

# PHYS 2010 (W20)

## Classical Mechanics

**2020.02.11**

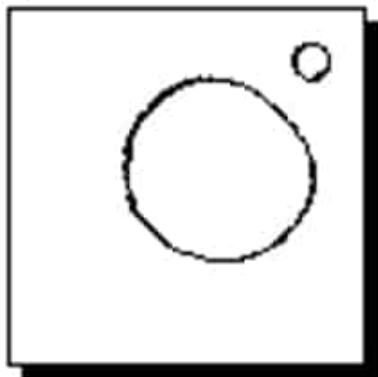
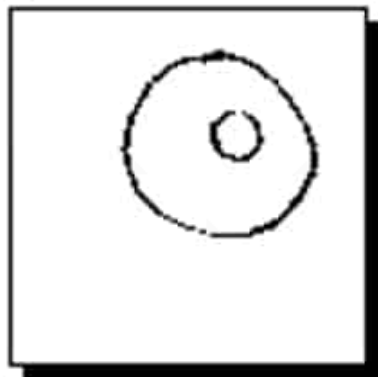
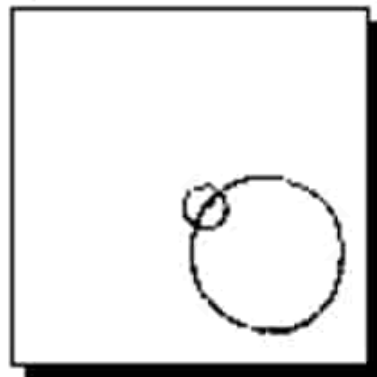
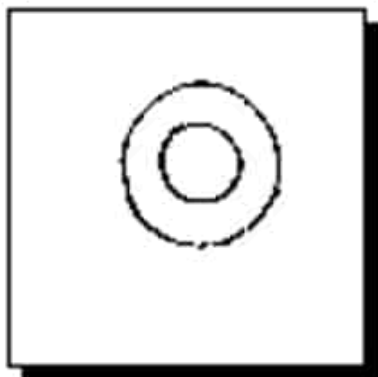
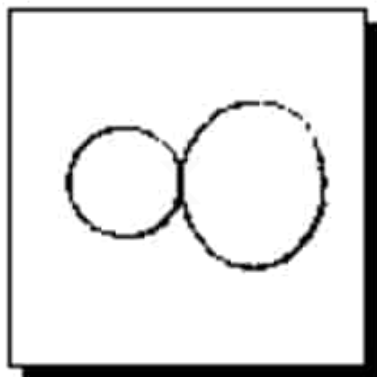
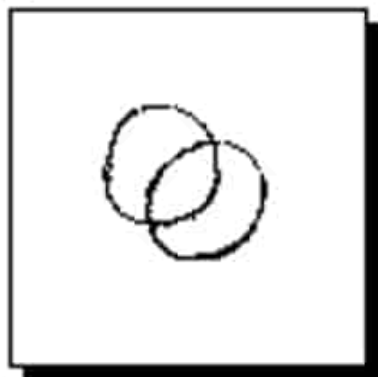
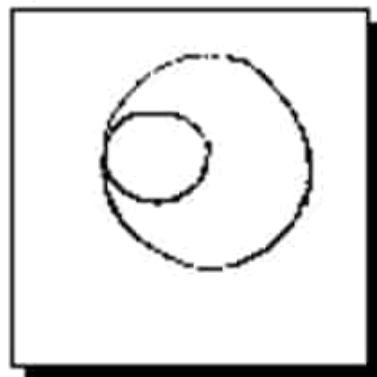
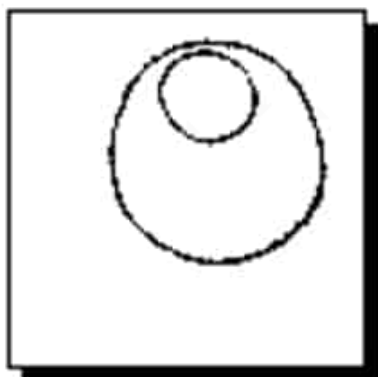
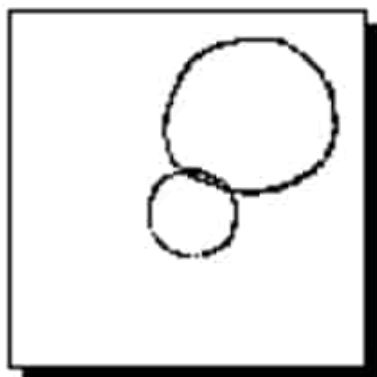
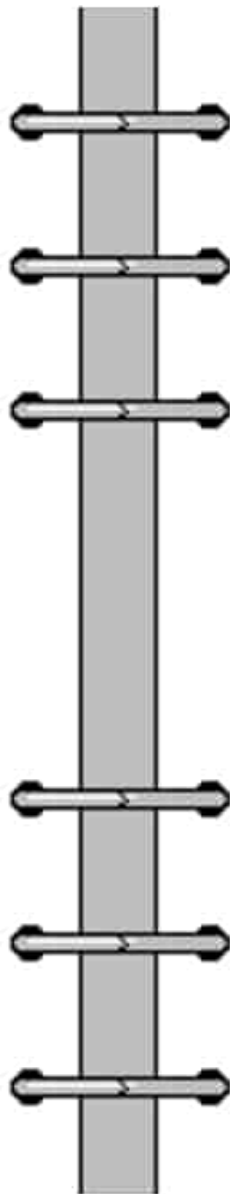
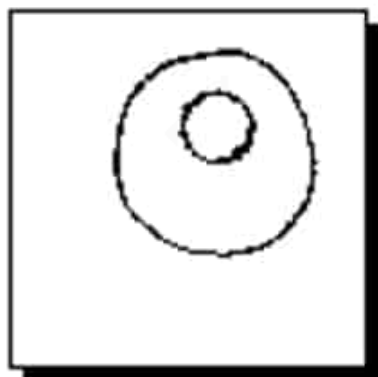
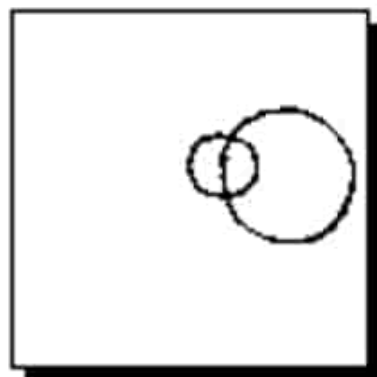
Relevant reading:

Knudsen & Hjorth: 15.ff, 16.2

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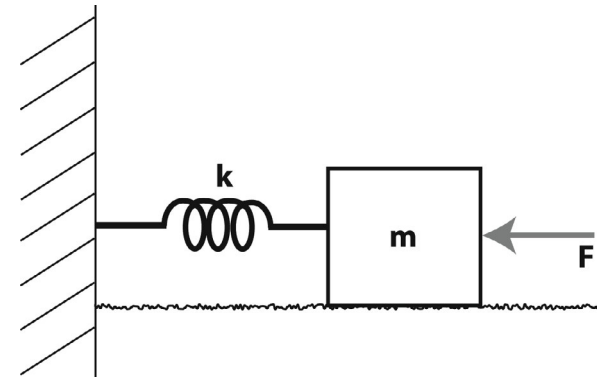
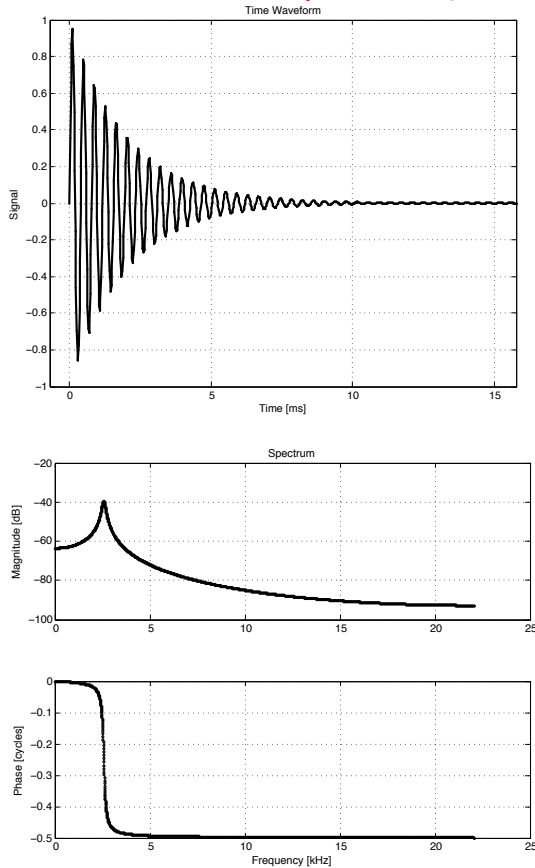
Ref.s:

Knudsen & Hjorth (2000), Fowles & Cassidy (2005)

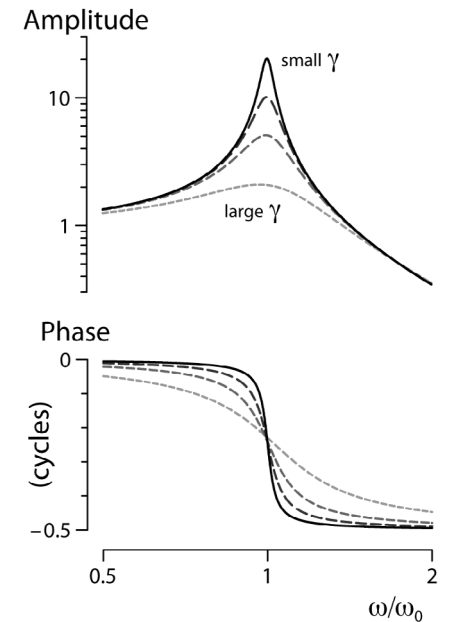
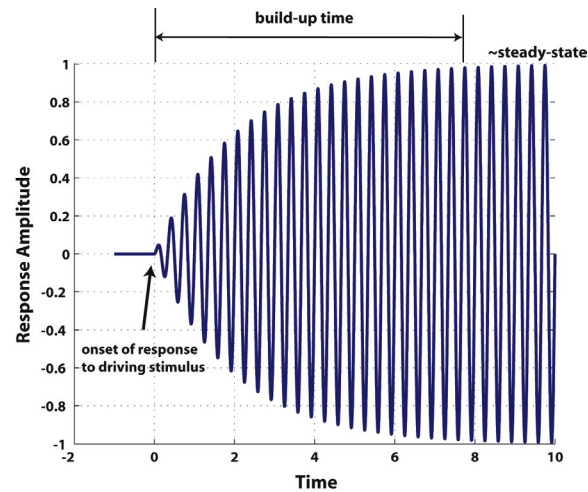


# Looking Ahead....

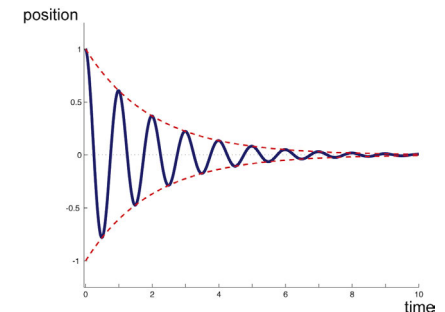
## Damped HO (DHO)



"Transient" responses...



- The *steady-state* response of the sinusoidally-driven harmonic oscillator acts like a band-pass filter
- Connection between *steady-state* response & impulse response



## SHO Revisted

Two essential features:

1. An inertial component, capable of carrying kinetic energy.
2. An elastic component, capable of storing elastic potential energy.

Two fundamental laws:

1. By Newton's law ( $F = ma$ ),

$$-kx = ma$$

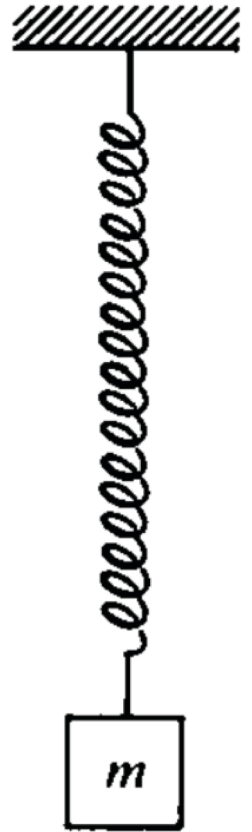
2. By conservation of total mechanical energy ( $E$ ),

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = E$$

Two associated differential equations:

$$m \frac{d^2x}{dt^2} + kx = 0$$

$$\frac{1}{2}m \left( \frac{dx}{dt} \right)^2 + \frac{1}{2}kx^2 = E$$





## Aside: Solving 2<sup>nd</sup> order ODEs

Numerous approaches to "solve":  $m \frac{d^2 x}{dt^2} + kx = 0$

$$x(t) = A \cos(\omega_o t + \phi)$$

👉 Guess a solution and plug it in, e.g.,  $x = A \sin(\omega t + \varphi_0)$

👉 Separate variables and integrate: (see Lec.05 notes)

The method of *separation of variables* works by putting all the  $x$ -values on one side of the equation and all the  $y$ -values on the other, giving

$$y dy = -x dx.$$

We then integrate each side separately:

$$\int y dy = - \int x dx,$$

👉 Rewrite as a system of 1<sup>st</sup> order ODEs  
(we'll return to this in a bit)

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -\omega_o^2 x$$

👉 Complex numbers!

Note: These methods are not mutually exclusive!

## SHO: Complex exponentials

Rewrite in terms of natural frequency

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

Now assume a solution in the form of a (possibly) complex exponential:

$$x = Ce^{pt}$$

→ Let us pause for one moment, as we just made a big connection point to other classes/topics....

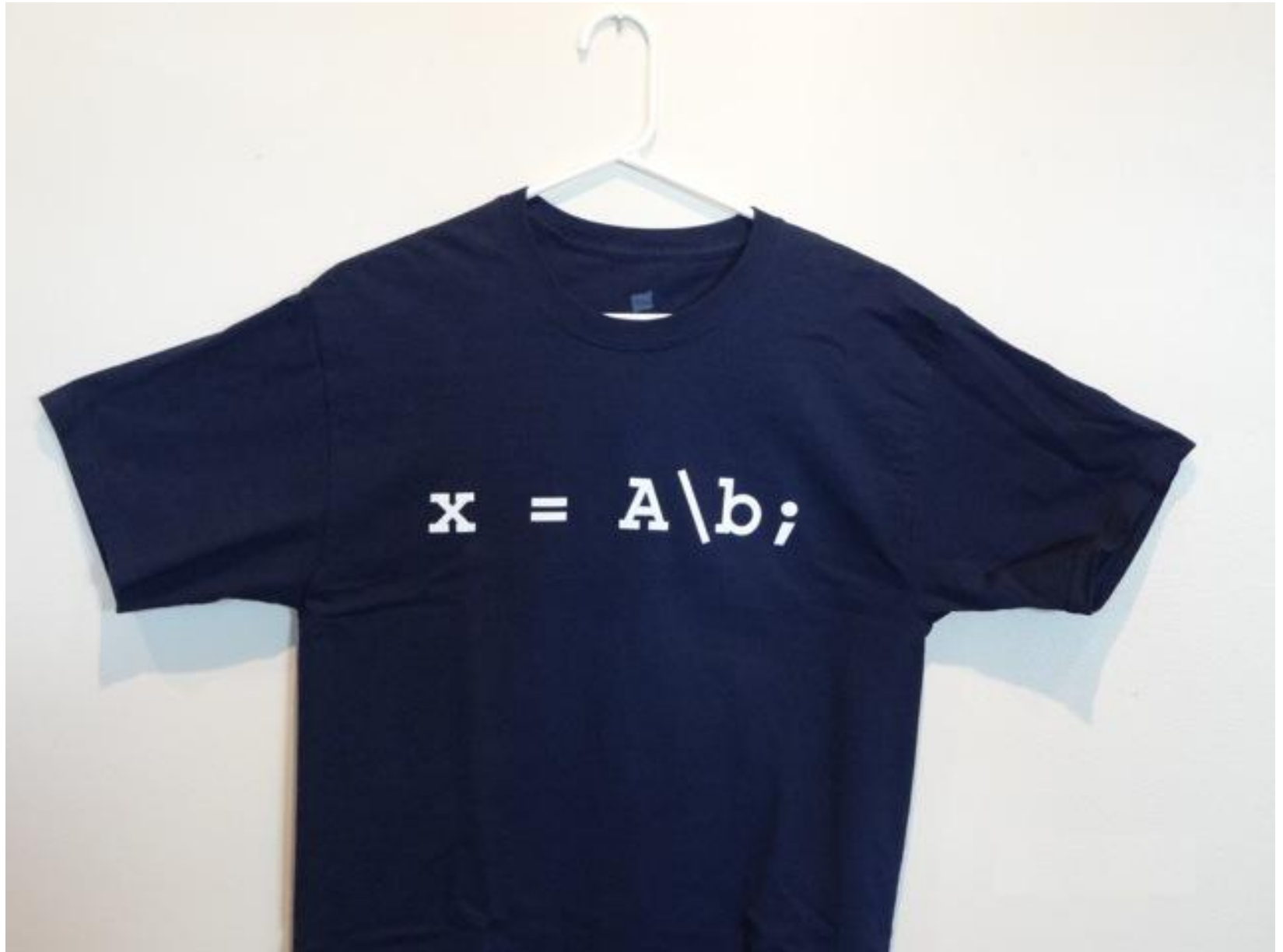
What mathematically does  $p$  represent?

**Eigenvalue!**

According to wikipedia:

*Eigenvalues and eigenvectors feature prominently in the analysis of linear transformations. The prefix eigen- is adopted from the German word eigen for "proper", "characteristic". Originally utilized to study principal axes of the rotational motion of rigid bodies, eigenvalues and eigenvectors have a wide range of applications, for example in stability analysis, vibration analysis, atomic orbitals, facial recognition, and matrix diagonalization.*

Do you speak Matlab?



## Eigenvalues (& Eigenvectors)

### Eigenvalue –

*A scalar associated with a given linear transformation of a vector space and having the property that there is some nonzero vector which when multiplied by the scalar is equal to the vector obtained by letting the transformation operate on the vector; especially : a root of the characteristic equation of a matrix*

[from Merriam-Webster]

- Etymology: *eigen* is German for ‘own’, ‘peculiar’
- Rich history in mathematics (Cauchy, Euler, Fourier, Hilbert, ....)
- Used in a wide variety of physics & engineering applications such as:
  - Classical mechanics (e.g., ‘principle axes’ in rigid body rotations)
  - Quantum mechanics (e.g., for solutions to Schrodinger’s equation, the eigenvalue of the wavefunction is the associated energy  $E$ )
  - Principal Component Analysis (PCA)
  - Image processing and ‘Eigenfaces’
  - Harmonic oscillator...

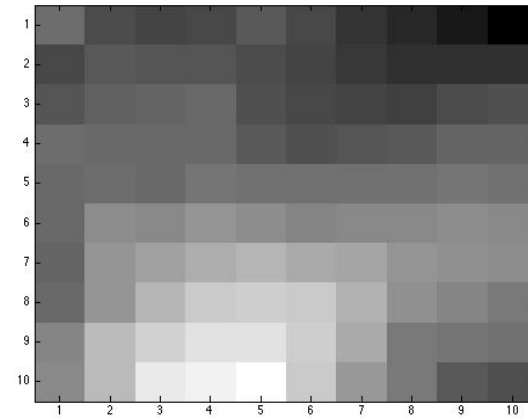
## Aside: Eigenfaces



- Keep in mind that an image is really just an array of numbers....

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$$

.... much like the matrix form for linear ODEs



- So why not try calculating eigenvalues and eigenvectors for an image?

$\mathbf{C} =$

115	108	106	107	111	107	103	100	97	91
107	111	110	110	108	106	104	102	102	102
110	112	113	114	109	107	106	105	108	109
115	114	114	114	111	109	110	111	113	113
114	115	114	117	116	116	116	116	117	116
114	122	121	124	122	120	121	121	122	121
113	124	126	129	131	128	127	124	123	122
114	124	131	135	136	135	130	123	120	118
120	132	137	140	140	136	128	118	117	116
121	132	142	144	147	135	125	118	111	109

## Aside: Eigenfaces

- Averaging the faces emphasizes common features
- Basic idea is the eigenvectors (the associated eigenvalues acting as their weighting) uniquely identify a face and thereby can be used for *facial recognition* algorithms in computer vision
- Goal is to obtain a 'low-dimensional' representation of a 'face' and is related to the more general notion of Principal Component Analysis (PCA)
- Requires determination of an 'average face', then comparing to a 'test image' to quantify the associated 'covariance'



→ In short, a key method for machine vision re facial recognition....

## SHO: Complex exponentials

Rewrite in terms of natural frequency

$$\frac{d^2x}{dt^2} + \omega^2x = 0$$

Now assume a solution in the form of a (possibly) complex exponential:

$$x = Ce^{pt}$$

Plug in assumed form of solution:

$$p^2Ce^{pt} + \omega^2Ce^{pt} = 0$$

Solving the ODE becomes an algebraic problem due to the assumptions we made!

Associated eigenvalues

Solve through:

$$p^2 + \omega^2 = 0$$

$$p^2 = -\omega^2$$

$$p = \pm j\omega$$

## SHO: Complex exponentials

$$x = C_1 e^{j\omega t} + C_2 e^{-j\omega t}$$

Plugging it back in...

$$\begin{aligned} x &= C e^{j(\omega t + \alpha)} + C e^{-j(\omega t + \alpha)} \\ &= 2C \cos(\omega t + \alpha) \\ &\equiv A \cos(\omega t + \alpha) \end{aligned}$$

Now there are a couple ways things could play out, but keep in mind the same basic issue it at play: there are **two free parameters** ( $C$  &  $\alpha$ , or  $C_1$  &  $C_2$ , or  $A$  and  $\alpha$ )

$$z = A \cos(\omega t + \alpha) + jA \sin(\omega t + \alpha)$$

$$x = \text{real part of } z \quad \text{where } z = A e^{j(\omega t + \alpha)}$$

Note: The imaginary part of  $z$  is not any less "physical". It still contains the two key pieces of information (i.e.,  $A$  and  $\alpha$  here)! Choosing the real part here is just a convention.



## SHO: Complex exponentials

$$\begin{aligned}x &= Ce^{i(\omega t + \alpha)} + Ce^{-i(\omega t + \alpha)} \\ &= 2C \cos(\omega t + \alpha) \\ &\equiv A \cos(\omega t + \alpha)\end{aligned}$$

Now there are a couple ways things could play out, but keep in mind the same basic issue it at play: there are **two free parameters** ( $C$  &  $\alpha$ , or  $C_1$  &  $C_2$ , or  $A$  and  $\alpha$ )

Note: For the SHO, those two free parameters (plus our general form of the solution) tell us everything about how the system will behave for all time(!!)

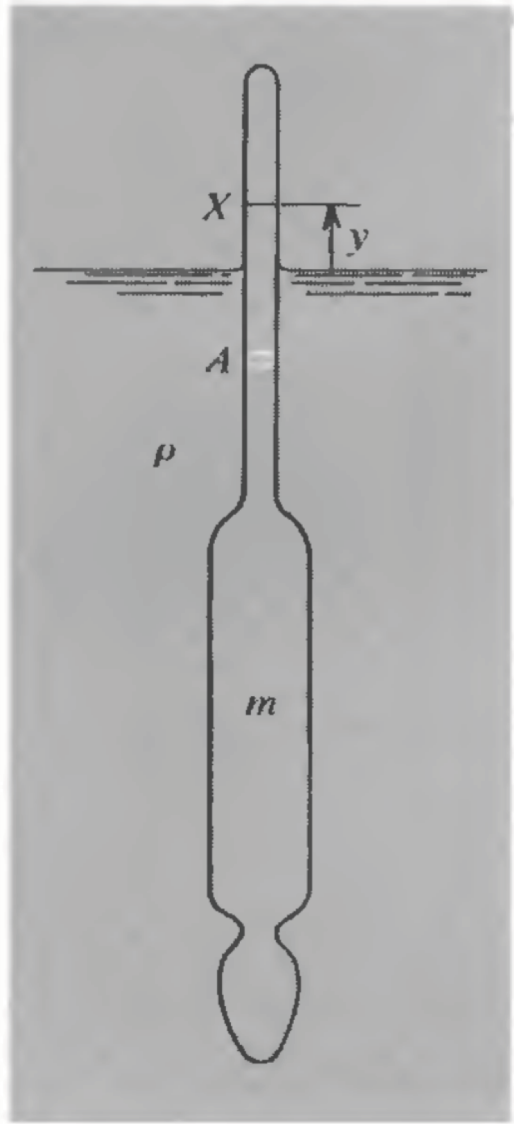
→ So what determines those two free parameters?

$$\frac{d^2x}{dt^2} + \omega^2x = 0$$

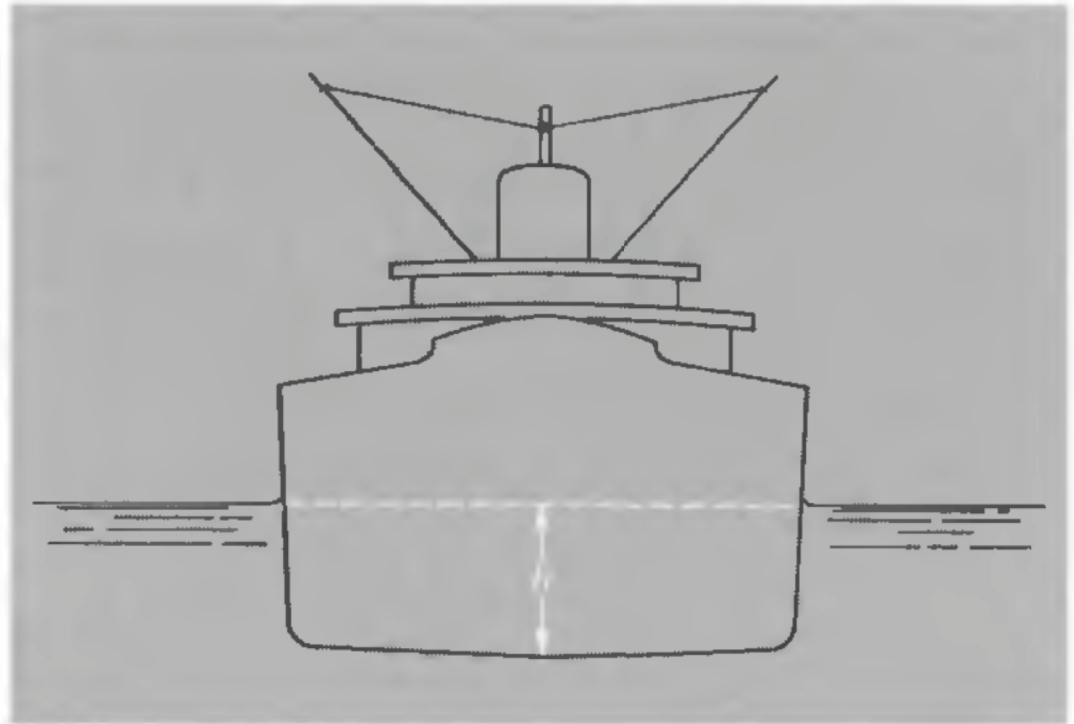
2<sup>nd</sup> order ODE requires two unique **initial conditions** (or two unique boundary conditions) to find a specific solution [e.g.,  $x(t=0) = x_0$  and  $v(t=0) = v_0$  ]

## SHO: A physical example (Floating Object)

### Hydrometer



### Ship in water (neglecting waves, rocking motions, etc..)



→ Force due to gravity is offset by the **buoyant** force, which acts as a **restorative force** (it is simply proportional to the weight of fluid displaced)

## SHO: A physical example (Floating Object)

Assume  $m$  is mass of hydrometer,  $\rho$  is the density of liquid, and only the neck (w/ cross-sectional area  $A$ ) sticks above the waterline. Upon setting up the coord system as shown, if  $y$  is the distance above the waterline relative to rest, then we have:

$$m \frac{d^2 y}{dt^2} = -g\rho A y$$

Stemming from our earlier analysis:

$$\omega = \sqrt{\frac{g\rho A}{m}}$$

$$T = 2\pi \sqrt{\frac{m}{g\rho A}}$$

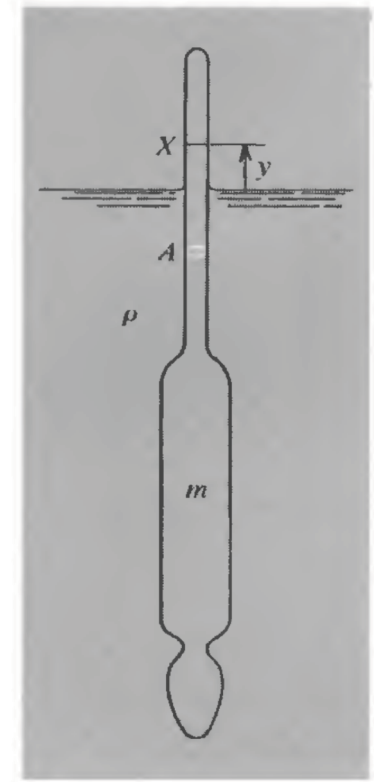
Some reasonable values:

$$m \approx 10^{-2} \text{ kg}$$

$$A \approx 2.5 \times 10^{-5} \text{ m}^2$$

$$g \approx 10 \text{ m/sec}^2$$

$$\rho \approx 1.2 \times 10^3 \text{ kg/m}^3$$



Yields:

$$T \approx 1 \text{ sec}$$

SHO: A physical example (Floating Object)

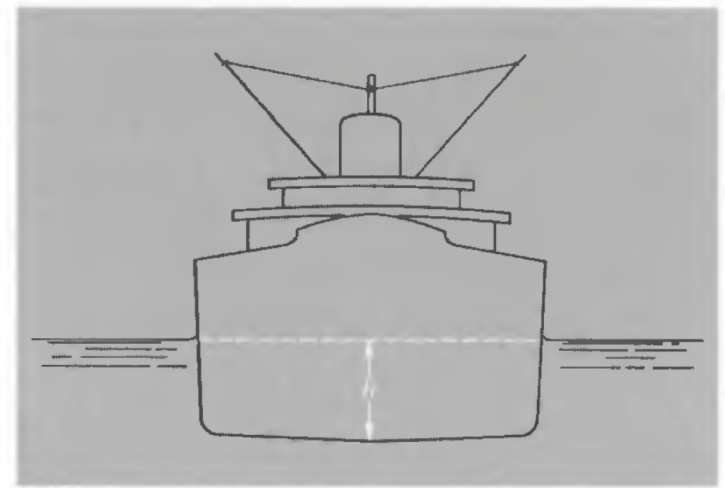
$$T = 2\pi \sqrt{\frac{m}{g\rho A}}$$

Assume ship bottom is flat. Then the "draft" of the ship ( $h$ , i.e., the distance from the waterline down to the bottom yields an estimate of the ships mass):

$$m = \rho Ah$$

$$T = 2\pi \sqrt{\frac{h}{g}}$$

→ For a large ship w/ a 10 m draft, the period of oscillation is ~6 s

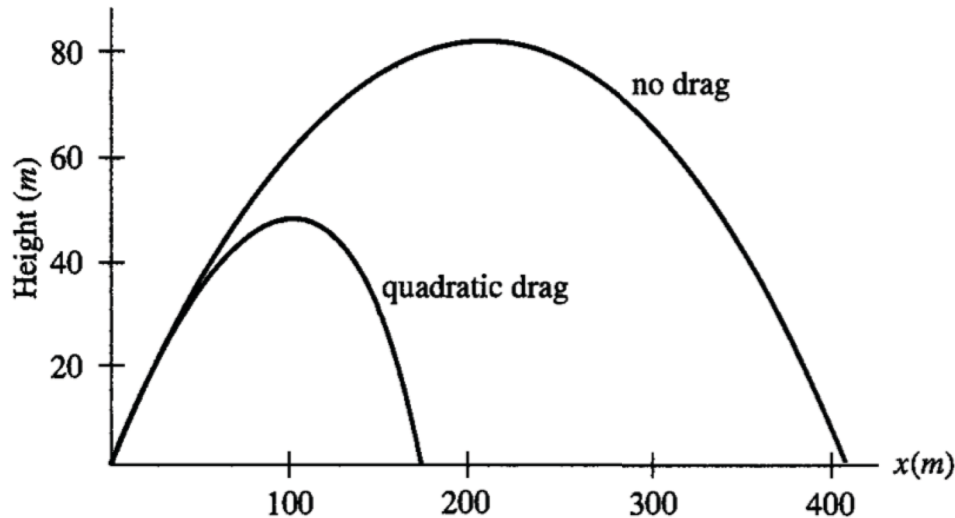


## Damping....

The "real world" is generally not as conservative as the canonical "simple" oscillator thus far considered....

Recall how messy things got for projectile motion w/ drag....

**Baseball trajectories**

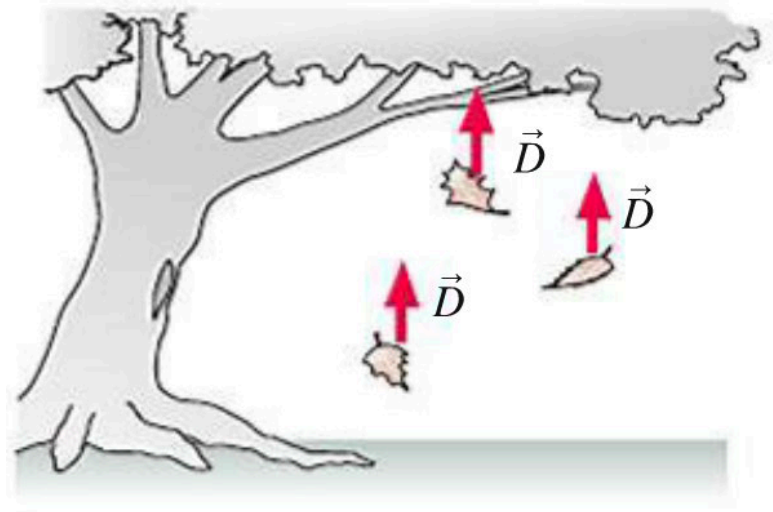


General form for resistive force of a "fluid" on an object

$$R(v) = b_1v + b_2v^2$$

.... nonetheless we carry on.

Air resistance is a significant force on falling leaves. It points opposite the direction of motion.



## Recall

$$R(\mathbf{v}) = b_1 v + b_2 v^2$$

Projectiles aside, we presently have a STRONG interest in keeping things linear at the moment, and thus will assume for simplicity  $b_1 \gg b_2$

$$\mathbf{f} = -f(v)\hat{\mathbf{v}}, \quad (2.1)$$

where  $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$  denotes the unit vector in the direction of  $\mathbf{v}$ , and  $f(v)$  is the magnitude of  $\mathbf{f}$ .

The function  $f(v)$  that gives the magnitude of the air resistance varies with  $v$  in a complicated way, especially as the object's speed approaches the speed of sound. However, at lower speeds it is often a good approximation to write<sup>1</sup>

$$f(v) = bv + cv^2 = f_{\text{lin}} + f_{\text{quad}} \quad (2.2)$$

where  $f_{\text{lin}}$  and  $f_{\text{quad}}$  stand for the linear and quadratic terms respectively,

$$f_{\text{lin}} = bv \quad \text{and} \quad f_{\text{quad}} = cv^2. \quad (2.3)$$

The physical origins of these two terms are quite different: The linear term,  $f_{\text{lin}}$ , arises from the viscous drag of the medium and is generally proportional to the viscosity of the medium and the linear size of the projectile (Problem 2.2). The quadratic term,  $f_{\text{quad}}$ , arises from the projectile's having to accelerate the mass of air with which it is continually colliding;  $f_{\text{quad}}$  is proportional to the density of the medium and the cross-sectional area of the projectile (Problem 2.4). In particular, for a spherical projectile (a cannonball, a baseball, or a drop of rain), the coefficients  $b$  and  $c$  in (2.2) have the form

$$b = \beta D \quad \text{and} \quad c = \gamma D^2 \quad (2.4)$$

## Damped HO

Eqn. of motion

$$m \frac{d^2 x}{dt^2} = -kx - bv$$

Making a  
change of  
variables:

$$\frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = 0$$

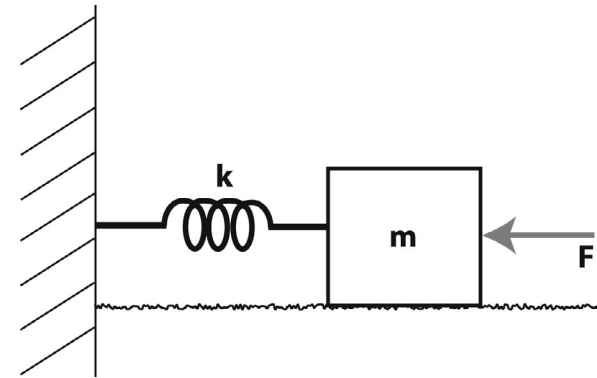
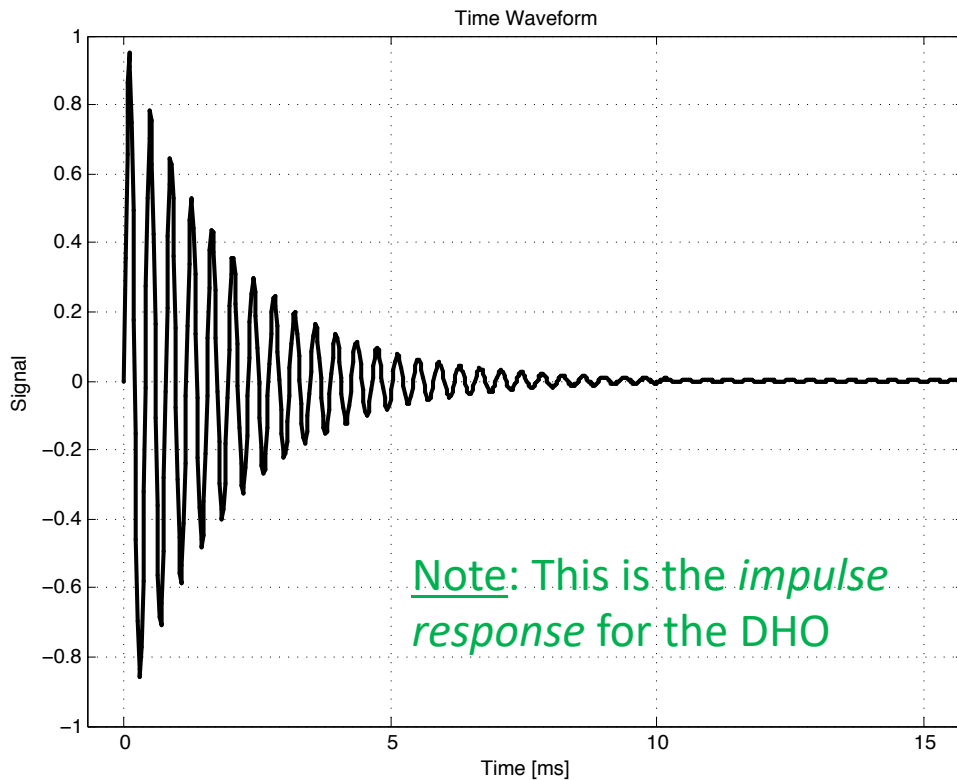
where

$$\gamma = \frac{b}{m} \quad \omega_0^2 = \frac{k}{m}$$

Now we must deal w/ a necessary reality: Despite solutions (possibly) being oscillatory, they **will not/cannot be sinusoidal**

## DHO

(given an "impulse" push at  $t = 0$ )



Now we must deal w/ a necessary reality: Despite solutions (possibly) being oscillatory, they will not/cannot be sinusoidal....

.... but intuitively you already knew this



## Damped HO

Eqn. of motion

$$m \frac{d^2 x}{dt^2} = -kx - bv$$

Making a  
change of  
variables:

$$\frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = 0$$

where (remember these!)

$$\gamma = \frac{b}{m} \quad \omega_0^2 = \frac{k}{m}$$

Now we must deal w/ a necessary reality: Despite solutions (possibly) being oscillatory, they will not/cannot be sinusoidal

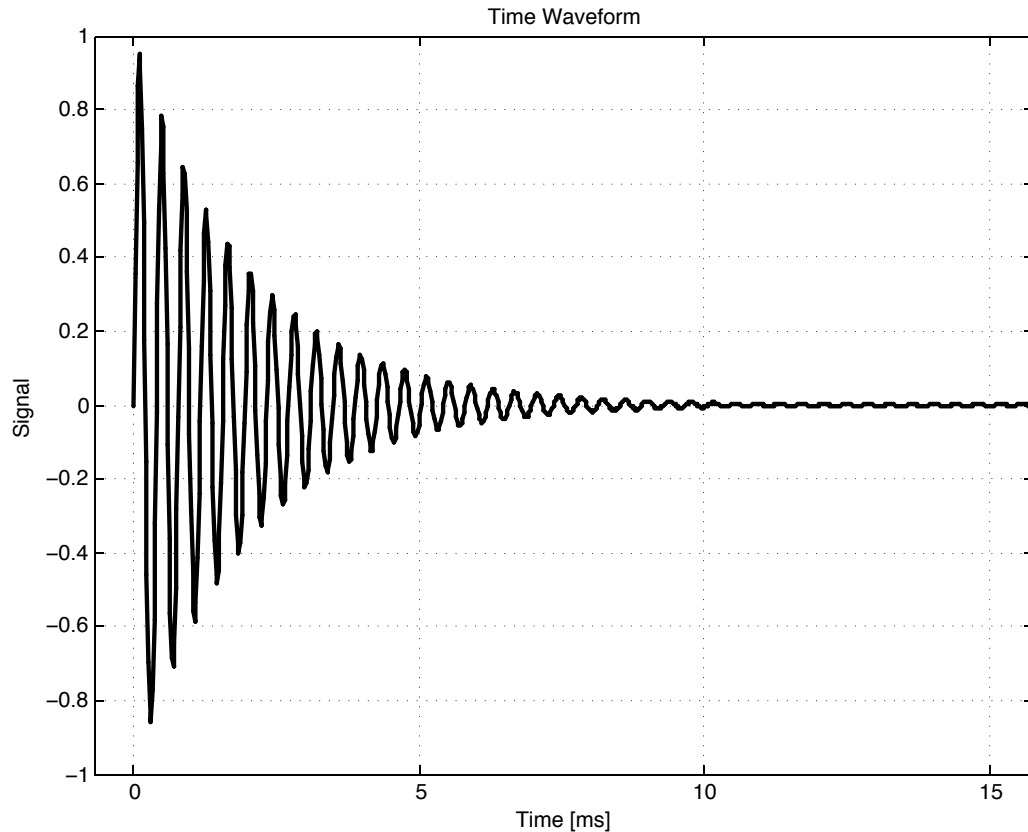
So we shift to a complex form....

$$\frac{d^2 z}{dt^2} + \gamma \frac{dz}{dt} + \omega_0^2 z = 0$$

With an assumed  
solution of the form

$$z = Ae^{j(pt+\alpha)}$$

## Complex Exponentials....



$$\frac{d^2 z}{dt^2} + \gamma \frac{dz}{dt} + \omega_0^2 z = 0$$

$$z = Ae^{j(p t + \alpha)}$$

→ Not only does this assumed form of solution capture the oscillation, it also describes the exponential decay/growth (all of which is encapsulated in the eigenvalues)

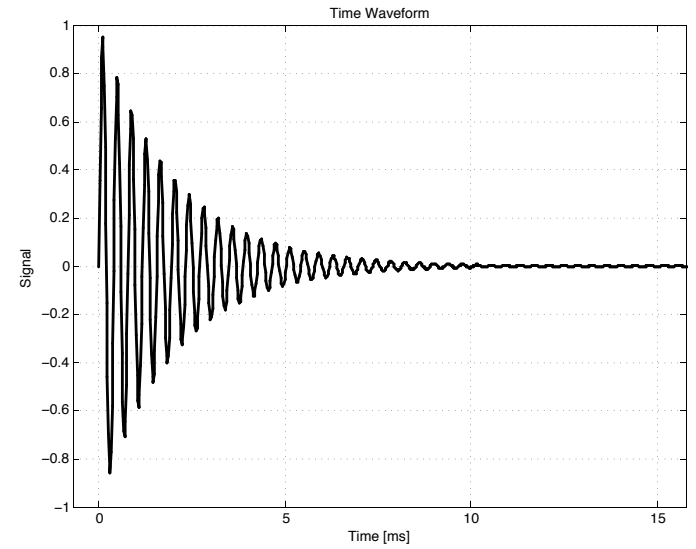
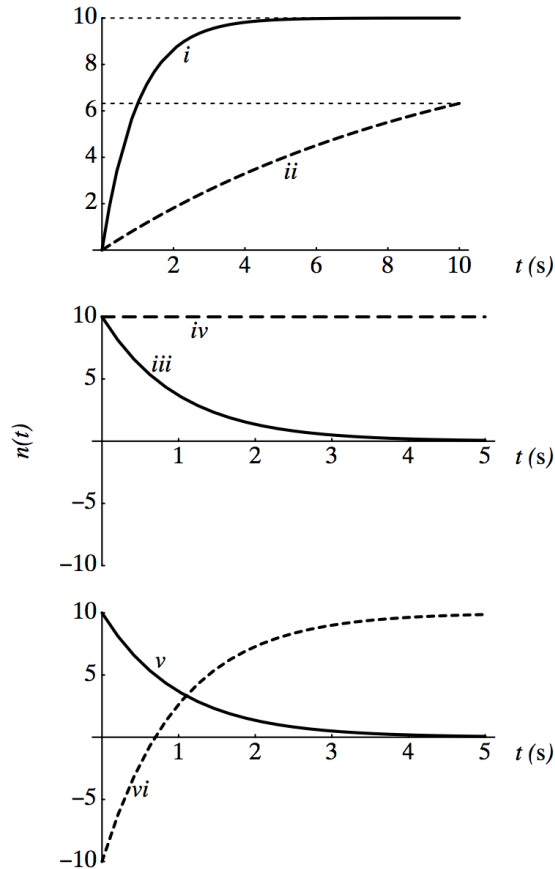
## Recall

**Problem 1.** A first-order, linear differential equation with constant coefficients and a constant inhomogeneous (drive or input) term has an exponential solution. Therefore, the solution can be written in the form

$$n(t) = n_{\infty} + \left(n_0 - n_{\infty}\right) e^{-t/\tau},$$

where  $n_0 = n(0)$  is the initial value of  $n(t)$  and  $n_{\infty} = \lim_{t \rightarrow \infty} n(t)$  is the final value of  $n(t)$ . The form of this solution can be verified by evaluating  $n(t)$  at  $t = 0$  and  $t \rightarrow \infty$ . Substitution into the differential equation shows that this solution satisfies the differential equation. The solutions for cases i-vi are shown in Figure 1. The solutions for part a (i and ii) have the same initial and final values but different time constants (by  $t = 10$  s, curve ii is just above 6 and has not yet reached its final value of 10). The solutions for part b (iii and iv) have the same initial values and different final values. Although curve iv was calculated with the same time constant as in iii, it doesn't make sense to compare the time constants of the curves, since curve iv isn't changing. The solutions for part c (v and vi) have different initial and final values and the same time constants.

# Recall



$$z = Ae^{j(\omega t + \alpha)}$$

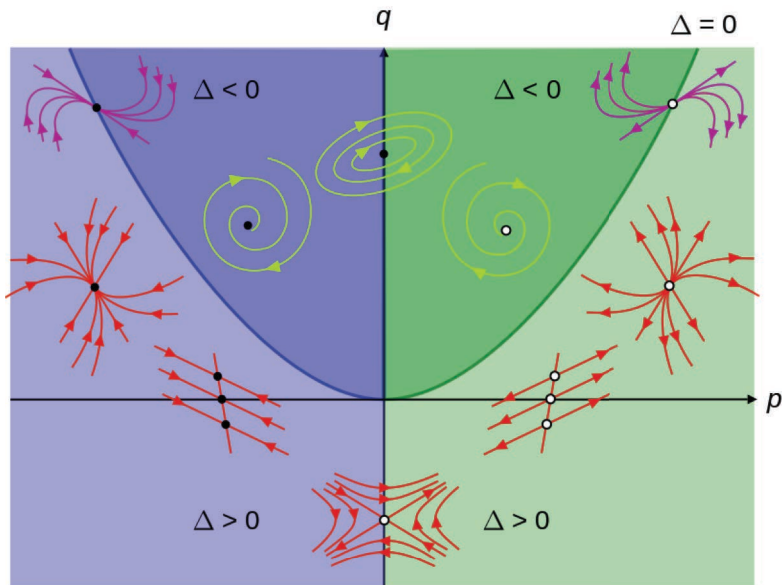
$$z = A \cos(\omega t + \alpha) + jA \sin(\omega t + \alpha)$$

**Figure 1.** Solutions to parts i-vi. In the upper panel, horizontal dotted lines are shown at the final value of 10 and for the value of  $n(t)$  at  $t = \tau$ , i.e., the line is at  $10(1 - e^{-1})$ .

## Damped HO (via complex exponentials)

### Note:

There are a lot of starting points w/ regard to aspects such as the assumed form of the solution (see right & below as different possible examples). They may lead in slightly different directions analysis-wise, but ultimately they lead to the same place. It is worthwhile to spend a bit of to convince yourself of such, especially as you learn new mathematical methods....



$$\begin{aligned} \frac{dx}{dt} &= Ax + By & p &= A + D \\ \frac{dy}{dt} &= Cx + Dy & q &= AD - BC \\ & & \Delta &= p^2 - 4q \end{aligned}$$

$$x = Ce^{pt}$$

$$z = Ae^{j(pt+\alpha)}$$

$$x(t) = Ae^{i(\omega t + \delta)}$$

$$x(t) = Ae^{-i(\omega t + \delta)}$$

$$x = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} c_1 e^{\lambda_1 t} + \begin{bmatrix} k_3 \\ k_4 \end{bmatrix} c_2 e^{\lambda_2 t}$$

## Damped HO (via complex exponentials)

Combining  
these two:

$$z = Ae^{j(pt+\alpha)} \quad \frac{d^2z}{dt^2} + \gamma \frac{dz}{dt} + \omega_0^2 z = 0$$

We obtain:

$$(-p^2 + jp\gamma + \omega_0^2)Ae^{j(pt+\alpha)} = 0$$

Or more  
succinctly:

$$-p^2 + jp\gamma + \omega_0^2 = 0$$

This is sometimes referred to  
as the *characteristic equation*

$$p = n + js$$

A handful of ways to deal w/  
this, such as rewriting in terms  
of real and imaginary parts and  
solving each separately:

$$\text{Real parts:} \quad -n^2 + s^2 - s\gamma + \omega_0^2 = 0$$

$$\text{Imaginary parts:} \quad -2ns + n\gamma = 0$$

Note: Another approach is to solve the char. eqn. via the quadratic formula  
(see additional slides at end)

## Damped HO (via complex exponentials)

$$z = Ae^{j(pt+\alpha)}$$

When the  
smoke clears:

$$s = \frac{\gamma}{2} \quad n^2 = \omega_0^2 - \frac{\gamma^2}{4}$$

$$p = n + js$$

Plugging back in:

$$\begin{aligned} z &= Ae^{j(nt+js t+\alpha)} \\ &= Ae^{-st}e^{j(nt+\alpha)} \end{aligned}$$

And subsequently, from  
our convention:

$$x = Ae^{-st} \cos(nt + \alpha)$$

Using variables  
from the ODE:

$$x = Ae^{-\gamma t/2} \cos(\omega t + \alpha)$$

→ The system doesn't  
even oscillate at the  
natural frequency!

where

$$\omega^2 = \omega_0^2 - \frac{\gamma^2}{4} = \frac{k}{m} - \frac{b^2}{4m^2}$$

## Damped HO: Loss of Energy

For the moment, let's assume  
(i.e., relatively small damping)

$$\gamma \ll \bar{\omega},$$

Recall that for SHO, the  
total energy is:

$$E = \frac{1}{2}kA^2$$

Thus for the damped  
case, we have:

$$A(t) = A_0 e^{-\gamma t/2}$$

Or more succinctly:

$$E(t) = E_0 e^{-\gamma t}$$

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = 0$$

$$\gamma = \frac{b}{m} \quad \omega_0^2 = \frac{k}{m}$$

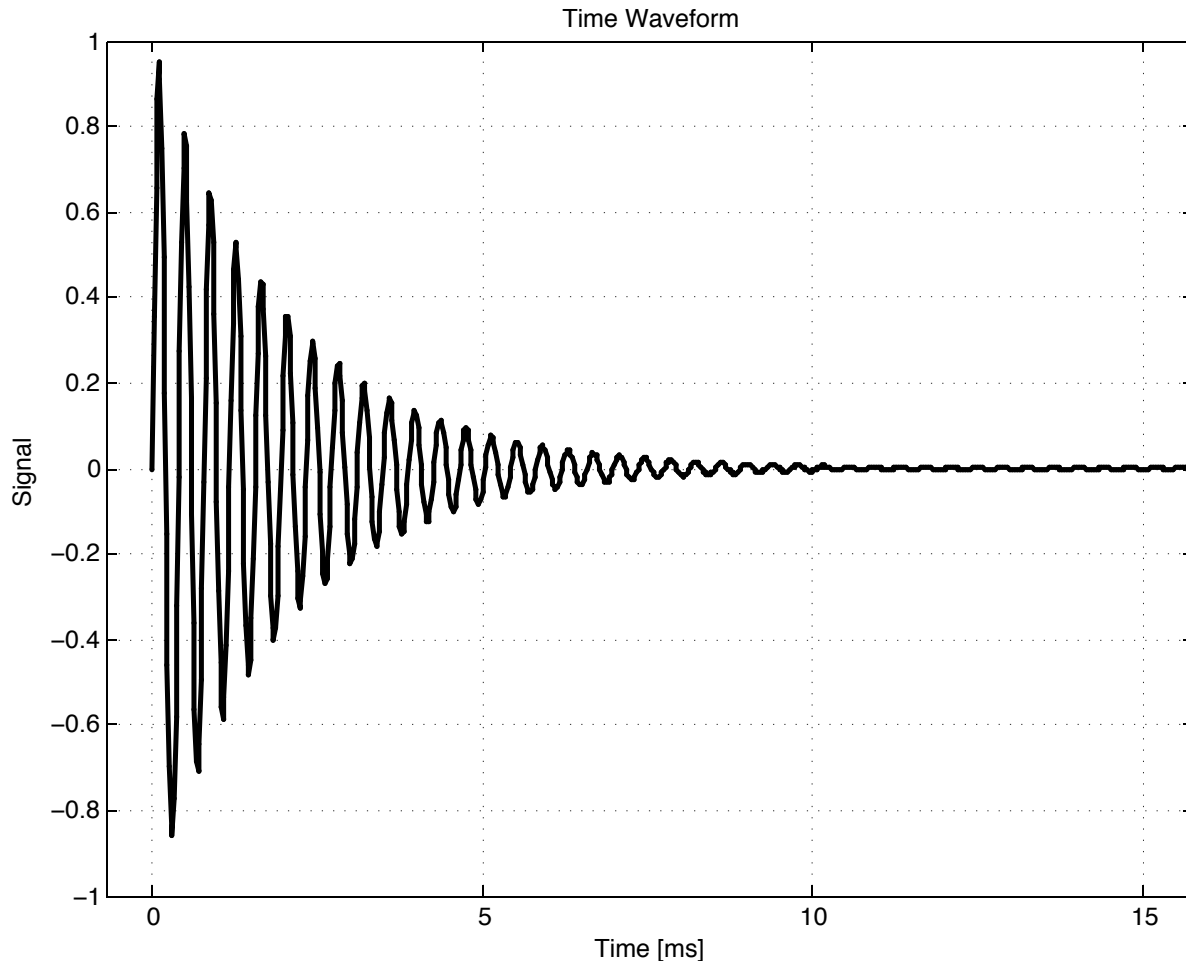
$$\omega^2 = \omega_0^2 - \frac{\gamma^2}{4} = \frac{k}{m} - \frac{b^2}{4m^2}$$

$$\longrightarrow E(t) = \frac{1}{2}kA_0^2 e^{-\gamma t}$$

→ Thus energy leaks out via  
an exponential decay due  
to the damping



## Damped HO: Loss of Energy



Note - Recall that we assumed

$$z = Ae^{j(pt+\alpha)}$$

... but a more common convention is

$$x = Ce^{pt}$$

$$x = Ae^{-\gamma t/2} \cos(\omega t + \alpha)$$

$$E(t) = E_0 e^{-\gamma t}$$

→ What about other relative damping cases?

(i.e., small vs medium vs large damping)

$$p = n + js$$

$$s = \frac{\gamma}{2} \quad n^2 = \omega_0^2 - \frac{\gamma^2}{4}$$

Thus it is more typical to find that the real part of the eigenvalue describes energy loss/gain (rather than the imaginary part, as is the case here)

## Reference: System of linear autonomous ODEs

- Let's consider a simple 2<sup>nd</sup> order system (all these ideas scale up for higher dimension systems)

$$\frac{dx}{dt} = ax + by$$

$$\frac{dy}{dt} = cx + dy$$

- Re-express in matrix/vector form:

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$$

- Let's make an assumption: solutions will have the form of (possibly complex) exponentials

$$x = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} c_1 e^{\lambda_1 t} + \begin{bmatrix} k_3 \\ k_4 \end{bmatrix} c_2 e^{\lambda_2 t}$$

This expression explicitly deals with the eigenvalues and eigenvectors of the system

## Reference: Eigen Decomposition

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$$

Characteristic equation:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

→ determinant (det) is scalar value associated with a square matrix

ODE as combination of eigenvalues and eigenvectors

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \quad \text{'secular equation'}$$

General solution:

$$x = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} c_1 e^{\lambda_1 t} + \begin{bmatrix} k_3 \\ k_4 \end{bmatrix} c_2 e^{\lambda_2 t}$$

→ Remember, we implicitly assume the solution has this exponential form!

## Reference: Finding Eigenvalues

Characteristic equation:  $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

Quadratic equation w/  
two roots (for a 2<sup>nd</sup>  
order system)

$$\lambda^2 - \lambda(a + d) + (ad - bc) = 0$$

Note that complex roots  
are possible

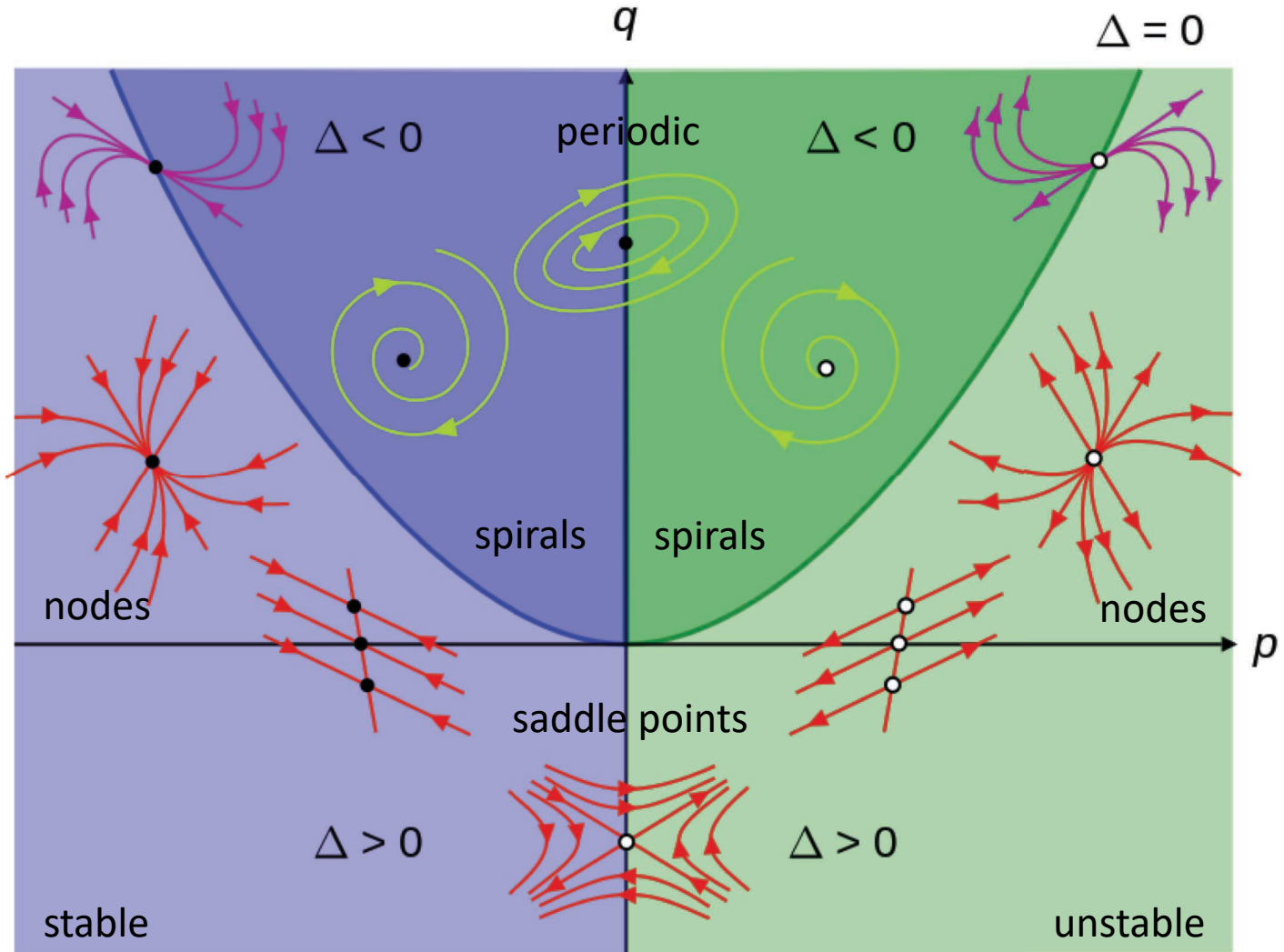
$$\lambda = \frac{(a + d) \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2}$$

$$x = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} c_1 e^{\lambda_1 t} + \begin{bmatrix} k_3 \\ k_4 \end{bmatrix} c_2 e^{\lambda_2 t}$$

→ Eigenvalues explicitly tell you how the solutions behave!

Reference: Classification of equilibrium points (linear autonomous 2<sup>nd</sup> order systems)

Orbits



$$\frac{dx}{dt} = Ax + By$$

$$\frac{dy}{dt} = Cx + Dy$$

$$p = A + D$$

$$q = AD - BC$$

$$\Delta = p^2 - 4q$$

Ex.

$$\frac{dx}{dt} = 5x - 3y$$

$$\frac{dy}{dt} = 2x - 4y$$

→ Only a single equilibrium point exists (at the origin). Stability?

$$\mathbf{A} = \begin{pmatrix} 5 & -3 \\ 2 & -4 \end{pmatrix}$$

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0$$

$$p = \text{Tr}(\mathbf{A}) = 5 + (-4) = 1$$

$$q = \det(\mathbf{A}) = 5(-4) - (-3)2 = -14$$

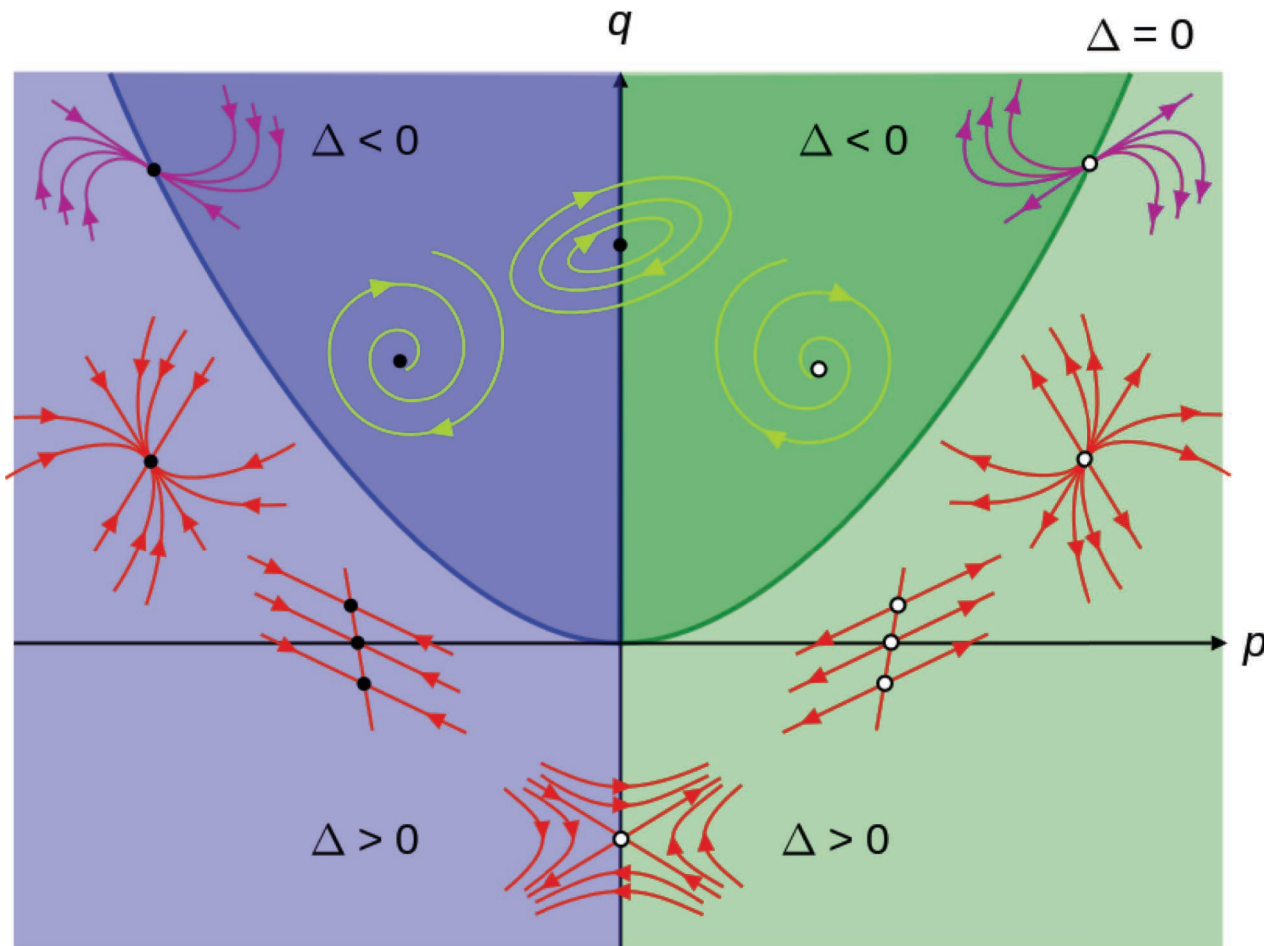
$$\lambda = \frac{1}{2} (1 \pm \sqrt{1 + 56})$$

$$x = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} c_1 e^{\lambda_1 t} + \begin{bmatrix} k_3 \\ k_4 \end{bmatrix} c_2 e^{\lambda_2 t}$$

$$\lambda = -3.27, 4.27$$

→ General solution is a linear combination of a (real-valued) exponentials, one converging and one diverging

Ex. (cont.)



$$p = \text{Tr}(\mathbf{A}) = 5 + (-4) = 1$$
$$q = \det(\mathbf{A}) = 5(-4) - (-3)2 = -14$$

$$\lambda = -3.27, 4.27$$

→ Solution curves approach the origin, then diverge away

→ Equilibrium point at origin (where the eigenvectors meet) is said to be a *saddle*

$$\frac{dx}{dt} = Ax + By$$
$$\frac{dy}{dt} = Cx + Dy$$

$$p = A + D$$
$$q = AD - BC$$
$$\Delta = p^2 - 4q$$

## Damped HO (Alternative Approach re complex exponentials)

Rewrite as a system of first order ODEs

$$\ddot{x} + \gamma\dot{x} + \omega_o^2 x = 0$$

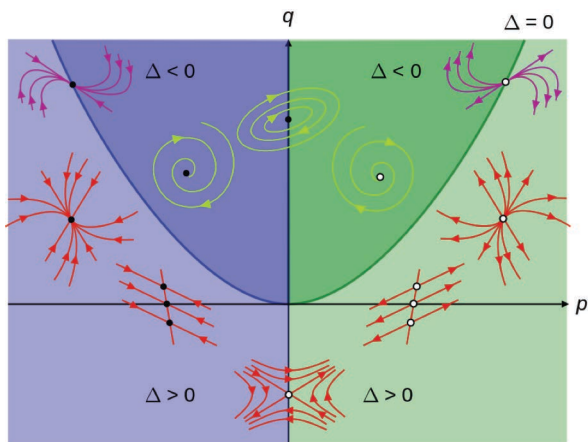
$$\frac{dx}{dt} = y$$

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\omega_o^2 & -\gamma \end{pmatrix}$$

$$\frac{dy}{dt} = -\omega_o^2 x - \gamma y$$

$$\lambda = \frac{1}{2} \left( -\gamma \pm \sqrt{\gamma^2 - 4\omega_o^2} \right)$$

$$p = -\gamma \quad q = \omega_o^2 (>0)$$



- What if  $\gamma$  is zero? Negative?
- Depending upon the sign and relative values of  $\gamma$  and  $\omega_o$ ,  $\lambda$  can be complex

→ Eigenvalues characterize behavior of all possible solution types!

$$x(t) = A e^{-\gamma t/2} e^{i(\omega t + \alpha)}$$



```

% ### LINode45EX.m ###          01.26.16
% Numerically integrate a general 2nd order linear autonomous system (w/
% const. coefficients)
% x' = a*x + b*y
% y' = c*x + d*y

clear
% -----
% User input (Note: All paramters are stored in a structure)
P.y0(1) = 1.0; % initial value for x
P.y0(2) = 1; % initial value for y
P.A= [-3.9 3; % matrix A to contain coefficients A= [a b
-2 1]; % c d]

% Integration limits
P.t0 = 0.0; % Start value
P.tf = 10.0; % Finish value
P.dt = 0.01; % time step
% -----
% +++
% determine some basic derived quantities
p= P.A(1,1)+ P.A(2,2); % Tr(A)
q= P.A(1,1)* P.A(2,2)-P.A(1,2)* P.A(2,1); % det(A)
disp(['Tr(A)= ' num2str(p), ' and det(A)= ',num2str(q)]);
eigV1= [0.5*(p+sqrt(p^2-4*q)) 0.5*(p-sqrt(p^2-4*q))]; % calc. eigenvalues directly
eigV2= eig(P.A); % calculate via Matlab's built-in routine
disp(['eigenvalues= ' num2str(eigV1(1)), ' and ',num2str(eigV1(2))]);

% +++
% use built-in ode45 to solve
[t y] = ode45('LINfunction', [P.t0:P.dt:P.tf],P.y0,[],P);

% -----
% visualize
% NOTE (re variable naming): x=y(1) and y=y(2)
figure(1); clf;
plot(t,y(:,1)); hold on; grid on;
xlabel('t'); ylabel('x(t)')
% Phase plane
figure(2); clf;
plot(y(:,1), y(:,2)); hold on; grid on;
xlabel('x(t)'); ylabel('y(t)')
% "solution space"
figure(3); clf;
plot(p,q,'rx','MarkerSize',9,'LineWidth',3); hold on; grid on;
if (abs(p)<1), pSpan= linspace(-1,1,100);
else pSpan= linspace(-1.5*p-0.1,1.5*p+0.1,100); end
qSpan= pSpan.^2/4;
plot(pSpan,qSpan,'k-', 'LineWidth',2); %ylim([-max(qSpan) max(qSpan)])
plot(pSpan,zeros(numel(pSpan),1),'b--', 'LineWidth',2);
xlabel('Tr(A)'); ylabel('det(A)')

```

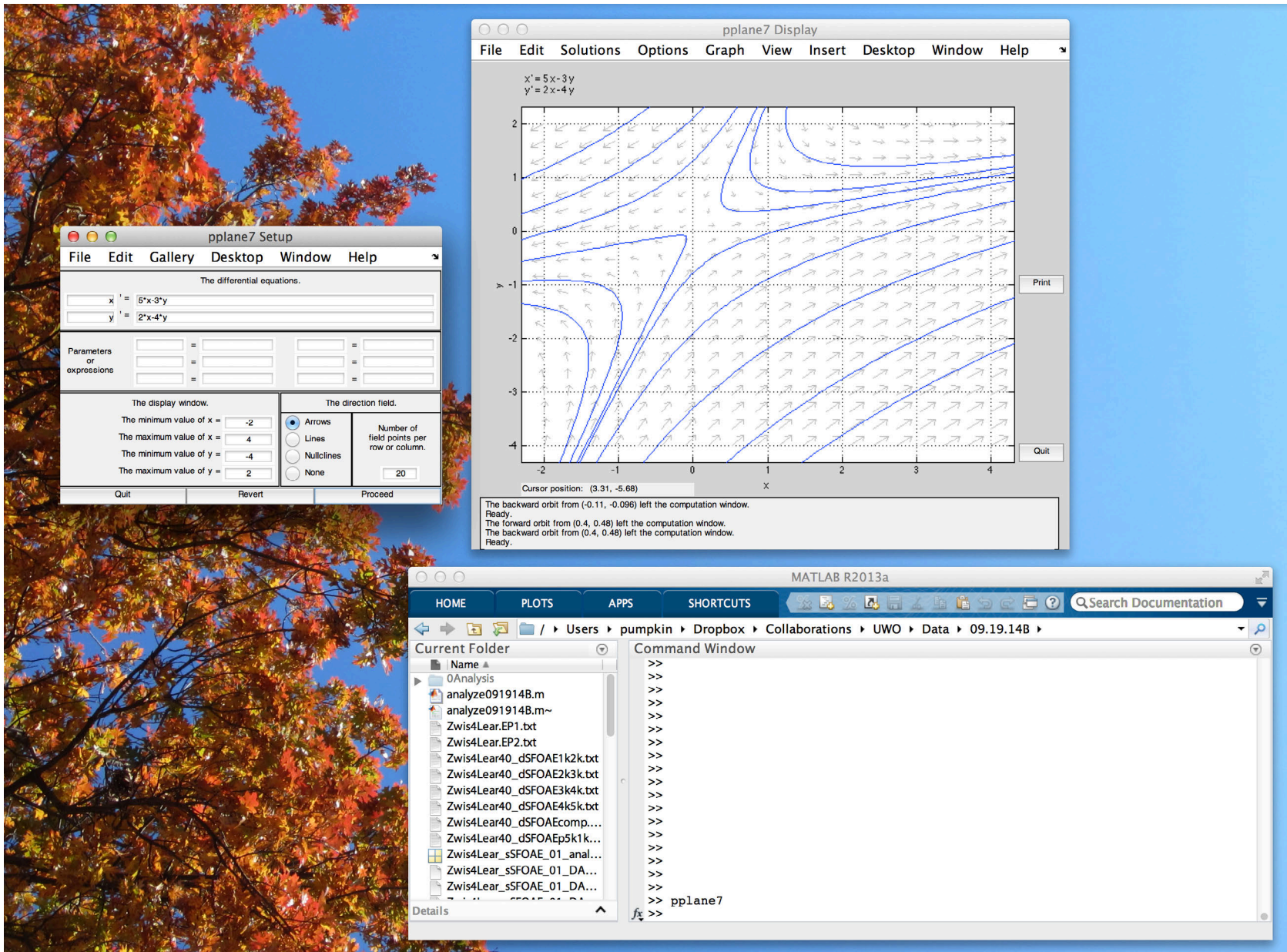
```

function [out1] = LINfunction(t,y,flag,P)
% -----
% y(1) ... x
% y(2) ... y
out1(1)= P.A(1,1)*y(1) + P.A(1,2)*y(2);
out1(2)= P.A(2,1)*y(1) + P.A(2,2)*y(2);
out1= out1';

```

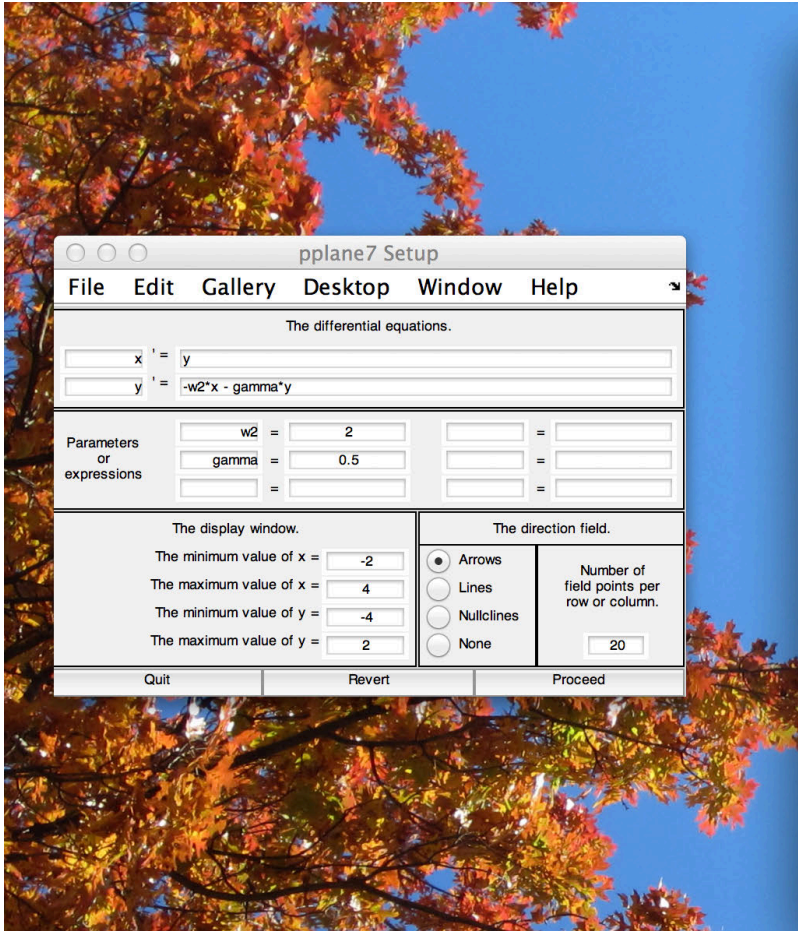
# Damped HO (Phase Plane Analysis)

→ Computationally, use our ode45 code or pplane to explore behavior of solution curves



# Damped HO (Phase Plane Analysis)

- $\gamma = 0.5$
- $\omega_0^2 = 2$



**pplane7 Setup**

File Edit Gallery Desktop Window Help

The differential equations.

$x' = y$

$y' = -w2*x - \text{gamma}*y$

Parameters or expressions

$w2 = 2$

$\text{gamma} = 0.5$

The display window.

The minimum value of  $x = -2$

The maximum value of  $x = 4$

The minimum value of  $y = -4$

The maximum value of  $y = 2$

The direction field.

Arrows

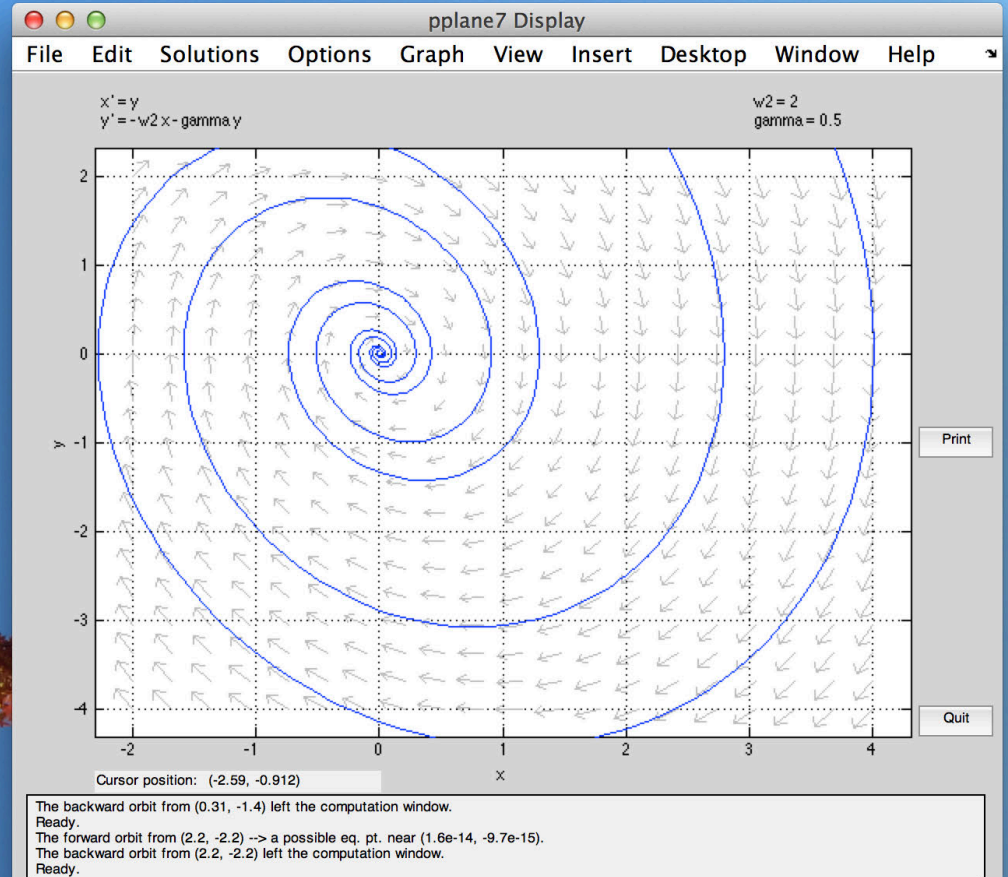
Lines

Nullclines

None

Number of field points per row or column: 20

Quit Revert Proceed





# Damped HO (Phase Plane Analysis)

- $\gamma = 2$
- $\omega_0^2 = 2$

pplane7 Setup

File Edit Gallery Desktop Window Help

The differential equations.

$x' = y$

$y' = -w2*x - gamma*y$

Parameters or expressions

$w2 = 2$

$gamma = 2$

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The minimum value of x = -2

The maximum value of x = 4

The minimum value of y = -4

The maximum value of y = 2

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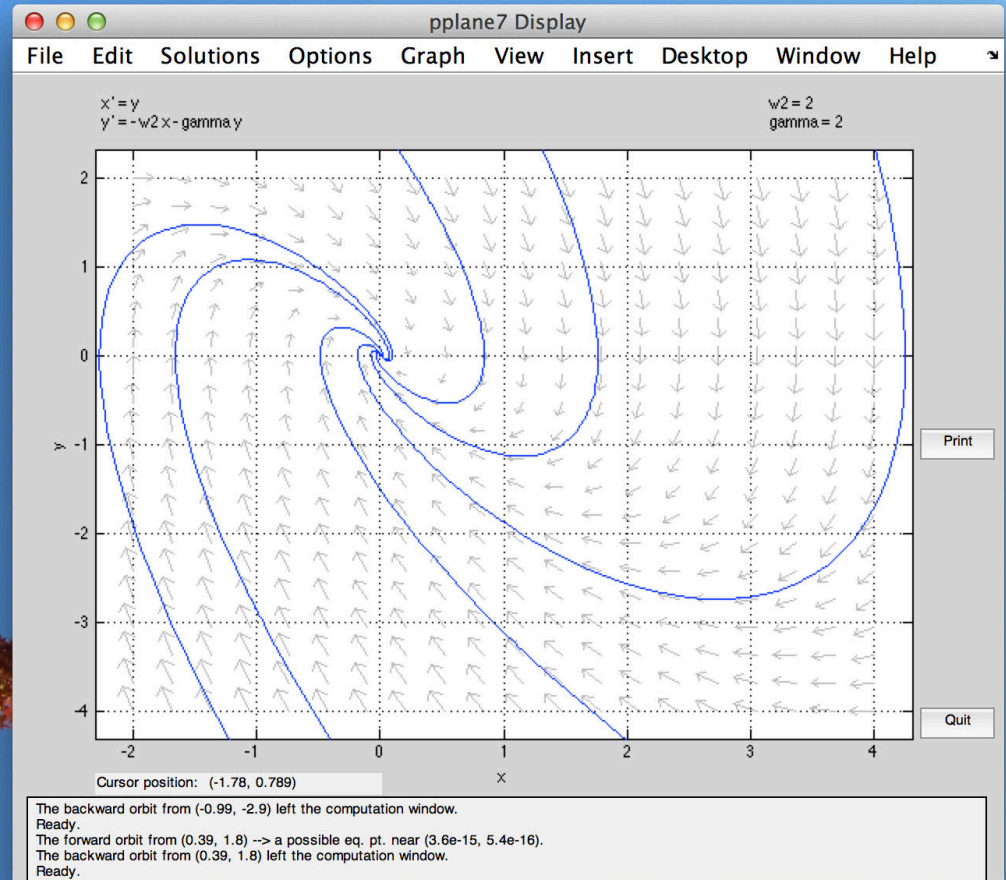
Lines

Nullclines

None

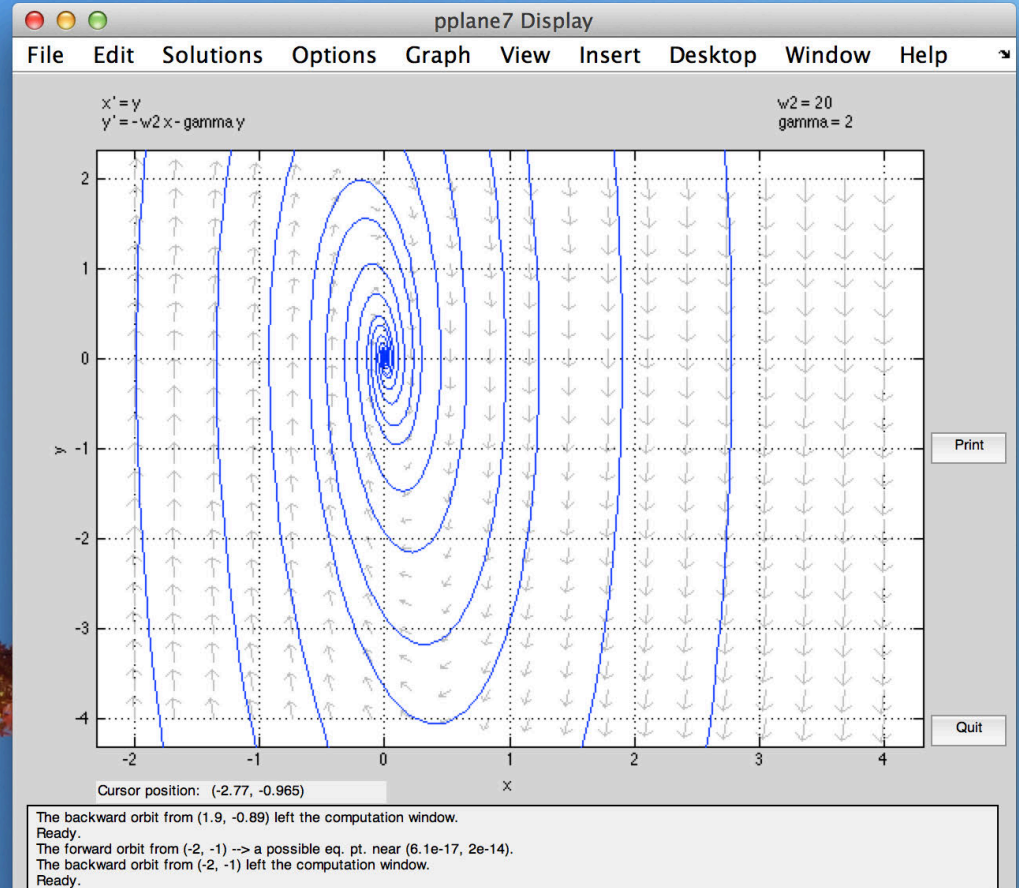
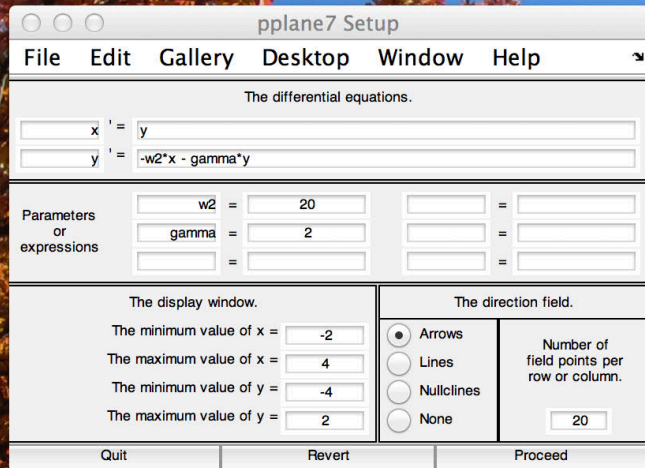
Number of field points per row or column. 20

Quit Revert Proceed



# Damped HO (Phase Plane Analysis)

- $\gamma = 2$
- $\omega_0^2 = 20$





# Damped HO (Phase Plane Analysis)

- $\gamma = 20$
- $\omega_0^2 = 20$

