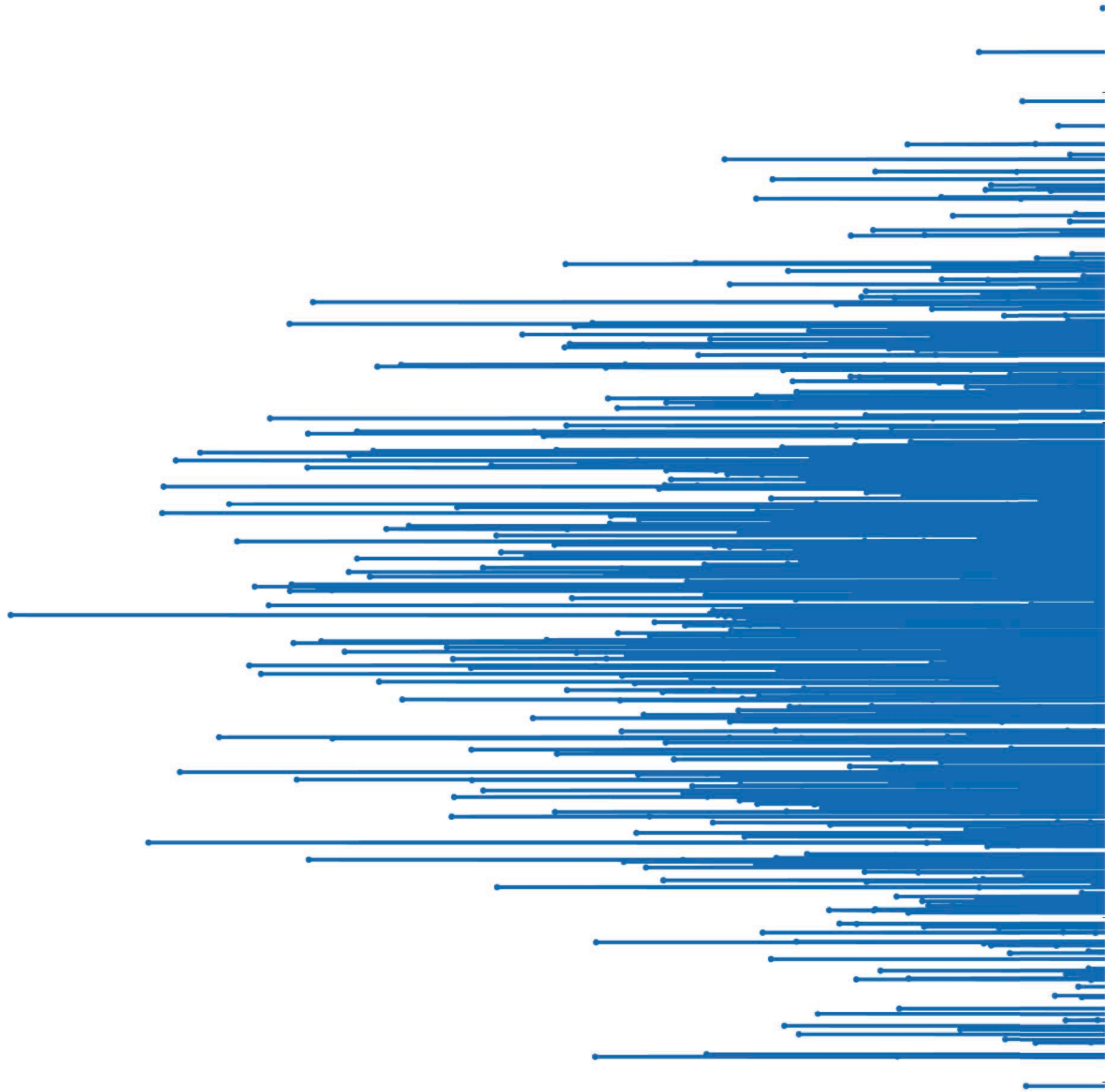
PHYS 2010 (W20)
Classical Mechanics



2020.02.06

Relevant reading:
Knudsen & Hjorth: 15.ff, 16.2

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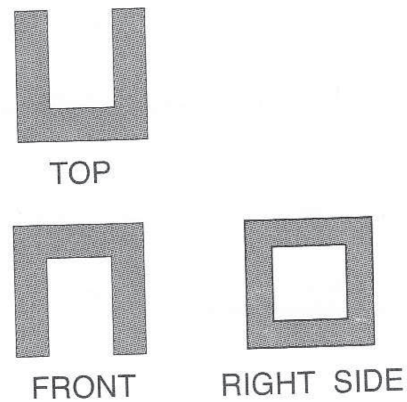
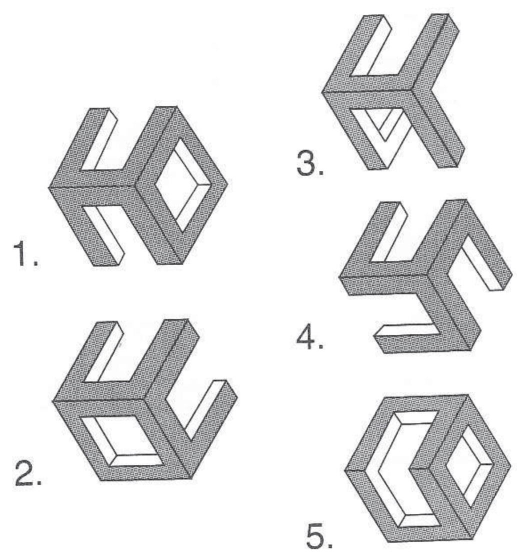
Ref.s:
Knudsen & Hjorth (2000), Fowles & Cassidy (2005), French (1971)

What's the Plan?

Which three-dimensional object at the bottom does the simple orthographic projection above it represent?





Class Reps

Marie Skłodowska *Curie*

Marie Maynard

Rosalind Franklin

CV Raman

Complex #s

Whether or not you have been introduced to this kind of analysis previously, you will be able to recognize that we are walking along a dividing line—or, more properly, a bridge—between geometry and algebra. If the quantities a and b are real numbers, as we have assumed in example c, then the combination $z = a + jb$ is what is known as a complex number. But in geometrical terms it can be regarded as a displacement along an axis at some angle θ to the x axis, such that $\tan\theta = b/a$

Definition:

$$i = \sqrt{-1} \ \ (= j)$$

Cartesian form:

$$z = a + ib$$

Euler's Formula:

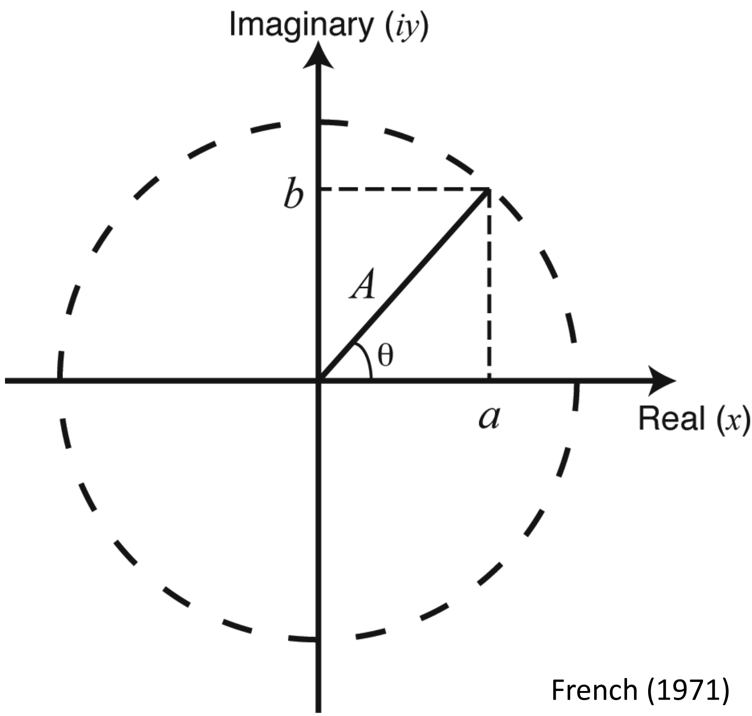
$$a + ib = Ae^{i\theta}$$

Magnitude

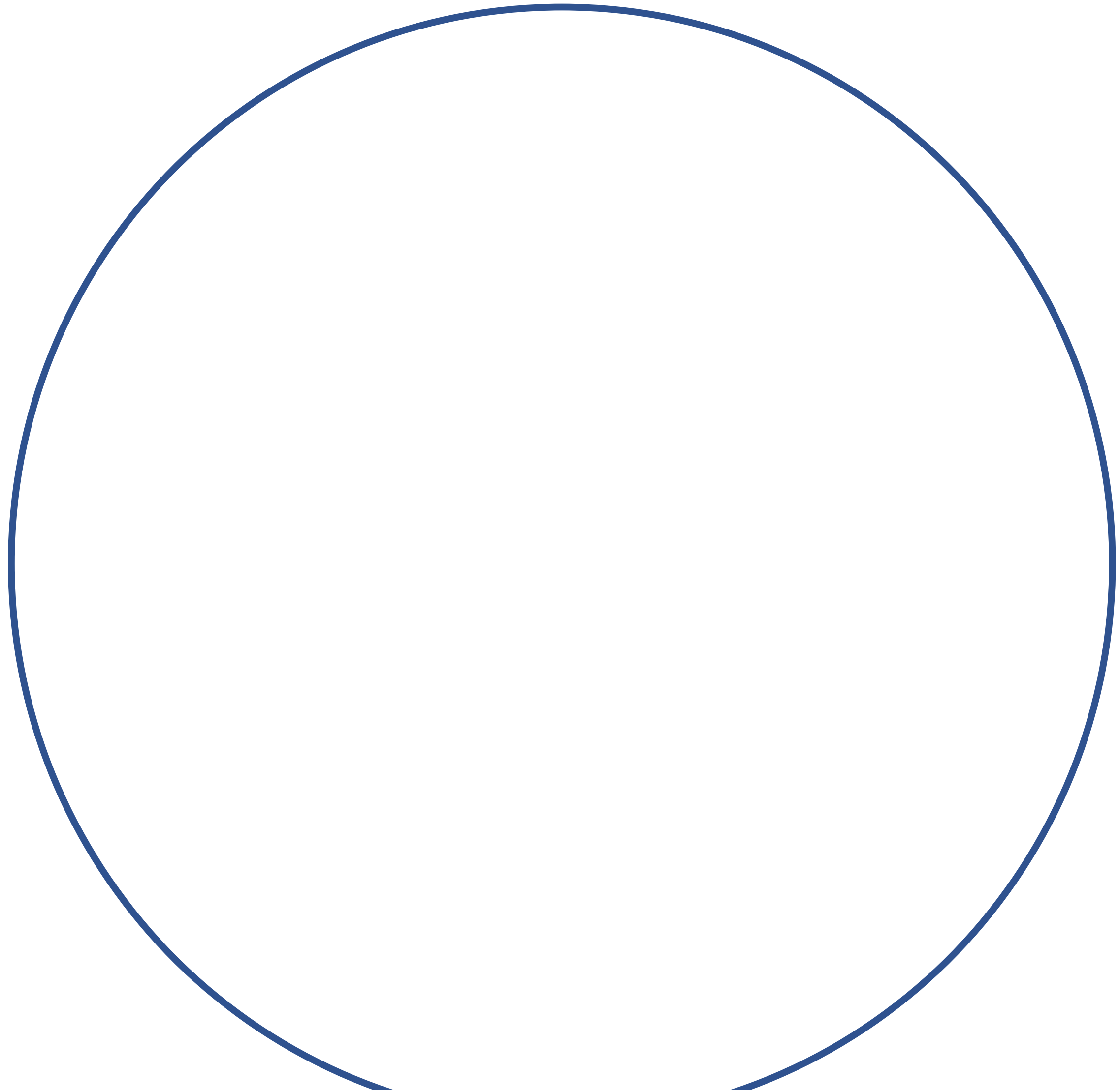
$$|a + ib| = A$$

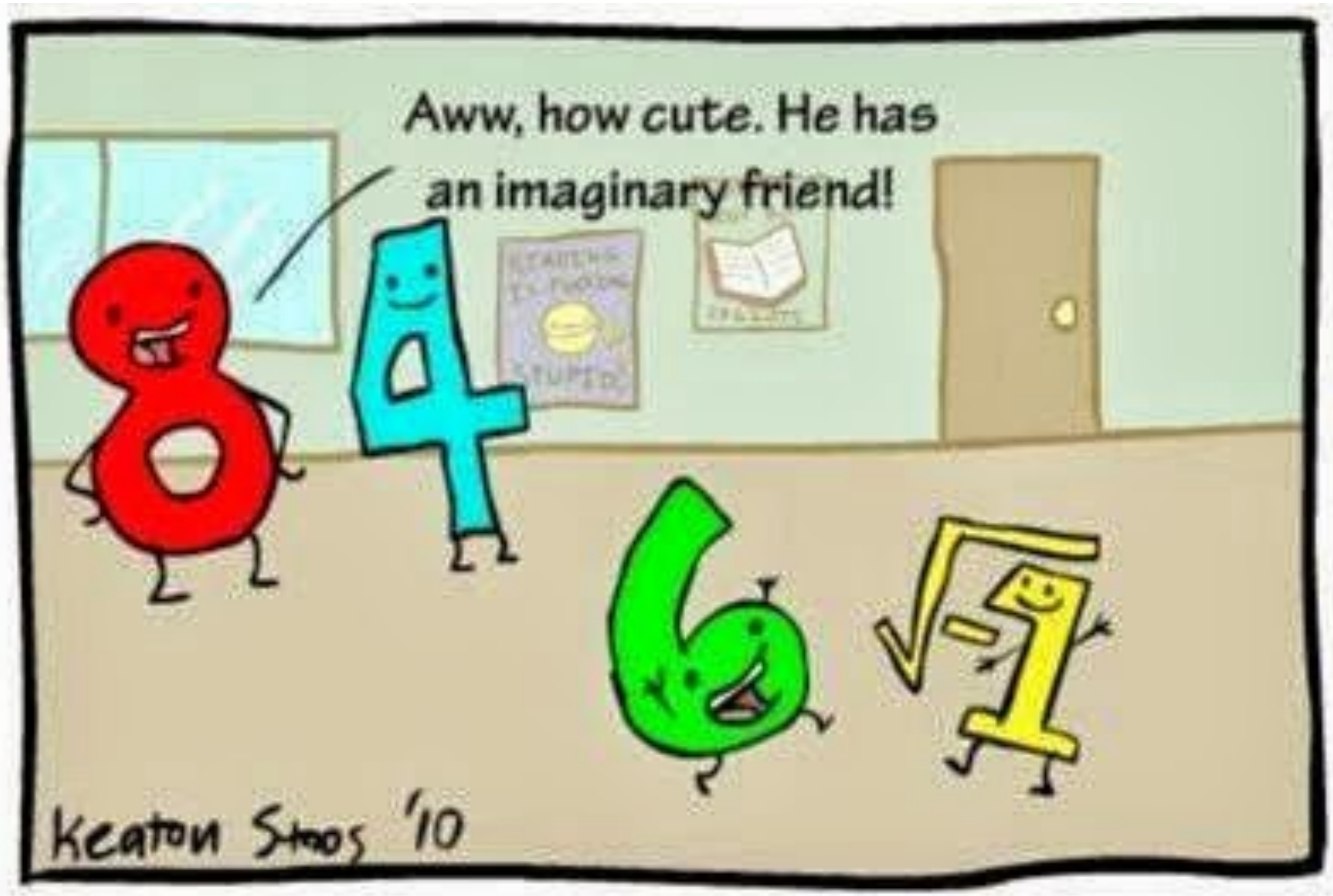
Phase

$$\angle(a + ib) = \theta$$



French (1971)




Aww, how cute. He has an imaginary friend!
Keaton Stoos '10

Jens M. Knudsen Poul G. Hjorth

Elements of Newtonian Mechanics

Including Nonlinear Dynamics

Jens M. Knudsen Poul G. Hjorth

Elements of Newtonian Mechanics

Including Nonlinear Dynamics

Complex Exponentials

[1]By Taylor's theorem,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots$$

Start w/ two key Taylor series expansions:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}\cdots \qquad \cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!}\cdots$$

Putting these two together (along w/ j):

$$\cos\theta + j\sin\theta = 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots$$

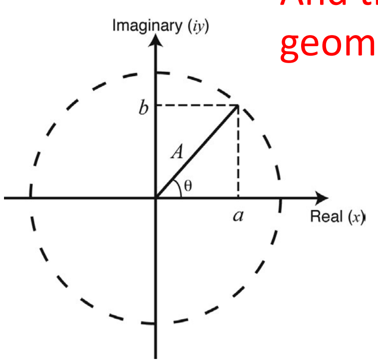
Or as:

$$\cos\theta + j\sin\theta = 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \cdots + \frac{(j\theta)^n}{n!} + \cdots$$

Leading to Euler's formula:

$$\cos\theta + j\sin\theta = e^{j\theta}$$

And the associated geometric version



French (1971)

<u>Using Complex Exponentials</u>

Equation of motion:

$$m\frac{d^2x}{dt^2} = -k_1 x$$

Consider our original approach:

If

$$x = A\cos(\omega t + \alpha)$$

then

$$\frac{dx}{dt} = -\omega A\sin(\omega t + \alpha)$$

$$\frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t + \alpha)$$

Instead, consider: $z = A\cos(\omega t + \alpha) + jA\sin(\omega t + \alpha)$

Or

$$z = Ae^{j(\omega t + \alpha)}$$

where x = real part of z

Then

$$\frac{dz}{dt} = j\omega Ae^{j(\omega t + \alpha)} = j\omega z$$

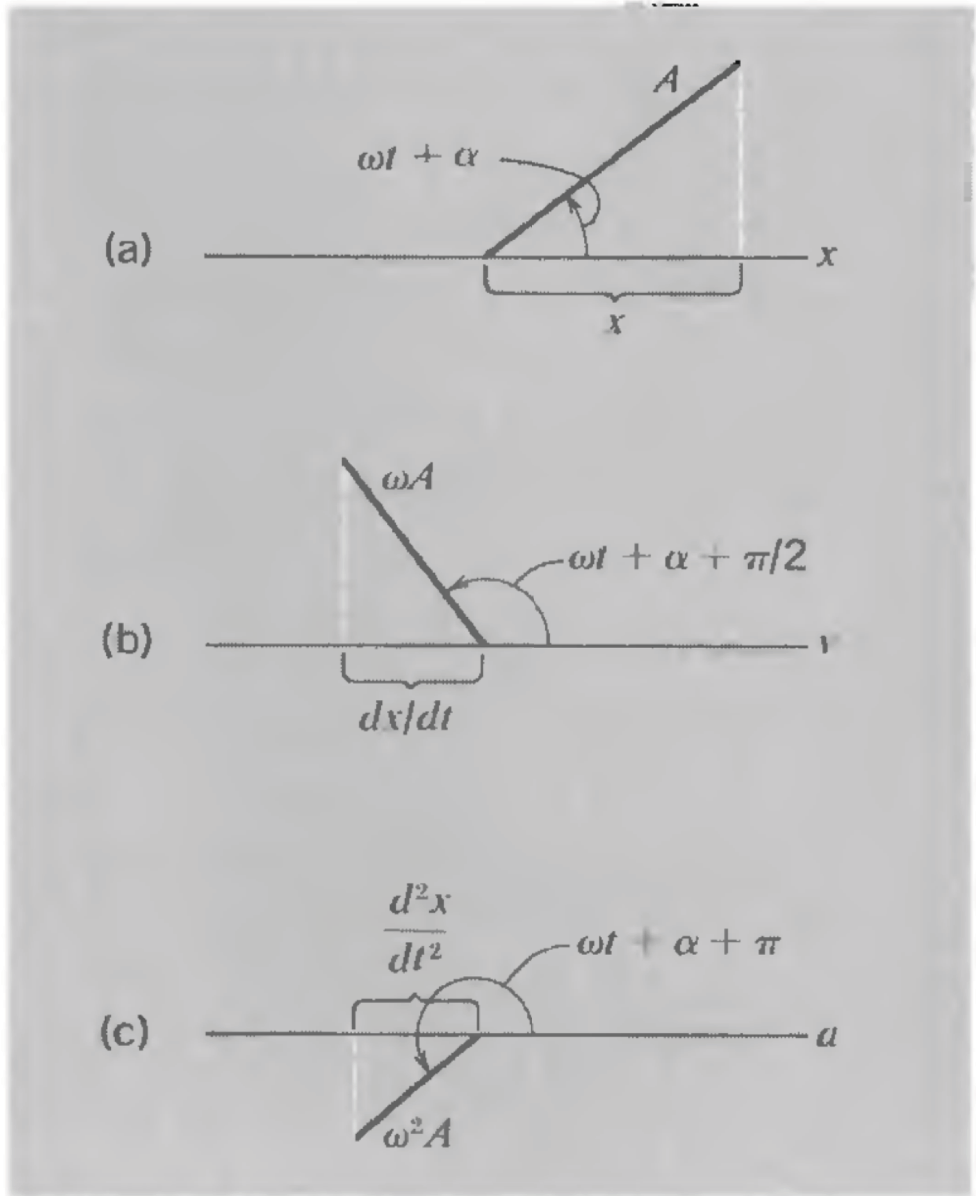
$$\frac{d^2z}{dt^2} = (j\omega)^2 Ae^{j(\omega t + \alpha)} = -\omega^2 z$$

→ We have turned an ODE problem dealing w/ sinusoids into an algebraic (and/or geometric) one

French (1971)

Using Complex Exponentials

Fig. 1–7 (a) Displacement vector z and its real projection x. (b) Velocity vector dz/dt and its real projection dx/dt. (c) Acceleration vector d^2z/dt^2 and its real projection d^2x/dt^2.

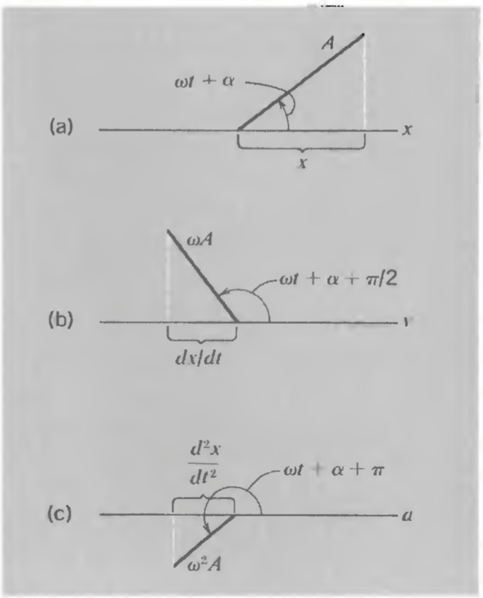


So from this viewpoint, there is a simple geometric representation (e.g., the acceleration always points in a direction opposite to the displacement)

French (1971)

<u>Using Complex Exponentials</u>

$$\cos\theta + j\sin\theta = e^{j\theta}$$



Aside from this geometric representation, why go to the trouble of imaginary #s?

Two reasons: (which we each develop further)

- Makes "harder" HO problems much more tractable (e.g., damped case, driven case)
- Much easier to "add" sinusoids together, an extremely useful "tool" to have in the toolbox...

(inverse) Fourier Transform

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{i\omega t} d\omega$$

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$$

French (1971)

Ex.

Given Euler's relation $e^{j\theta} = \cos\theta + j\sin\theta$, find

(a) The geometric representation of $e^{-j\theta}$.

(b) The exponential representation of $\cos\theta$.

(c) The exponential representation of $\sin\theta$.

French (1971)

Ex.

To take successive derivatives of $e^{j\theta}$ with respect to θ, one merely multiplies by j:

$$\frac{d}{d\theta}(Ae^{j\theta}) = jAe^{j\theta}$$

Show that this prescription works if the sinusoidal representation $e^{j\theta} = \cos\theta + j\sin\theta$ is used.

French (1971)

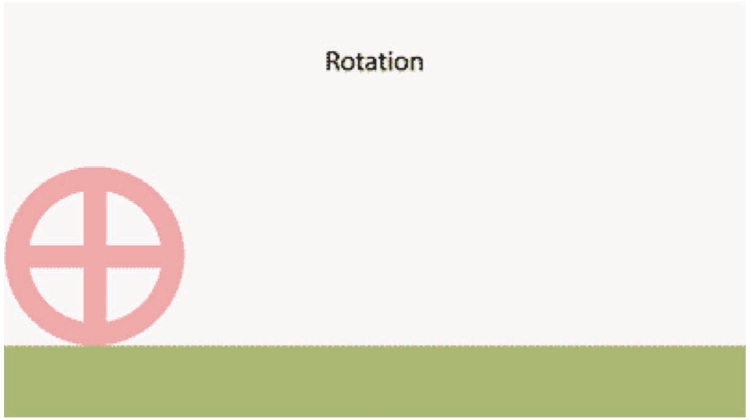
Superposition & Linearity

Brief excursion to Wikipedia (re *superposition principle*)

The superposition principle, also known as superposition property, states that, for all linear systems, the net response caused by two or more stimuli is the sum of the responses that would have been caused by each stimulus individually. So that if input A produces response X and input B produces response Y then input (A + B) produces response (X + Y).

→ Hard to overemphasize how profoundly important this idea is!!

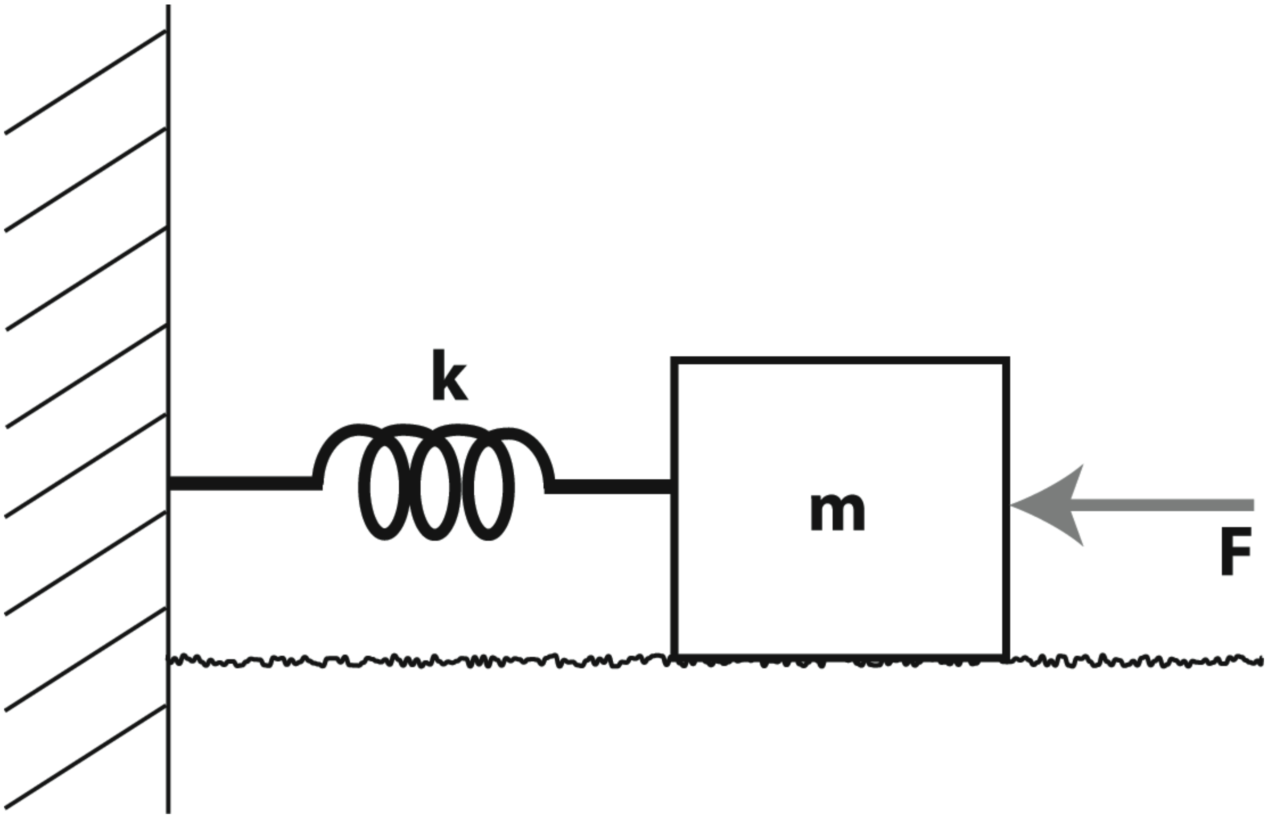
Superposition of almost plane waves (diagonal lines) from a distant source and waves from the wake of the ducks. Linearity holds only approximately in water and only for waves with small amplitudes relative to their wavelengths.



Rolling motion as superposition of two motions. The rolling motion of the wheel can be described as a combination of two separate motions: translation without rotation, and rotation without translation.

Wikipedia

Superposition & Linearity



→ When dealing with linear oscillators (or linear systems in general), superposition takes a domineering position in how we approach analysis and modeling

Superposition

Let us consider an example:

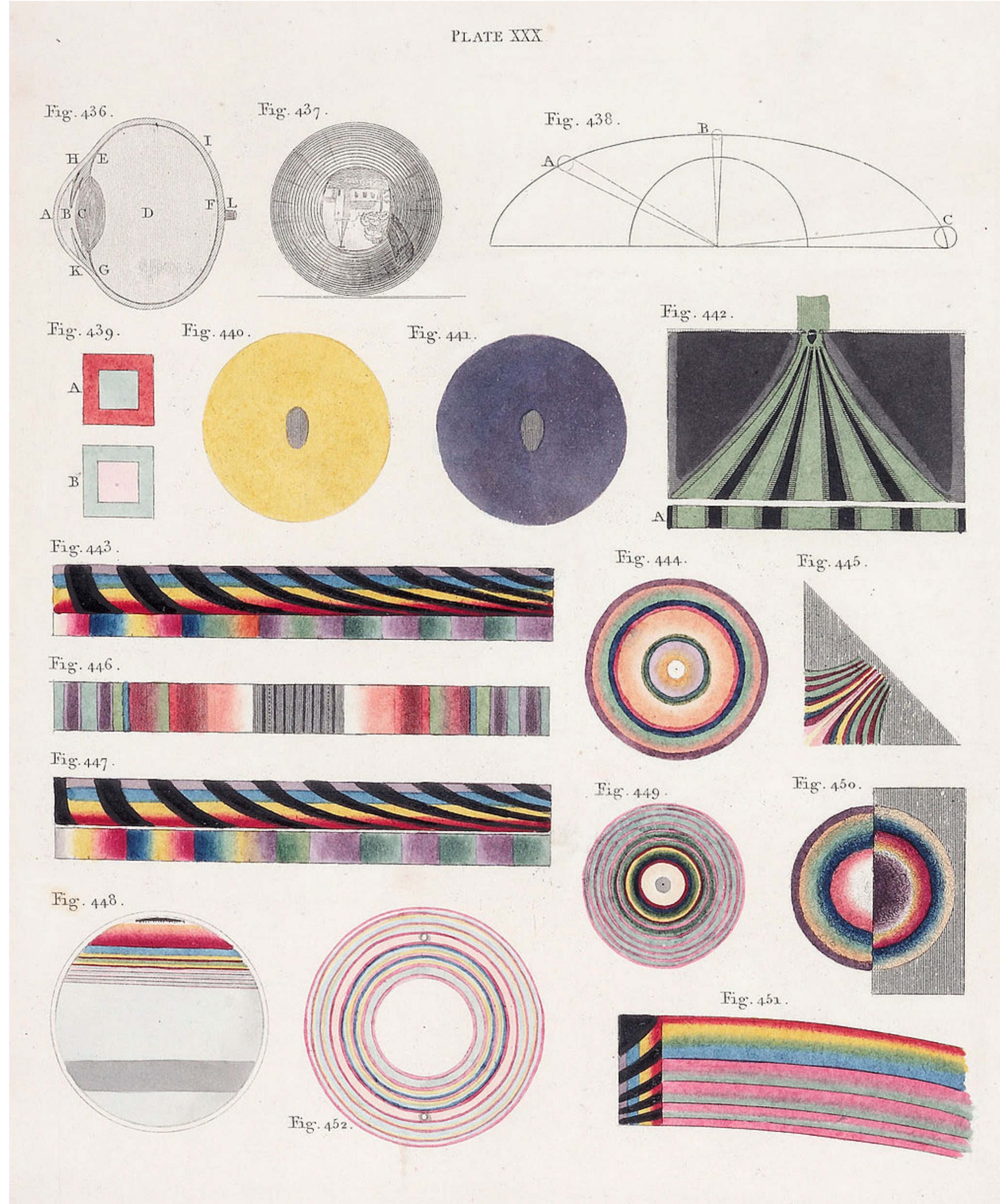
Interference between identical but spatially separated "sources"

***Fig. 2–2** Array to detect phase difference as function of microphone position in the superposition of signals from two loudspeakers.*

French (1971)

Superposition

Thomas Young (1773-1829)

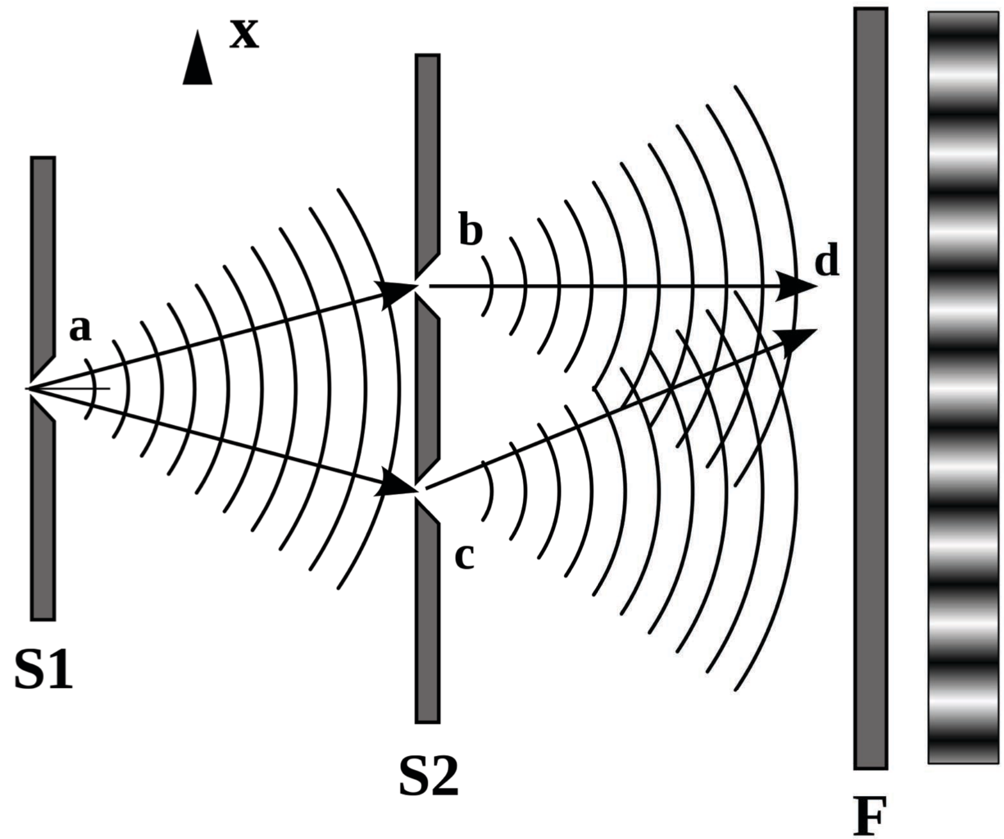


Wikipedia

Superposition

Thomas Young (1773-1829)

"Double-slit experiment"



→ Wave theory of light

(via *phase interference*)

Wikipedia

Superposition

Each source:

$$x_1 = A_1 \cos(\omega t + \alpha_1)$$
$$x_2 = A_2 \cos(\omega t + \alpha_2)$$

Their sum (at the mic)

$$x = x_1 + x_2 = A_1 \cos(\omega t + \alpha_1) + A_2 \cos(\omega t + \alpha_2)$$

$$= A \cos(\omega t + \alpha)$$

A tad messy to solve for the constants there....

If instead we used complex exponentials:

$$z_1 = A_1 e^{j(\omega t + \alpha_1)}$$
$$z_2 = A_2 e^{j(\omega t + \alpha_2)}$$

Then:

$$z = z_1 + z_2 = A_1 e^{j(\omega t + \alpha_1)} + A_2 e^{j(\omega t + \alpha_2)}$$

$$= e^{j(\omega t + \alpha_1)}[A_1 + A_2 e^{j(\alpha_2 - \alpha_1)}]$$

→ This latter equation, while seemingly intimidating, tells us a lot!

French (1971)

Superposition

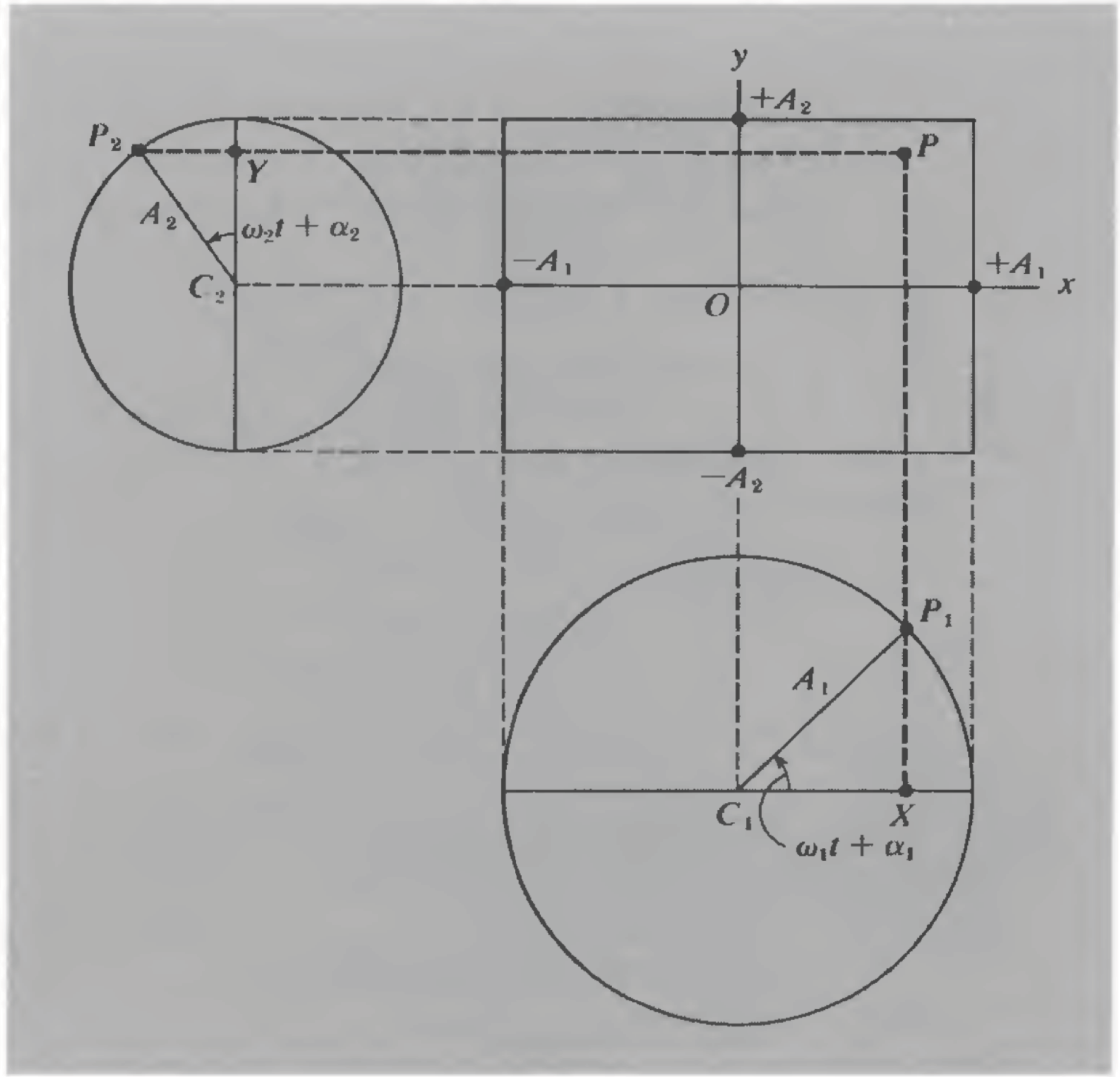


Fig. 2–8 Geometrical representation of the superposition of simple harmonic vibrations at right angles.

French (1971)

Superposition

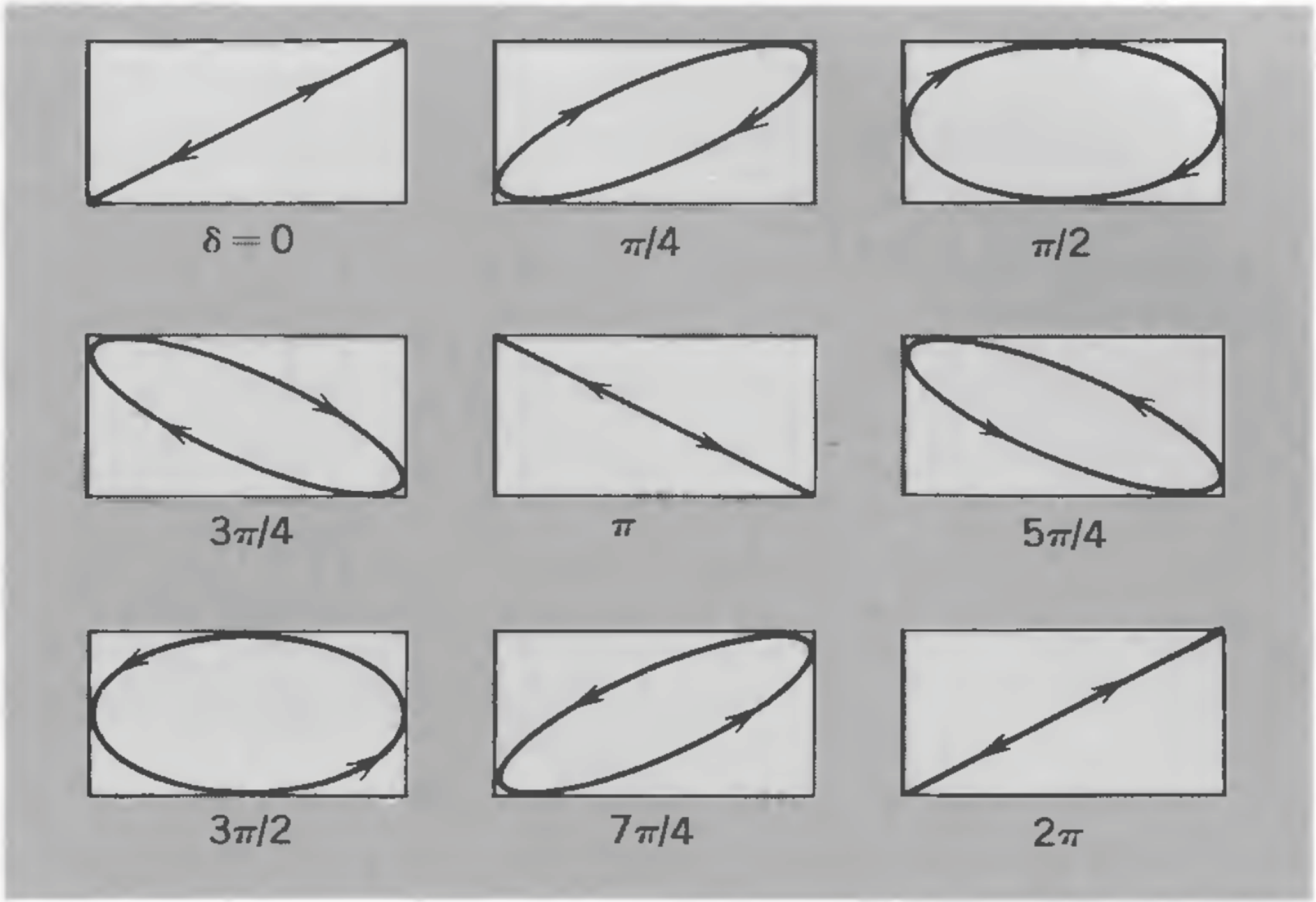


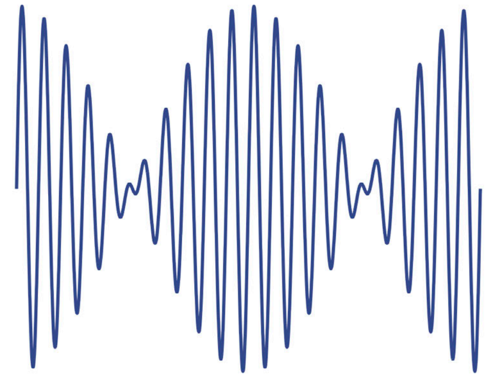
Fig. 2–10 Superposition of two perpendicular simple harmonic motions of the same frequency for various initial phase differences.

French (1971)

Superposition: "Beats" (adding two different frequencies)

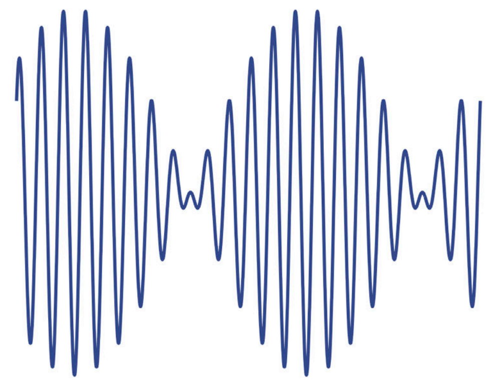
$$x(t) = A_1 \sin(2\pi f_1 t + \phi_1) + A_2 \sin(2\pi f_2 t + \phi_2)$$

$f_1=1,\ f_2=1.1$
$A_1=1,\ A_2=1$
$\phi_1=0,\ \phi_2=0$



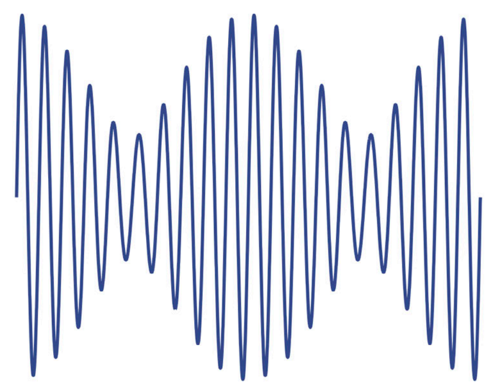
→ Changing (relative) phase affects summation

$f_1=1,\ f_2=1.1$
$A_1=1,\ A_2=1$
$\phi_1=\pi/2,\ \phi_2=0$



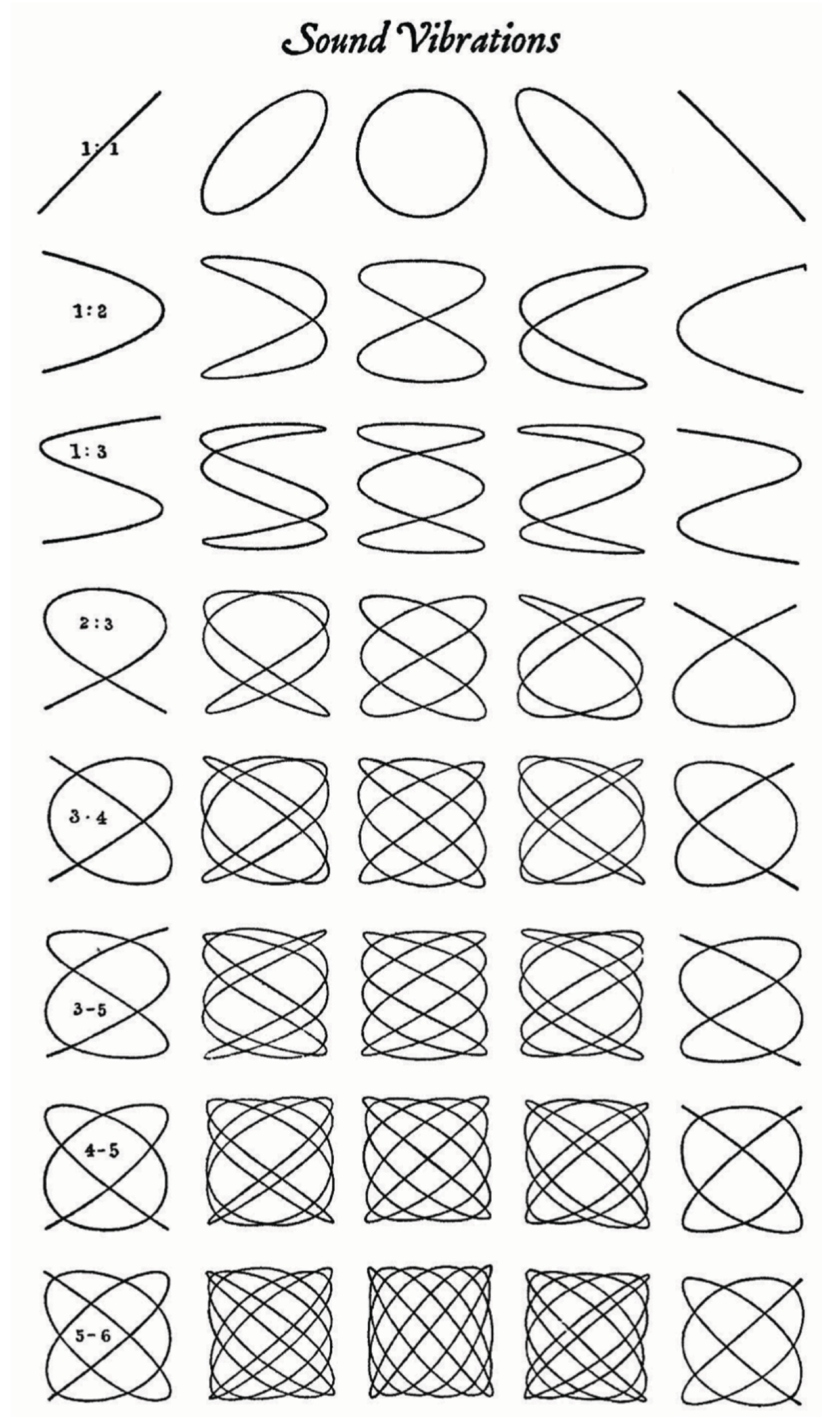
→ Changing (relative) amplitudes affects summation

$f_1=1,\ f_2=1.1$
$A_1=2,\ A_2=1$
$\phi_1=0,\ \phi_2=0$

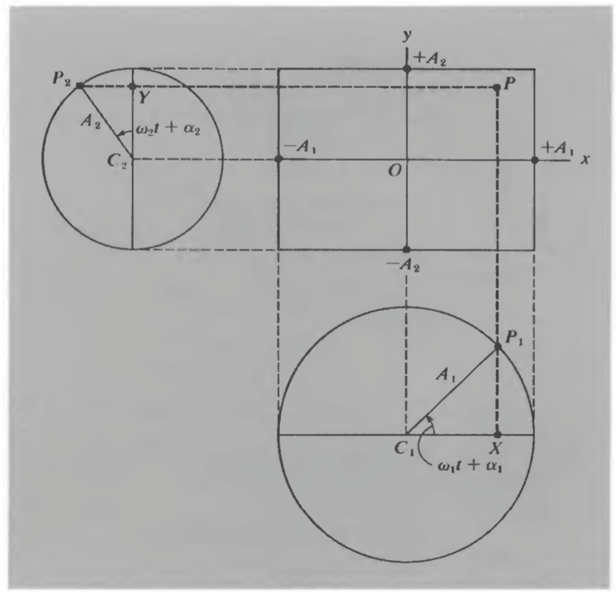


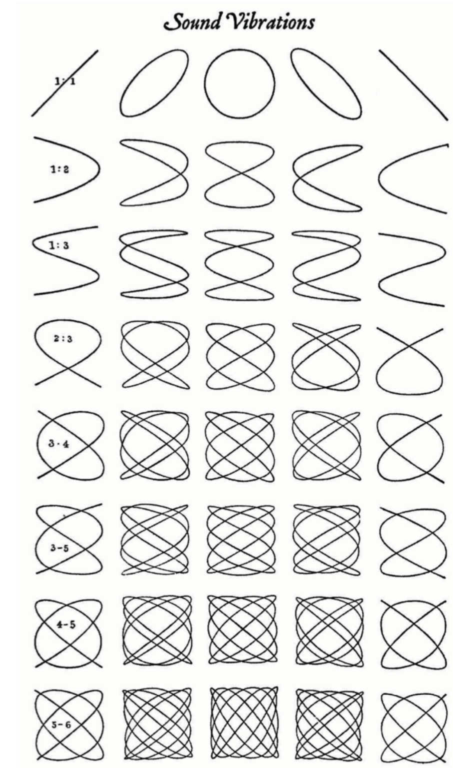
Tangent I (re Adding Sinusoids): Lissajou

Jules Antoine Lissajous (1822-1880)



Tangent I (re Adding Sinusoids): Lissajou





PHYS 2030 (Winter 2018) - HW 4

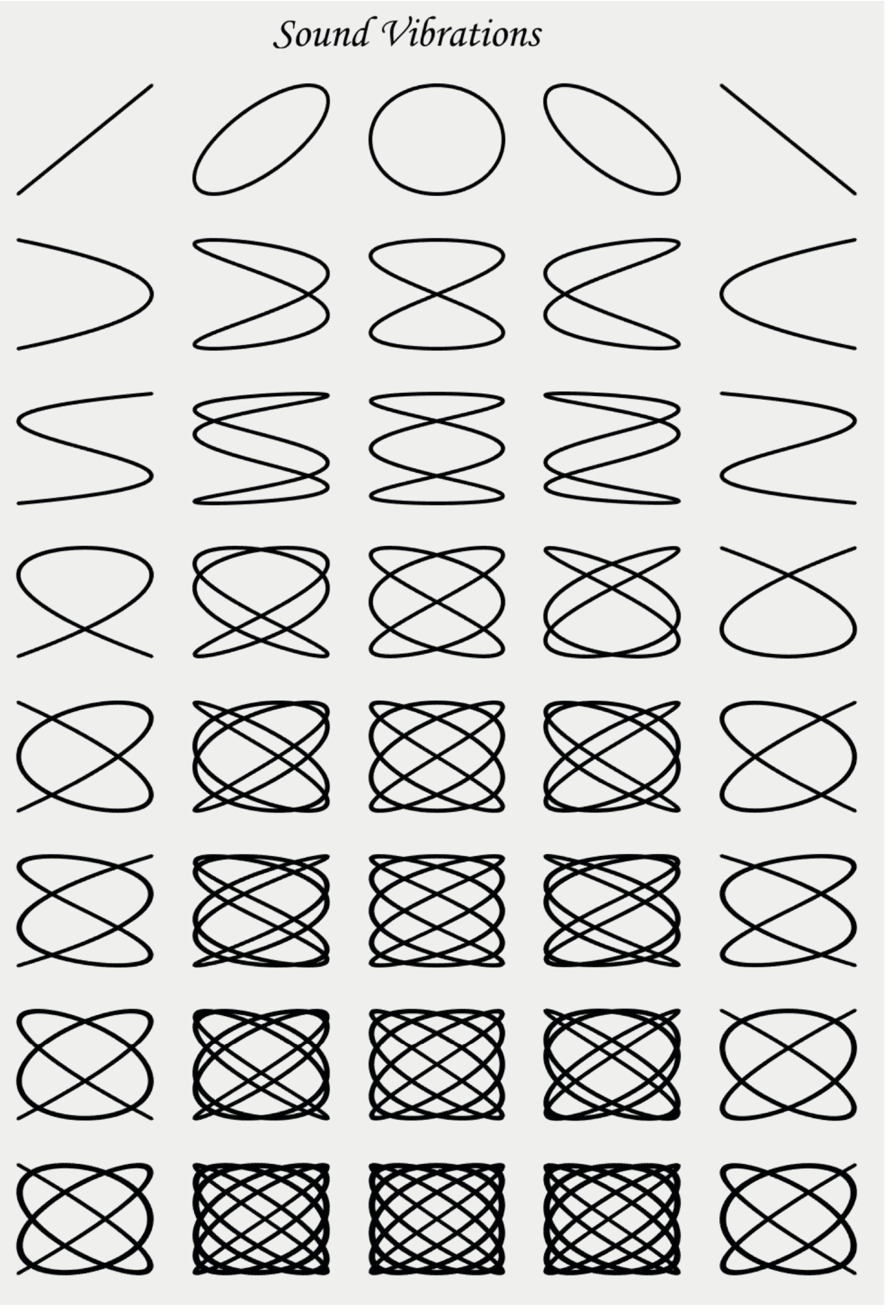
1. [2 points] The following image was found on a greeting card. Write a Matlab code to reproduce the figure as closely as possible. Your answer should include your code, as well as the produced figure. Your code should plot all of them in the same `Figure` window (i.e., `subplot` might be useful!). Also provide a brief explanation as to how such curves might be produced through mechanical means[1].

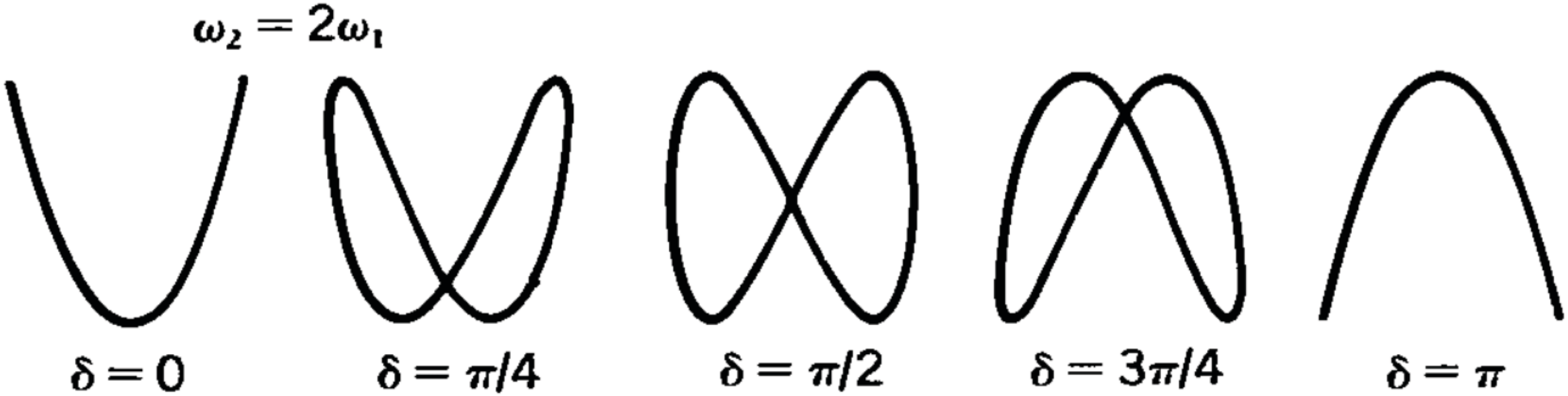
---

[1] If you are unsure how to start, perhaps look up a bit of history on *Nathaniel Bowditch*.

```
% ### EXlissajou2B.m ###      2018.02.14      C. Bergevin
clear
% ======================================================
In.tMax= 100;   % max. time to compute out too {100}
In.tS= 1000;    % total # of time steps {1000}
In.delta= pi*[0 0.25 0.5 0.75 1;  % phase diff. vals
              0.75 0.8750 1 1.125 1.25;
              1 0.25 0.5 0.75 0;
              0.5 1.25 1 0.75 1.5;
              0.8750 0.8125 0.75 0.6875 0.625;
              1 0.25 0.5 0.75 0;
              1.5 0.75 1 1.25 0.5;
              0.25 0.515 0.5 0.485 0.75];
In.freqs= [1 1;1 2;1 3;2 3;3 4;3 5;4 5;5 6];    % freqs (a and b) for two sinusoids
In.mag= [1 1];  % mags. (A and B) for two sinusoids {[1 1]}
In.flip= [1 -1 -1 1 1 1 1 1];  % flip for x re "2018.02 lissajou.pdf"
% ======================================================
t= linspace(0,In.tMax,In.tS);
figure(1); clf; hold on; title('Sound Vibrations')
nH= size(In.delta,2);  % # of phases to compute/plot
nV= size(In.freqs,1);  % # of ratios to compute/plot
indx= 1;    % dummy indexer for subplot
% ---
for nn=1:nV
    for mm=1:nH
        x= In.mag(1)*sin(In.freqs(nn,1)*t + In.delta(nn,mm));
        y= In.mag(2)*sin(In.freqs(nn,2)*t);
        subplot(nV,nH,indx);
        plot(y,x,'k-','LineWidth',2);
        indx= indx+ 1;
        ax = gca; ax.Visible = 'off';
    end
end
% --- snippet below is for the title (borrowed some bits from mathworks.com)
ha= axes('Position',[-0.135 -0.02 1 1],'Xlim',[0 1],'Ylim',[0
1],'Box','off','Visible','off','Units',...
    'normalized','clipping','off');
text(0.5,0.98,'Sound Vibrations','FontSize',24,'FontWeight','bold','FontName','Monotype Corsiva');
```

Tangent I: Lissajou



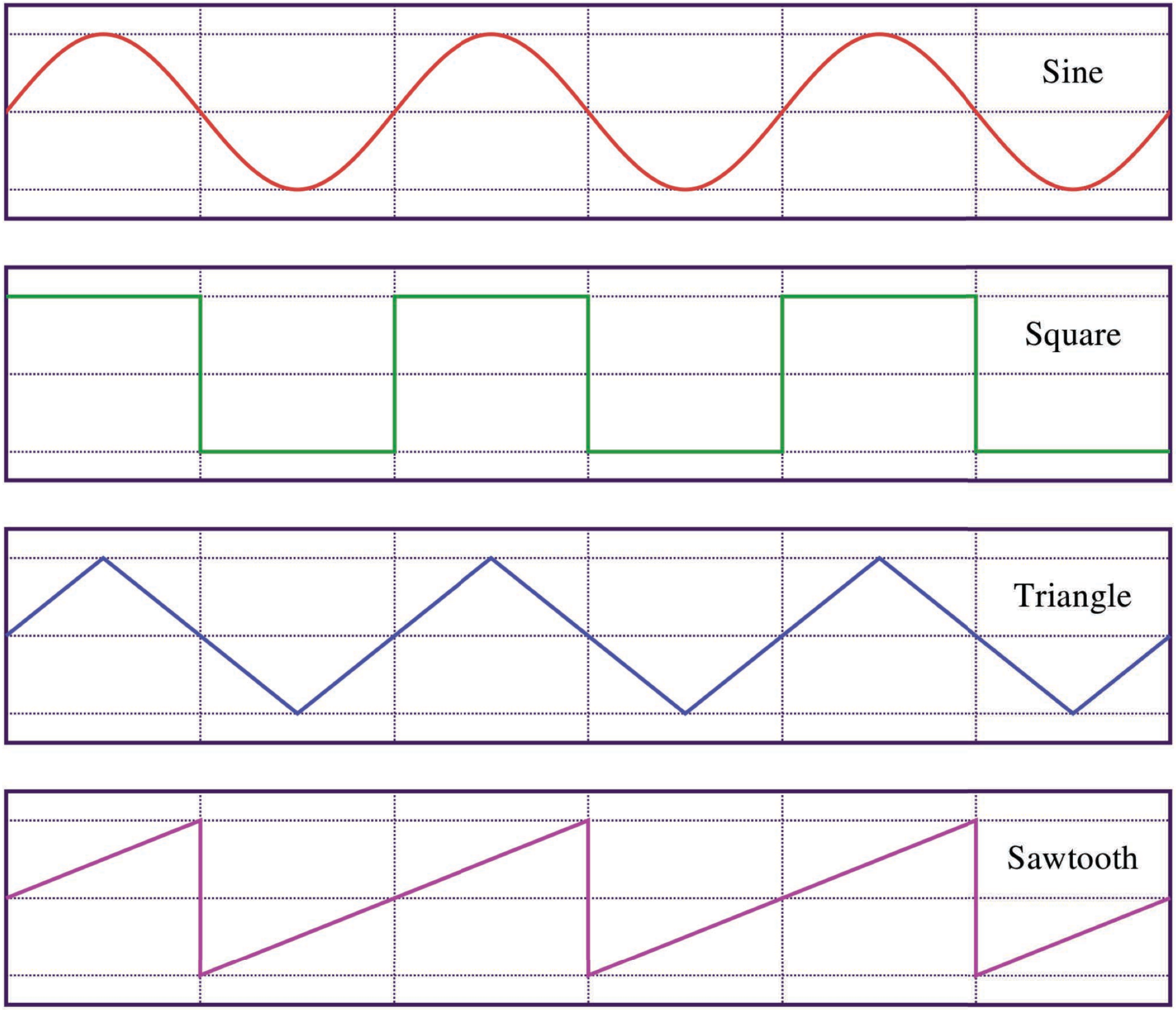


Tattoo design for physics nerds?

Fig. 2–13 Lissajous figures for $\omega_2 = 2\omega_1$ with various initial phase differences.

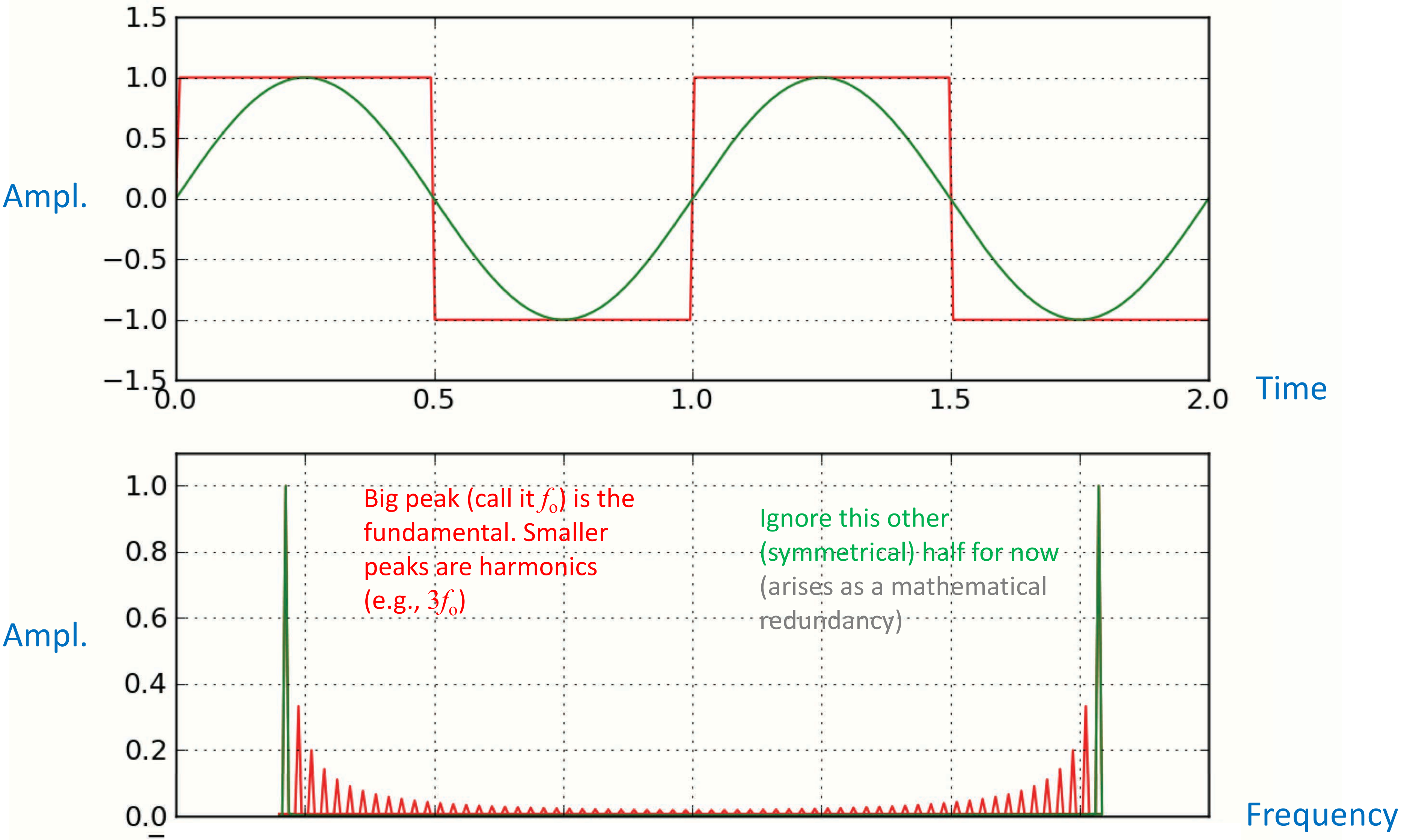
French (1971)

Tangent II (re Adding Sinusoids): Square Waves



Wikipedia

Tangent II (re Adding Sinusoids): Square Waves



wikipedia (square wave)

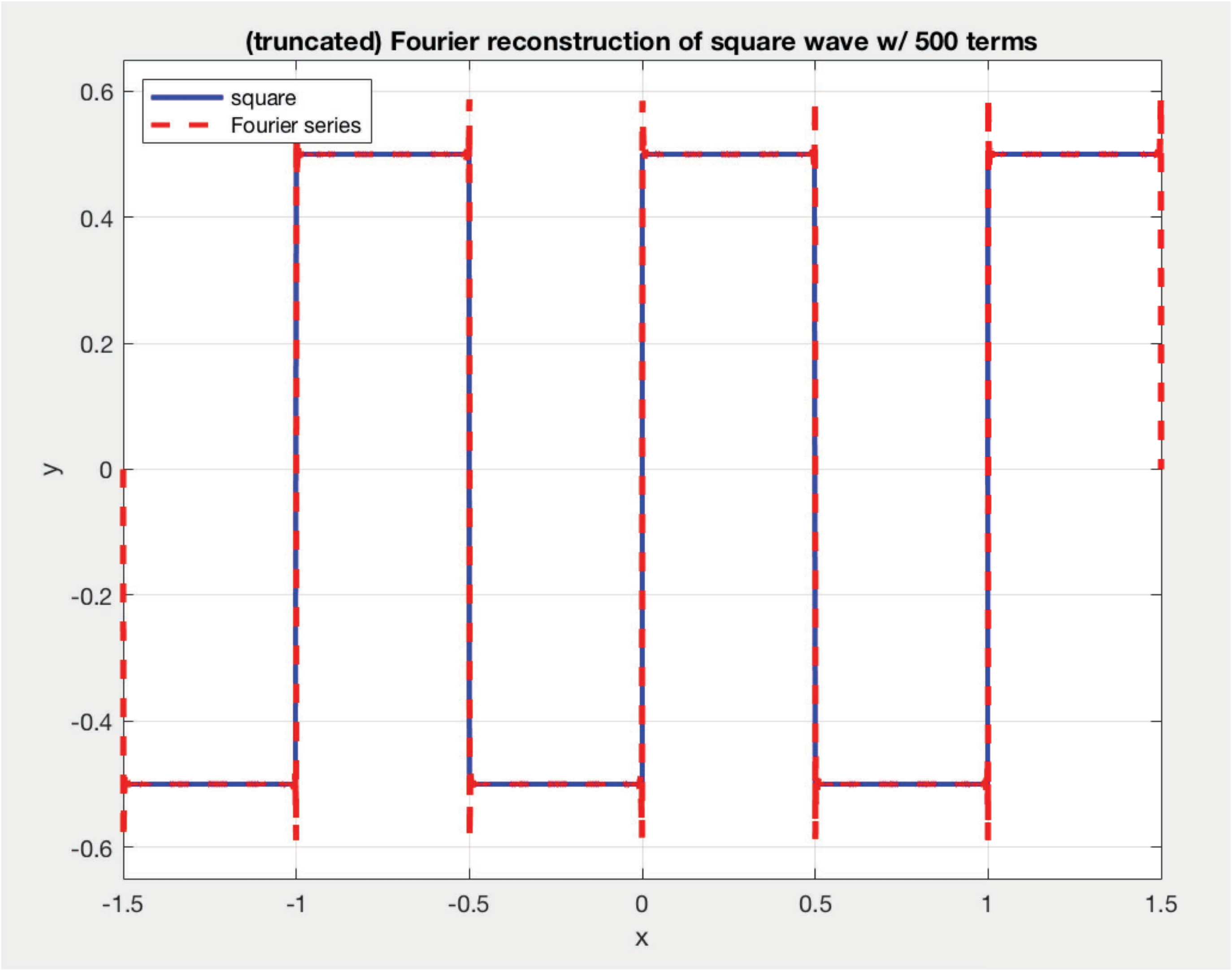
```
% ### EXsquareW.m ###    2017.02.19 C.Bergevin

% Visually demonstrate the build-up of a square wave
% by adding successive (user-specified) terms of the Fourier series
% expansion; also quantifies the Gibbs phenomonon

clear
% ==============================================================
P.order= [1 2 4 8 15 25 100 500];    % array of # of terms to compute {[1 2 4 8 15 25 100 500]}
P.tau= 1;    % period {1}
P.A= 1;      % peak-to-peak amplitude {1}
P.M= 10000;    % total # of points per interval (must be even?) {1000}
P.pause= 0.5;    % time to pause between displaying new iterates [s] {0.5}
% ==============================================================
t= linspace(-1.5*P.tau,1.5*P.tau,3*P.M);  % time array
squareT= repmat(P.A*([zeros(P.M/2,1);ones(P.M/2,1)]-0.5),3,1)';    % create sawtooth baseline
% --- use a loop to add in the terms
for mm=1:numel(P.order)
    tempN= P.order(mm);
    squareF= 0;    % dummy initial indexer
    for nn= 1:tempN
        nextTerm= (2*P.A/pi)*(1/(2*nn-1))*sin((2*nn-1)*(2*pi/P.tau)*t);  % create next term in
series
        squareF= squareF+ nextTerm;
    end
    % --- estimate "overshoot"
    [M,indx]= max(squareF);
    disp(['Overshoot ratio ~',num2str(M/max(squareT))]);
    % --- visualize
    figure(1); clf;
    h1= plot(t,squareT,'b-','LineWidth',2); hold on; grid on; xlabel('x'); ylabel('y');
ylim(0.65*P.A*[-1 1]);
    h2= plot(t,squareF,'r--','LineWidth',2); legend([h1 h2],'square','Fourier
series','Location','NorthWest');
    title(['(truncated) Fourier reconstruction of square wave w/ ',num2str(tempN),' terms']);
    pause(P.pause);
end
```

Tangent II (re Adding Sinusoids): Square Waves

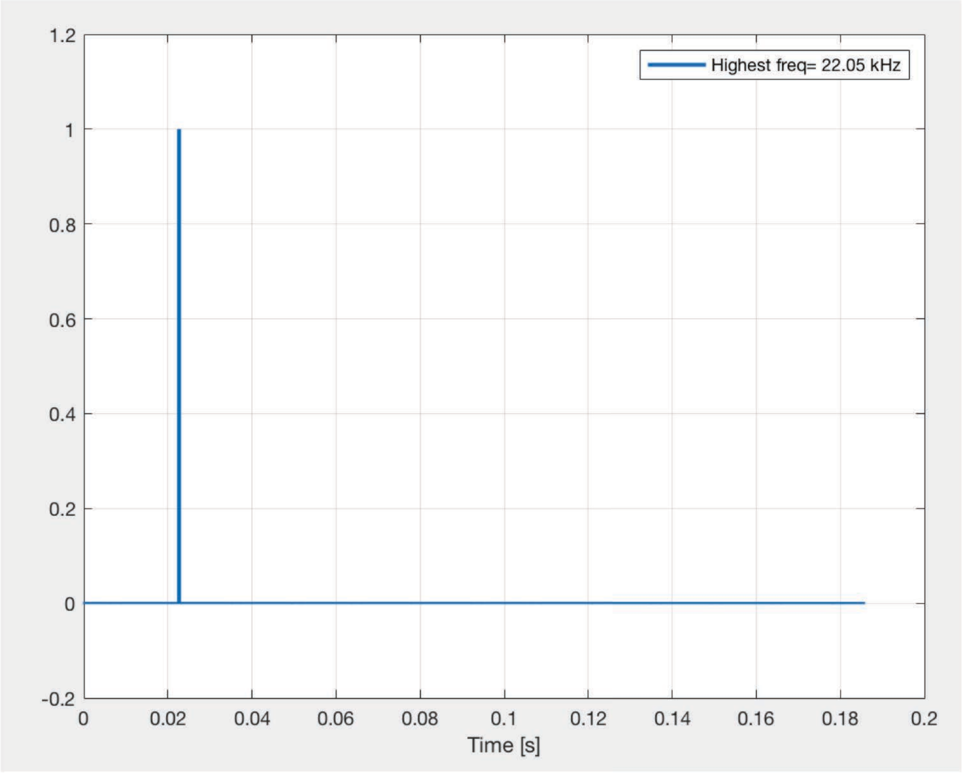
EXsquareW.m



Tangent III (re Adding Sinusoids): Impulses

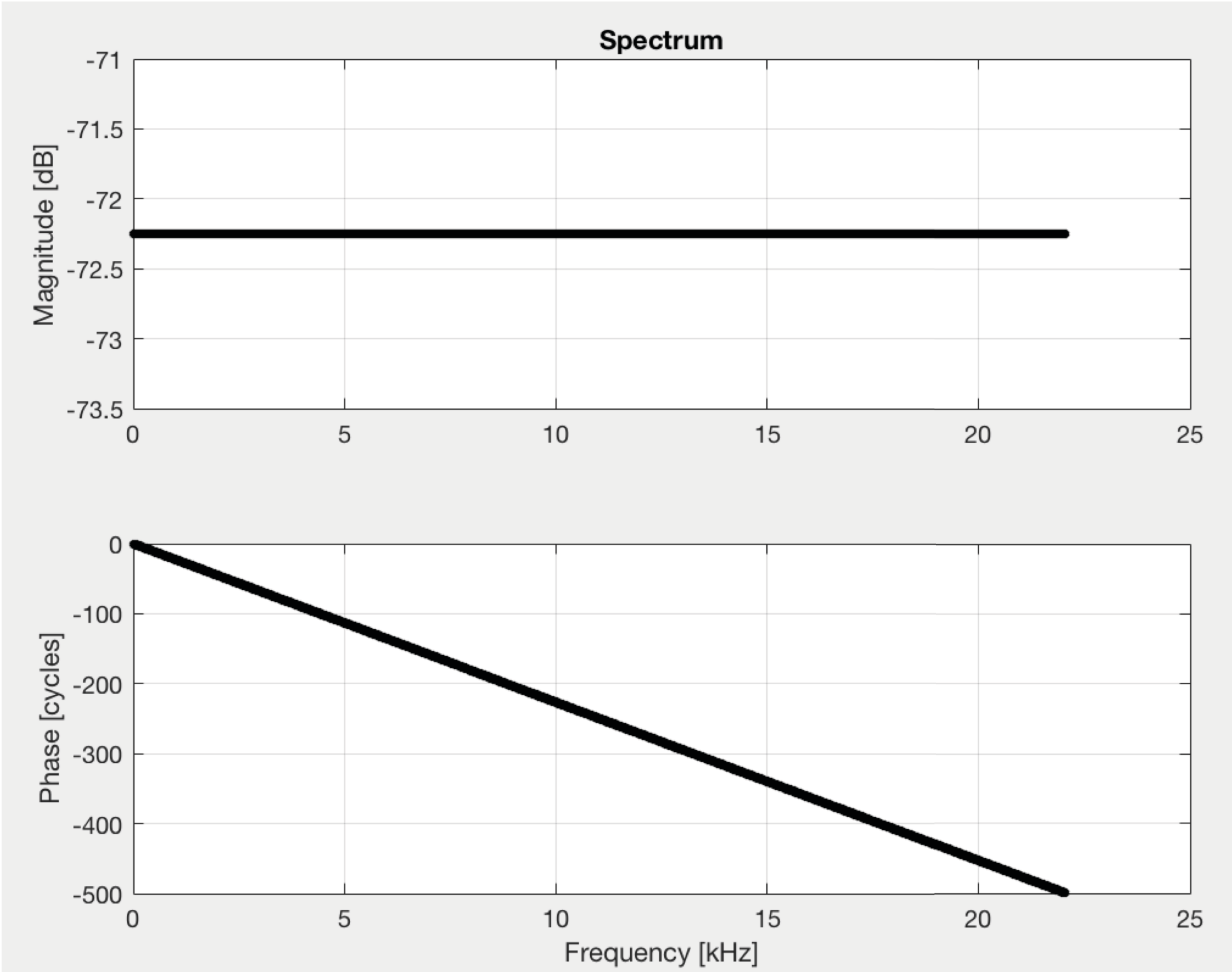
$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$



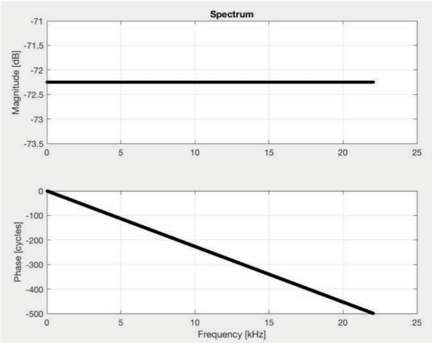
Delta functions are much more ubiquitous in your lives (and science) than you might think/realize......

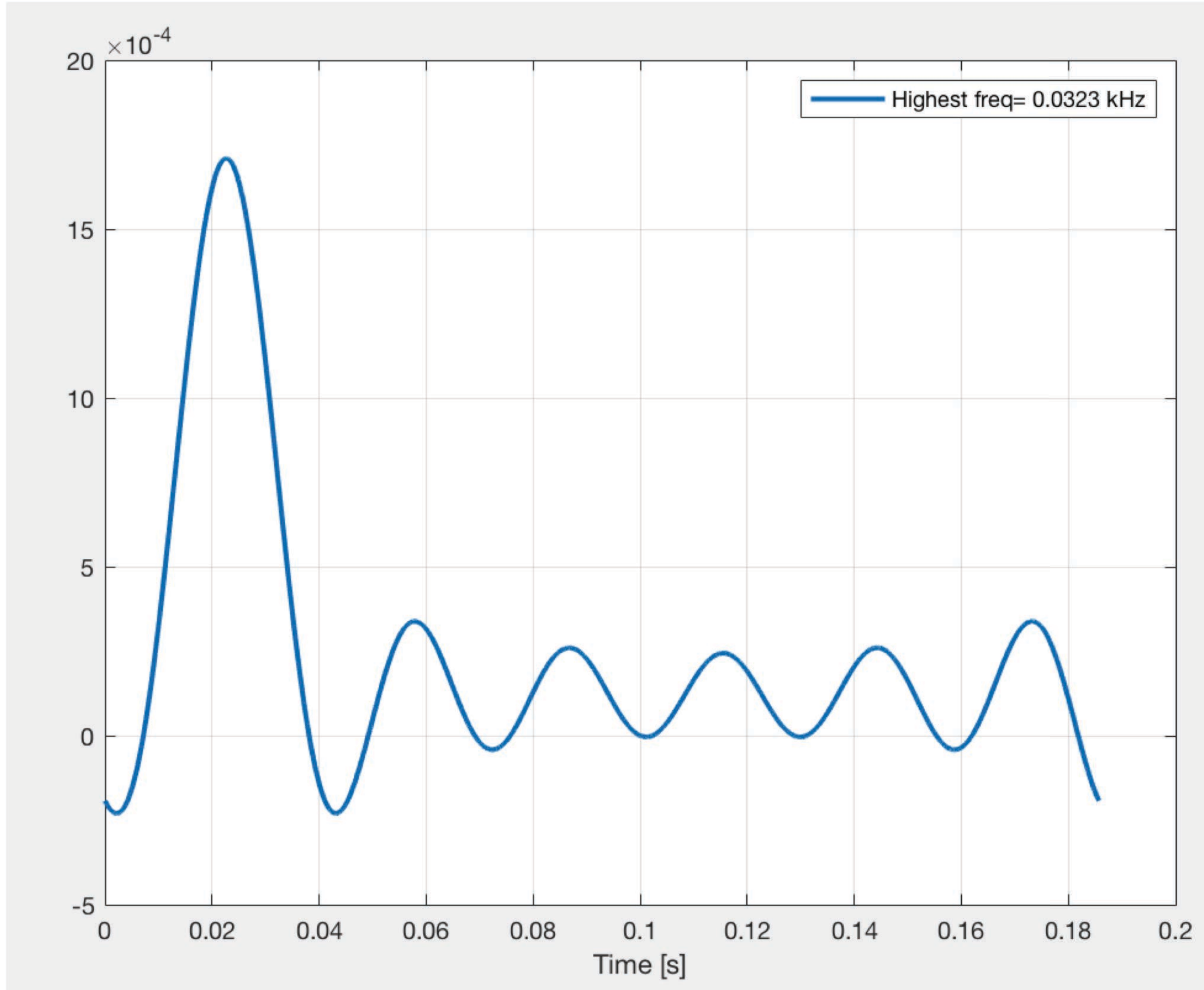
```
% ### EXbuildImpulse2.m ###       11.03.14
clear; clf;
% --------------------------------
SR= 44100;          % sample rate [Hz]
Npoints= 8192;      % length of fft window (# of points) [should ideally be 2^N]
                    % [time window will be the same length]
INDXon= 1000;      % index at which click turns 'on' (i.e., go from 0 to 1)
INDXoff= 1001;    % index at which click turns 'off' (i.e., go from 1 to 0)
% --------------------------------
dt= 1/SR;  % spacing of time steps
freq= [0:Npoints/2];     % create a freq. array (for FFT bin labeling)
freq= SR*freq./Npoints;
t=[0:1/SR:(Npoints-1)/SR]; % create an appropriate array of time points
% build signal
clktemp1= zeros(1,Npoints); clktemp2= ones(1,INDXoff-INDXon);
signal= [clktemp1(1:INDXon-1) clktemp2 clktemp1(INDXoff:end)];
% ------------------------------
% *******
% plot "final" time waveform of signal
if 1==1
    figure(3); clf; plot(t*1000,signal,'ko-','MarkerSize',5)
    grid on; hold on; xlabel('Time [ms]'); ylabel('Signal'); title('Time Waveform')
end
% *******
% now compute/plot FFT of the signal
sigSPEC= rfft(signal);
% MAGNITUDE
figure(1); clf;
subplot(211); plot(freq/1000,db(sigSPEC),'ko-','MarkerSize',3)
hold on; grid on; ylabel('Magnitude [dB]'); title('Spectrum (or "Look Up Table")')
% PHASE
subplot(212); plot(freq/1000,cycs(sigSPEC),'ko-','MarkerSize',3)
xlabel('Frequency [kHz]'); ylabel('Phase [cycles]'); grid on;
% *******
% now make animation of click getting built up, using the info from the FFT
sum= zeros(1,numel(t)); % (initial) array for reconstructed waveform
inclV=[1:30,floor(linspace(31,floor(0.9*numel(freq)),100)),...
    floor(linspace(0.9*numel(freq),numel(freq),20))];
figure(2); clf; grid on;
for nn=1:numel(freq)
    sum= sum+ abs(sigSPEC(nn))*cos(2*pi*freq(nn)*t + angle(sigSPEC(nn)));
    if ismember(nn,inclV),  plot(t,sum,'LineWidth',2); grid on; xlabel('Time [s]');
    legend(['Highest freq= ',num2str(freq(nn)/1000),' kHz'])
    pause(3/(nn));  end
end
```



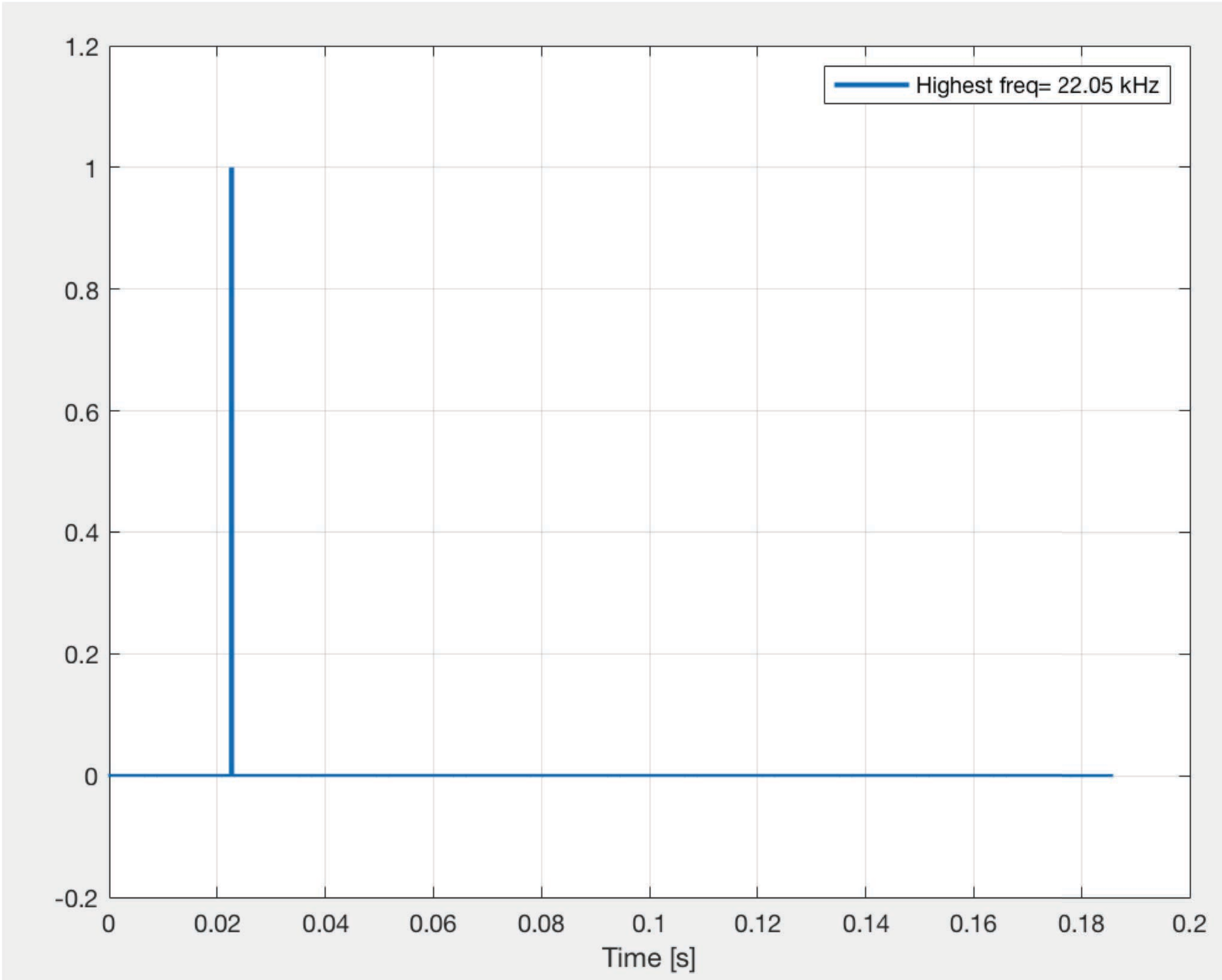
Spectrum
Magnitude [dB]
-71
-71.5
-72
-72.5
-73
-73.5
0
5
10
15
20
25
Phase [cycles]
0
-100
-200
-300
-400
-500
0
5
10
15
20
25
Frequency [kHz]

EXbuildImpulse2.m

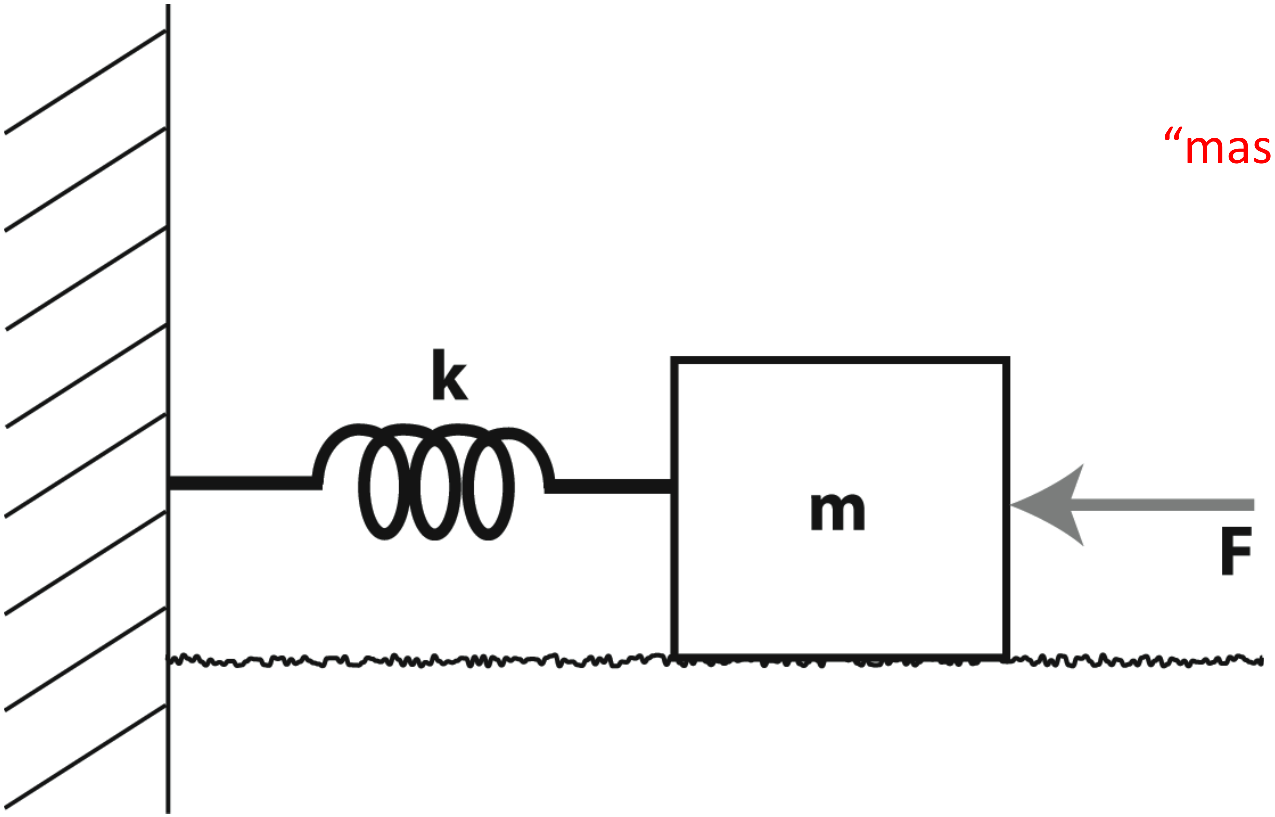




EXbuildImpulse2.m



Tying this all back together....



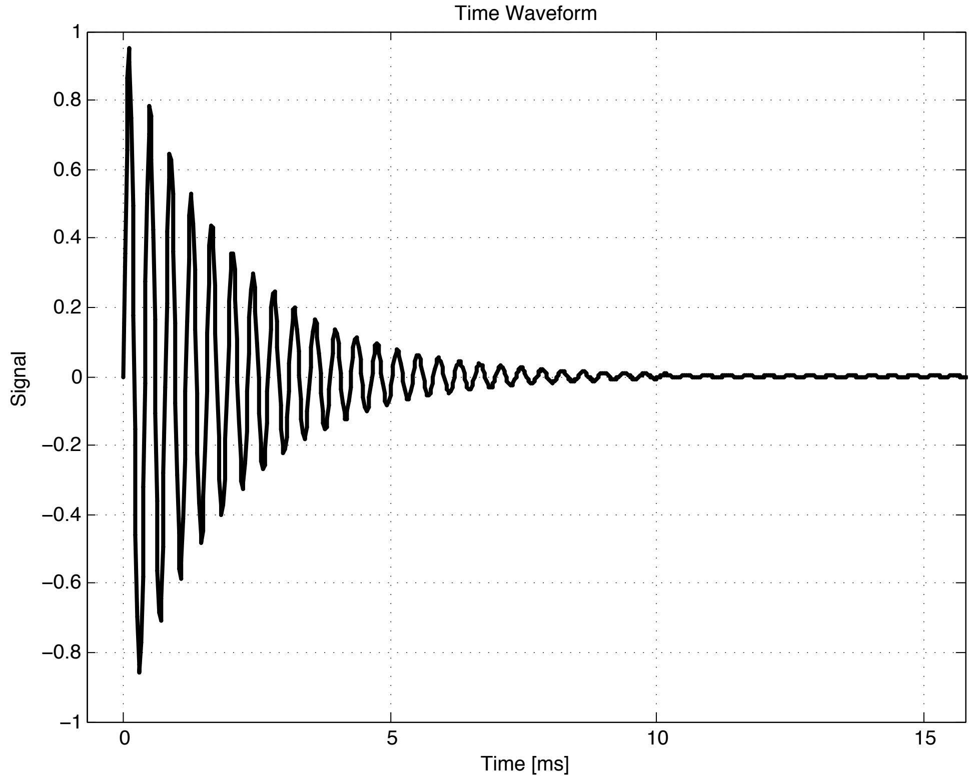
"mass-on-a-spring"

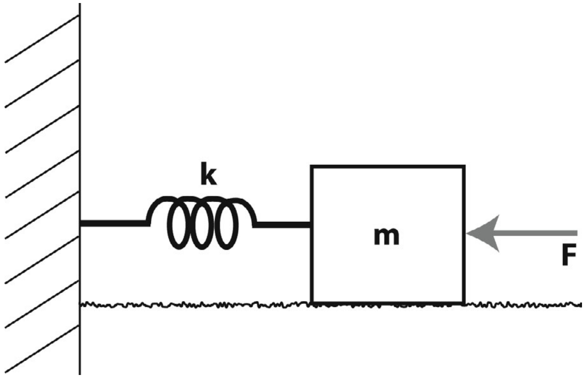
- One of the more fundamental/canonical problems in all areas of physics...

Looking Ahead....

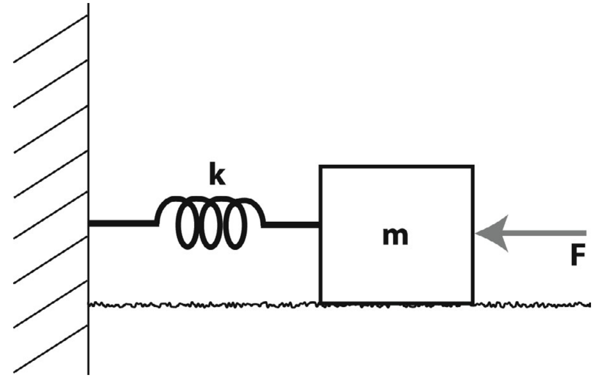
Damped HO

(given an "impulse" push at $t = 0$)



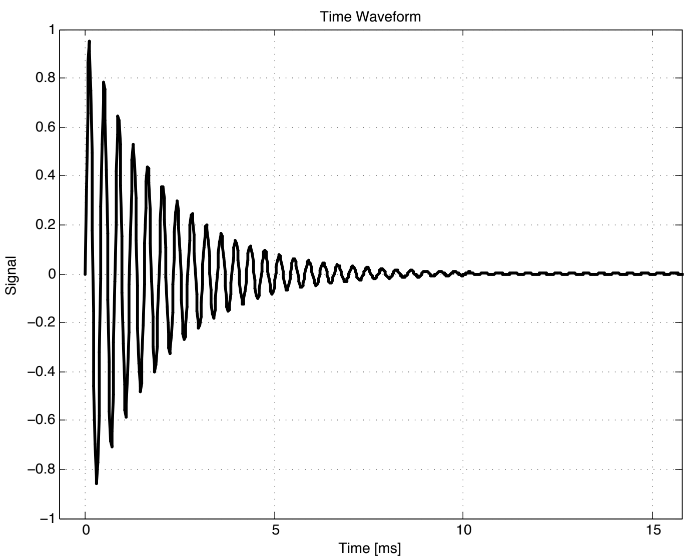


Looking Ahead....

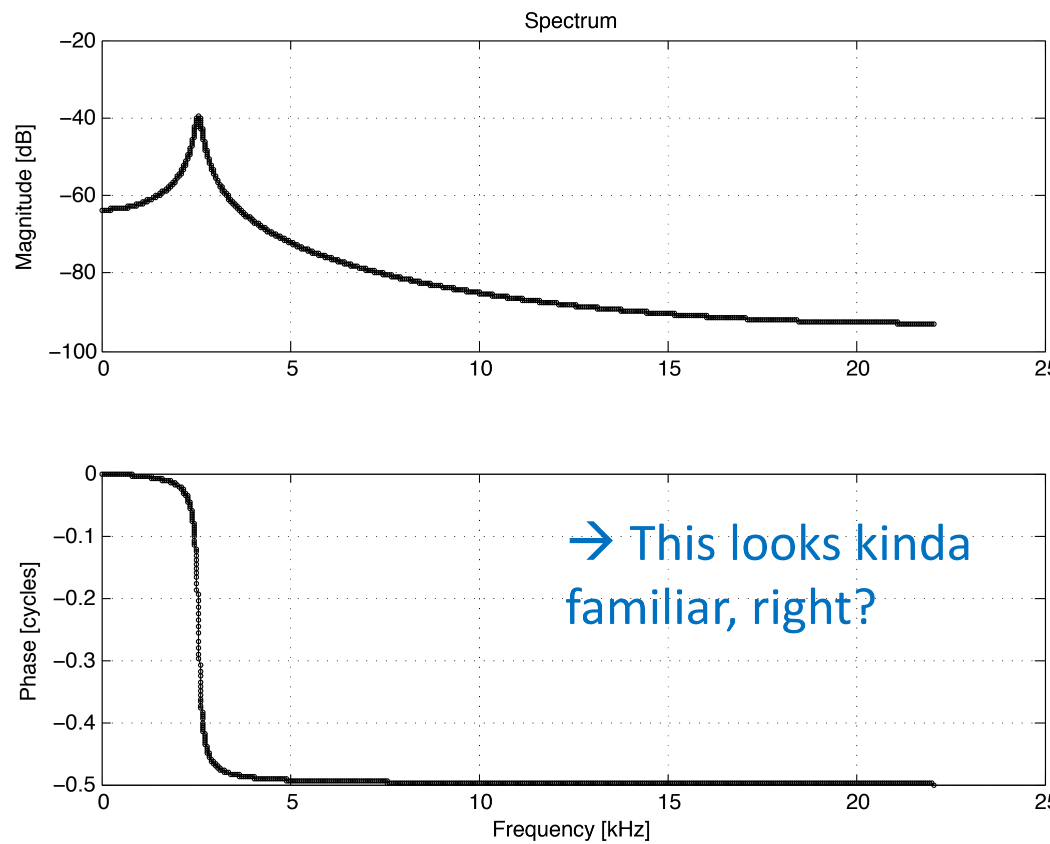


Damped HO

(given an "impulse" push at $t = 0$)



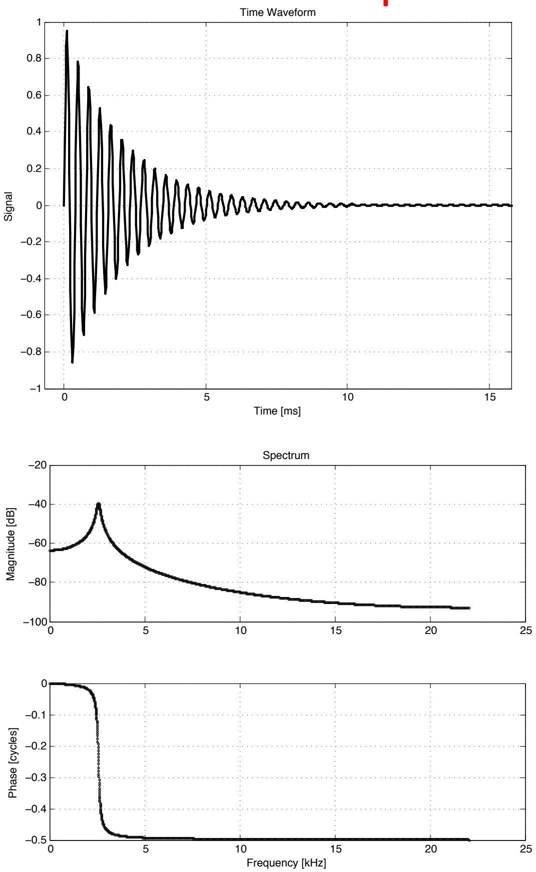
Corresponding "spectral" version

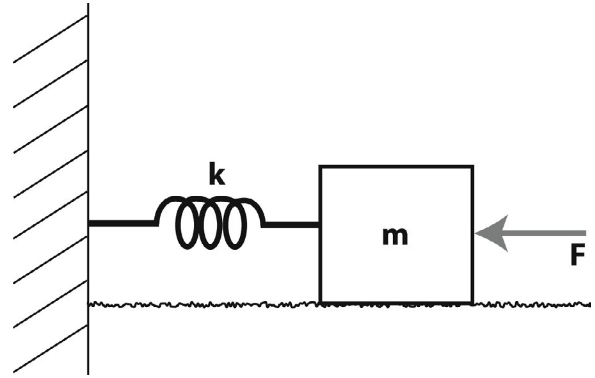


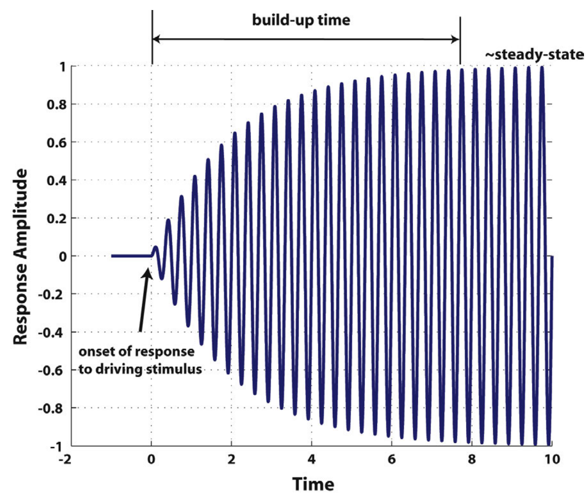
→ This looks kinda familiar, right?

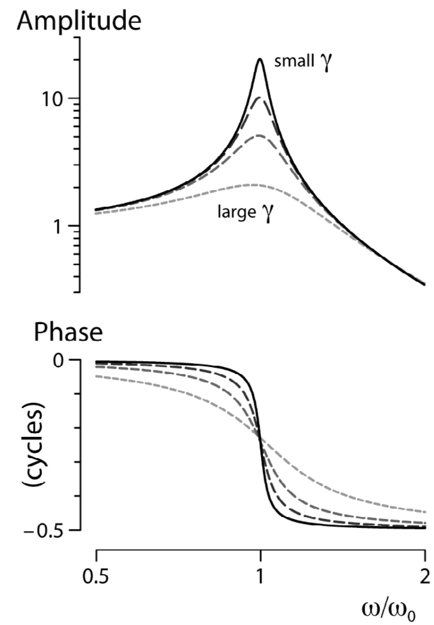
Looking Ahead....

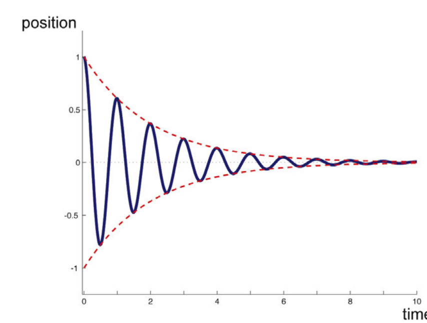
Damped HO











- The *steady-state* response of the sinusoidally-driven harmonic harmonic oscillator acts like a *band-pass filter*

- Connection between *steady-state* response & *impulse response*

SHO Revisted

Two essential features:

1. An inertial component, capable of carrying kinetic energy.
2. An elastic component, capable of storing elastic potential energy.

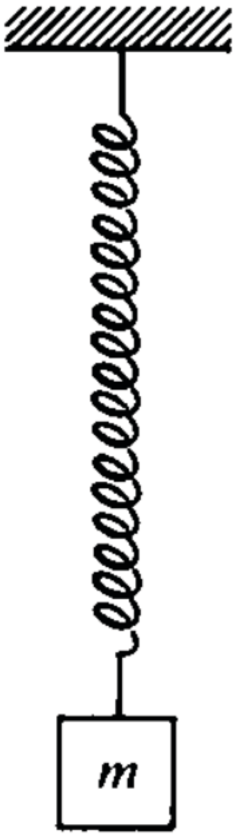
Two fundamental laws:

1. By Newton's law ($F = ma$),

$$-kx = ma$$

2. By conservation of total mechanical energy (E),

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = E$$



Two associated differential equations:

$$m\frac{d^2x}{dt^2} + kx = 0 \qquad \tfrac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \tfrac{1}{2}kx^2 = E$$

French (1971)

<u>SHO</u>: Complex exponentials

Rewrite in terms of natural frequency

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

Now assume a solution in the form of a (possibly) complex exponential:

$$x = Ce^{pt}$$

→ Let us pause for one moment, as we just made a big connection point to other classes/topics....

What mathematically does p represent?

Eigenvalue!

According to wikipedia:

Eigenvalues and eigenvectors feature prominently in the analysis of linear transformations. The prefix eigen- is adopted from the German word eigen for "proper", "characteristic". Originally utilized to study principal axes of the rotational motion of rigid bodies, eigenvalues and eigenvectors have a wide range of applications, for example in stability analysis, vibration analysis, atomic orbitals, facial recognition, and matrix diagonalization.

French (1971)