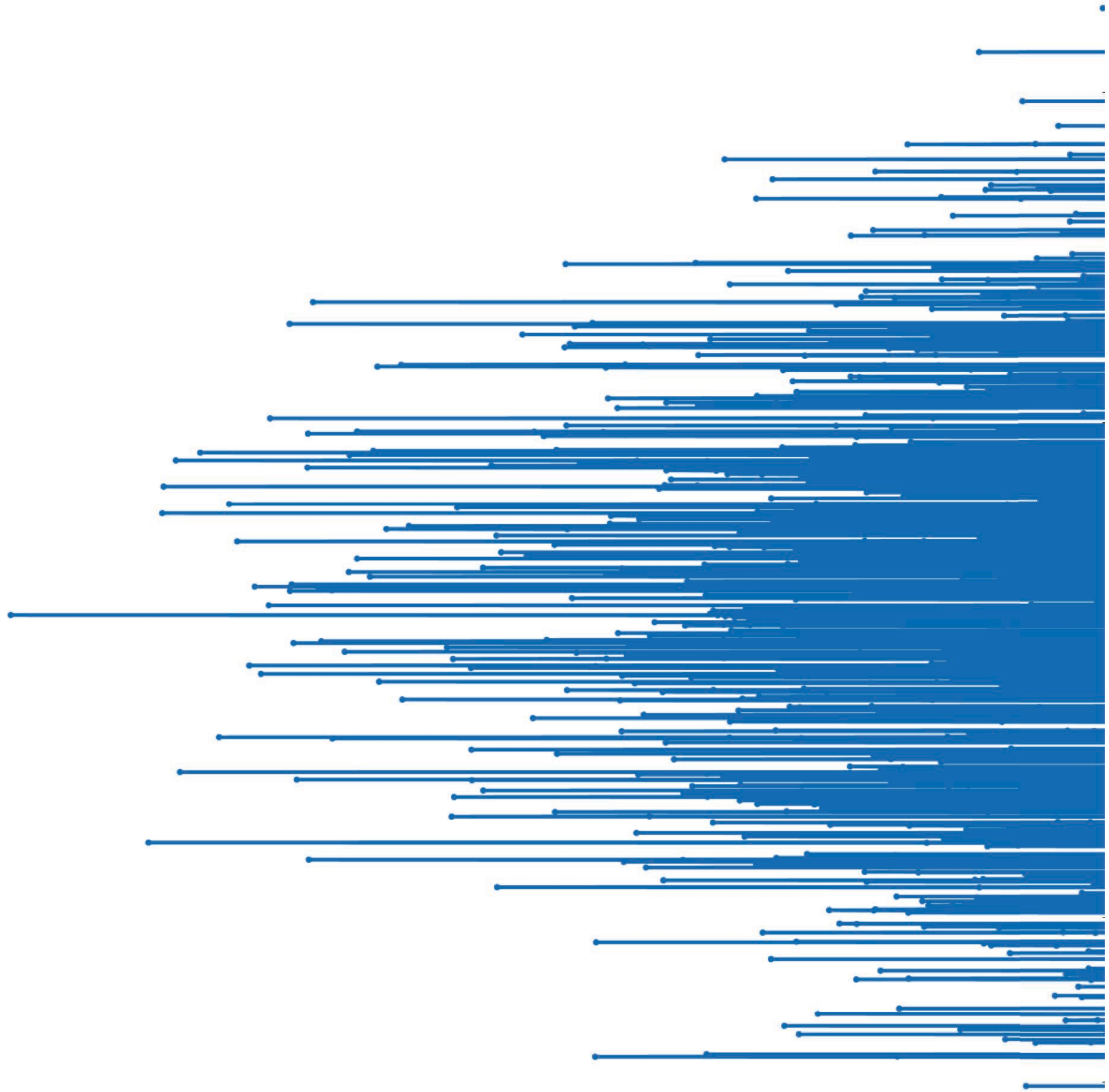
PHYS 2010 (W20)
Classical Mechanics



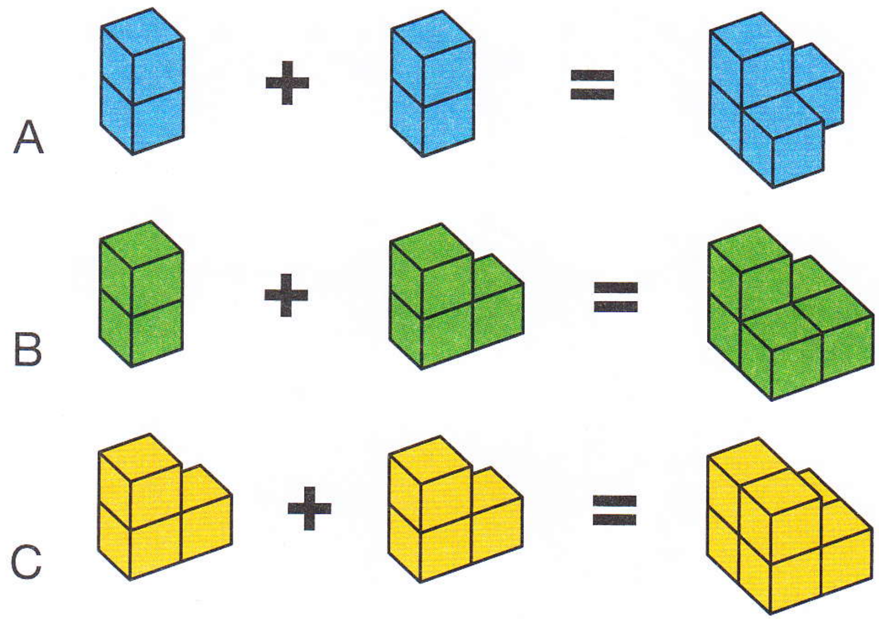
2020.01.28

Relevant reading:
Knudsen & Hjorth: 8.4-8.8

Christopher Bergevin
York University, Dept. of Physics & Astronomy
Office: Petrie 240 Lab: Farq 103
cberge@yorku.ca

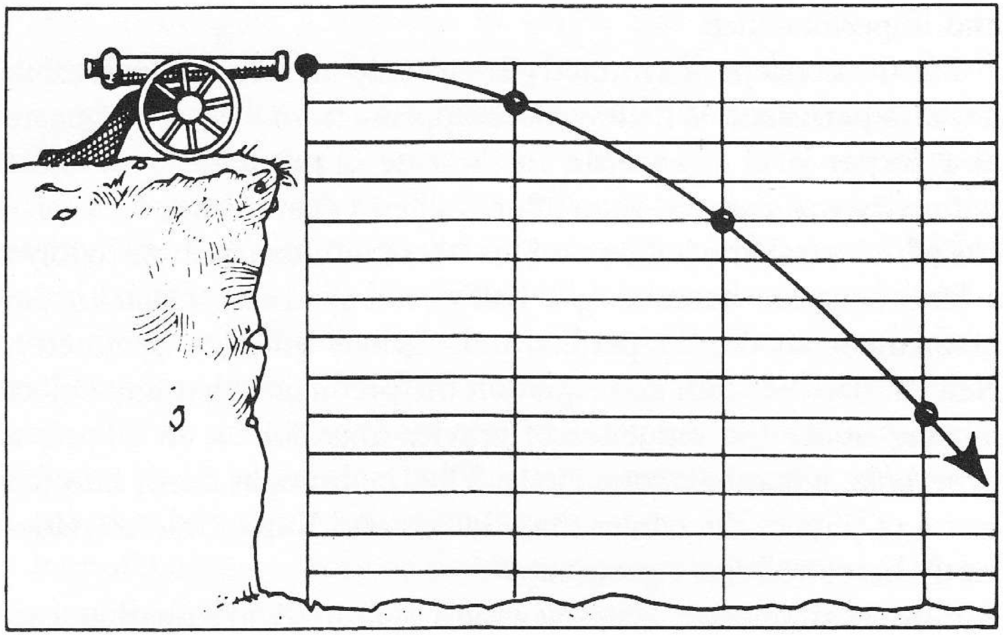
Ref.s:
Knudsen & Hjorth (2000), Fowles & Cassidy (2005)

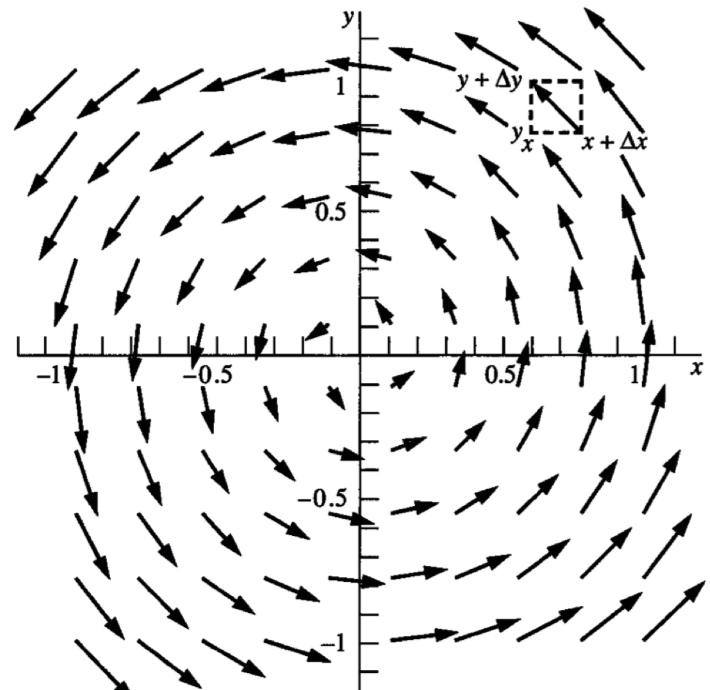
78. Cubic Sums



Which equation is not correct?

A B C





y
1
y + Δy
y
x
x + Δx
0.5
−1
−0.5
0.5
1
x
−0.5
−1

div grad curl and all that

an informal text on vector calculus

third edition

h. m. schey

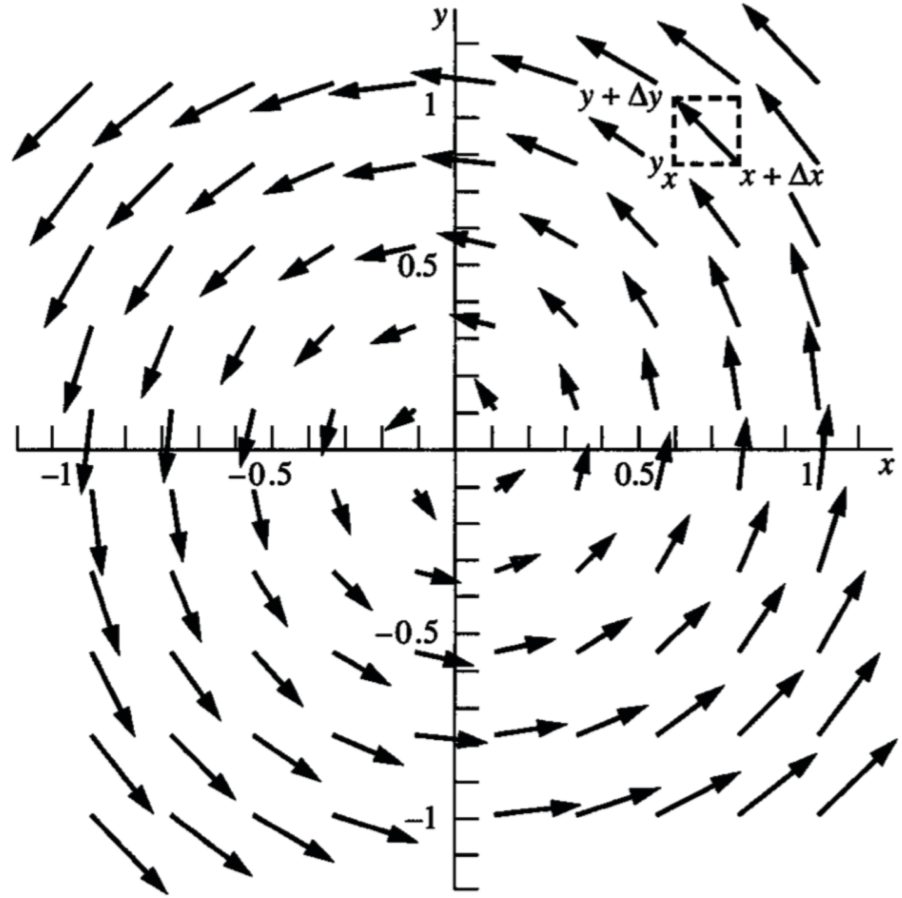
<u>The return....</u>

Consider this force field
(which seemingly just depends upon "location")

$$\mathbf{F}(x, y)$$

$F_x = -by$ and $F_y = +bx$, where b is some constant

Is it conservative?



$$\oint \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0 \, .$$ says no.

But can we develop a more intuitive *physical* explanation?

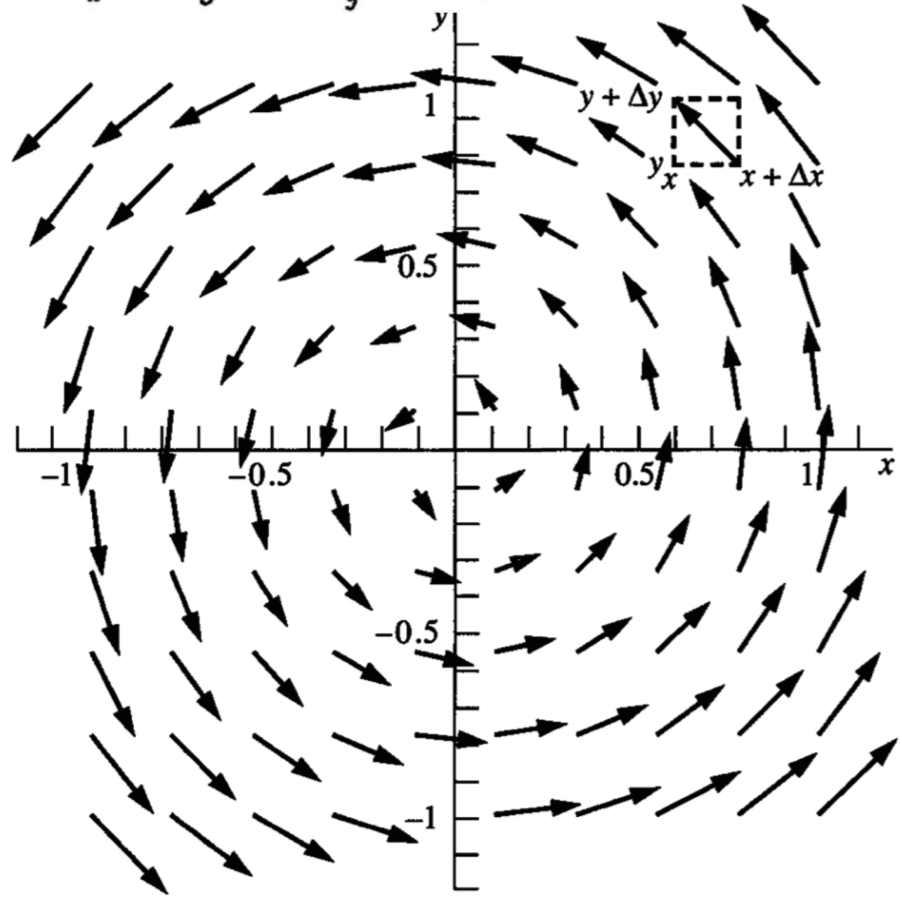
And how do we tie all this back to the notion of a *potential field*?

$$-\frac{dV(x)}{dx} = F(x)$$

Fowles & Cassidy

(Non-)Conservative Force Fields

$F_x = -by$ and $F_y = +bx$,

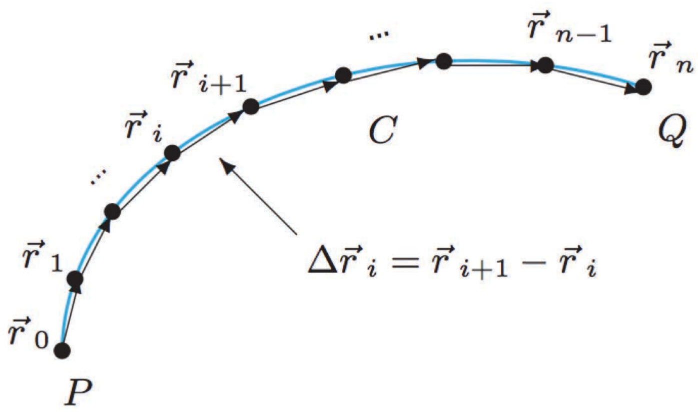


The only way in which we could assign a unique value to the potential energy would be if the closed-loop work integral vanished. In such cases, the work done along a path from A to B would be path-independent and would equal both the potential energy loss and the kinetic energy gain. The total energy of the particle would be a constant, independent of its location in such a force field! We, therefore, must find the constraint that a particular force must obey if its closed-loop work integral is to vanish.

Fowles & Cassidy

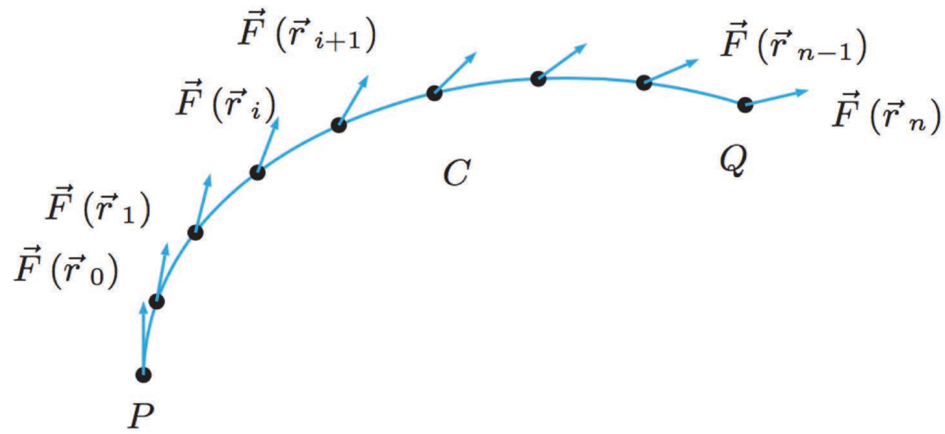
Recall: Line Integrals

So we have a (vector) curve (C)...



The curve C, oriented from P to Q, approximated by straight line segments represented by displacement vectors $\Delta\vec{r}_i = \vec{r}_{i+1} - \vec{r}_i$

... and a vector field (F)



The vector field $\vec{F}$ evaluated at the points with position vector $\vec{r}_i$ on the curve C oriented from P to Q

Now add up the combo of the two...

$$\sum_{i=0}^{n-1} \vec{F}(\vec{r}_i) \cdot \Delta\vec{r}_i.$$

Hughes-Hallet et al

Let's calculate the work associated w/ taking a test particle around the closed loop as shown:

$$W = \oint \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_x^{x+\Delta x} F_x(y)\,dx + \int_y^{y+\Delta y} F_y(x+\Delta x)\,dy$$

$$+ \int_{x+\Delta x}^{x} F_x(y+\Delta y)\,dx + \int_{y+\Delta y}^{y} F_y(x)\,dy$$

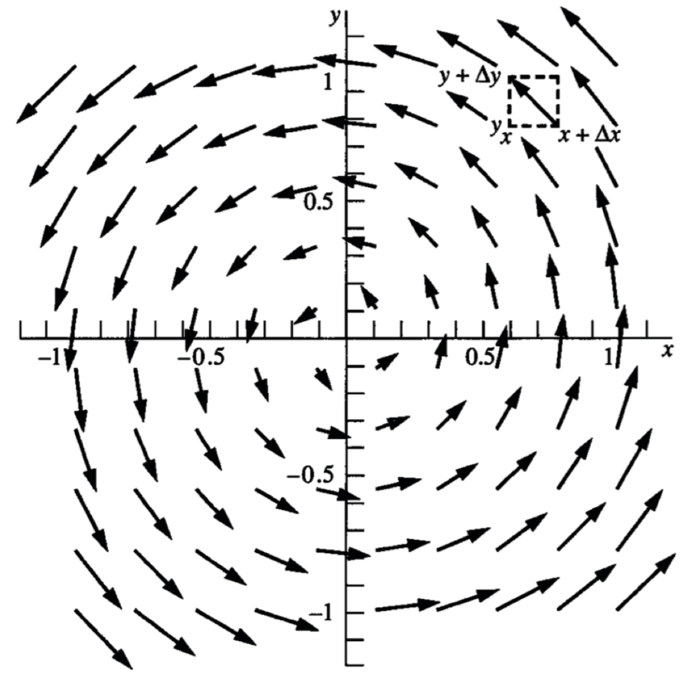
$$= \int_y^{y+\Delta y} (F_y(x+\Delta x) - F_y(x))\,dy$$

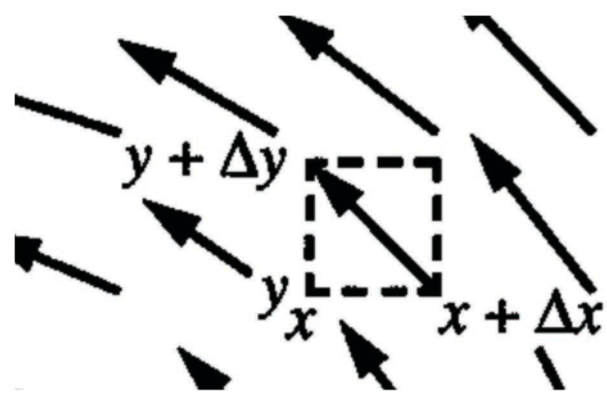
$$+ \int_x^{x+\Delta x} (F_x(y) - F_x(y+\Delta y))\,dx$$

$$= (b(x+\Delta x) - bx)\Delta y + (b(y+\Delta y) - by)\Delta x$$

$$= 2b\Delta x\,\Delta y$$

$F_x = -by$ and $F_y = +bx$;



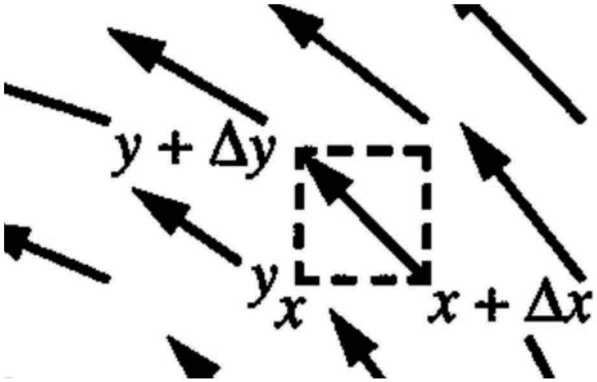


Fowles & Cassidy

(Non-)Conservative Force Fields

$F_x = -by$ and $F_y = +bx$;

$$W = \oint \mathbf{F} \cdot d\mathbf{r} \quad = 2b\Delta x\, \Delta y$$



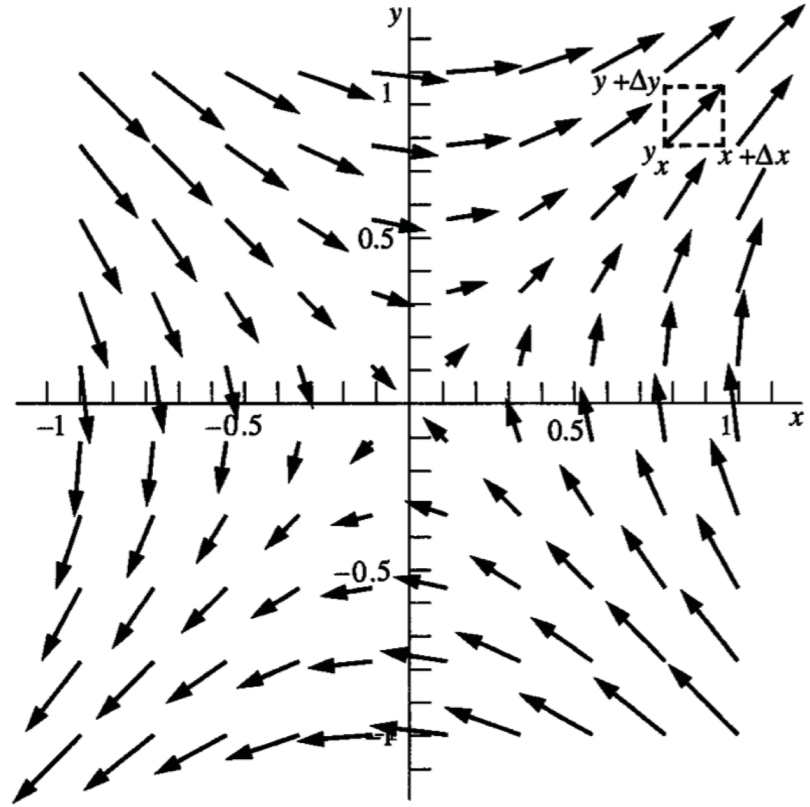
The work done is nonzero and is proportional to the area of the loop, $\Delta A = \Delta x \cdot \Delta y$, which was chosen in an arbitrary fashion. If we divide the work done by the area of the loop and take limits as $\Delta A \to 0$, we obtain the value $2b$. The result is dependent on the precise nature of this particular nonconservative force field.

Recall: $F_x = -by$ and $F_y = +bx$, where b is some constant

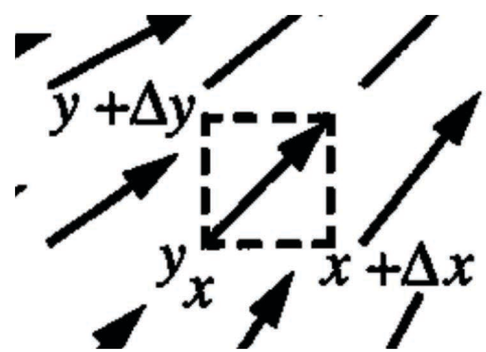
→ What if we make a teeny tiny algebraic change to our vector field? $F_x = by$ and $F_y = bx$.

<u>(Non-)Conservative Force Fields</u>

$$F_x = by \text{ and } F_y = bx.$$



Here we have gotten rid of "circulation", but maintained the magnitude at any given point....



.... and now for any closed loop, the work done per unit area traversing it has disappeared!

[Leaving this as an exercise. Do it both algebraically (via the integral) and graphically!]

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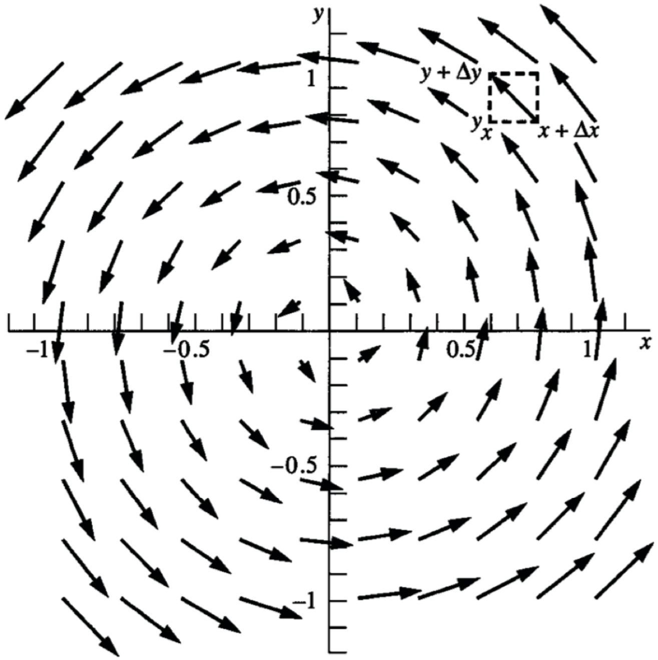
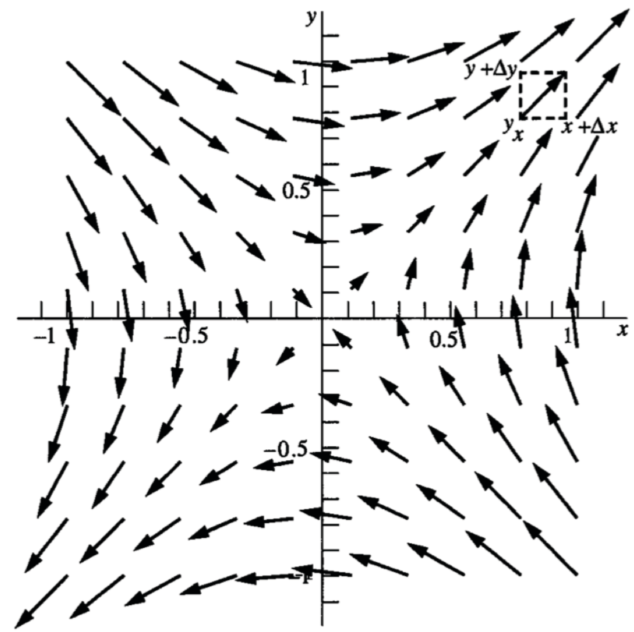
(Non-)Conservative Force Fields

$F_x = -by$ and $F_y = +bx$;



$F_x = by$ and $F_y = bx$.



This all suggests some geometric property of the field will determine whether it is conservative or not!

→ Let us derive the constraint upon $\mathbf{F}$ to ensure such

Fowles & Cassidy

(Non-)Conservative Force Fields

Note that the integrands are reminiscent of terms for a Taylor series expansion...

$$= \int_y^{y+\Delta y} (F_y(x+\Delta x) - F_y(x))\,dy$$
$$+ \int_x^{x+\Delta x} (F_x(y) - F_x(y+\Delta y))\,dx$$

So let's make a 1st order Taylor series expansion....

$$F_x(y+\Delta y) = F_x(y) + \frac{\partial F_x}{\partial y}\Delta y$$
$$F_y(x+\Delta x) = F_y(x) + \frac{\partial F_y}{\partial x}\Delta x$$

... and plug back into our integral:

$$\oint \mathbf{F}\cdot d\mathbf{r} = \int_y^{y+\Delta y}\left(\frac{\partial F_y}{\partial x}\Delta x\right)dy - \int_x^{x+\Delta x}\left(\frac{\partial F_x}{\partial y}\Delta y\right)dx$$

$$= \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\Delta x\Delta y = 2b\Delta x\Delta y$$

So we just need this term to be zero!

$$\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

For $F_x = -by$ and $F_y = +bx$;

Fowles & Cassidy

(Non-)Conservative Force Fields

This is familiar...

$$\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

$$\oint \mathbf{F} \cdot d\mathbf{r} = \int_S \text{curl}\, \mathbf{F} \cdot \hat{\mathbf{n}}\, da$$

$$\text{curl}\, \mathbf{F} = \mathbf{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

Theorem 20.1: Stokes' Theorem

If S is a smooth oriented surface with piecewise smooth, oriented boundary C, and if $\vec{F}$ is a smooth vector field on an open region containing S and C, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_S \text{curl}\, \vec{F} \cdot d\vec{A}\,.$$

The orientation of C is determined from the orientation of S according to the right-hand rule.

Fowles & Cassidy

Recall: Green Theorem

We now have two ways of seeing that a vector field $\vec{F}$ in the plane is path-dependent. We can evaluate $\int_C \vec{F} \cdot d\vec{r}$ for some closed curve and find it is not zero, or we can show that $\partial F_2/\partial x - \partial F_1/\partial y \neq 0$. It's natural to think that

$$\int_C \vec{F} \cdot d\vec{r} \quad \text{and} \quad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

might be related. The relation is called Green's Theorem.

Again, stating without proof:

Theorem 18.3: Green's Theorem

Suppose C is a piecewise smooth simple closed curve that is the boundary of a region R in the plane and oriented so that the region is on the left as we move around the curve. See Figure 18.44. Suppose $\vec{F} = F_1\vec{i} + F_2\vec{j}$ is a smooth vector field on an open region containing R and C. Then

$$\int_C \vec{F} \cdot d\vec{r} = \int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy.$$

Hughes-Hallet et al

<u>Recall</u>: "The curl test"

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0 \,.$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy.$$

The Curl Test for Vector Fields in 2-Space

Suppose $\vec{F} = F_1\vec{i} + F_2\vec{j}$ is a vector field with continuous partial derivatives such that

- The domain of $\vec{F}$ has the property that every closed curve in it encircles a region that lies entirely within the domain. In particular, the domain of $\vec{F}$ has no holes.
- $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = 0.$

Then $\vec{F}$ is path-independent, so $\vec{F}$ is a gradient field and has a potential function.

The Curl Test for Vector Fields in 3-Space

Suppose $\vec{F}$ is a vector field on 3-space with continuous partial derivatives such that

- The domain of $\vec{F}$ has the property that every closed curve in it can be contracted to a point in a smooth way, staying at all times within the domain.
- $\operatorname{curl} \vec{F} = \vec{0}\,.$

Then $\vec{F}$ is path-independent, so $\vec{F}$ is a gradient field and has a potential function.

Hughes-Hallet et al

Conservative Force Fields

Now if we want $\mathbf{F}$ such that: $$\oint \mathbf{F} \cdot d\mathbf{r} = \int_S \operatorname{curl} \mathbf{F} \cdot \hat{\mathbf{n}}\, da = 0$$

Then it turns out to be sufficient to have:

$$F_x = -\frac{\partial V}{\partial x} \qquad F_y = -\frac{\partial V}{\partial y} \qquad F_z = -\frac{\partial V}{\partial z}$$

For example, consider then the $\mathbf{z}$ component:

$$\frac{\partial F_x}{\partial y} = -\frac{\partial^2 V}{\partial y \partial x} \qquad \frac{\partial F_y}{\partial x} = -\frac{\partial^2 V}{\partial x \partial y} = -\frac{\partial^2 V}{\partial y \partial x} \qquad \therefore \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0$$

Just one catch: Need to assume V is everywhere continuous and differentiable!

Fowles & Cassidy

Conservative Force Fields

So when:

$$\text{curl } \mathbf{F} = 0$$

That is when:

$$\nabla \times \mathbf{F} = \mathbf{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) = 0$$

Then we can express the force as:

$$\mathbf{F} = -\mathbf{i}\frac{\partial V}{\partial x} - \mathbf{j}\frac{\partial V}{\partial y} - \mathbf{k}\frac{\partial V}{\partial z}$$

Note:

$$\nabla \times \nabla V \equiv 0$$

Or:

$$\mathbf{F} = -\nabla V$$

where

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

Compare our 1-D vs higher-dimensional cases:

$$-\frac{dV(x)}{dx} = F(x) \qquad\qquad \mathbf{F} = -\nabla V$$

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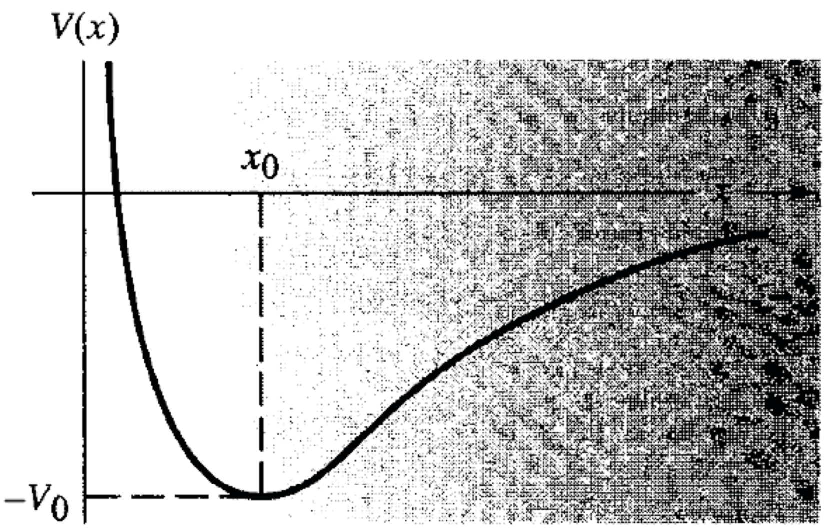
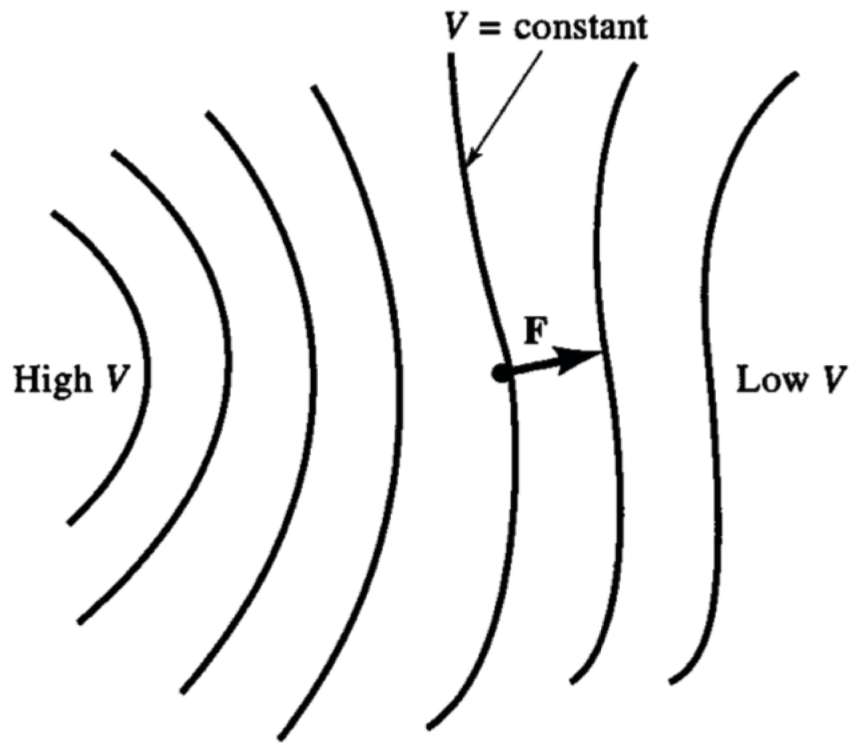
Conservative Force Fields

Compare our 1-D vs higher-dimensional cases:

$$-\frac{dV(x)}{dx} = F(x)$$

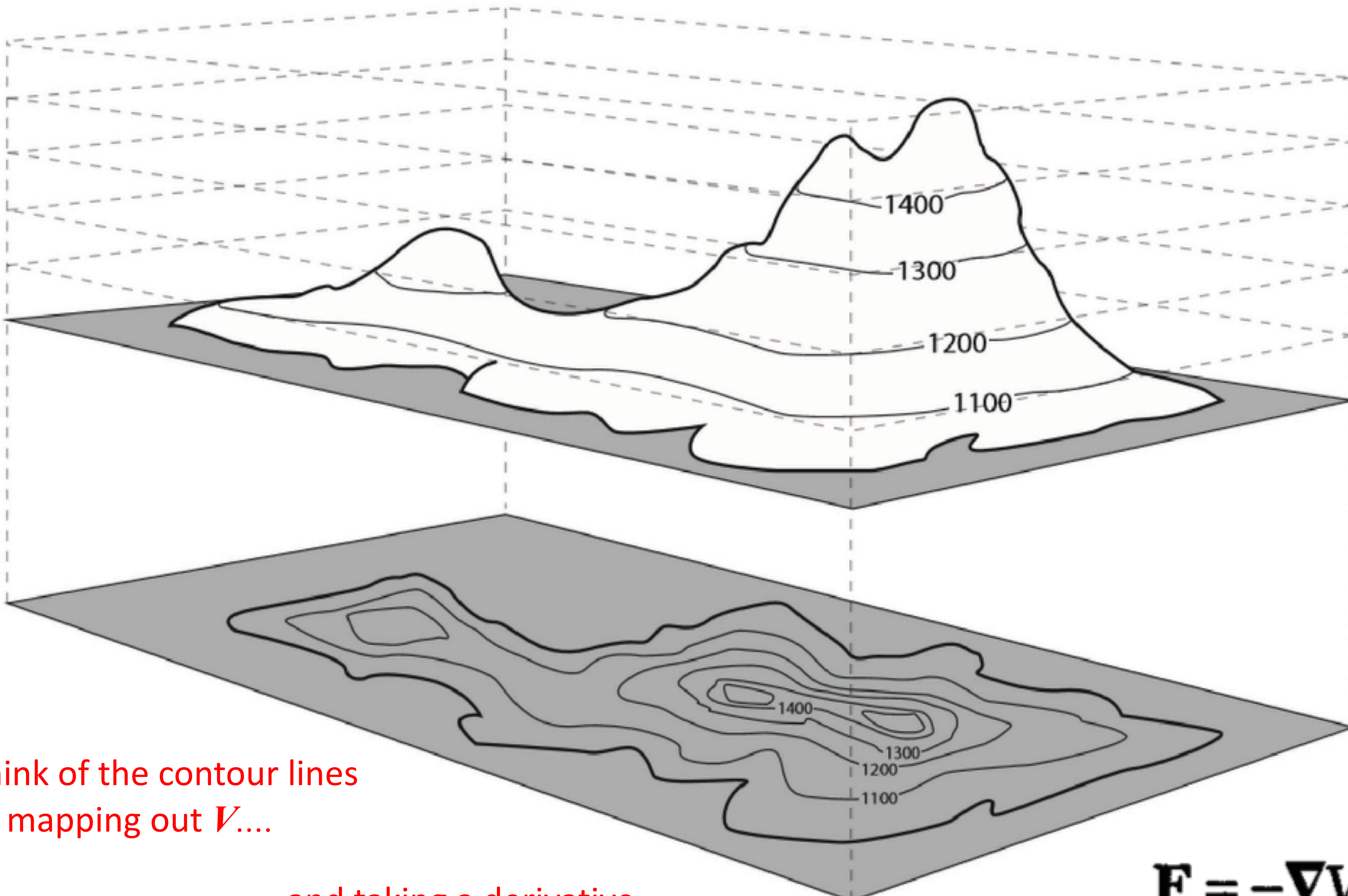
$$\mathbf{F} = -\nabla V$$





Fowles & Cassidy

<u>Recall</u>: Topographic Maps



Think of the contour lines
as mapping out V....

... and taking a derivative
of such would yield F

$$\mathbf{F} = -\nabla V$$

https://mapscaping.com/blogs/geo-candy/explaining-topographic-maps

Conservation of Energy (REVISTED)

Assume $\mathbf{F}$ is a conservative force → Work done moving a particle form A to B is:

$$\int_A^B \mathbf{F}\cdot d\mathbf{r} = -\int_A^B \nabla V(\mathbf{r})\cdot d\mathbf{r} = -\int_{A_x}^{B_x} \frac{\partial V}{\partial x}dx - \int_{A_y}^{B_y} \frac{\partial V}{\partial y}dy - \int_{A_z}^{B_z} \frac{\partial V}{\partial z}dz$$

$$= -\int_A^B dV(\mathbf{r}) = -\Delta V = V(A) - V(B)$$

The last step illustrates the fact that $\nabla V \cdot d\mathbf{r}$ is an *exact* differential equal to dV.

And the work done by (any) force is equal to net change in kinetic energy:

$$\int_A^B \mathbf{F}\cdot d\mathbf{r} = \Delta T = -\Delta V$$

$$\therefore \Delta(T+V) = 0$$

$$\therefore T(A) + V(A) = T(B) + V(B) = E = \text{constant}$$

→ So we end up at the same place (as expected) as our former 1-D case

(we have just generalized to higher dimensions)

Fowles & Cassidy

Nonconservative Force Fields?

Assume $\mathbf{F}'$ is NOT conservative → There is no potential function V →

$\mathbf{F}' \cdot d\mathbf{r}$ is not an exact differential

But what is conservative ($\mathbf{F}$) and non-conservative forces ($\mathbf{F}'$) are both at play?

Work done over an increment is: $(\mathbf{F} + \mathbf{F}') \cdot d\mathbf{r} = -dV + \mathbf{F}' \cdot d\mathbf{r} = dT$

Work-energy theorem becomes: $\int_A^B \mathbf{F}' \cdot d\mathbf{r} = \Delta(T + V) = \Delta E$

→ So total energy is not constant, but increases or decreases depending upon $\mathbf{F}'$

Note: If the force is dissipative (e.g., drag, air resistance), then: $\mathbf{F}' \cdot d\mathbf{r} < 0$

Largely in part due to the fact that $\mathbf{F}'$ would have to be opposite the direction of motion!

Fowles & Cassidy

Looking Ahead/Beyond 2010.....

"... to reduce the theory of mechanics, and the art of solving the associated problems, to general formulae, whose simple development provides all the equations necessary for the solution of each problem. . . . to unite, and present from one point of view, the different principles which have, so far, been found to assist in the solution of problems in mechanics; by showing their mutual dependence and making a judgement of their validity and scope possible. . . . No diagrams will be found in this work. The methods that I explain in it require neither constructions nor geometrical or mechanical arguments, but only the algebraic operations inherent to a regular and uniform process. Those who love Analysis will, with joy, see mechanics become a new branch of it and will be grateful to me for having extended its field."

—Joseph Louis de Lagrange, *Avertissement for Mechanique Analytique*, 1788

Hamilton's *Variational Principle*

$$\delta J = \delta \int_{t_1}^{t_2} L \, dt = 0$$

the "*Lagrangian*"

$$L = T - V$$

→ Rather than use vectors and forces, the idea was/is to use scalars and energy....

Fowles & Cassidy

Ex.

Consider the (2-D) potential energy function:

$$V(\mathbf{r}) = V_0 - \tfrac{1}{2} k\delta^2 e^{-r^2/\delta^2}$$

where $\mathbf{r} = \mathbf{i}x + \mathbf{j}y$ and V_0, k, and δ are constants.

a. Make a plot of $V(\mathbf{r})$ [including equipotential lines]

b. Determine the associated force function $\mathbf{F}(\mathbf{r})$

c. Describe what V_o means, both in terms of $V(\mathbf{r})$ and $\mathbf{F}$

d. Plot $\mathbf{F}(\mathbf{r})$

e. Determine the force for small displacements Δ ($<< \delta$)

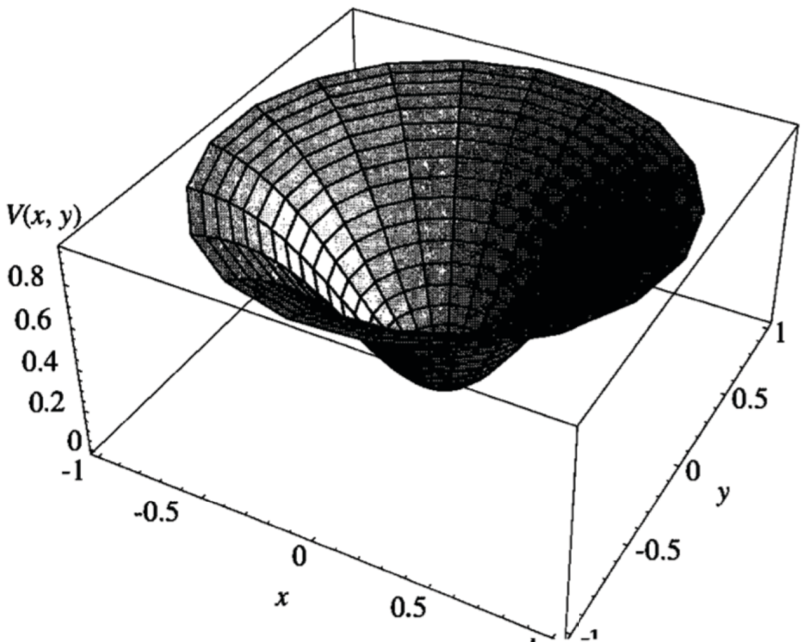
f. Suppose a particle of mass m is moving in that field and passes through the origin at $t=0$ with speed v_o . What will its speed be at a small distance away (given by $\mathbf{r} = \mathbf{e}_R\, \Delta$, where $\Delta << \delta$) from the origin?

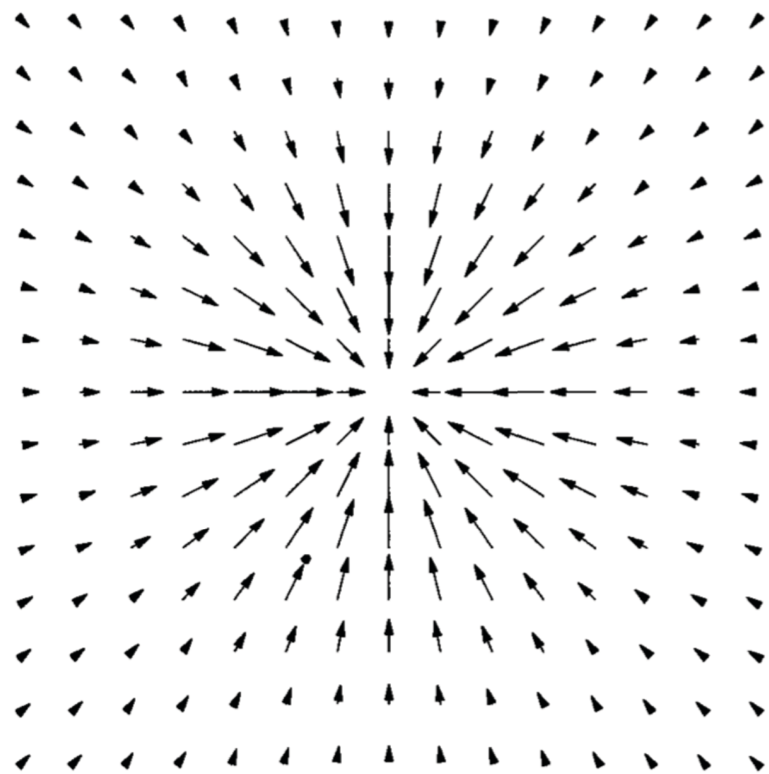
Fowles & Cassidy

Ex. SOLS

$$V(x,y) = V_0 - \tfrac{1}{2}k\delta^2 e^{-(x^2+y^2)/\delta^2}$$

$$\mathbf{F} = -\nabla V = -\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}\right)V(x,y)$$
$$= -k(\mathbf{i}x + \mathbf{j}y)e^{-(x^2+y^2)/\delta^2}$$
$$= -k\mathbf{r}e^{-r^2/\delta^2}$$





Fowles & Cassidy

Ex. SOLS

Conservative force means that:

$$E = T + V = \text{constant}$$

$$E = \tfrac{1}{2}mv^2 + V(\mathbf{r}) = \tfrac{1}{2}mv_0^2 + V(0)$$

$$v^2 = v_0^2 + \frac{2}{m}[V(0) - V(\mathbf{r})]$$

$$= v_0^2 + \frac{2}{m}\left[\left(V_0 - \tfrac{1}{2}k\delta^2\right) - \left(V_0 - \tfrac{1}{2}k\delta^2 e^{-\Delta^2/\delta^2}\right)\right]$$

$$= v_0^2 - \frac{k\delta^2}{m}[1 - e^{-\Delta^2/\delta^2}]$$

$$\approx v_0^2 - \frac{k\delta^2}{m}[1 - (1 - \Delta^2/\delta^2)]$$

$$= v_0^2 - \frac{k}{m}\Delta^2$$

Recall:

$$\exp x := \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

Fowles & Cassidy