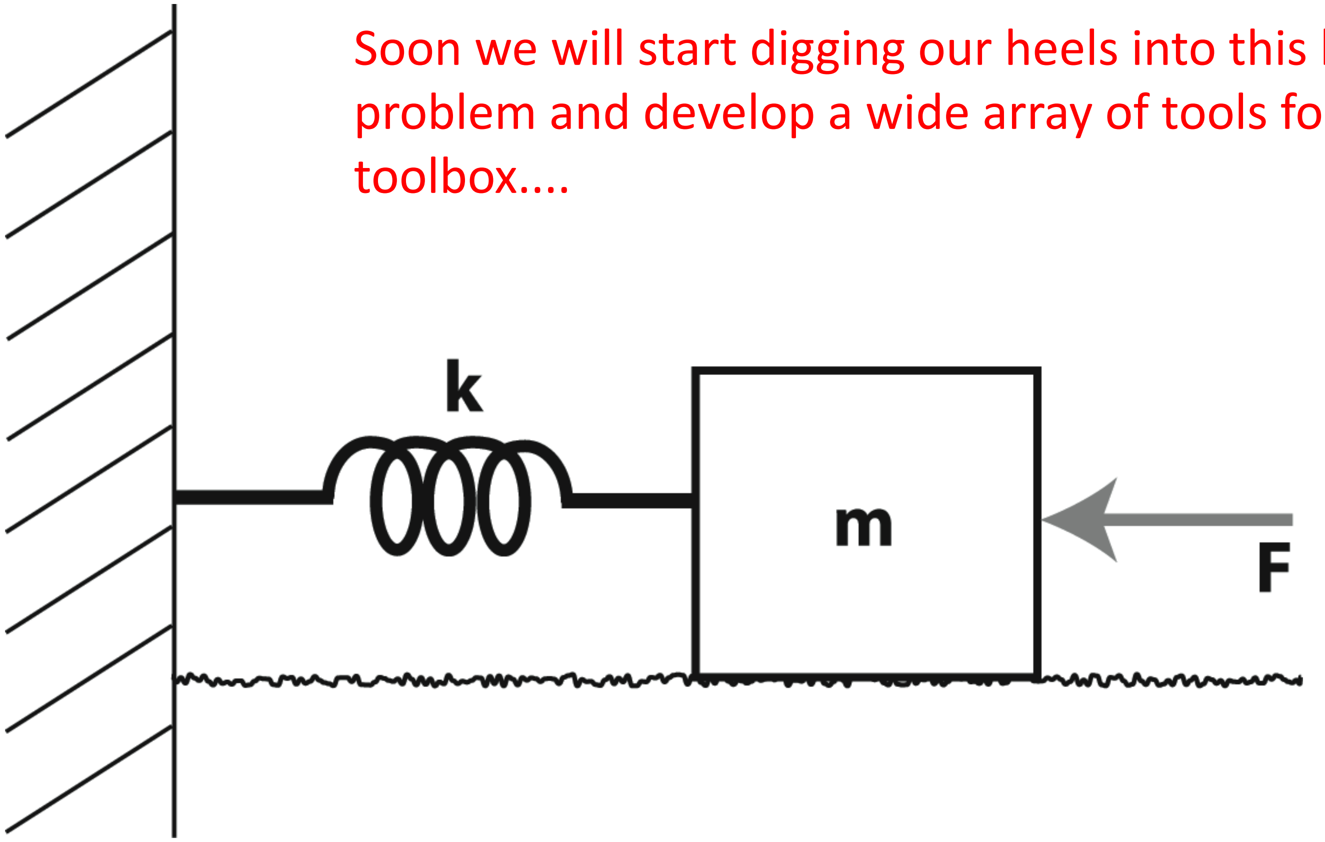
→ An example of an application of classical mechanics applied to systems that aren't really "classical" per se...

(see also Lord Rayleigh and elastic scattering re "Why is the sky blue?")





Looking Ahead....



Soon we will start digging our heels into this key 1-D problem and develop a wide array of tools for our toolbox....

.... but first let us wrap up some loose ends (projectile motion in higher dimensions w/ drag) so to set the stage for later topics/approaches)

Recall: Projectile Motion 2) Vector calculation

$$\mathbf{r}(t) = \mathbf{i}bt + \mathbf{j}\left(ct - \frac{gt^2}{2}\right) + \mathbf{k}0$$

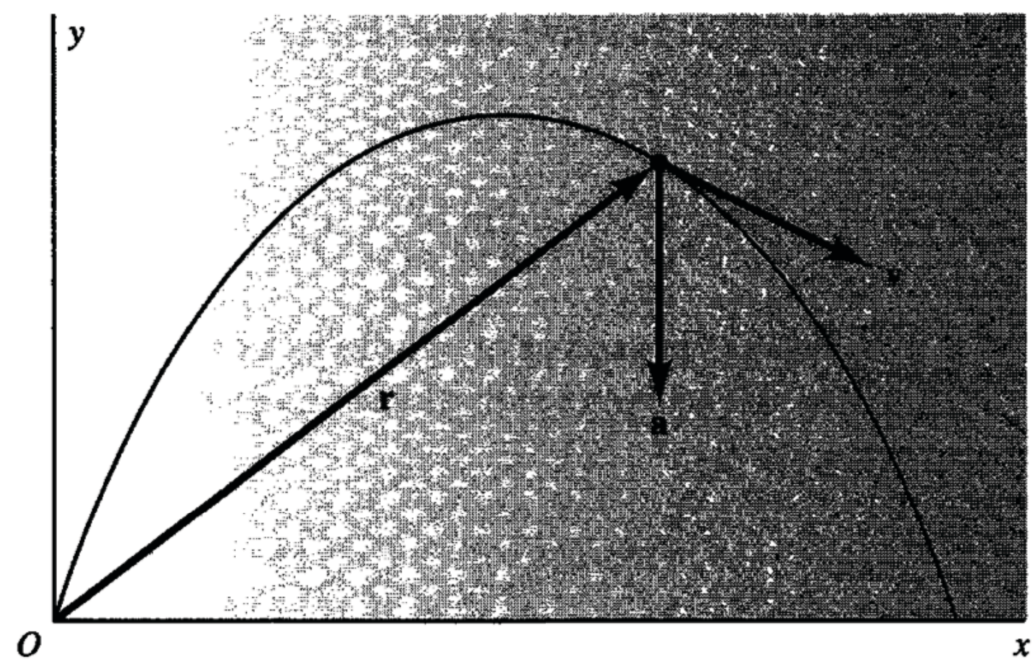
$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i}b + \mathbf{j}(c - gt)$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -\mathbf{j}g$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -\mathbf{k}gt + \mathbf{v}_0$$

Note the (slight) change in coord system as assumed at start of lecture!

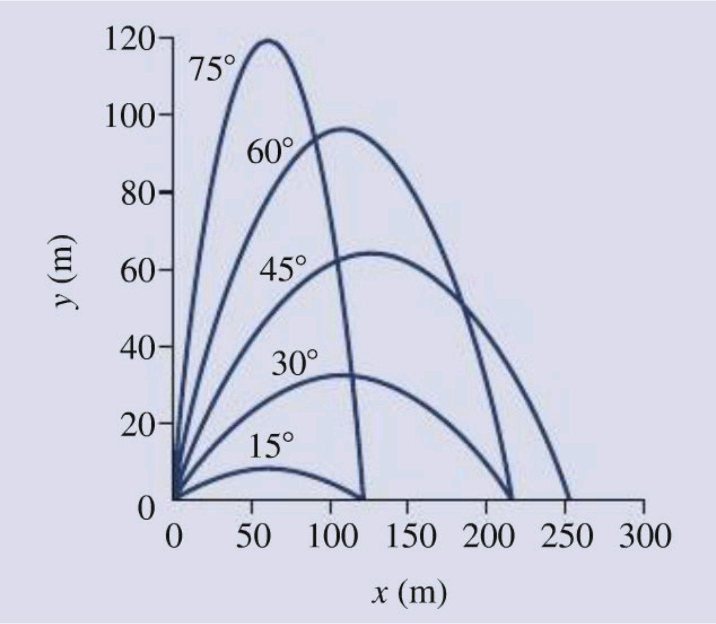
$$v = [b^2 + (c - gt)^2]^{1/2}$$



Fowles & Cassidy

<u>Recall</u> (re Projectile Motion)

$$x = \frac{v_0^2}{g} \sin 2\theta_0 \qquad \text{(horizontal range)}$$



A harder problem: What happens if there is “drag” (i.e., air resistance)?

Non-conservative Forces

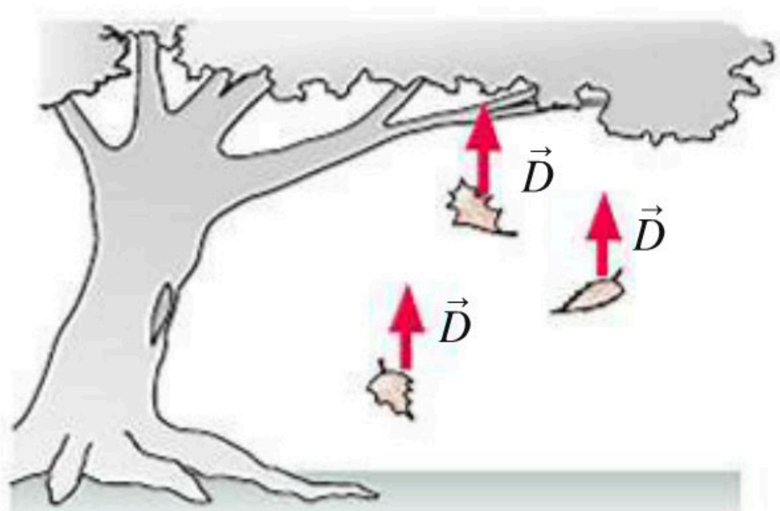
Non-conservative Forces

Before considering higher dimensional descriptions, let's consider 1-D forces that depend upon velocity....

Recall re *drag* and *terminal velocity*:

$$v = \frac{mg}{k}\left(1 - e^{-kt/m}\right)$$

Air resistance is a significant force on falling leaves. It points opposite the direction of motion.



Knight (2013)

Note: A heavier object does fall faster!

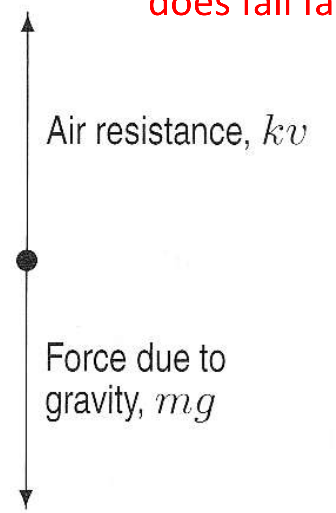


Figure 11.44: Forces acting on a falling object

Ex.

If I drop a bowling ball, a spoon, and a book at the same time from the same height, do they fall at the same rate?

If you ask people around you, what will they say? I bet the will say one of the following answers:

- Heaver objects fall faster. If you drop a heavy and light object together, the heavy one will get to the ground first.
- This is trick question. I remember in physics that everything falls the same. You can't trick me twice.

https://www.wired.com/2013/10/do-heavier-objects-really-fall-faster/

Non-conservative Forces

Before considering higher dimensional descriptions, let's consider 1-D forces that depend upon velocity....

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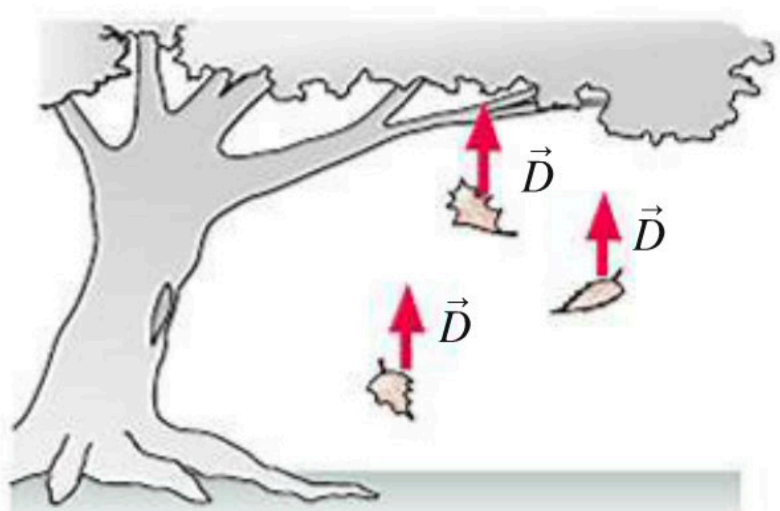

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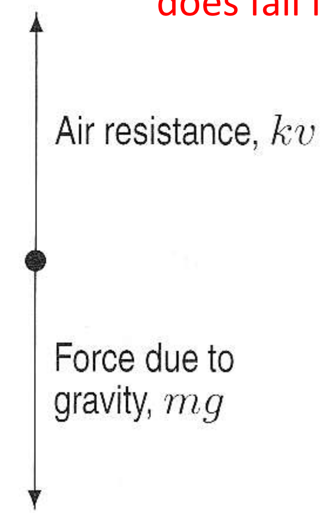


Figure 11.44: Forces acting on a falling object

$$F_0 + F(v) = m\frac{dv}{dt}$$

$$F_0 + F(v) = mv\frac{dv}{dx}$$

→ Let us return to such w/ an added degree of formalism

Ex. 1-D Linear drag (no gravity)

Note: These cases give us a chance to highlight/practice solving ODEs

Consider: $$-c_1 v = m\frac{dv}{dt}$$

For example, this might be a block launched along a horizontal surface with velocity v_o some "air resistance"

Now we use separation of variables and integrate to solve for t:

Fowles & Cassidy

Aside: Separation of Variables

Consider: $$\frac{dy}{dx} = -\frac{x}{y},$$

The method of *separation of variables* works by putting all the x-values on one side of the equation and all the y-values on the other, giving

$$y\,dy = -x\,dx.$$

We then integrate each side separately:

$$\int y\,dy = -\int x\,dx,$$

Hughes-Hallet et al

<u>Aside</u>: Separation of Variables → "Justification"

Suppose a differential equation can be written in the form

$$\frac{dy}{dx} = g(x)f(y).$$

Provided $f(y) \neq 0$, we write $f(y) = 1/h(y)$, so the right-hand side can be thought of as a fraction,

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}.$$

If we multiply through by $h(y)$, we get

$$h(y)\frac{dy}{dx} = g(x).$$

Thinking of y as a function of x, so $y = y(x)$, and $dy/dx = y'(x)$, we can rewrite the equation as

$$h(y(x)) \cdot y'(x) = g(x).$$

Now integrate both sides with respect to x:

$$\int h(y(x)) \cdot y'(x)\, dx = \int g(x)\, dx.$$

The form of the integral on the left suggests that we use the substitution $y = y(x)$. Since $dy = y'(x)\, dx$, we get

$$\int h(y)\, dy = \int g(x)\, dx.$$

If we can find antiderivatives of h and g, then this gives the equation of the solution curve.

Note that transforming the original differential equation,

$$\frac{dy}{dx} = \frac{g(x)}{h(y)},$$

into

$$\int h(y)\, dy = \int g(x)\, dx$$

looks as though we have treated dy/dx as a fraction, cross-multiplied, and then integrated. Although that's not exactly what we have done, you may find this a helpful way of remembering the method. In fact, the dy/dx notation was introduced by Leibniz to allow shortcuts like this (more specifically, to make the chain rule look like cancellation).

Hughes-Hallet et al

Ex. 1-D Linear drag (no gravity)

Note: These cases give us a chance to highlight/practice solving ODEs

Consider: $$-c_1 v = m\frac{dv}{dt}$$

For example, this might be a block launched along a horizontal surface with velocity v_o some "air resistance"

Now we use separation of variables and integrate to solve for t:

$$t = \int_{v_0}^{v} -\frac{m\,dv}{c_1 v} = -\frac{m}{c_1}\ln\left(\frac{v}{v_0}\right)$$

Rearranging and solving for v:

$$v = v_0 e^{-c_1 t/m}$$

Note that if we hadn't neglected gravity (i.e., a constant in the ODE), things wouldn't be too different...

$$v = \frac{mg}{k}\left(1 - e^{-kt/m}\right)$$

Integrating again:

$$x = \int_0^t v_0 e^{-c_1 t/m}\,dt = \frac{mv_0}{c_1}\left(1 - e^{-c_1 t/m}\right)$$

Block asymptotically approaches a limiting value:

$$x_{lim} = mv_0/c_1.$$

Fowles & Cassidy

<u>Ex.</u> 1-D Quadratic drag (no gravity)

Now we have a nonlinear drag:

$$-c_2 v^2 = m\frac{dv}{dt}$$

Again, separation of variables and integrate to solve for t:

$$t = \int_{v_n}^{v} \frac{-m\,dv}{c_2 v^2} = \frac{m}{c_2}\left(\frac{1}{v} - \frac{1}{v_0}\right)$$

Rearranging and solving for v:

$$v = \frac{v_0}{1+kt}$$

where $k = c_2 v_0 / m.$

Integrating again:

$$x(t) = \int_0^t \frac{v_0\,dt}{1+kt} = \frac{v_0}{k}\ln(1+kt)$$

→ Interestingly, the block never stops moving in this case(!?!)

Good rule-of-thumb: Nonlinear things are commonly weird/unintuitive!

Fowles & Cassidy

Ex. 1-D Quadratic drag (w/ gravity)

Let x be vertical distance (positive when going upwards, away from Earth's surface)

For simplicity, assume the object is either dropped or thrown downwards:
(otherwise we need to be more crafty with the sign of the drag term due to the squaring!)

$$m\frac{dv}{dt} = mg - c_2 v^2 = mg\left(1 - \frac{c_2}{mg}v^2\right)$$

Simplifying a bit:

$$\frac{dv}{dt} = g\left(1 - \frac{v^2}{v_t^2}\right)$$

where

$$v_t = \sqrt{\frac{mg}{c_2}}$$

Now we are getting into trickier ODEs to deal w/ solving....
(a "Table of Integrals" helps!!)

$$t - t_0 = \int_{v_0}^{v} \frac{dv}{g\left(1 - \frac{v^2}{v_t^2}\right)} = \tau\left(\tanh^{-1}\frac{v}{v_t} - \tanh^{-1}\frac{v_0}{v_t}\right)$$

where

$$\tau = \frac{v_t}{g} = \sqrt{\frac{m}{c_2 g}}$$

Fowles & Cassidy

Ex. 1-D Quadratic drag (w/ gravity)

$$t - t_0 = \int_{v_0}^{v} \frac{dv}{g\left(1 - \frac{v^2}{v_t^2}\right)} = \tau\left(\tanh^{-1}\frac{v}{v_t} - \tanh^{-1}\frac{v_0}{v_t}\right)$$

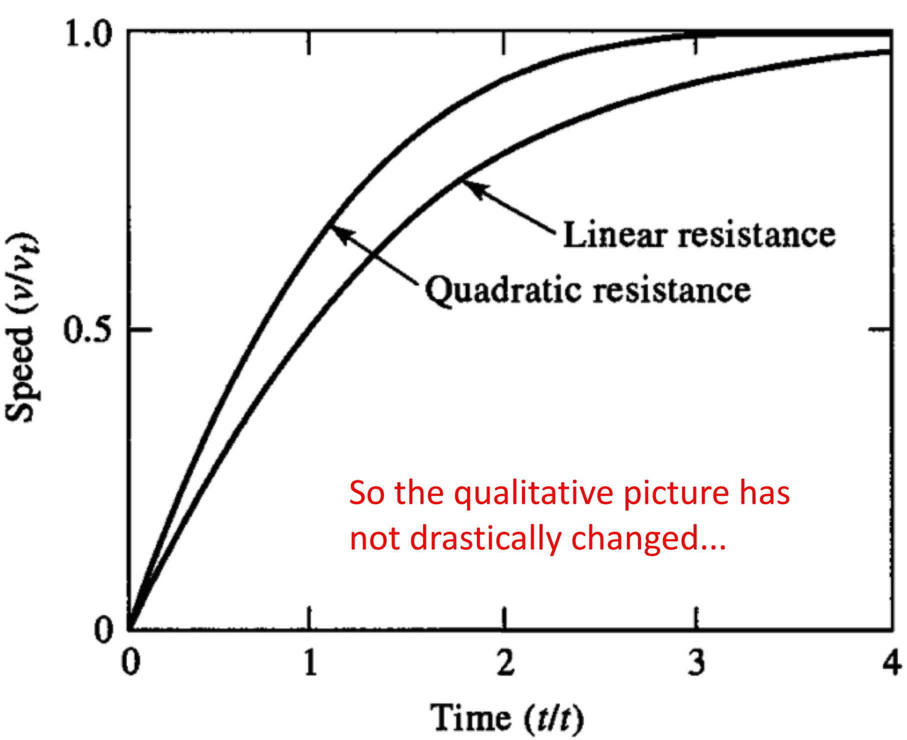
Solving for v: $$v = v_t \tanh\left(\frac{t - t_0}{\tau} - \tanh^{-1}\frac{v_0}{v_t}\right)$$

Note: A bit hairy! (a hyperbolic tangent of an inverse hyperbolic tangent!)

Now if the ball is dropped at t=0 (i.e., v_o=0):

$$v = v_t \tanh\frac{t}{\tau} = v_t\left(\frac{e^{2t/\tau} - 1}{e^{2t/\tau} + 1}\right)$$

$$v_t = \sqrt{\frac{mg}{c_2}}$$



So the qualitative picture has not drastically changed...

Here the object reaches its terminal velocity relatively quicker....

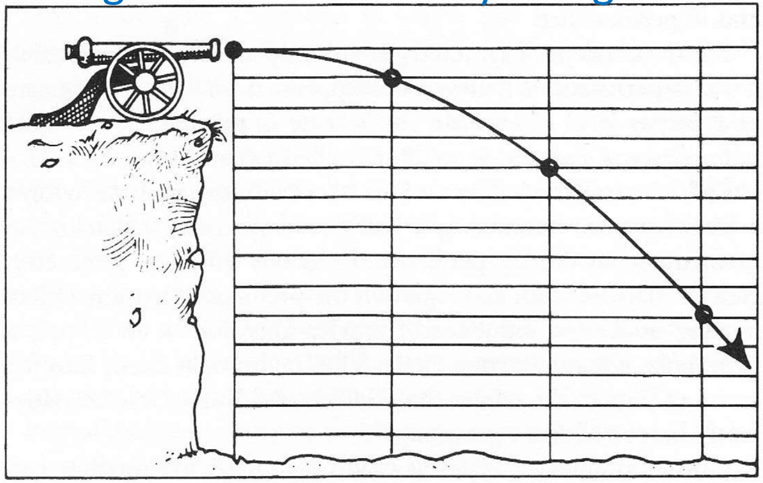
.... and a heavier object still falls faster!

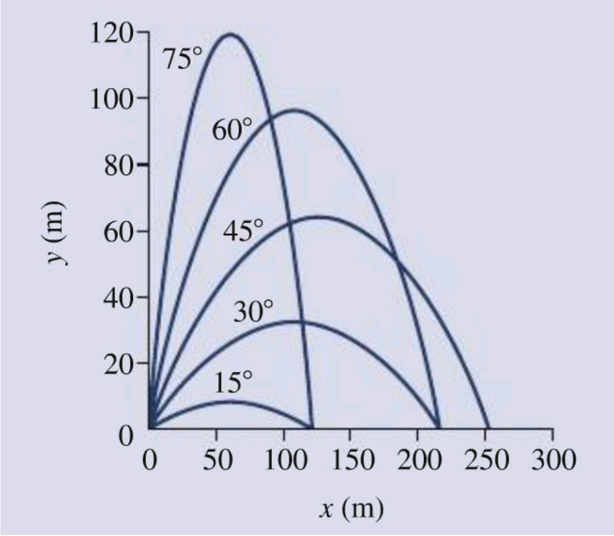
Fowles & Cassidy

<u>Looking Ahead</u>: Higher Dimensions & Projectile Motion

Niccolò Tartaglia (1499-1557)

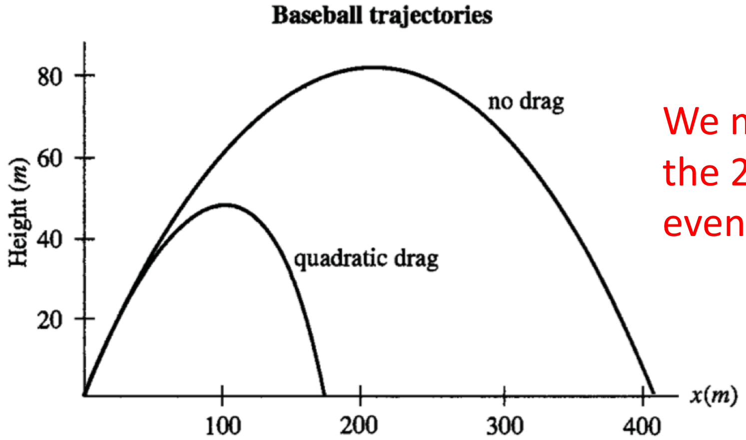
No drag case was relatively straightforward...





1-D nonlinear drag case less so....

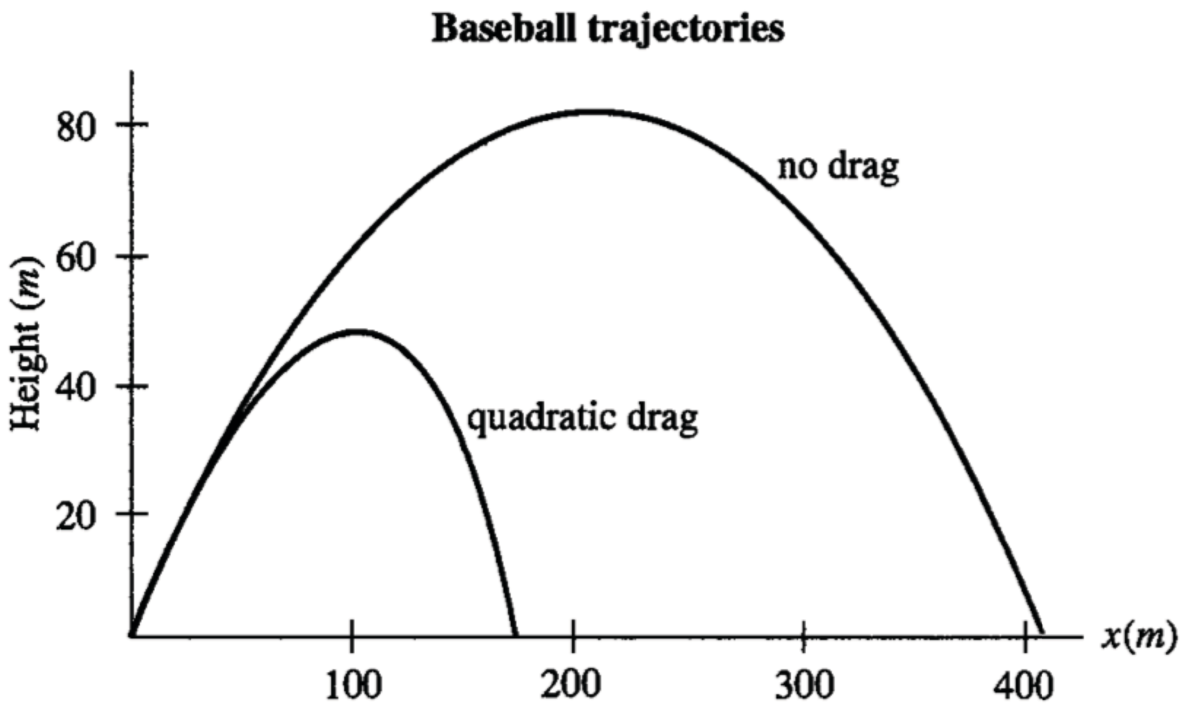
$$m\frac{dv}{dt} = mg - c_2 v^2 = mg\left(1 - \frac{c_2}{mg}v^2\right)$$



Fowles & Cassidy

We might expect the 2-D case to be even "messier"!

Projectile Motion w/ Drag: Higher Dimensions



Fowles & Cassidy

→ Before delving into this (non-conservative) problem, let's return to motion in higher dimensions by virtue of conservative forces....

Force & Work: Higher Dimensions

Recall: Work in 1-D

$$W = \int_{x_0}^{x} F(x)\,dx = -\int_{x_0}^{x} dV = -V(x) + V(x_0) = T - T_0$$

Force in 2-D
(vector form)

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

Force in 2-D
(scalar/component form in Cartesian coords)

$$F_x = m\ddot{x}$$
$$F_y = m\ddot{y}$$
$$F_z = m\ddot{z}$$

Note: In many "real world" cases, these eqns. are typically too complex to solve analytically

Let us assume mass is a constant and that $\mathbf{F}$ does not explicitly depend upon time

First take dot product w/ $\mathbf{v}$:

$$\mathbf{F} \cdot \mathbf{v} = \frac{d\mathbf{p}}{dt} \cdot \mathbf{v} = \frac{d(m\mathbf{v})}{dt} \cdot \mathbf{v}$$

Now (via chain rule):

$$d(\mathbf{v} \cdot \mathbf{v})/dt = 2\mathbf{v} \cdot \dot{\mathbf{v}}$$

Note: Here we took a derivative of a dot product

Review: Derivative of Vectors I

Starting point: A generic vector $\mathbf{A}$
(in Cartesian coords here)

$$\mathbf{A}(u) = \mathbf{i}A_x(u) + \mathbf{j}A_y(u) + \mathbf{k}A_z(u)$$

Derivative re a scalar:
(by definition)

$$\frac{d\mathbf{A}}{du} = \lim_{\Delta u \to 0} \frac{\Delta \mathbf{A}}{\Delta u} = \lim_{\Delta u \to 0} \left(\mathbf{i}\frac{\Delta A_x}{\Delta u} + \mathbf{j}\frac{\Delta A_y}{\Delta u} + \mathbf{k}\frac{\Delta A_z}{\Delta u} \right)$$

$\Delta A_x = A_x(u + \Delta u) - A_x(u)$ and so on.

Or more compactly:

$$\frac{d\mathbf{A}}{du} = \mathbf{i}\frac{dA_x}{du} + \mathbf{j}\frac{dA_y}{du} + \mathbf{k}\frac{dA_z}{du}$$

Carrying on...

$$\frac{d}{du}(\mathbf{A} + \mathbf{B}) = \frac{d\mathbf{A}}{du} + \frac{d\mathbf{B}}{du}$$

... and on

$$\frac{d(n\mathbf{A})}{du} = \frac{dn}{du}\mathbf{A} + n\frac{d\mathbf{A}}{du}$$

$$\frac{d(\mathbf{A} \cdot \mathbf{B})}{du} = \frac{d\mathbf{A}}{du} \cdot \mathbf{B} + \mathbf{A} \cdot \frac{d\mathbf{B}}{du}$$

$$\frac{d(\mathbf{A} \times \mathbf{B})}{du} = \frac{d\mathbf{A}}{du} \times \mathbf{B} + \mathbf{A} \times \frac{d\mathbf{B}}{du}$$

Fowles & Cassidy

Force & Work: Higher Dimensions

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Force in 2-D (vector form)

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Note: Here we took a derivative of a dot product

$$\mathbf{F}\cdot\mathbf{v} = \frac{d}{dt}\left(\frac{1}{2}m\mathbf{v}\cdot\mathbf{v}\right) = \frac{dT}{dt}$$

Or:

$$\mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = \frac{dT}{dt}$$

$$\therefore \int \mathbf{F}\cdot d\mathbf{r} = \int dT = T_f - T_i = \Delta T$$

Arriving back at the familiar (now in vector form) *work-energy theorem*

Fowles & Cassidy

Force & Work: Higher Dimensions

$$\therefore \int \mathbf{F} \cdot d\mathbf{r} = \int dT = T_f - T_i = \Delta T$$

Left-hand term is a *line integral*...

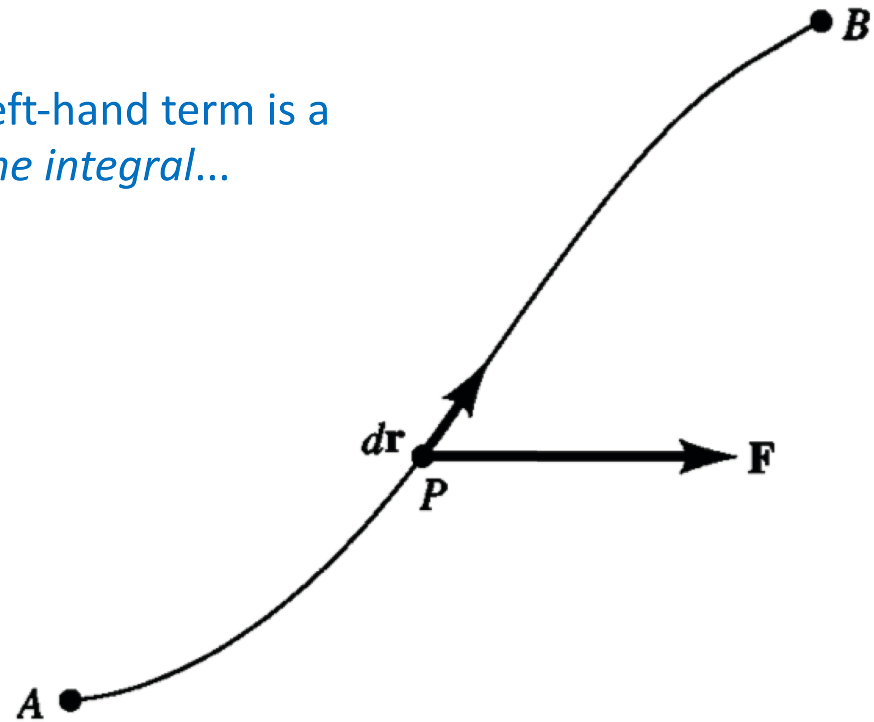


Figure 4.1.1 The work done by a force $\mathbf{F}$ is the line integral $\int_A^B \mathbf{F} \cdot d\mathbf{r}$.

"The line integral represents the work done on the particle by the force $\mathbf{F}$ as the particle moves along its trajectory from A to B. The right-hand side of the equation is the net change in the kinetic energy of the particle. $\mathbf{F}$ is the net sum of all vector forces acting on the particle; hence, the equation states that the work done on a particle by the net force acting on it, in moving from one position in space to another, is equal to the difference in the kinetic energy of the particle at those two positions."

→ Clearly, as we (eventually) move into higher dimensions, vector calculus is going to come into play!

Fowles & Cassidy

Review: 1-D Conservative Force Fields

So we needed to be able to find that potential energy function V

Recall some basics relating (conservative) forces and energy in 1-D that we previously derived:

$$F(x) = m\ddot{x}$$

Here we just needed F to depend upon x only (not v, t, etc...)

Roping in energy:

$$-\frac{dV(x)}{dx} = F(x)$$

$$F(x) = mv\frac{dv}{dx} = \frac{m}{2}\frac{d(v^2)}{dx} = \frac{dT}{dx}$$

Putting the pieces together:

$$W = \int_{x_0}^{x} F(x)\,dx = -\int_{x_0}^{x} dV = -V(x) + V(x_0) = T - T_0$$

Or more succinctly:

$$\int F_x dx = -\Delta V = V(A) - V(B)$$

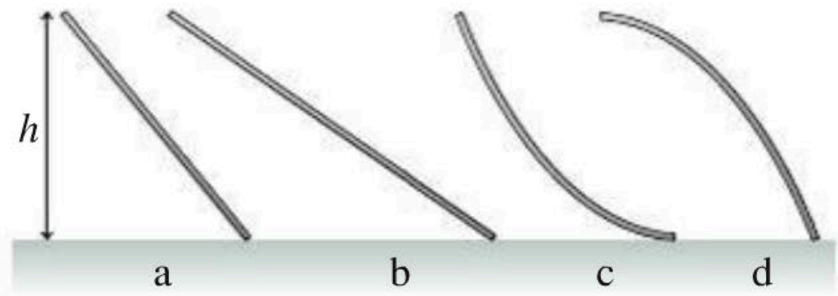
$$E_{tot} = V(A) + T(A) = V(B) + T(B) = \text{constant}$$

Put another way, for the conservative force, the "path" does not matter, only the end points (A and B)

Fowles & Cassidy

Ex

STOP TO THINK 10.2 A small child slides down the four frictionless slides a–d. Each has the same height. Rank in order, from largest to smallest, her speeds v_a to v_d at the bottom.



Knight (2013)

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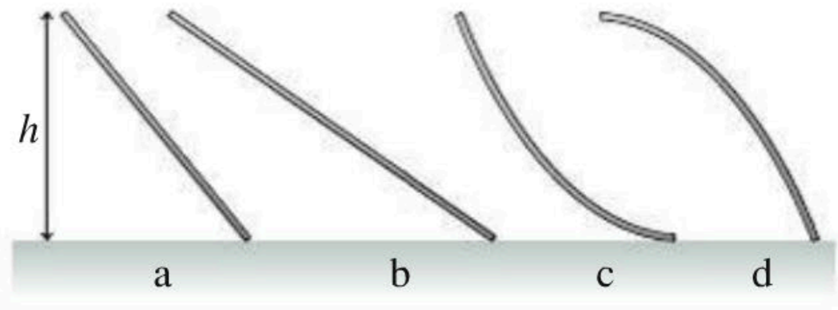

All the same

But be careful re what you are being asked!

What if we were asked to rank re how long it took to get to the bottom?

Knight (2013)

Conservative Forces → Scaling Up to Higher Dimensions

For 1-D it was sufficient to state:

We now restrict ourselves to the special case where the force F depends on position only, $F = F(x)$. (In particular, F is not dependent on time).

And indeed we did go a bit further (re higher dimensions):

A force field is called *conservative* if the force vector $\mathbf{F}$ of the field depends only on the position $\mathbf{r}$ of the particle and the work integral $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path of integration, depending only on the initial point A and the final point B, of the path.

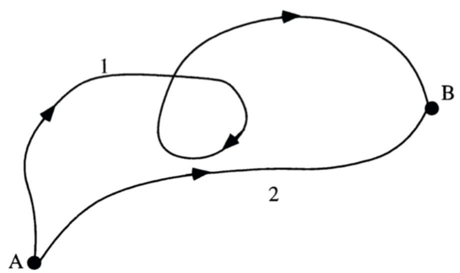

Fig. 8.2. Defining a conservative force field. The particle is moved from A to B along the path 1 or along the path 2

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0 \, .$$

→ Can we firm this up, so to have a clearer picture as to whether a given *force "field"* is in fact *conservative* or not?

Just as we constrained ourselves to 1-D earlier, let us generalize our "higher dimensional" argument to focus on 2-D for clarity...

Knudsen & Hjorth