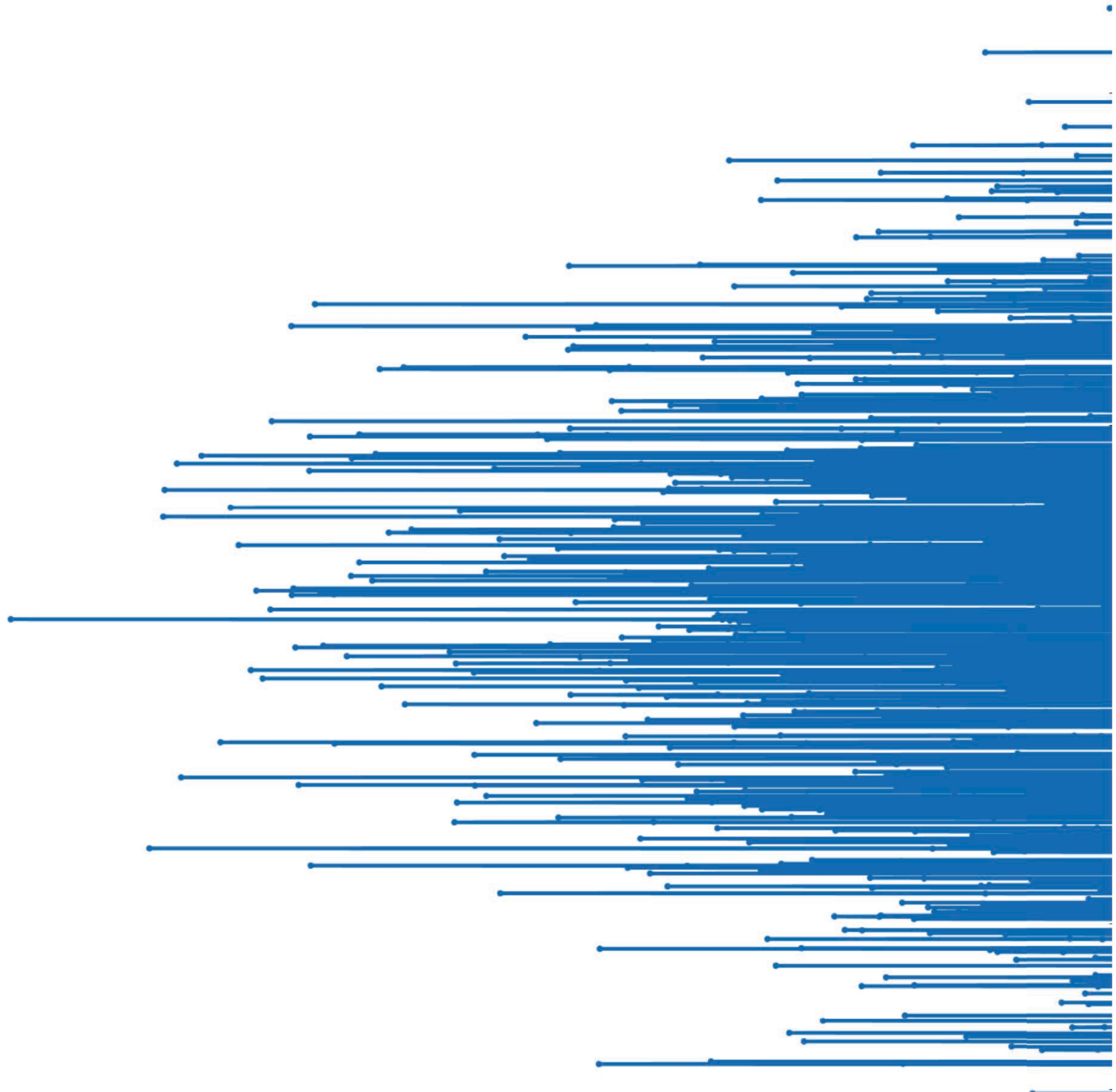
# PHYS 2010 (W20)
# Classical Mechanics



**2020.01.23**

Relevant reading:
Knudsen & Hjorth: 8.2-8.3

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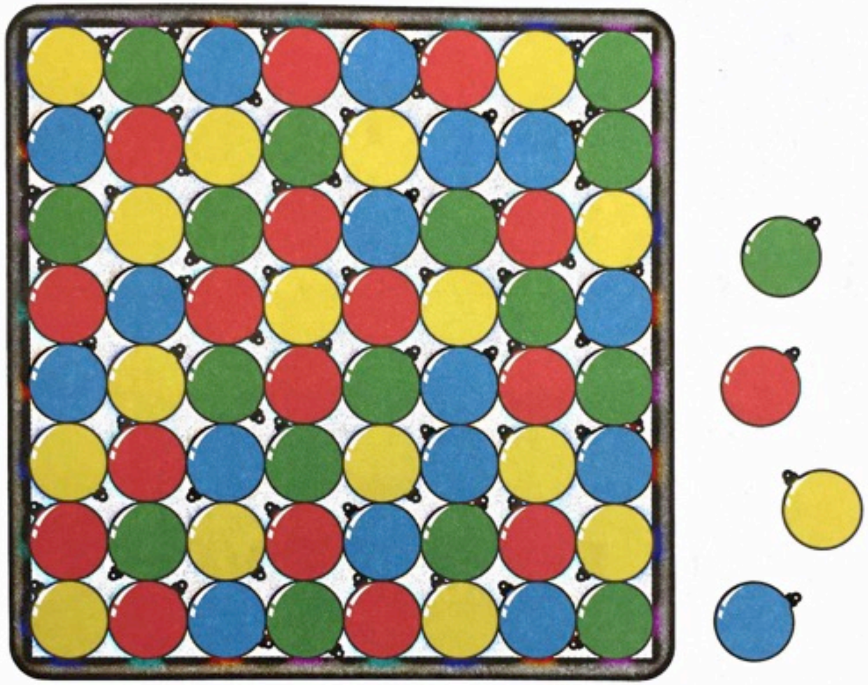
Refs:
Knudsen & Hjorth (2000), Fowles & Cassidy (2005); Hughes-Hallet et al. (2013)

# 237. Christmas Tree Ornaments

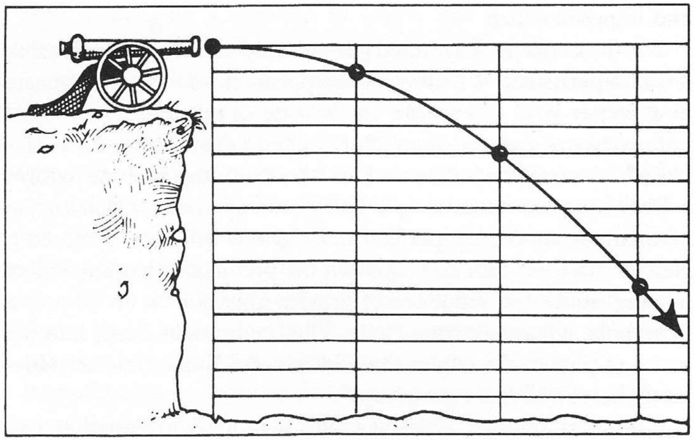


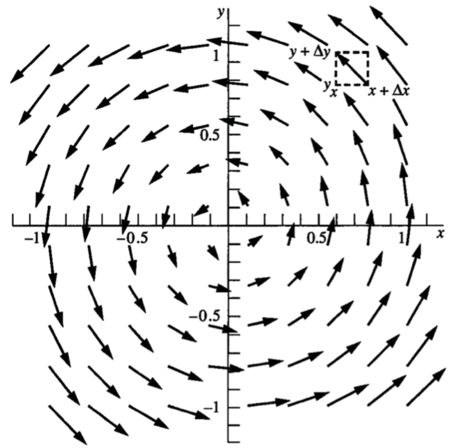
There are 64 Christmas tree ornaments packed into the box. True or false: It is possible to add 4 more and then repack all 68 into the box without the ornaments overlapping each other.

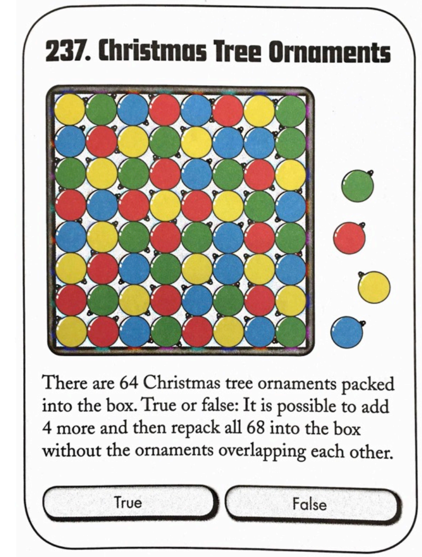
True

False

# Speaking of packing more in.....





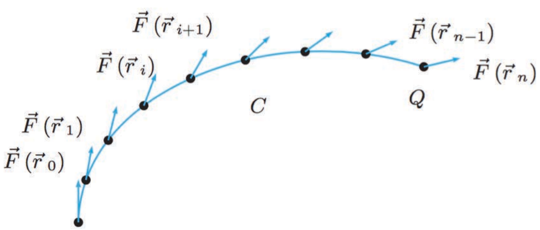




Figure 18.3: The vector field $\vec F$ evaluated at the points with position vector $\vec r_i$ on the curve $C$ oriented from $P$ to $Q$

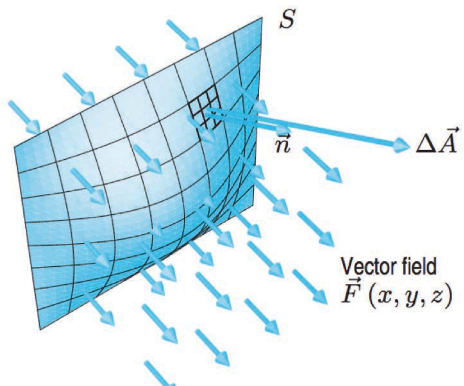



Figure 19.7: Flux of a vector field through a curved surface $S$

**Fundamental Theorem of Calculus for Line Integrals**

$$\int_C \operatorname{grad} f \cdot d\vec r = f(Q) - f(P).$$

**Stokes' Theorem**

$$\int_S \operatorname{curl} \vec F \cdot d\vec A = \int_C \vec F \cdot d\vec r.$$

**Divergence Theorem**

$$\int_W \operatorname{div} \vec F \, dV = \int_S \vec F \cdot d\vec A.$$

Hughes-Hallet et al

# Conservative Forces → Scaling Up to Higher Dimensions

For 1-D it was sufficient to state:

We now restrict ourselves to the special case where the force $F$ depends on position only, $F = F(x)$. (In particular, $F$ is not dependent on time).

And indeed we did go a bit further (re higher dimensions):

A force field is called *conservative* if the force vector $\mathbf{F}$ of the field depends only on the position $\mathbf{r}$ of the particle and the work integral $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path of integration, depending only on the initial point A and the final point B, of the path.

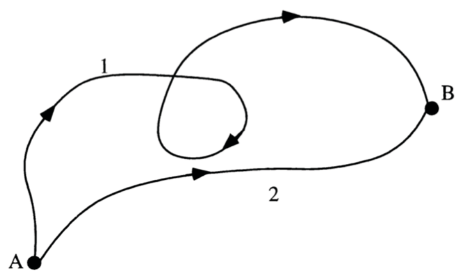



**Fig. 8.2.** Defining a conservative force field. The particle is moved from A to B along the path 1 or along the path 2

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0 .$$

→ Can we firm this up, so to have a clearer picture as to whether a given *force "field"* is in fact *conservative* or not?

(and in doing so, review some essential bits of vector calculus)

Just as we constrained ourselves to 1-D earlier, let us generalize our "higher dimensional" argument to chiefly focus on 2-D (and sometimes 3-D) for clarity...

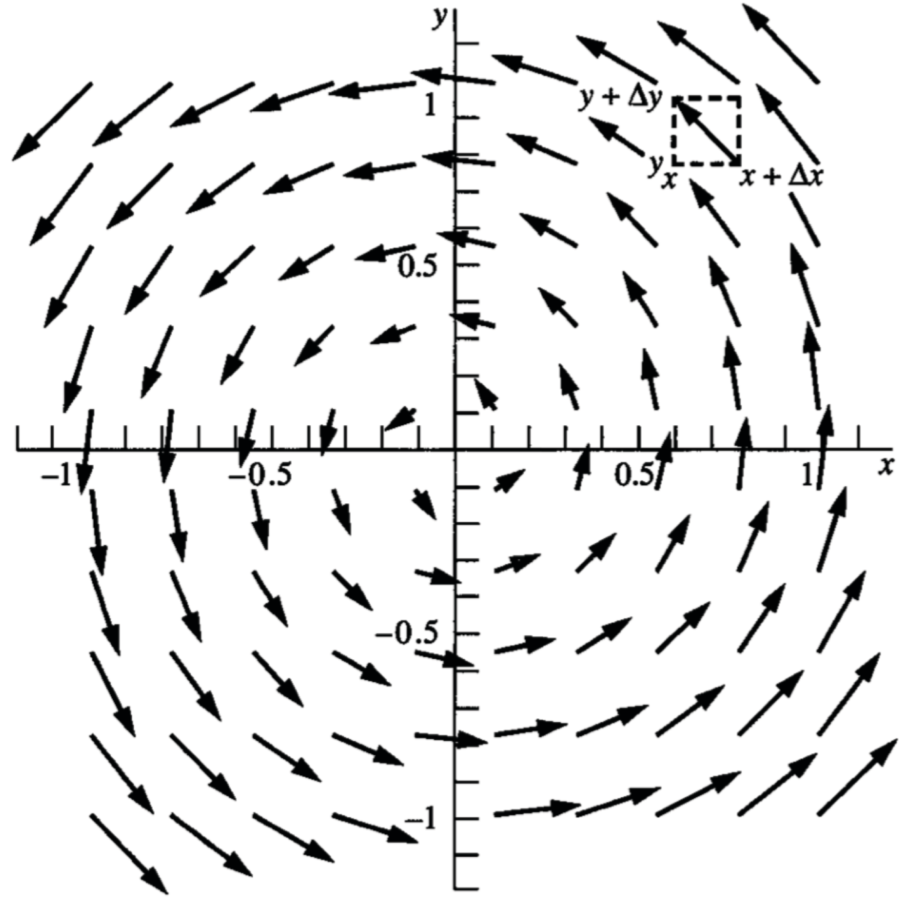
Knudsen & Hjorth

Ex. (re Conservative Force Field?)

Consider this force field
(which seemingly just depends upon "location")

$$\mathbf{F}(x, y)$$

$F_x = -by$ and $F_y = +bx$, where $b$ is some constant




Is it conservative?

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0 .$$ says no.

But can we develop a more intuitive *physical* explanation?

And how do we tie all this back to the notion of a *potential field*?

$$-\frac{dV(x)}{dx} = F(x)$$

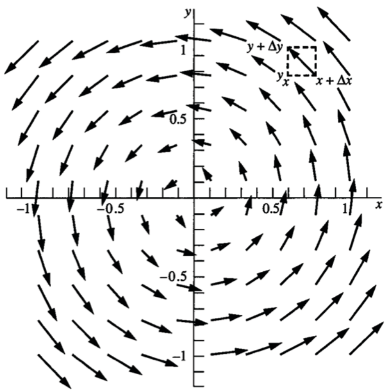
Fowles & Cassidy

<u>Conservative Force Fields</u> (WOT version)

This situation, at first glance, does not appear to be so unusual. After all, when you drop a ball in a gravitational force field, it falls and gains kinetic energy, with an accompanying loss of an equal amount of potential energy. The question here is, can we even define a potential energy function for this circulating particle such that it would lose an amount of "potential energy" equal to the kinetic energy it gained, thus preserving its overall energy, as it travels from one point to another? That is not the case here. If we were to calculate the work done on this particle in tracing out some path that came back on itself (such as the rectangular path indicated by the dashed line in Figure 4.1.2), we would obtain a nonzero result! In traversing such a loop over and over again, the particle would continue to gain kinetic energy equal to the nonzero value of work done per loop. But if the particle could be assigned a potential energy dependent only upon its $(x, y)$ position, then its change in potential energy upon traversing the closed loop would be zero. It should be clear that there is no way in which we could assign a unique value of potential energy for this particle at any particular point on the $xy$ plane. Any value assigned would depend on the previous history of the particle. For example, how many loops has the particle already made before arriving at its current position?

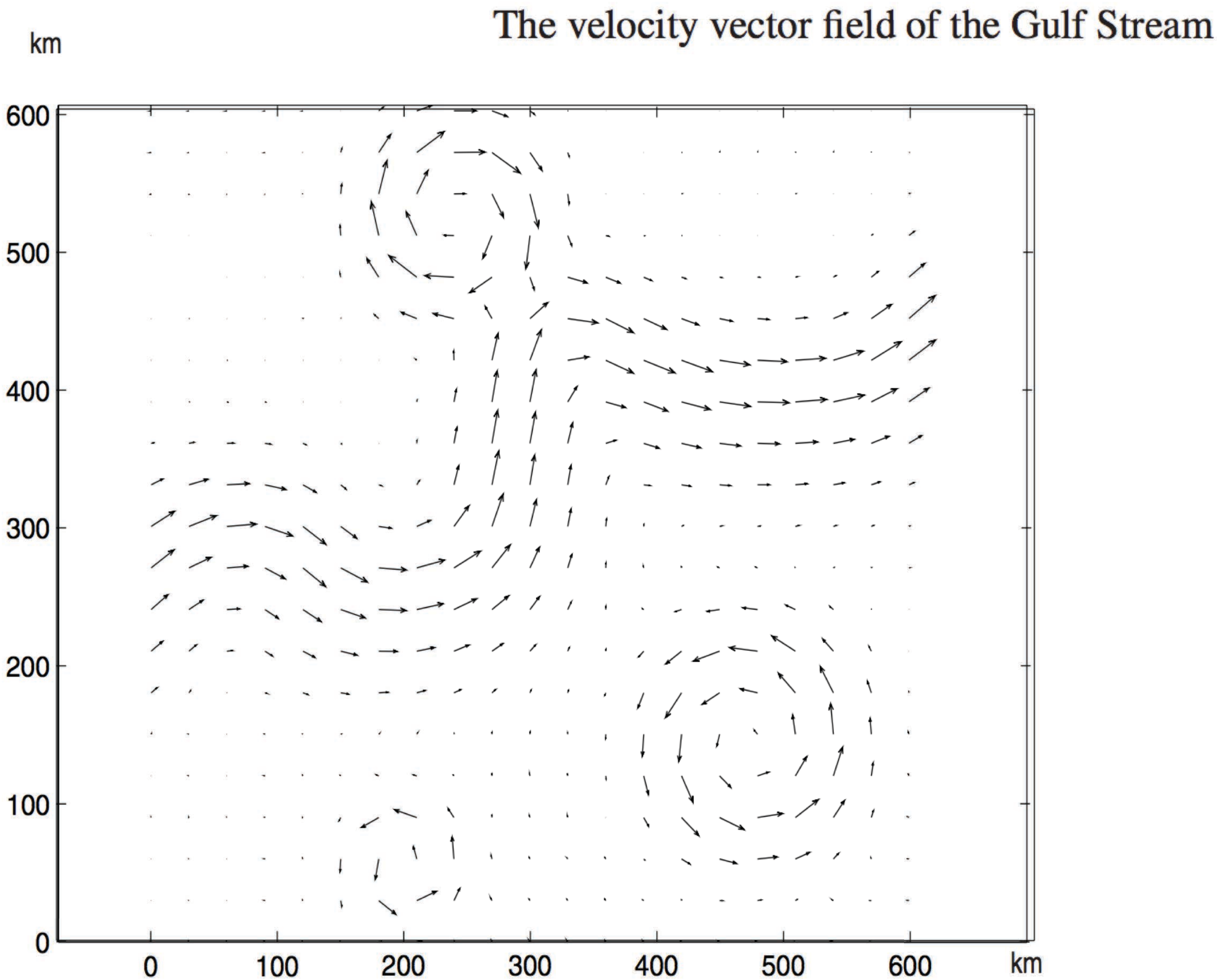
→ To get a proper grasp on these pieces, we are going to need to weave some vector calc pieces (e.g., Stoke's Theorem) into the narrative...

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0 \, .$$




Fowles & Cassidy

Review: Vector Fields

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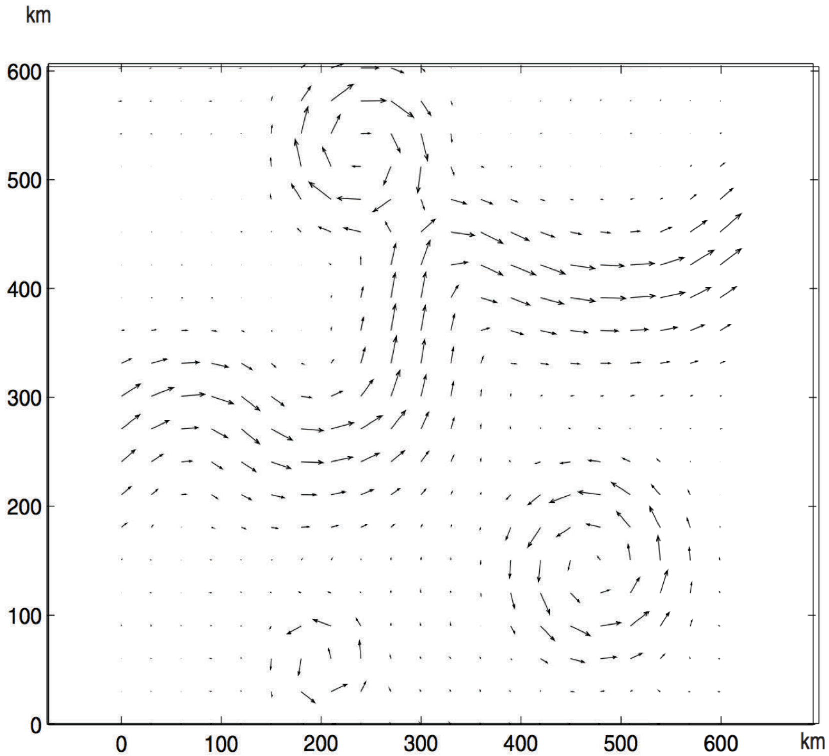
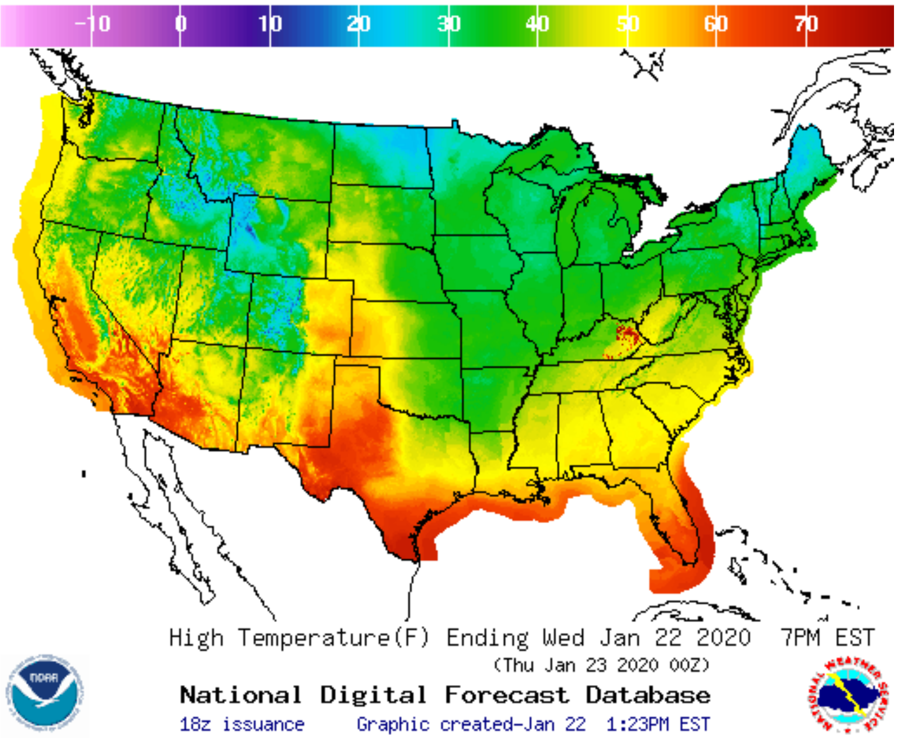
# Review: Vector Fields vs Scalar Fields

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Review: Vector Fields

The gravitational field of the earth

A **vector field** in 2-space is a function $\vec{F}(x, y)$ whose value at a point $(x, y)$ is a 2-dimensional vector. Similarly, a vector field in 3-space is a function $\vec{F}(x, y, z)$ whose values are 3-dimensional vectors.

Hughes-Hallet et al

<u>Ex.</u> Vector Fields

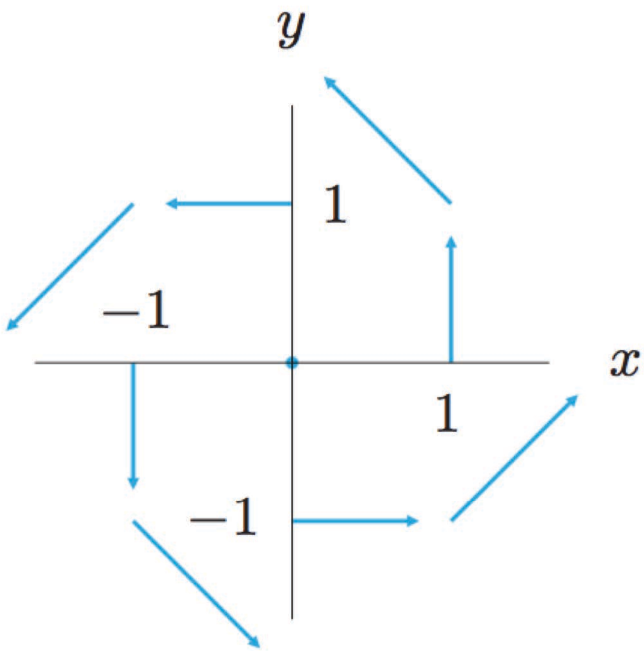
Consider:

$$\vec{F}(x, y) = -y\vec{i} + x\vec{j}$$

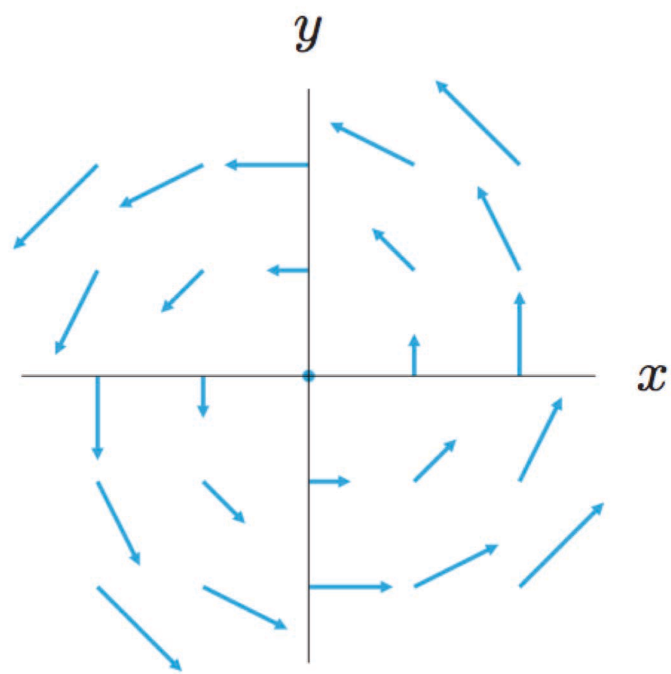
Could express as a table of #s/vectors....

| $x$ \ $y$ | $-1$ | $0$ | $1$ |
|---|---|---|---|
| $-1$ | $\vec{i} - \vec{j}$ | $-\vec{j}$ | $-\vec{i} - \vec{j}$ |
| $0$ | $\vec{i}$ | $\vec{0}$ | $-\vec{i}$ |
| $1$ | $\vec{i} + \vec{j}$ | $\vec{j}$ | $-\vec{i} + \vec{j}$ |

... or make a rough plot....




... or as a better plot!

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Ex. Vector Fields

Formula for this (3-D) field?

Newton's Law of Gravitation (magnitude)

$$\|\vec{F}(\vec{r})\| = \frac{GMm}{\|\vec{r}\|^2},$$

Vector form:

$$\vec{F}(\vec{r}) = \frac{GMm}{\|\vec{r}\|^2}\left(-\frac{\vec{r}}{\|\vec{r}\|}\right) = \frac{-GMm\vec{r}}{\|\vec{r}\|^3}$$

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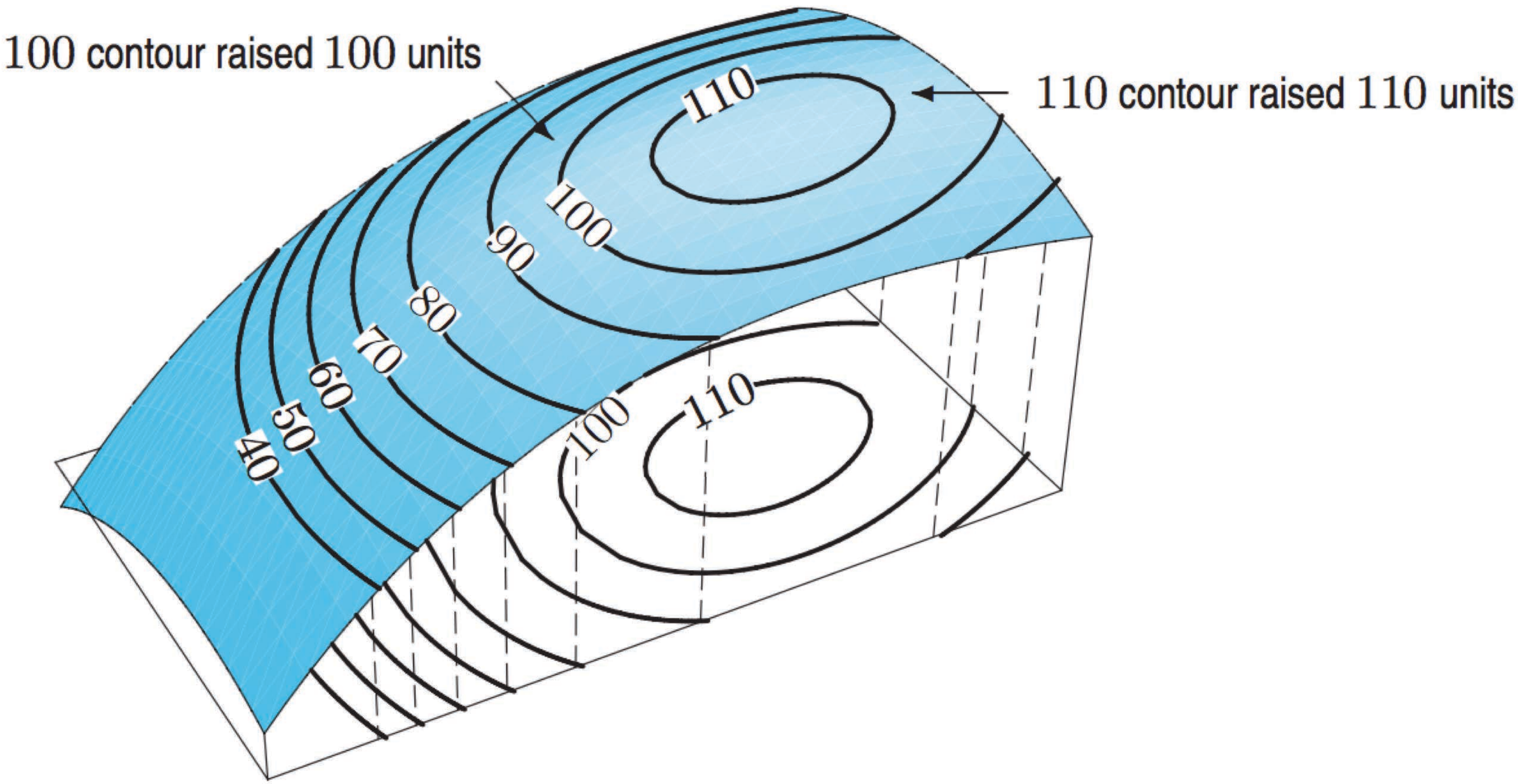
<u>Aside</u>: Contour Maps

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Aside: Contour Maps

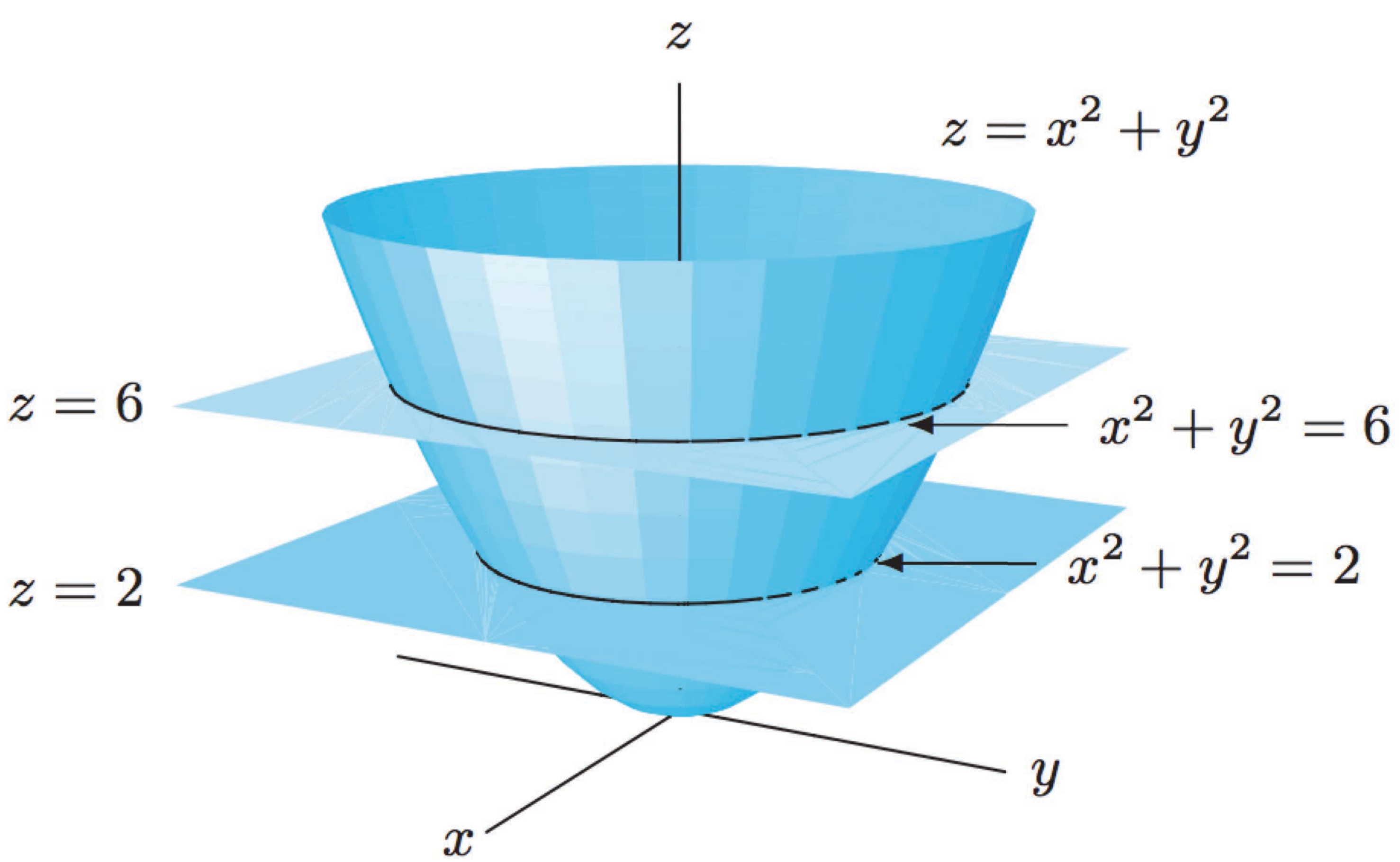


**Figure 12.40:** The graph of $f(x, y) = x^2 + y^2$

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## Review: Gradient Vector Fields

Recall: (re derivatives in 1-D, or of vectors re a scalar)

$$\frac{d\mathbf{A}}{du} = \mathbf{i}\frac{dA_x}{du} + \mathbf{j}\frac{dA_y}{du} + \mathbf{k}\frac{dA_z}{du}$$

$$-\frac{dV(x)}{dx} = F(x)$$

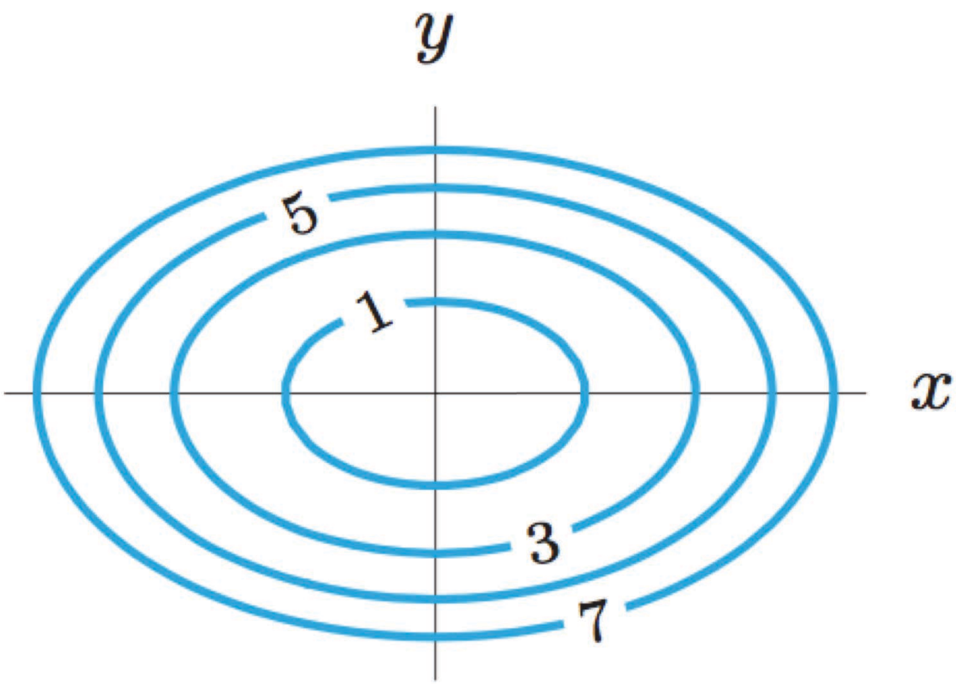


**Figure 17.26:** The contour map of $f(x, y) = x^2 + 2y^2$

Can we determine the ***field*** that describes the "gradient" of this contour map?

→ That is, we want a derivative of a vector field....

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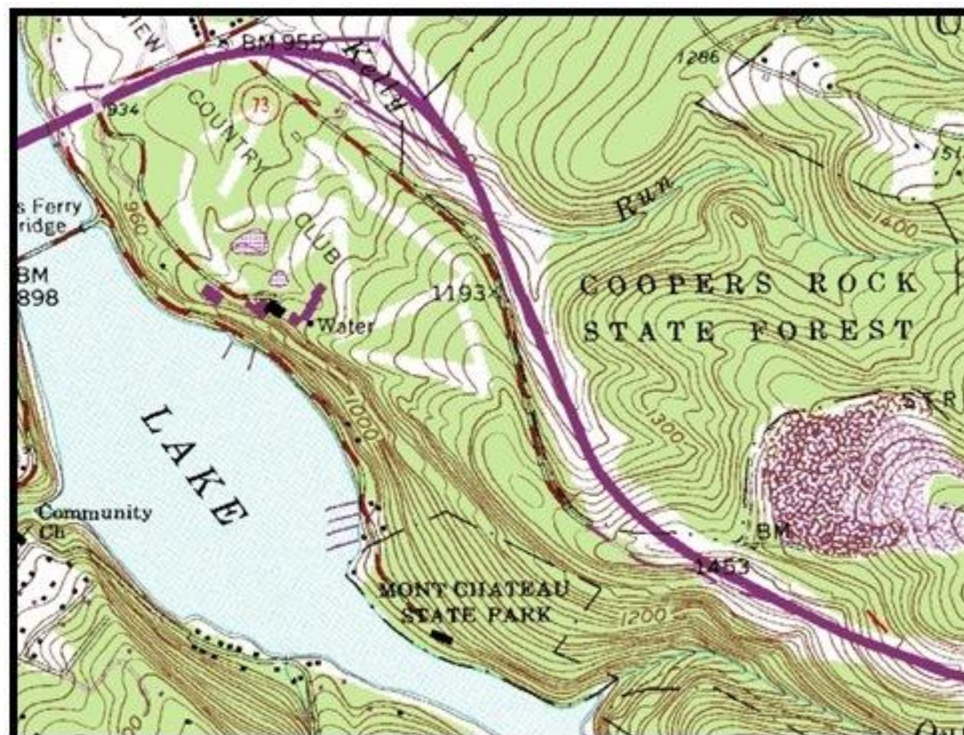
Intuitive aside....

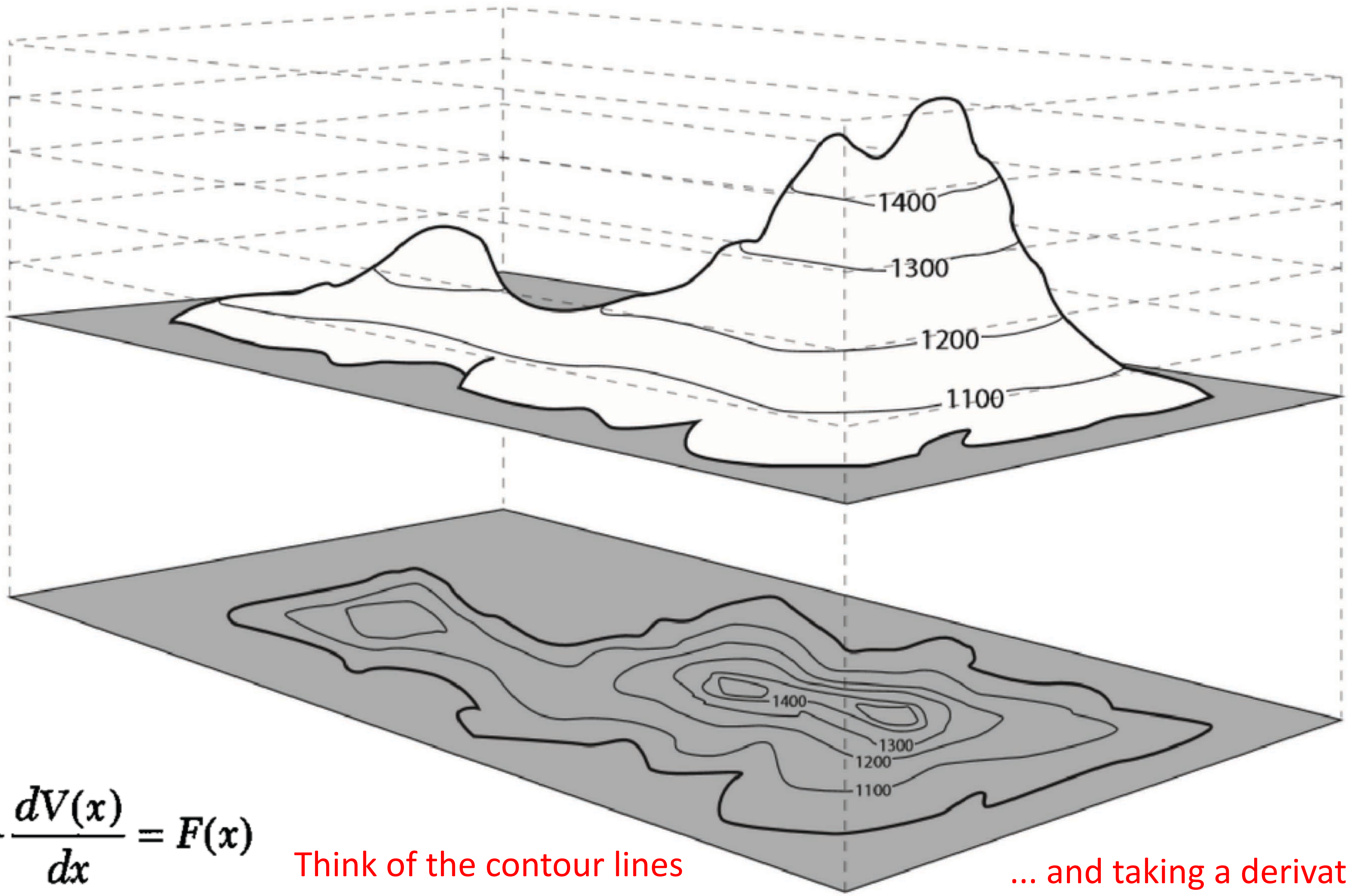

http://weirdrussia.com/2016/01/28/the-tuva-ethnic-group-a-secret-culture-behind-siberian-forests/

Aside: Topographic Maps


https://mapscaping.com/blogs/geo-candy/explaining-topographic-maps

# Aside: Topographic Maps



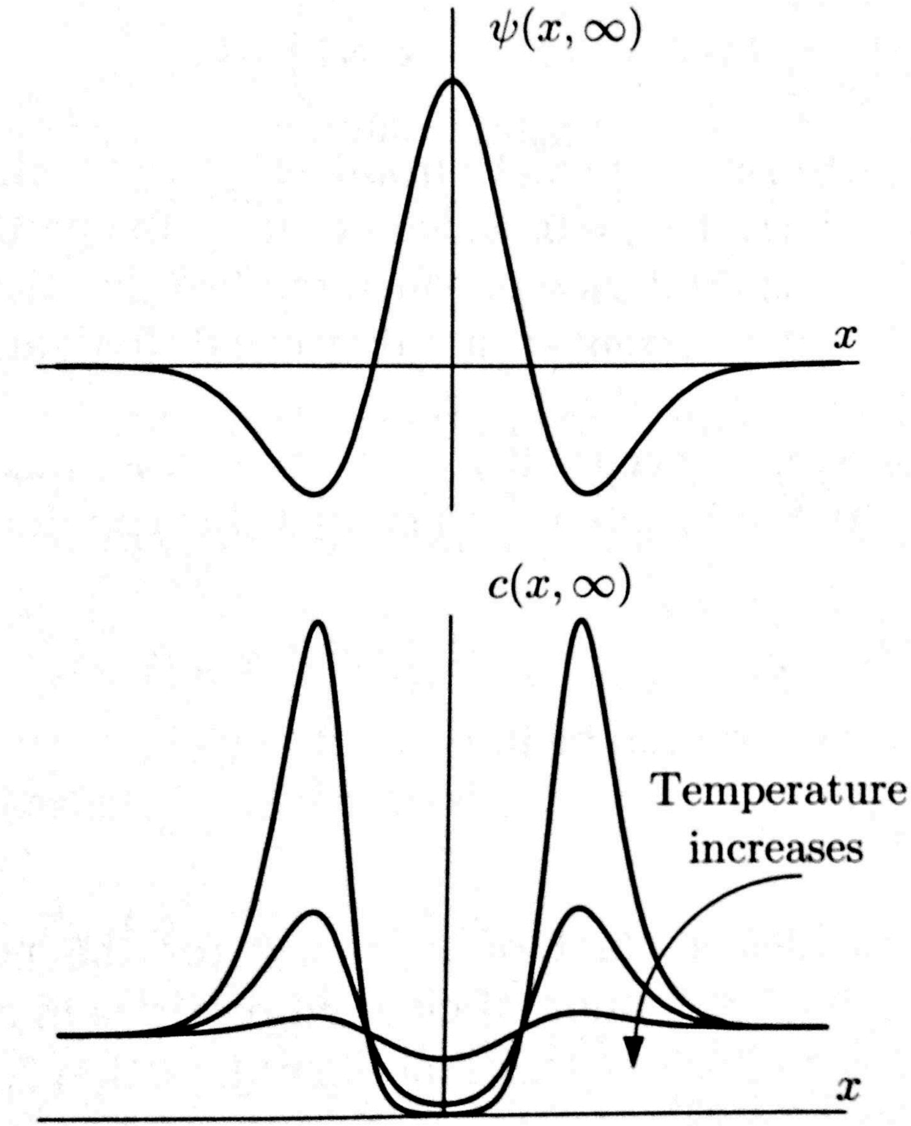

$$-\frac{dV(x)}{dx} = F(x)$$

Think of the contour lines as mapping out $V$....

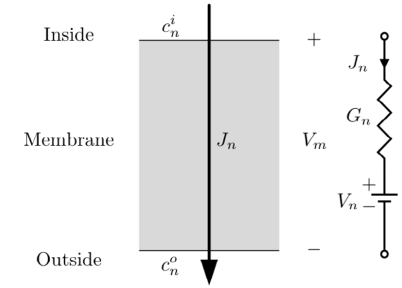
... and taking a derivative of such would yield $F$

https://mapscaping.com/blogs/geo-candy/explaining-topographic-maps

# Aside: Electrodiffusive Analog




**Model of Steady-State Electrodiffusion through Membranes**




Nernst Equilibrium Potential $V_n = \frac{RT}{z_n F} \ln \frac{c_n^o}{c_n^i}$

Electrical Conductivity $G_n = \frac{1}{\int_o^d \frac{dx}{u_n z_n^2 F^2 c_n(x)}} \geq 0$

Analog to gravitational potential energy

(no negative concentrations!)

$$\frac{\partial^2 \psi(x,t)}{\partial x^2} = -\frac{1}{\epsilon} \sum_n z_n F c_n(x,t)$$

**Figure 7.6** The spatial distribution of electric potential and ion concentration at electrodiffusive equilibrium for different temperatures.

Weiss (1996)

Looking *Downstream*: Force Fields

"The term force field simply means that if a small test particle were to be placed at any point $(x,y,)$ on the $xy$-plane, it would experience a force $\mathbf{F}$ "

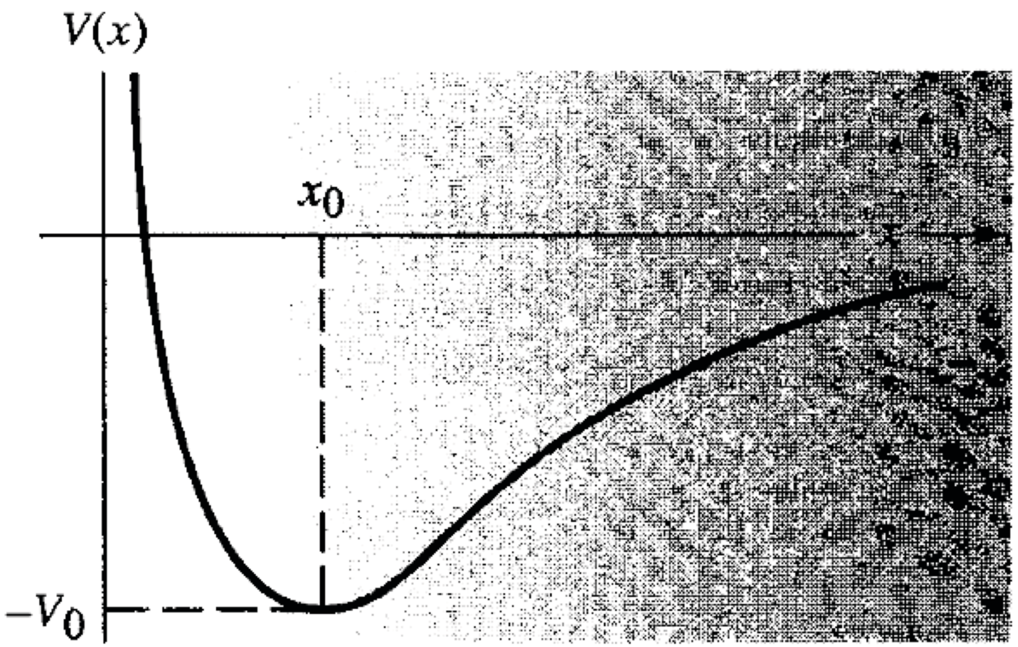
[1] A test particle is one whose mass is small enough that its presence does not alter its environment. Conceptually, we might imagine it placed at some point in space to serve as a "test probe" for the suspected presence of forces. The forces are "sensed" by observing any resultant acceleration of the test particle. We further imagine that its presence does not disturb the sources of those forces.

Recall: **1-D** "Morse Function"

$$V(x) = V_0\left[1 - e^{-(x-x_0)/\delta}\right]^2 - V_0$$

$$F(x) = -\frac{dV(x)}{dx} = -\frac{2V_0}{\delta^2}(x - x_0)$$

→ So the connection between force and potential fields in kinda intuitive....




Fowles & Cassidy

Determining Gradient Fields.....


The map here though is a 2-D vector field....

$$-\frac{dV(x)}{dx} = F(x)$$

So we need a vectorized version of this "gradient"....

.... graphically this is straightforward!

Ex: Gradient Vector Fields

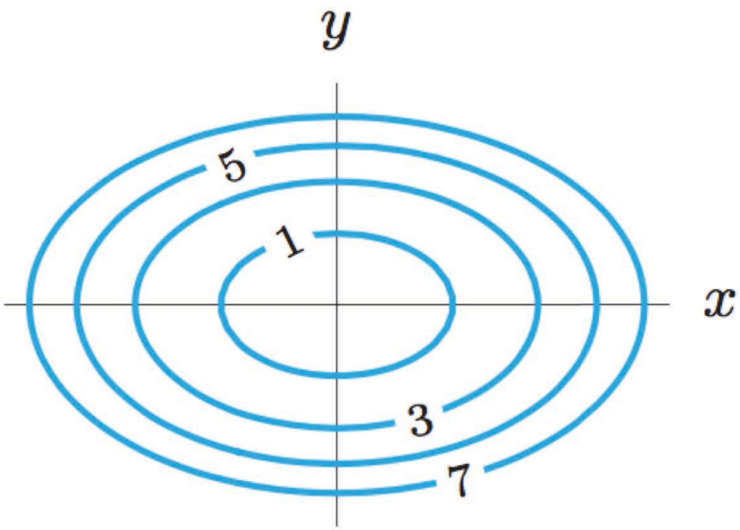



**Figure 17.26:** The contour map of $f(x, y) = x^2 + 2y^2$

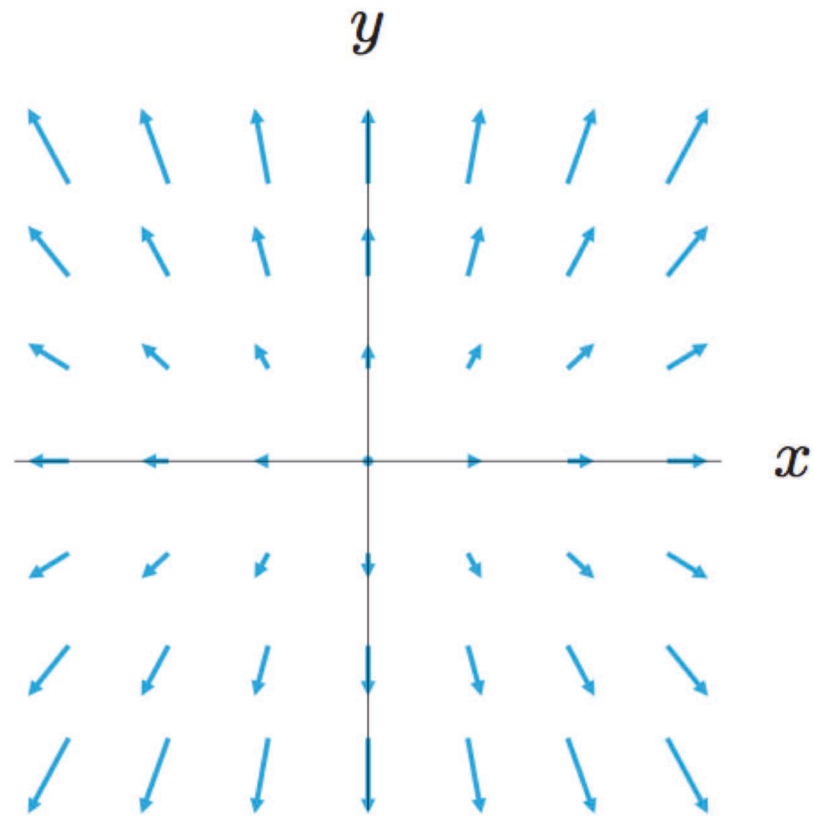



**Figure 17.29:** grad $f$

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## Review: Gradient Vectors

$$\text{grad}\, f = \nabla f.$$

In 2-D: $\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j}$

In 3-D: $\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$

**The Gradient Vector** of a differentiable function $f$ at the point $(a, b)$ is

$$\text{grad}\, f(a,b) = f_x(a,b)\vec{i} + f_y(a,b)\vec{j}$$

### Geometric Properties of the Gradient Vector in the Plane

If $f$ is a differentiable function at the point $(a, b)$ and $\text{grad}\, f(a,b) \neq \vec{0}$, then:

- The direction of $\text{grad}\, f(a,b)$ is
  - Perpendicular[1] to the contour of $f$ through $(a, b)$;
  - In the direction of the maximum rate of increase of $f$.
- The magnitude of the gradient vector, $\|\text{grad}\, f\|$, is
  - The maximum rate of change of $f$ at that point;
  - Large when the contours are close together and small when they are far apart.

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## Review: Derivative of Vectors II

Derivative of a vector re a scalar:

$$\frac{d\mathbf{A}}{du} = \mathbf{i}\frac{dA_x}{du} + \mathbf{j}\frac{dA_y}{du} + \mathbf{k}\frac{dA_z}{du}$$

**Types of matrix derivative**

| Types | Scalar | Vector | Matrix |
|---|---|---|---|
| Scalar | $\frac{\partial y}{\partial x}$ | $\frac{\partial \mathbf{y}}{\partial x}$ | $\frac{\partial \mathbf{Y}}{\partial x}$ |
| Vector | $\frac{\partial y}{\partial \mathbf{x}}$ | $\frac{\partial \mathbf{y}}{\partial \mathbf{x}}$ | |
| Matrix | $\frac{\partial y}{\partial \mathbf{X}}$ | | |

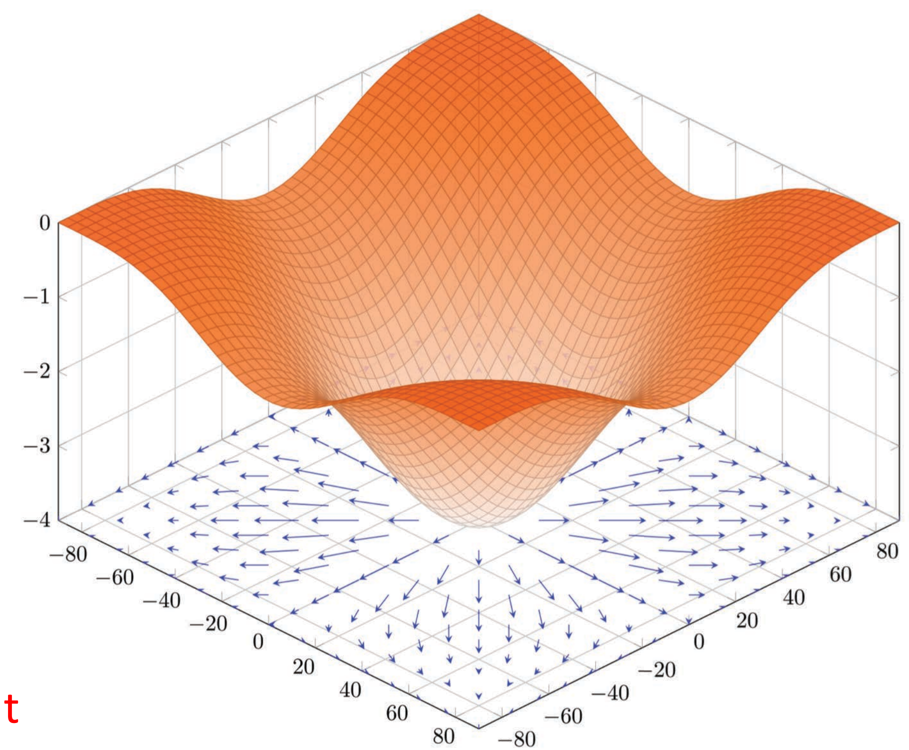
Thank you wikipedia! (re *matrix calculus*)

Gradient of a scalar field:

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Ex.

The gradient of the function $f(x,y) = -(\cos^2 x + \cos^2 y)^2$ depicted as a projected vector field on the bottom plane.




→ While now we have another useful tool in our toolbox to characterize *change*....

... the question still remains about the derivative of a vector field

Ex: Gradient Vector Fields

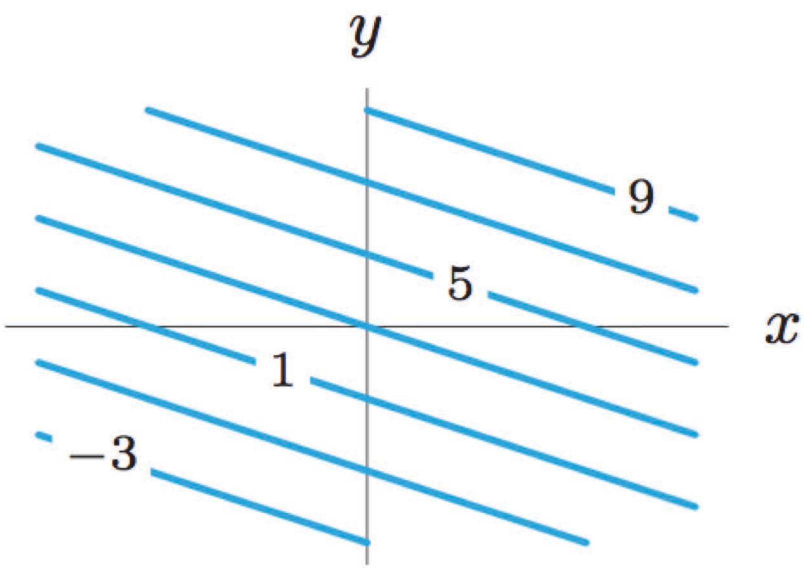



**Figure 17.28:** The contour map of $h(x, y) = x + 2y + 3$

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

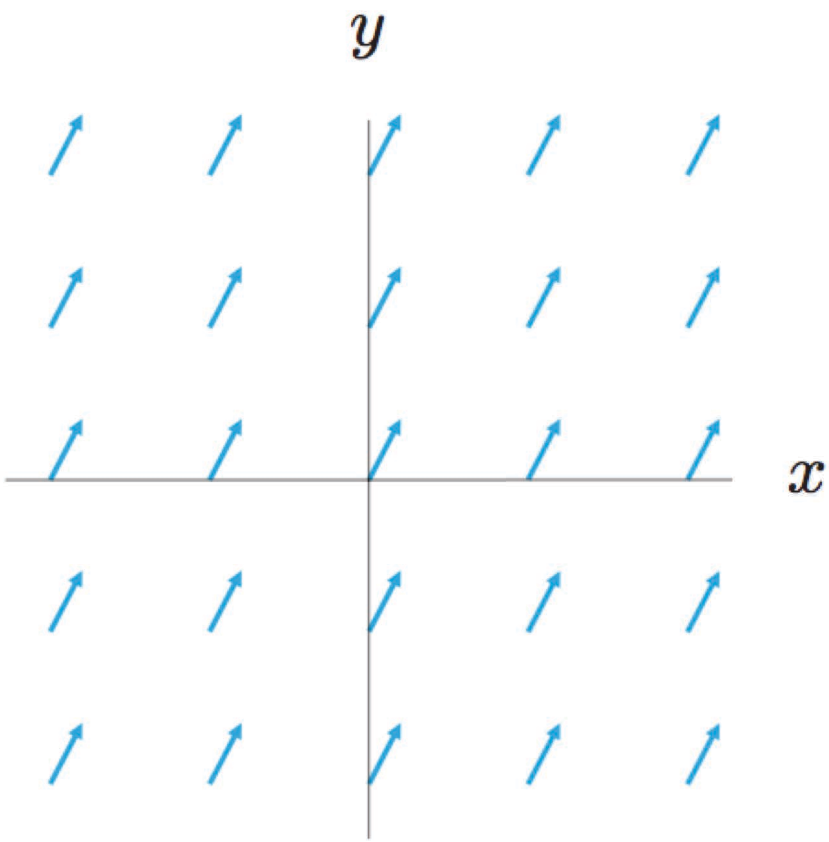



**Figure 17.31:** grad $h$

Note: While *grad h* is a vector field, *h* itself is a scalar field

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Ex: Gradient Vector Fields

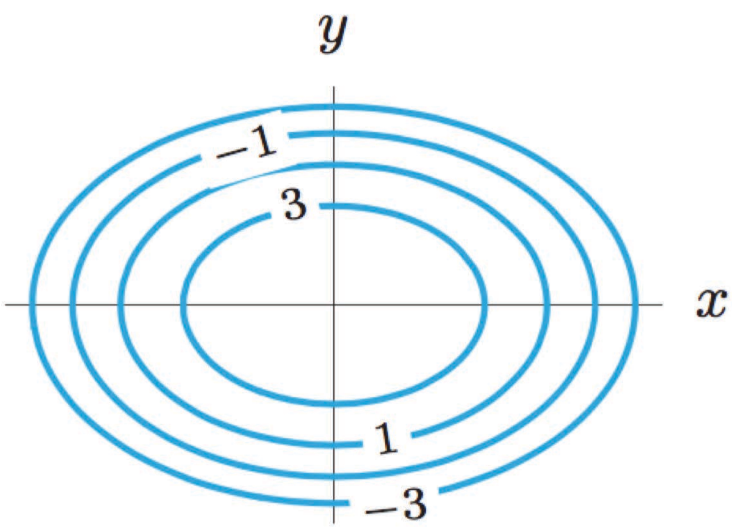


**Figure 17.27:** The contour map of $g(x, y) = 5 - x^2 - 2y^2$

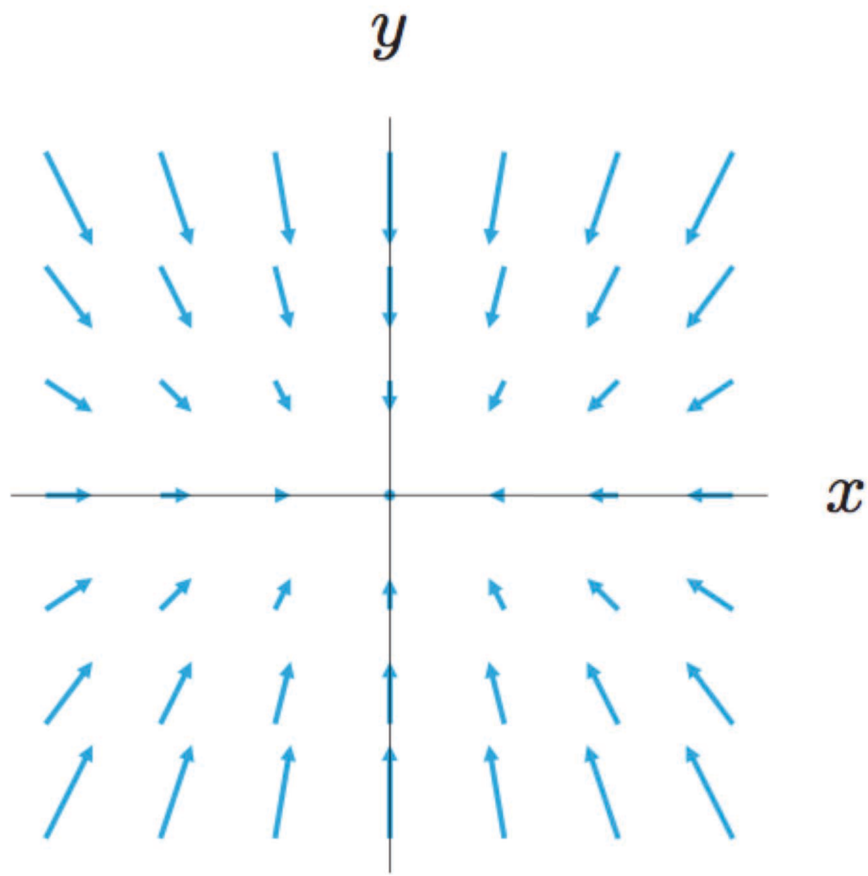


**Figure 17.30:** grad $g$

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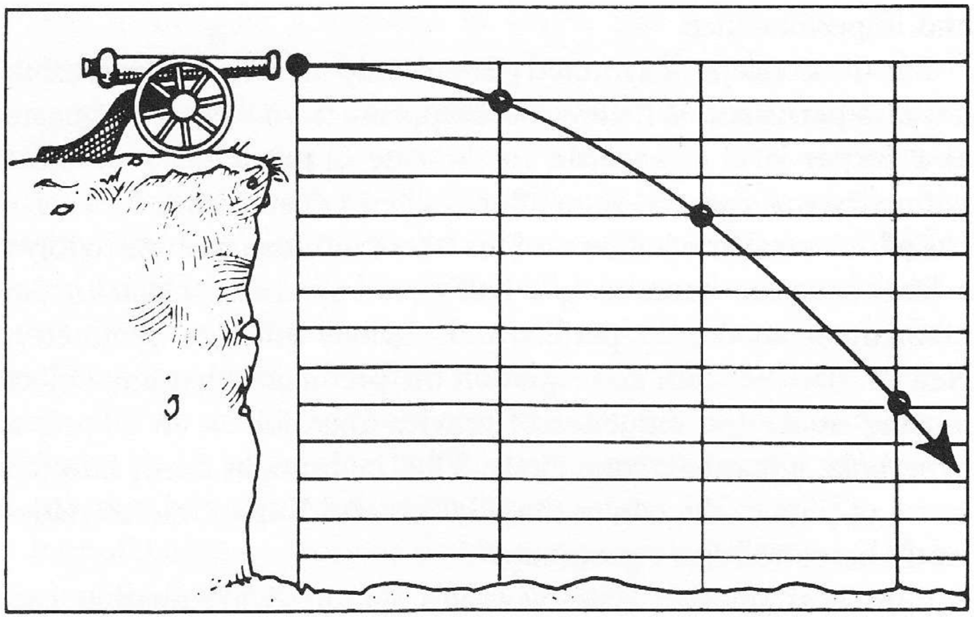
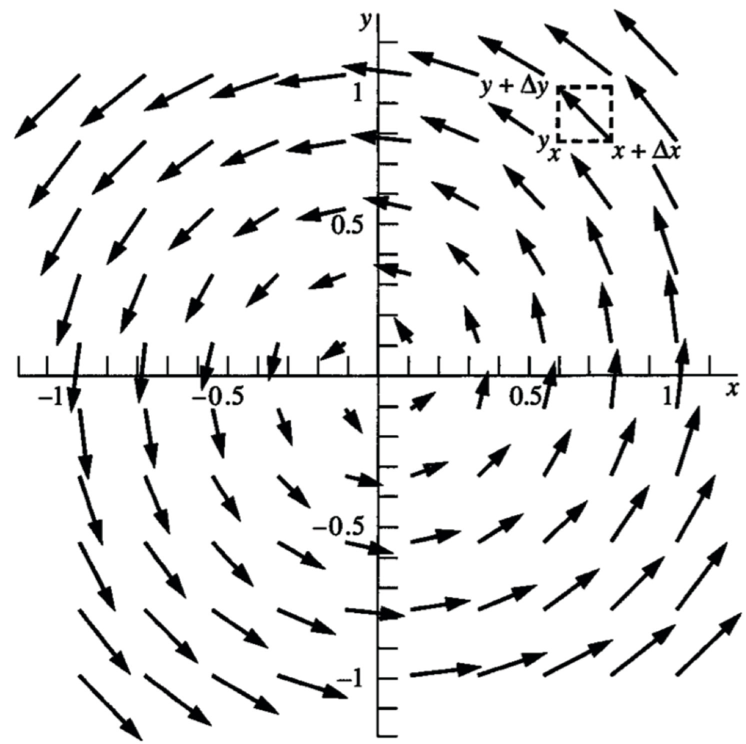
# Reminder!



von Baeyer




Fowles & Cassidy

Review: Line Integrals

$$\therefore \int \mathbf{F}\cdot d\mathbf{r} = \int dT = T_f - T_i = \Delta T$$

Recall our stated stipulation:

(re a conservative field)

$$\oint \mathbf{F}\cdot \mathrm{d}\mathbf{r} = 0\,.$$

→ To further make sense of this, let us further review some pieces so to ultimately tie back to gradient vector fields...

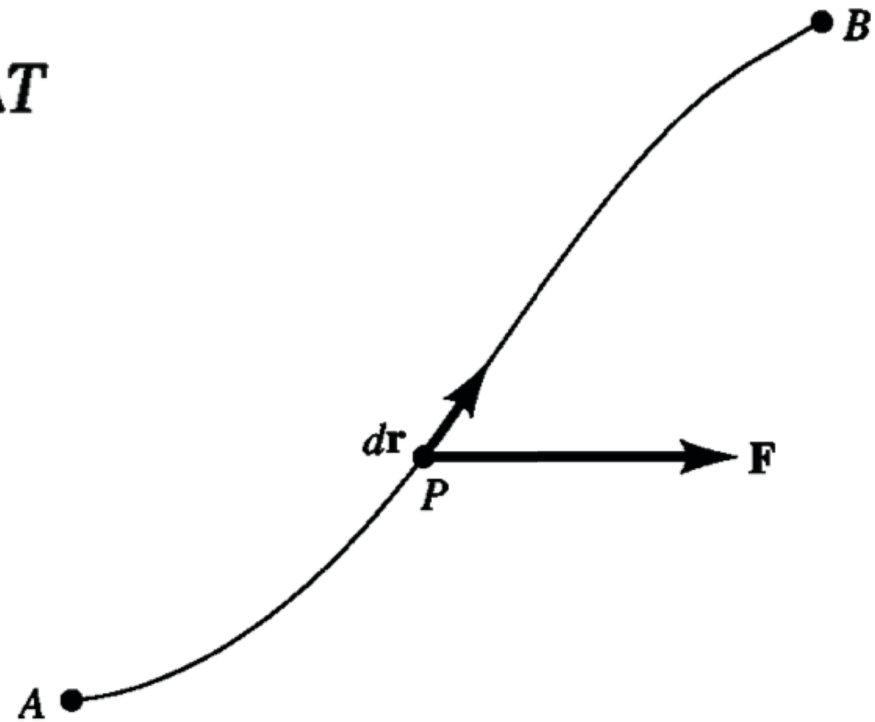
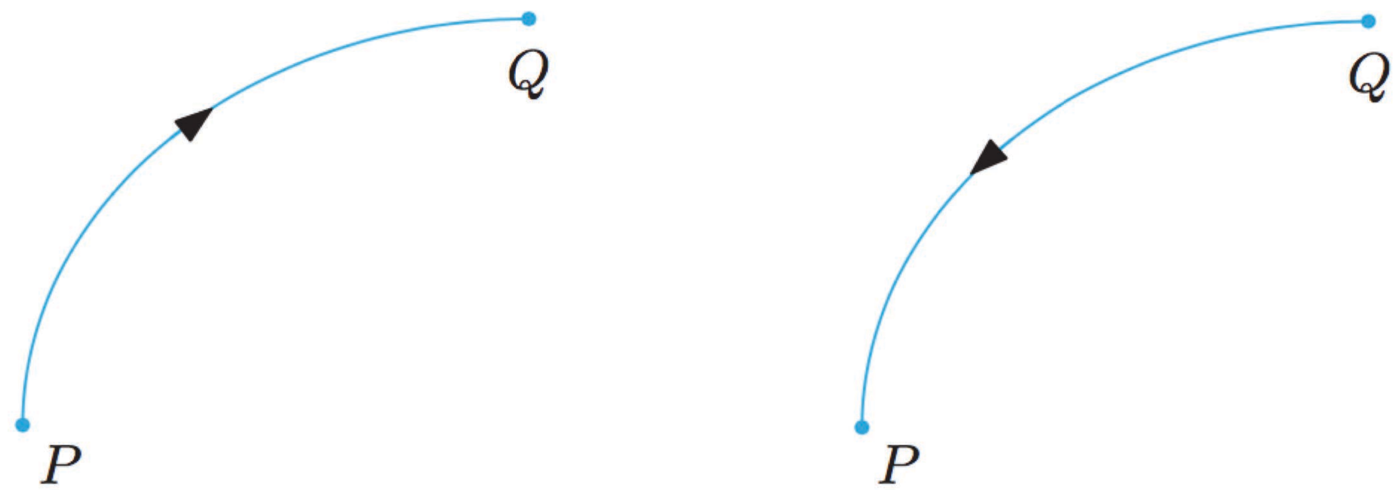



**Figure 4.1.1** The work done by a force $\mathbf{F}$ is the line integral $\int_A^B \mathbf{F}\cdot d\mathbf{r}$.

Fowles & Cassidy

# Review: Line Integrals

A curve is said to be **oriented** if we have chosen a direction of travel on it.

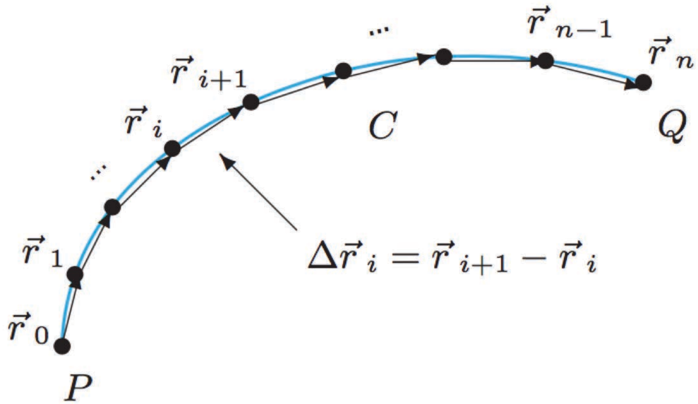



A curve with two different orientations represented by arrowheads

Hughes-Hallet et al

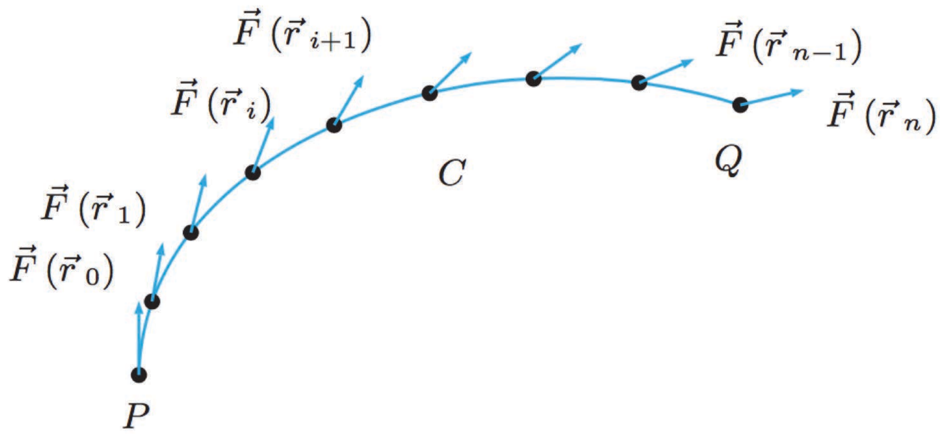
<u>Review</u>: Line Integrals

So we have a (vector) curve ($C$ )...




The curve $C$, oriented from $P$ to $Q$, approximated by straight line segments represented by displacement vectors $\Delta\vec{r}_i = \vec{r}_{i+1} - \vec{r}_i$

... and a vector field ($F$)




The vector field $\vec{F}$ evaluated at the points with position vector $\vec{r}_i$ on the curve $C$ oriented from $P$ to $Q$

Now add up the combo of the two...

$$\sum_{i=0}^{n-1} \vec{F}(\vec{r}_i) \cdot \Delta\vec{r}_i.$$

Hughes-Hallet et al

# Review: Line Integrals

Now add up the combo of the two...

$$\sum_{i=0}^{n-1} \vec{F}(\vec{r}_i) \cdot \Delta\vec{r}_i.$$

... and take the limit

The **line integral** of a vector field $\vec{F}$ along an oriented curve $C$ is

$$\int_C \vec{F} \cdot d\vec{r} = \lim_{\|\Delta\vec{r}_i\| \to 0} \sum_{i=0}^{n-1} \vec{F}(\vec{r}_i) \cdot \Delta\vec{r}_i.$$

A useful interpretation:

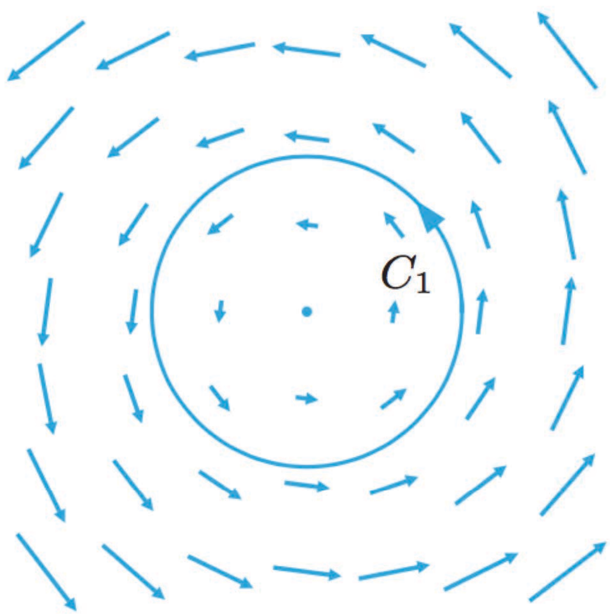
In general, the line integral of a vector field $\vec{F}$ along a curve $C$ measures the extent to which $C$ is going with $\vec{F}$ or against it.

Hughes-Hallet et al

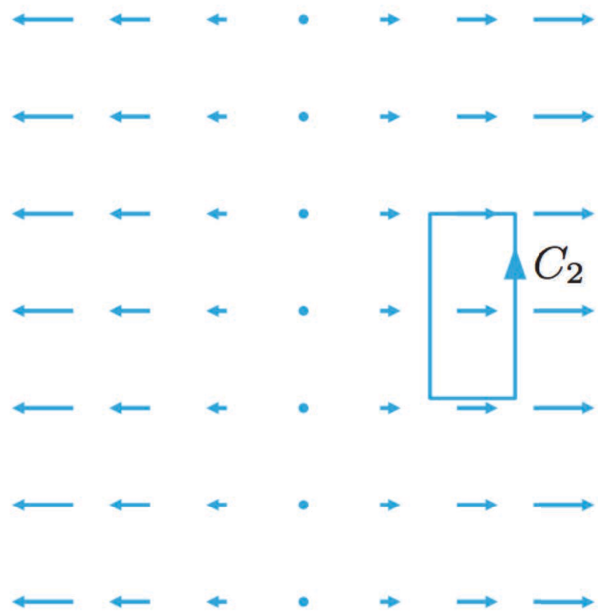
Review: Circulation (i.e., line integrals on closed "loops")

$$\int_C \vec{F} \cdot d\vec{r}$$

If $C$ is an oriented closed curve, the line integral of a vector field $\vec{F}$ around $C$ is called the **circulation** of $\vec{F}$ around $C$.




A circulating flow




A flow with zero circulation

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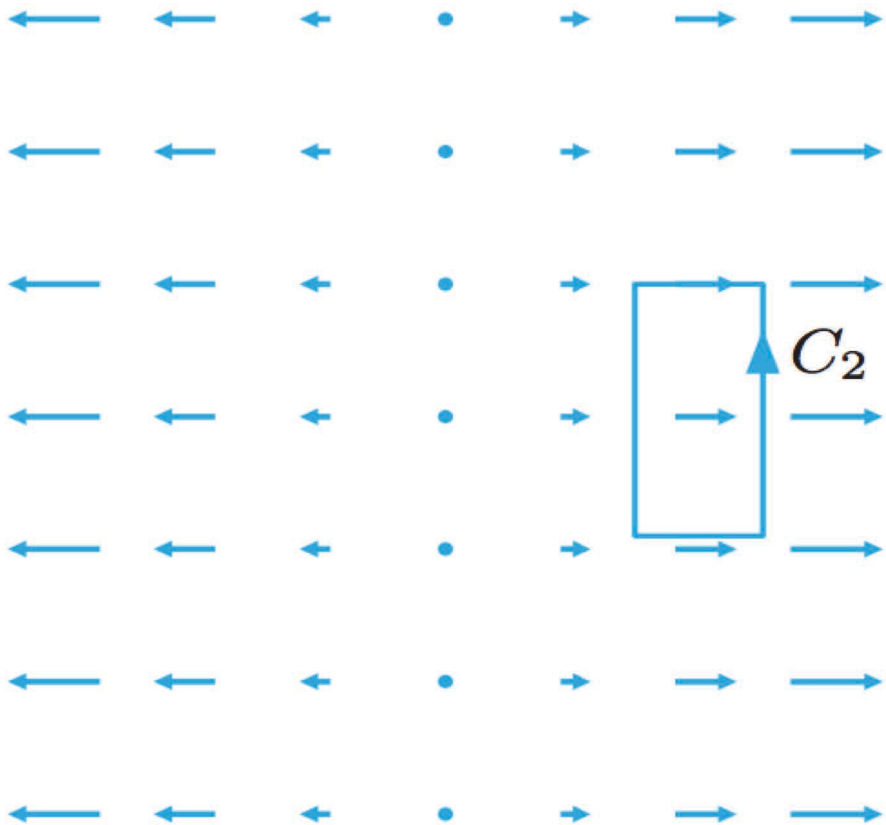
Review: Line Integrals




A flow with zero circulation

→ Note that this particular field has the interesting property that the circulation is zero for ALL possible closed loops.....

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Review: Fundamental Theorem of Calculus (in higher dimensions)

For 1-D: $$\int_a^b f'(t)\,dt = f(b) - f(a)$$

For 2-D (or higher):

**Theorem 18.1: The Fundamental Theorem of Calculus for Line Integrals**

Suppose $C$ is a piecewise smooth oriented path with starting point $P$ and ending point $Q$. If $f$ is a function whose gradient is continuous on the path $C$, then

$$\int_C \operatorname{grad} f \cdot d\vec{r} = f(Q) - f(P).$$

HOWEVER: There is a major catch here, one that relates back directly to the physics of 2010.... (courtesy of the "if" and the appearance of $\operatorname{grad} f$)

... let us first unpackage the bit above a bit more

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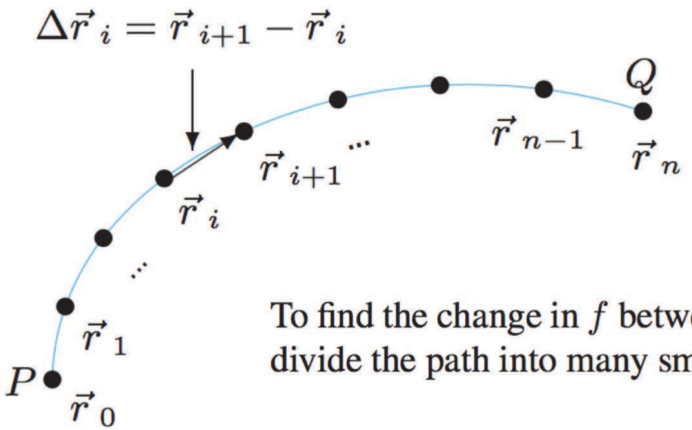
# Review: Fundamental Theorem of Calculus

**Theorem 18.1: The Fundamental Theorem of Calculus for Line Integrals**

Suppose $C$ is a piecewise smooth oriented path with starting point $P$ and ending point $Q$. If $f$ is a function whose gradient is continuous on the path $C$, then

$$\int_C \operatorname{grad} f \cdot d\vec{r} = f(Q) - f(P).$$




To find the change in $f$ between two points $P$ and $Q$, we choose a smooth path $C$ from $P$ to $Q$, then divide the path into many small pieces.

First we estimate the change in $f$ as we move through a displacement $\Delta\vec{r}_i$ from $\vec{r}_i$ to $\vec{r}_{i+1}$. Suppose $\vec{u}$ is a unit vector in the direction of $\Delta\vec{r}_i$. Then the change in $f$ is given by

$$\begin{aligned} f(\vec{r}_{i+1}) - f(\vec{r}_i) &\approx \text{Rate of change of } f \times \text{Distance moved in direction of } \vec{u} \\ &= f_{\vec{u}}(\vec{r}_i)\|\Delta\vec{r}_i\| \\ &= \operatorname{grad} f \cdot \vec{u}\,\|\Delta\vec{r}_i\| \\ &= \operatorname{grad} f \cdot \Delta\vec{r}_i. \qquad \text{since } \Delta\vec{r}_i = \|\Delta\vec{r}_i\|\vec{u} \end{aligned}$$

Therefore, summing over all pieces of the path, the total change in $f$ is given by

$$\text{Total change} = f(Q) - f(P) \approx \sum_{i=0}^{n-1} \operatorname{grad} f(\vec{r}_i) \cdot \Delta\vec{r}_i.$$

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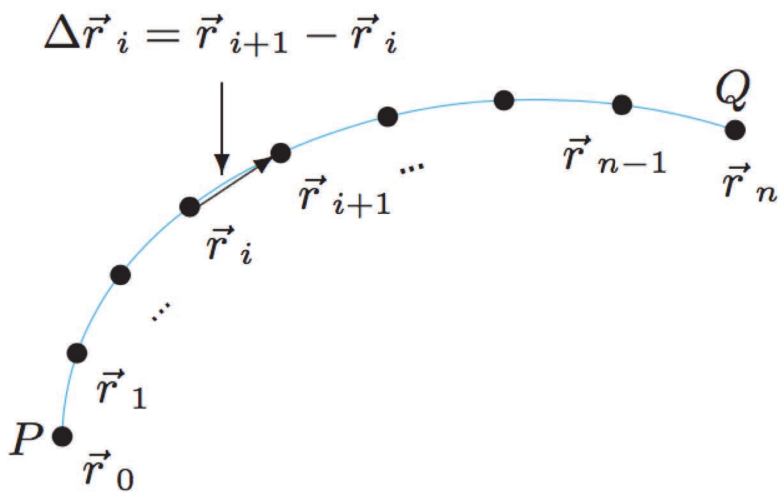
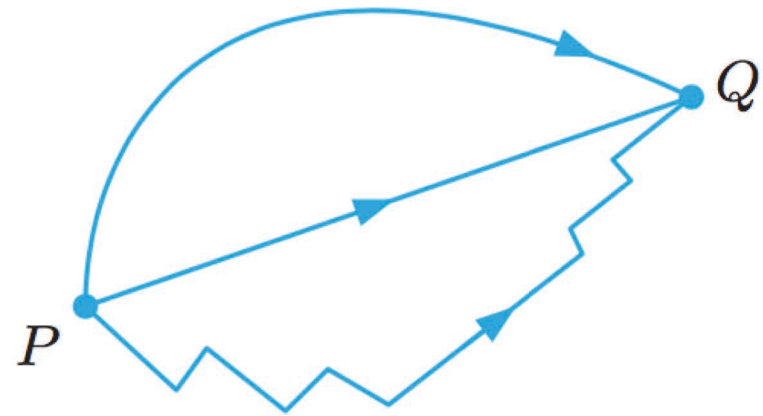
Review: Heading into the notion of a *conservative* field....




There are many different paths from $P$ to $Q$: all give the same value of $\int_C \text{grad } f \cdot d\vec{r}$

A vector field $\vec{F}$ is said to be **path-independent**, or **conservative**, if for any two points $P$ and $Q$, the line integral $\int_C \vec{F} \cdot d\vec{r}$ has the same value along any piecewise smooth path $C$ from $P$ to $Q$ lying in the domain of $\vec{F}$.

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Review: Conservative Fields & Gradients...

So let's come back to that "if"...

**Theorem 18.1: The Fundamental Theorem of Calculus for Line Integrals**

Suppose $C$ is a piecewise smooth oriented path with starting point $P$ and ending point $Q$. If $f$ is a function whose gradient is continuous on the path $C$, then

$$\int_C \operatorname{grad} f \cdot d\vec{r} = f(Q) - f(P).$$

Another way to frame the assumption:

Now suppose that $\vec{F}$ is any continuous gradient field, so $\vec{F} = \operatorname{grad} f$

$$\int_C \vec{F} \cdot d\vec{r} = f(Q) - f(P)$$

If $\vec{F}$ is a continuous gradient vector field, then $\vec{F}$ is path-independent.

## Why Do We Care About Path-Independent, or Conservative, Vector Fields?

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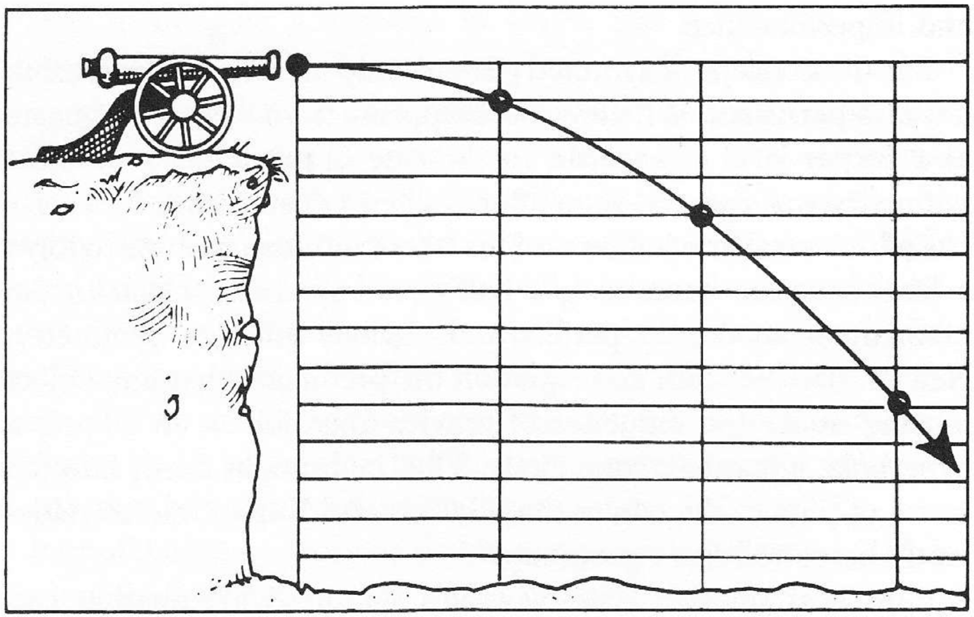
# Reminder!



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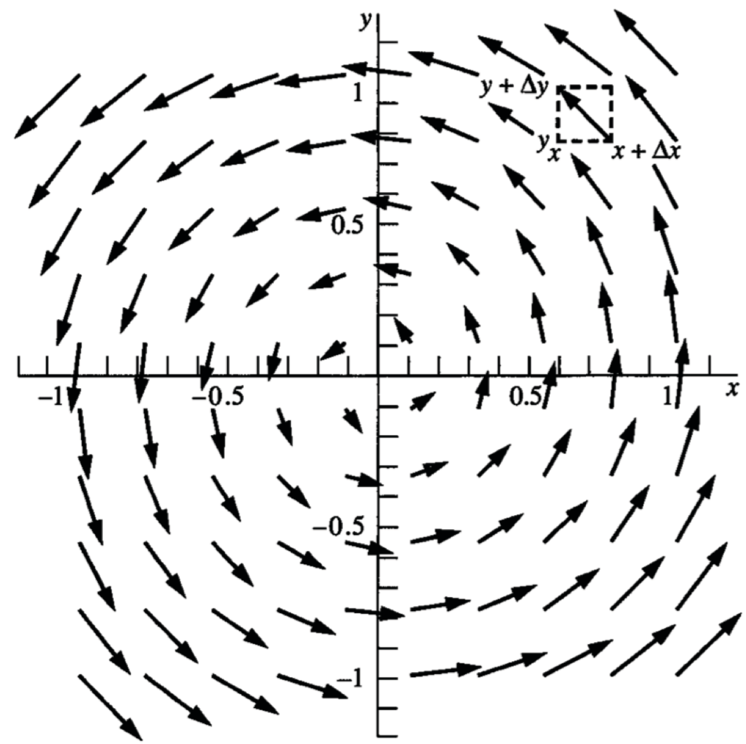


Fowles & Cassidy

## Why Do We Care About Path-Independent, or Conservative, Vector Fields?

Recall:
"The term force field simply means that if a small test particle were to be placed at any point $(x,y,)$ on the $xy$-plane, it would experience a force $\mathbf{F}$ "

Many of the fundamental vector fields of nature are path-independent—for example, the gravitational field and the electric field of particles at rest. The fact that the gravitational field is path-independent means that the work done by gravity when an object moves depends only on the starting and ending points and not on the path taken. For example, the work done by gravity (computed by the line integral) on a bicycle being carried to a sixth floor apartment is the same whether it is carried up the stairs in a zig-zag path or taken straight up in an elevator.

When a vector field is path-independent, we can define the *potential energy* of a body. When the body moves to another position, the potential energy changes by an amount equal to the work done by the vector field, which depends only on the starting and ending positions. If the work done had not been path-independent, the potential energy would depend both on the body's current position *and* on how it got there, making it impossible to define a useful potential energy.

$$-\frac{dV(x)}{dx} = F(x)$$

→ But we still have yet to define how to get $V$ from $F$ (or vice versa) in higher dimensions...

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## Review: Conservative Fields & Gradients...

**Theorem 18.2: Path-independent Fields Are Gradient Fields**

If $\vec{F}$ is a continuous path-independent vector field on an open region $R$, then $\vec{F} = \operatorname{grad} f$ for some $f$ defined on $R$.

Thus it follows:

A continuous vector field $\vec{F}$ defined on an open region is path-independent if and only if $\vec{F}$ is a gradient vector field.

Useful jargon:

If a vector field $\vec{F}$ is of the form $\vec{F} = \operatorname{grad} f$ for some scalar function $f$, then $f$ is called a **potential function** for the vector field $\vec{F}$.

And stating without proof (though the key necessary pieces have been laid out):

A vector field is path-independent if and only if $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed curve $C$.

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Review: Does a given vector field have a potential function?

## How to Tell If a Vector Field Is Path-Dependent Algebraically: The Curl

Again, stating without proof (though the key necessary pieces have been laid out):

If $\vec{F}(x, y) = F_1\vec{i} + F_2\vec{j}$ is a gradient vector field with continuous partial derivatives, then

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0.$$

If $\vec{F}(x, y) = F_1\vec{i} + F_2\vec{j}$ is an arbitrary vector field, then we define the 2-dimensional or scalar **curl** of the vector field $\vec{F}$ to be

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}.$$

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<u>Review</u>: Derivative of Vectors III

Derivative of a vector re a scalar:

$$\frac{d\mathbf{A}}{du} = \mathbf{i}\frac{dA_x}{du} + \mathbf{j}\frac{dA_y}{du} + \mathbf{k}\frac{dA_z}{du}$$

Gradient of a scalar field:

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Curl of a vector field:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\boldsymbol{\imath}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\boldsymbol{\jmath}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\boldsymbol{k}} = \begin{bmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{bmatrix}$$

# <u>Review</u>: Green Theorem

We now have two ways of seeing that a vector field $\vec{F}$ in the plane is path-dependent. We can evaluate $\int_C \vec{F} \cdot d\vec{r}$ for some closed curve and find it is not zero, or we can show that $\partial F_2/\partial x - \partial F_1/\partial y \neq 0$. It's natural to think that

$$\int_C \vec{F} \cdot d\vec{r} \quad \text{and} \quad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

might be related. The relation is called Green's Theorem.

Again, stating without proof:

**Theorem 18.3: Green's Theorem**

Suppose $C$ is a piecewise smooth simple closed curve that is the boundary of a region $R$ in the plane and oriented so that the region is on the left as we move around the curve. See Figure 18.44. Suppose $\vec{F} = F_1\vec{i} + F_2\vec{j}$ is a smooth vector field on an open region containing $R$ and $C$. Then

$$\int_C \vec{F} \cdot d\vec{r} = \int_R \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy.$$

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Review: "The curl test"

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0\,.$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_R \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy.$$

## The Curl Test for Vector Fields in 2-Space

Suppose $\vec{F} = F_1\vec{i} + F_2\vec{j}$ is a vector field with continuous partial derivatives such that

- The domain of $\vec{F}$ has the property that every closed curve in it encircles a region that lies entirely within the domain. In particular, the domain of $\vec{F}$ has no holes.
- $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = 0.$

Then $\vec{F}$ is path-independent, so $\vec{F}$ is a gradient field and has a potential function.

## The Curl Test for Vector Fields in 3-Space

Suppose $\vec{F}$ is a vector field on 3-space with continuous partial derivatives such that

- The domain of $\vec{F}$ has the property that every closed curve in it can be contracted to a point in a smooth way, staying at all times within the domain.
- $\operatorname{curl} \vec{F} = \vec{0}\,.$

Then $\vec{F}$ is path-independent, so $\vec{F}$ is a gradient field and has a potential function.

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Review: How much further do we need to go???

→ Yet more vector concepts come to play... (e.g., flux integrals, "sources" vs "sinks", divergence)

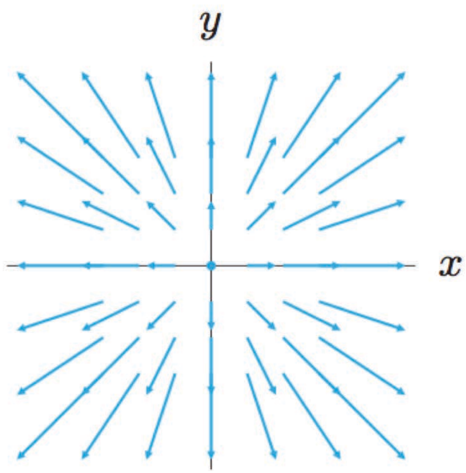


Figure 19.29: Vector field showing a source

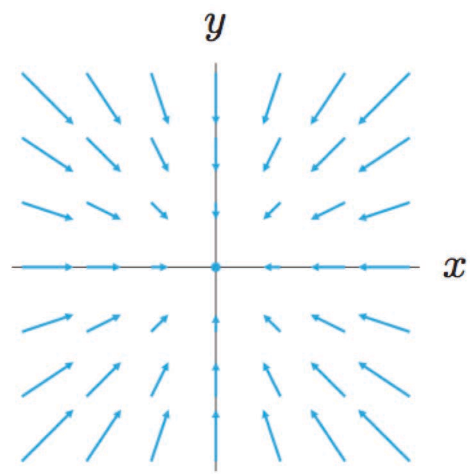


Figure 19.30: Vector field showing a sink

... but we'd like to get back to the "physics". So let's just highlight the essentials:

**Cartesian Coordinate Definition of Divergence**

If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, then

$$\text{div}\,\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

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<u>Review</u>: Divergence & Flux Integrals

## Geometric Definition of Divergence

The **divergence**, or **flux density**, of a smooth vector field $\vec{F}$, written **div** $\vec{F}$, is a scalar-valued function defined by

$$\text{div}\,\vec{F}(x, y, z) = \lim_{\text{Volume}\to 0} \frac{\int_S \vec{F} \cdot d\vec{A}}{\text{Volume of } S}.$$

Here $S$ is a sphere centered at $(x, y, z)$, oriented outward, that contracts down to $(x, y, z)$ in the limit.

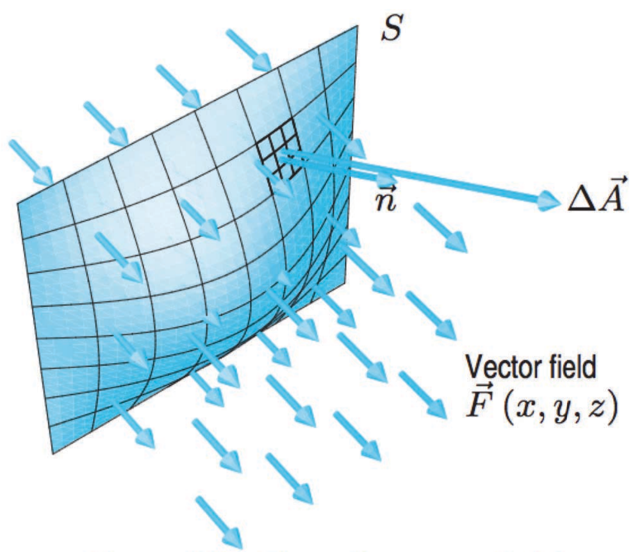


**Figure 19.7:** Flux of a vector field through a curved surface $S$

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# Review: Divergence & Flux Integrals

**Geometric Definition of Divergence**

The **divergence**, or **flux density**, of a smooth vector field $\vec{F}$, written **div**$\vec{F}$, is a scalar-valued function defined by

$$\operatorname{div}\vec{F}(x,y,z) = \lim_{\text{Volume}\to 0} \frac{\int_S \vec{F}\cdot d\vec{A}}{\text{Volume of } S}.$$

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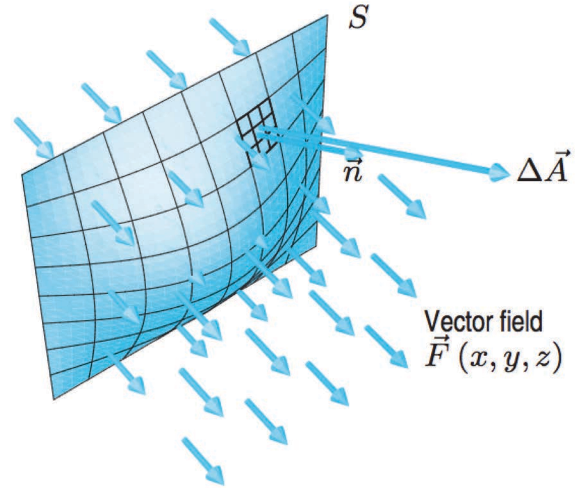


**Figure 19.7:** Flux of a vector field through a curved surface $S$

The **flux integral** of the vector field $\vec{F}$ through the oriented surface $S$ is

$$\int_S \vec{F}\cdot d\vec{A} = \lim_{\|\Delta\vec{A}\|\to 0} \sum \vec{F}\cdot\Delta\vec{A}.$$

If $S$ is a closed surface oriented outward, we describe the flux through $S$ as the flux out of $S$.

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# Review: Derivative of Vectors IV

Derivative of a vector re a scalar:

$$\frac{d\mathbf{A}}{du} = \mathbf{i}\frac{dA_x}{du} + \mathbf{j}\frac{dA_y}{du} + \mathbf{k}\frac{dA_z}{du}$$

Gradient of a scalar field:

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Curl of a vector field:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

Divergence of a vector field:

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot (F_1\vec{i} + F_2\vec{j} + F_3\vec{k}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

This one returns a scalar!

## Review: Divergence Theorem

The Divergence Theorem is a multivariable analogue of the Fundamental Theorem of Calculus; it says that the integral of the flux density over a solid region equals the flux integral through the boundary of the region.

$$\int_a^b f'(t)\,dt = f(b) - f(a) \qquad \int_C \operatorname{grad} f \cdot d\vec{r} = f(Q) - f(P)$$

### Theorem 19.1: The Divergence Theorem

If $W$ is a solid region whose boundary $S$ is a piecewise smooth surface, and if $\vec{F}$ is a smooth vector field on an open region containing $W$ and $S$, then

$$\int_S \vec{F} \cdot d\vec{A} = \int_W \operatorname{div} \vec{F}\, dV,$$

where $S$ is given the outward orientation.

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# Review: Stokes' Theorem

The Divergence Theorem says that the integral of the flux density over a solid region is equal to the flux through the surface bounding the region. Similarly, Stokes' Theorem says that the integral of the circulation density over a surface is equal to the circulation around the boundary of the surface.

**Theorem 20.1: Stokes' Theorem**

If $S$ is a smooth oriented surface with piecewise smooth, oriented boundary $C$, and if $\vec{F}$ is a smooth vector field on an open region containing $S$ and $C$, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_S \operatorname{curl} \vec{F} \cdot d\vec{A}.$$

The orientation of $C$ is determined from the orientation of $S$ according to the right-hand rule.

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# <u>Review</u>: Summary *re Vector Calculus*

"three multivariable versions of the Fundamental Theorem of Calculus"

**Fundamental Theorem of Calculus for Line Integrals**

$$\int_C \operatorname{grad} f \cdot d\vec{r} = f(Q) - f(P).$$

**Stokes' Theorem**

$$\int_S \operatorname{curl} \vec{F} \cdot d\vec{A} = \int_C \vec{F} \cdot d\vec{r}\,.$$

**Divergence Theorem**

$$\int_W \operatorname{div} \vec{F}\, dV = \int_S \vec{F} \cdot d\vec{A}\,.$$

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# div grad curl and all that

## an informal text on vector calculus

third edition

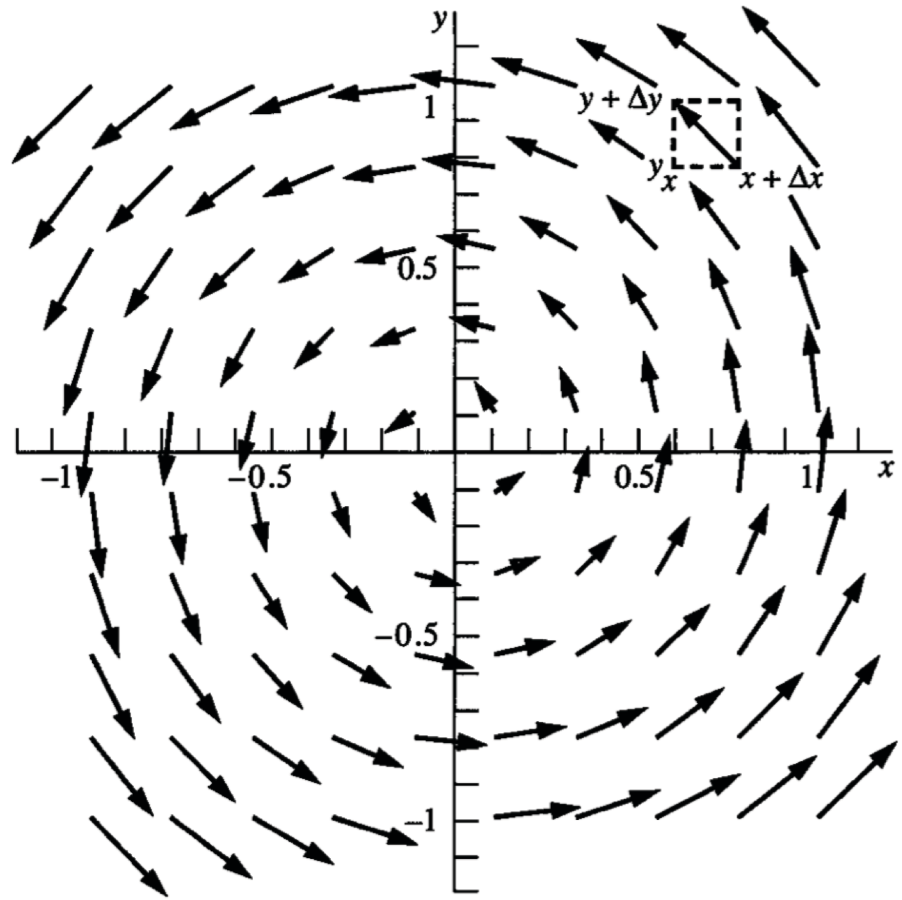
h. m. schey

The return....

Consider this force field
(which seemingly just depends upon "location")

$$\mathbf{F}(x, y)$$

$F_x = -by$ and $F_y = +bx$, where $b$ is some constant




Is it conservative?

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0\,.$$ says no.

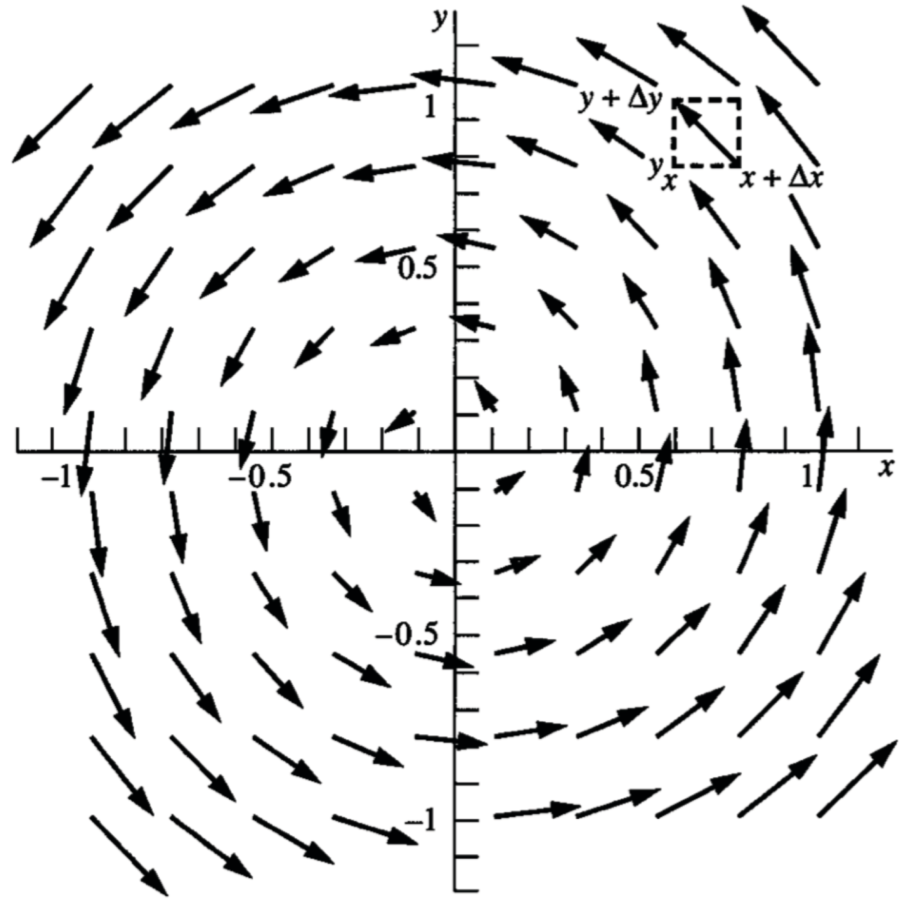
But can we develop a more intuitive *physical* explanation?

And how do we tie all this back to the notion of a *potential field*?

$$-\frac{dV(x)}{dx} = F(x)$$

Fowles & Cassidy

# (Non-)Conservative Force Fields




The only way in which we could assign a unique value to the potential energy would be if the closed-loop work integral vanished. In such cases, the work done along a path from *A* to *B* would be path-independent and would equal both the potential energy loss and the kinetic energy gain. The total energy of the particle would be a constant, independent of its location in such a force field! We, therefore, must find the constraint that a particular force must obey if its closed-loop work integral is to vanish.

Fowles & Cassidy