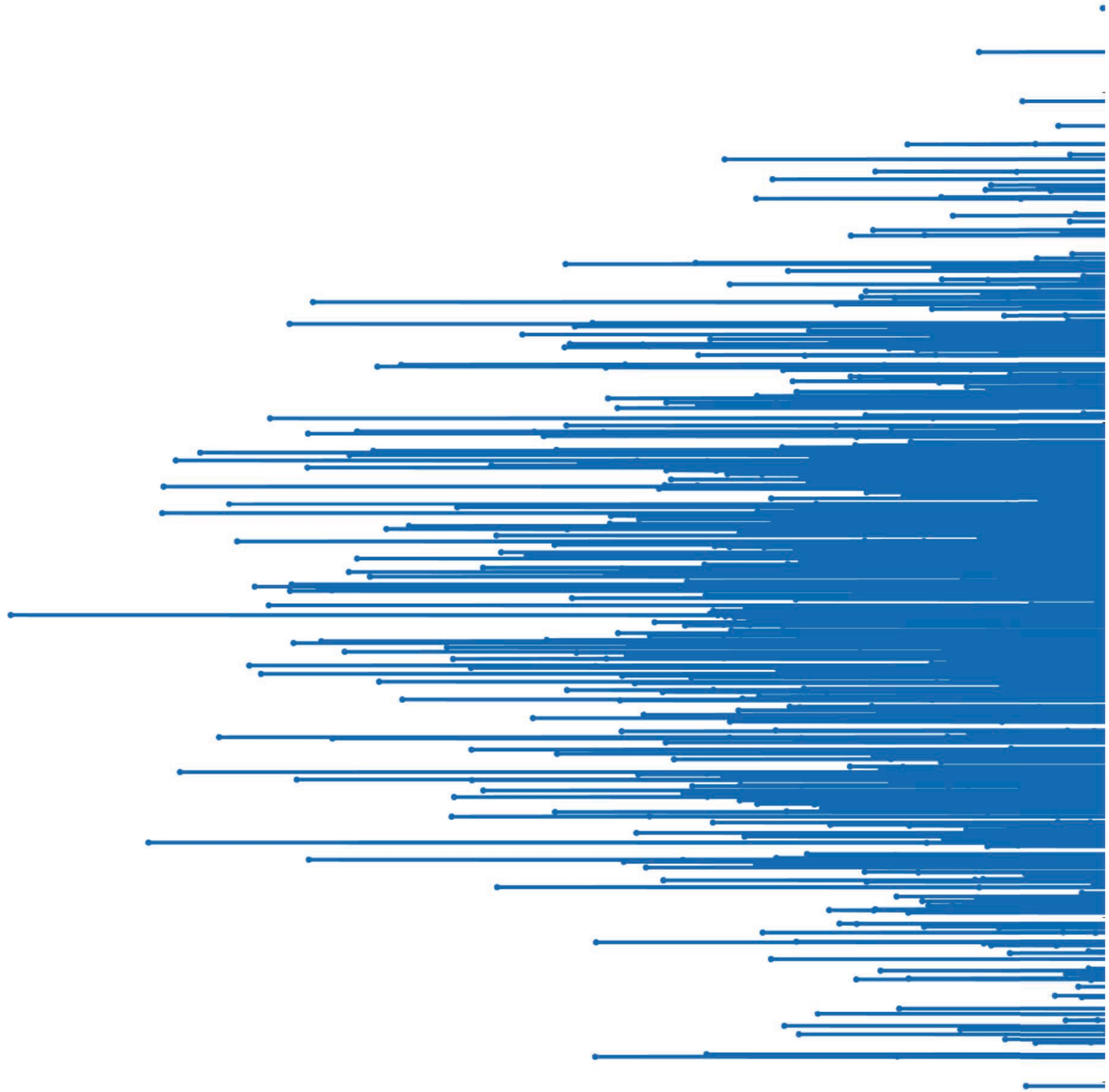
PHYS 2010 (W20)
Classical Mechanics



2020.02.27

Relevant reading:

Knudsen & Hjorth: 15.ff, 16.2

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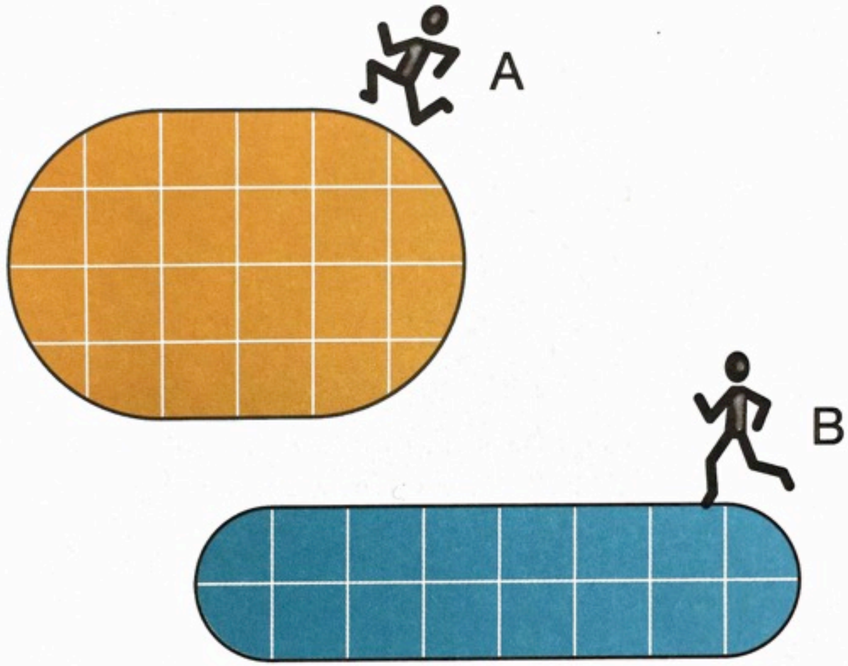
Ref.s:
Knudsen & Hjorth (2000), Fowles & Cassidy (2005)

233. Two Sprinters

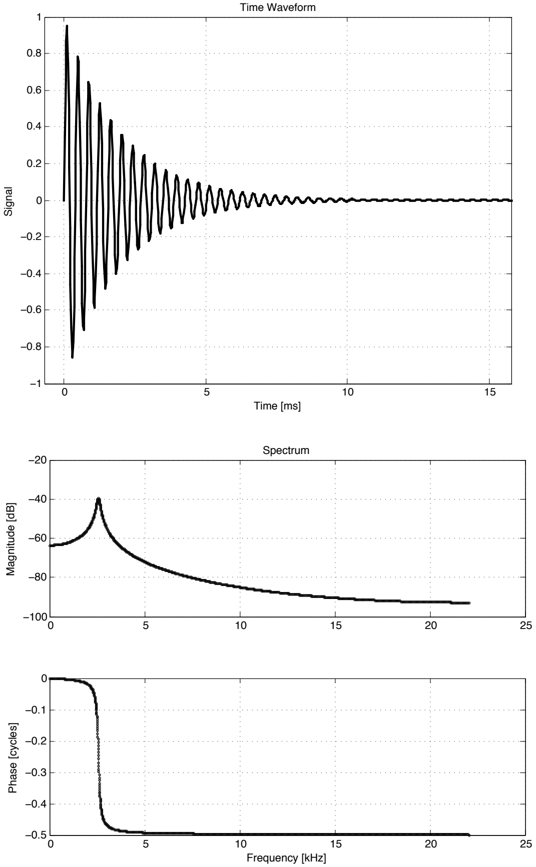


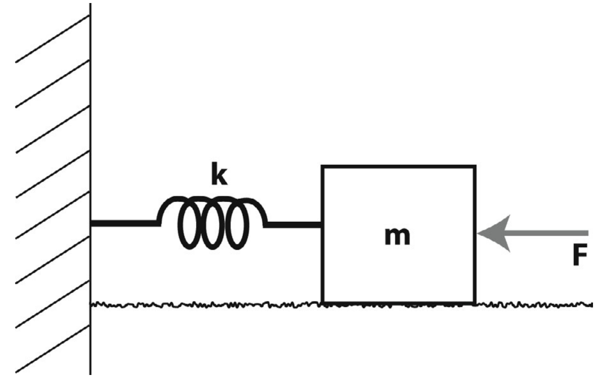
Which sprinter will run a longer distance to make the full circle and get to the start position (or will they run equal distances)?

A | B | Equal

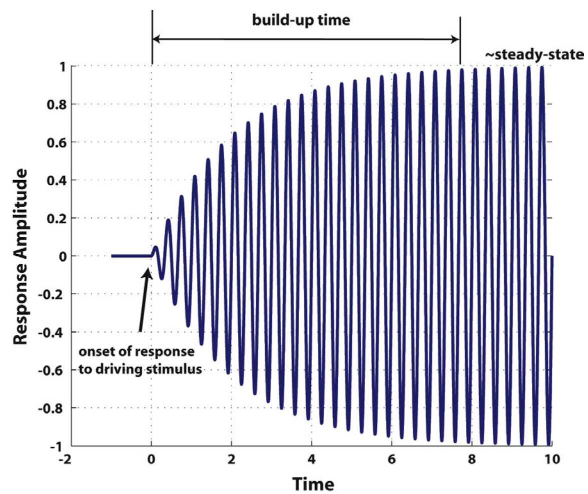
Looking Ahead....

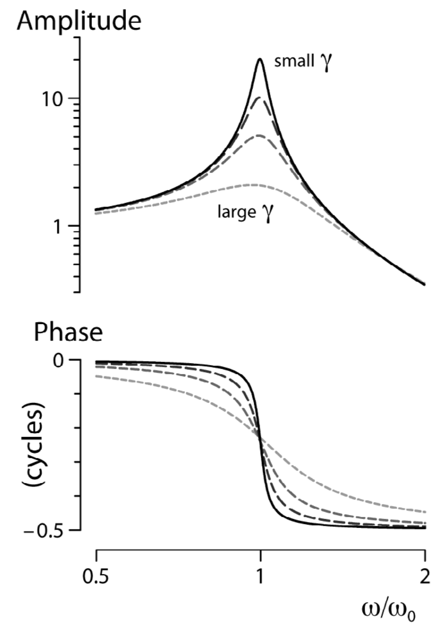
Damped HO (DHO)





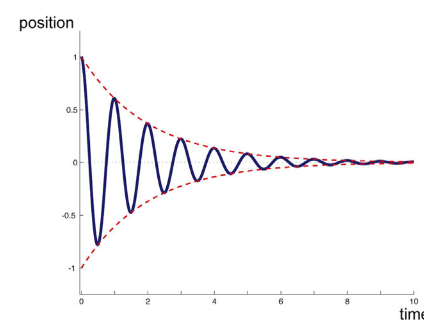
"Transient" responses...





- The *steady-state* response of the sinusoidally-driven harmonic harmonic oscillator acts like a *band-pass filter*

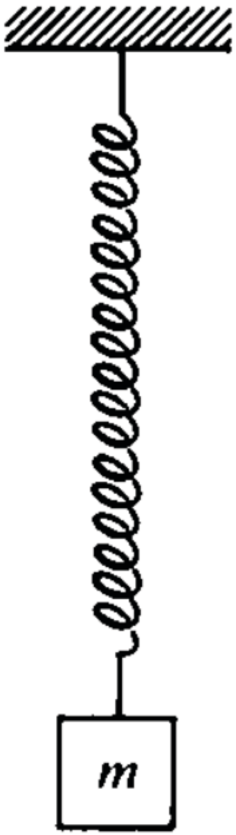
- Connection between *steady-state* response & *impulse response*



SHO Revisted

Two essential features:

1. An inertial component, capable of carrying kinetic energy.
2. An elastic component, capable of storing elastic potential energy.



Two fundamental laws:

1. By Newton's law ($F = ma$),

$$-kx = ma$$

2. By conservation of total mechanical energy (E),

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = E$$

Two associated differential equations:

$$m\frac{d^2x}{dt^2} + kx = 0 \qquad \qquad \tfrac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \tfrac{1}{2}kx^2 = E$$

French (1971)

SHO: Complex exponentials

Rewrite in terms of natural frequency

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

Now assume a solution in the form of a (possibly) complex exponential:

$$x = Ce^{pt}$$

Plug in assumed form of solution:

$$p^2Ce^{pt} + \omega^2Ce^{pt} = 0$$

Solving the ODE becomes an algebraic problem due to the assumptions we made!

Solve through:

$$p^2 + \omega^2 = 0$$

Associated eigenvalues

$$p^2 = -\omega^2$$

$$p = \pm j\omega$$

French (1971)

SHO: Complex exponentials

$$x = C_1 e^{j\omega t} + C_2 e^{-j\omega t}$$

Plugging it back in...

$$x = Ce^{j(\omega t + \alpha)} + Ce^{-j(\omega t + \alpha)}$$
$$= 2C\cos(\omega t + \alpha)$$
$$\equiv A\cos(\omega t + \alpha)$$

Now there are a couple ways things could play out, but keep in mind the same basic issue it at play: there are **two free parameters** (C & α, or C_1 & C_2, or A and α)

$$z = A\cos(\omega t + \alpha) + jA\sin(\omega t + \alpha)$$

$$x = \text{real part of } z \qquad \text{where} \quad z = Ae^{j(\omega t + \alpha)}$$

Note: The imaginary part of z is not any less "physical". It still contains the two key pieces of information (i.e., A and α here)! Choosing the real part here is just a convention.

French (1971)

SHO: Complex exponentials

$$x = Ce^{j(\omega t+\alpha)} + Ce^{-j(\omega t+\alpha)}$$
$$= 2C\cos(\omega t + \alpha)$$
$$\equiv A\cos(\omega t + \alpha)$$

Now there are a couple ways things could play out, but keep in mind the same basic issue it at play: there are **two free parameters** (C & α, or C_1 & C_2, or A and α)

Note: For the SHO, those two free parameters (plus our general form of the solution) tell us everything about how the system will behave for all time(!!)

→ So what determines those two free parameters?

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

2nd order ODE requires two unique **initial conditions** (or two unique boundary conditions) to find a specific solution [e.g., $x(t=0) = x_o$ and $v(t=0) = v_o$]

French (1971)

French (1971)

Damped HO

Eqn. of motion

$$m \frac{d^2x}{dt^2} = -kx - bv$$

Making a change of variables:

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + {\omega_0}^2 x = 0$$

where

$$\gamma = \frac{b}{m} \qquad {\omega_0}^2 = \frac{k}{m}$$

Now we must deal w/ a necessary reality: Despite solutions (possibly) being oscillatory, they **will not/cannot be sinusoidal**

French (1971)

Damped HO

Eqn. of motion

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Making a change of variables:

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0{}^2x = 0$$

where (remember these!)

$$\gamma = \frac{b}{m} \qquad \omega_0{}^2 = \frac{k}{m}$$

Now we must deal w/ a necessary reality: Despite solutions (possibly) being oscillatory, they will not/cannot be sinusoidal

So we shift to a complex form....

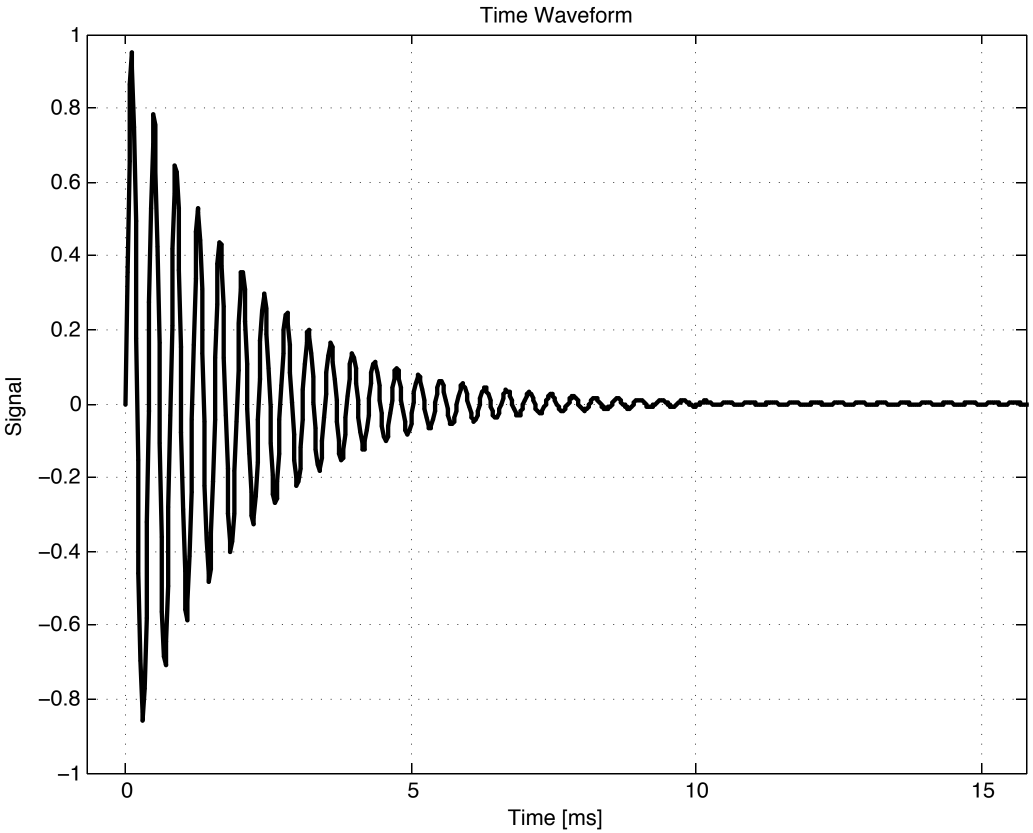
$$\frac{d^2z}{dt^2} + \gamma\frac{dz}{dt} + \omega_0{}^2z = 0$$

With an assumed solution of the form

$$z = Ae^{j(pt+\alpha)}$$

French (1971)

Complex Exponentials....



$$\frac{d^2z}{dt^2} + \gamma \frac{dz}{dt} + \omega_0{}^2 z = 0$$

$$z = Ae^{j(pt+\alpha)}$$

→ Not only does this assumed form of solution capture the oscillation, it also describes the exponential decay/growth

(all of which is encapsulated in the eigenvalues)

French (1971)

Damped HO (via complex exponentials)

Note:
There are a lot of starting points w/ regard to aspects such as the assumed form of the solution (see right & below as different possible examples). They may lead in slightly different directions analysis-wise, but ultimately they lead to the same place. It is worthwhile to spend a bit of to convince yourself of such, especially as you learn new mathematical methods....

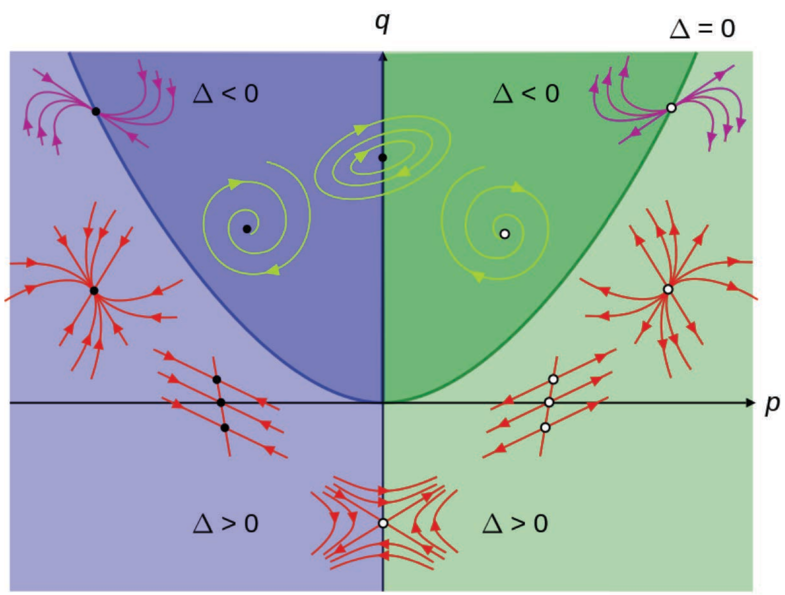
$$x = Ce^{pt}$$

$$z = Ae^{j(pt+\alpha)}$$

$$x(t) = Ae^{i(\omega t+\delta)}$$

$$x(t) = Ae^{-i(\omega t+\delta)}$$

$$x = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} c_1 e^{\lambda_1 t} + \begin{bmatrix} k_3 \\ k_4 \end{bmatrix} c_2 e^{\lambda_2 t}$$



$$\frac{dx}{dt} = Ax + By \qquad p = A + D$$
$$q = AD - BC$$
$$\frac{dy}{dt} = Cx + Dy \qquad \Delta = p^2 - 4q$$

French (1971)

Damped HO (via complex exponentials)

Combining these two:

$$z = Ae^{j(pt+\alpha)} \qquad \frac{d^2z}{dt^2} + \gamma\frac{dz}{dt} + {\omega_0}^2 z = 0$$

We obtain:

$$(-p^2 + jp\gamma + {\omega_0}^2)Ae^{j(pt+\alpha)} = 0$$

Or more succinctly:

$$-p^2 + jp\gamma + {\omega_0}^2 = 0$$

This is sometimes referred to as the *characteristic equation*

$$p = n + js$$

A handful of ways to deal w/ this, such as rewriting in terms of real and imaginary parts and solving each separately:

Real parts: $-n^2 + s^2 - s\gamma + {\omega_0}^2 = 0$

Imaginary parts: $-2ns + n\gamma = 0$

Note: Another approach is to solve the char. eqn. via the quadratic formula (see additional slides at end)

French (1971)

Damped HO (via complex exponentials)

$$z = Ae^{j(pt+\alpha)}$$

When the smoke clears:

$$s = \frac{\gamma}{2} \qquad n^2 = \omega_0{}^2 - \frac{\gamma^2}{4} \qquad p = n + js$$

Plugging back in:

$$z = Ae^{j(nt+jst+\alpha)} = Ae^{-st}e^{j(nt+\alpha)}$$

And subsequently, from our convention:

$$x = Ae^{-st}\cos(nt + \alpha)$$

Using variables from the ODE:

$$x = Ae^{-\gamma t/2}\cos(\omega t + \alpha)$$

where

$$\omega^2 = \omega_0{}^2 - \frac{\gamma^2}{4} = \frac{k}{m} - \frac{b^2}{4m^2}$$

→ The system doesn't even oscillate at the natural frequency!

French (1971)

Damped HO: Loss of Energy

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0{}^2 x = 0$$

For the moment, let's assume $\gamma \ll \omega,$

(i.e., relatively small damping)

$$\gamma = \frac{b}{m} \qquad \omega_0{}^2 = \frac{k}{m}$$

$$\omega^2 = \omega_0{}^2 - \frac{\gamma^2}{4} = \frac{k}{m} - \frac{b^2}{4m^2}$$

Recall that for SHO, the total energy is: $E = \frac{1}{2}kA^2$

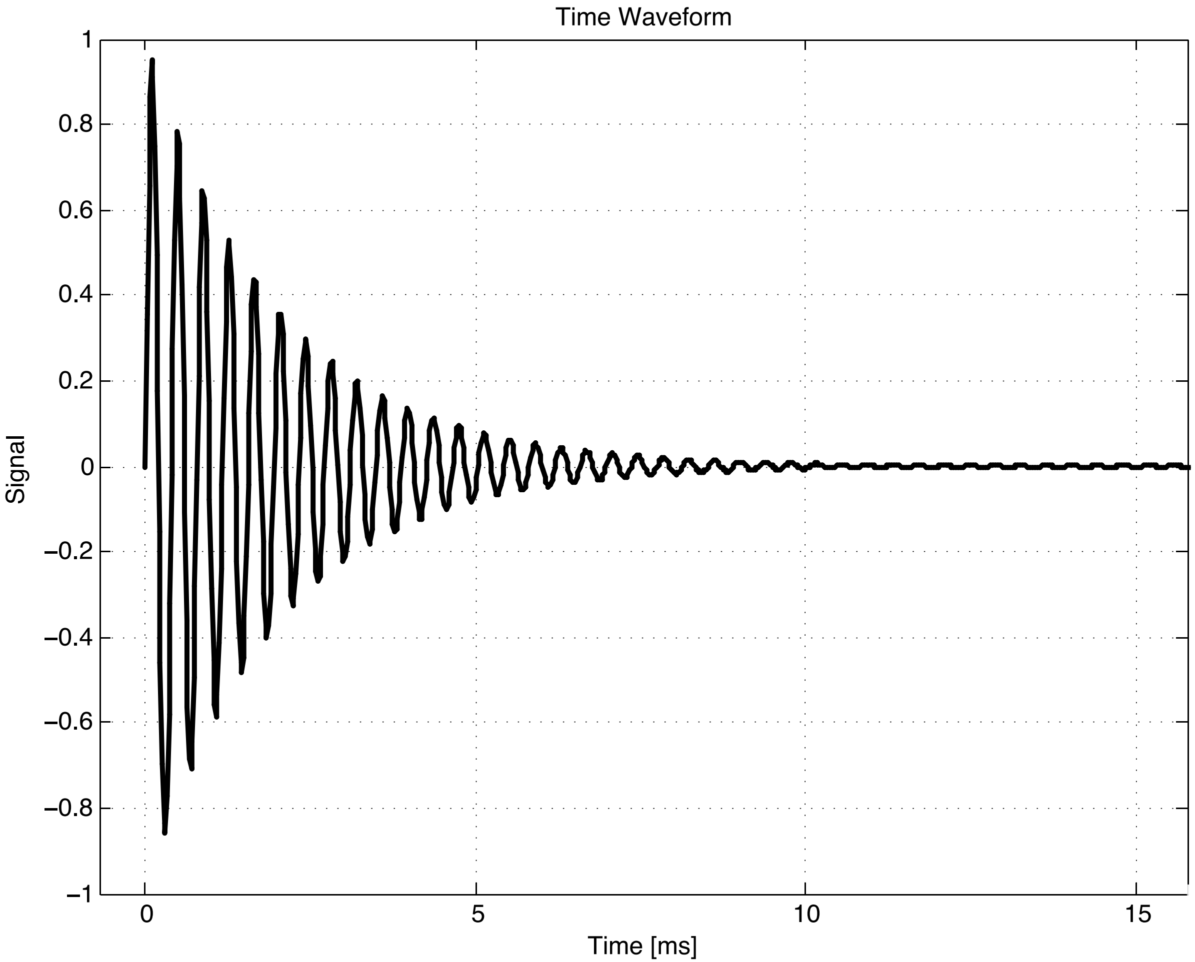
Thus for the damped case, we have: $A(t) = A_0 e^{-\gamma t/2}$ → $E(t) = \frac{1}{2}kA_0{}^2 e^{-\gamma t}$

Or more succinctly: $E(t) = E_0 e^{-\gamma t}$

→ Thus energy leaks out via an exponential decay due to the damping

French (1971)

Damped HO: Loss of Energy



$$x = Ae^{-\gamma t/2}\cos(\omega t + \alpha)$$

$$E(t) = E_0 e^{-\gamma t}$$

→ What about other relative damping cases?
(i.e., small vs medium vs large damping)

$$p = n + js$$

$$s = \frac{\gamma}{2} \qquad n^2 = {\omega_0}^2 - \frac{\gamma^2}{4}$$

Note - Recall that we assumed

$$z = Ae^{j(pt+\alpha)}$$

... but a more common convention is

$$x = Ce^{pt}$$

Thus it is more typical to find that the real part of the eigenvalue describes energy loss/gain (rather than the imaginary part, as is the case here)

Reference: System of linear autonomous ODEs

$$\frac{dx}{dt} = ax + by$$

$$\frac{dy}{dt} = cx + dy$$

- Let's consider a simple 2nd order system (all these ideas scale up for higher dimension systems)

- Re-express in matrix/vector form:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$$

- Let's make an assumption: solutions will have the form of (possibly complex) exponentials

$$x = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} c_1 e^{\lambda_1 t} + \begin{bmatrix} k_3 \\ k_4 \end{bmatrix} c_2 e^{\lambda_2 t}$$

This expression explicitly deals with the **eigenvalues** and **eigenvectors** of the system

wikipedia (phase space)

Reference: Eigen Decomposition

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$$

Characteristic equation:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

→ determinant (det) is scalar value associated with a square matrix

ODE as combination of eigenvalues and eigenvectors

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

'secular equation'

General solution:

$$x = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} c_1 e^{\lambda_1 t} + \begin{bmatrix} k_3 \\ k_4 \end{bmatrix} c_2 e^{\lambda_2 t}$$

→ Remember, we implicitly assume the solution has this exponential form!

Reference: Finding Eigenvalues

Characteristic equation:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0$$

Quadratic equation w/ two roots (for a 2nd order system)

$$\lambda^2 - \lambda(a+d) + (ad - bc) = 0$$

Note that complex roots are possible

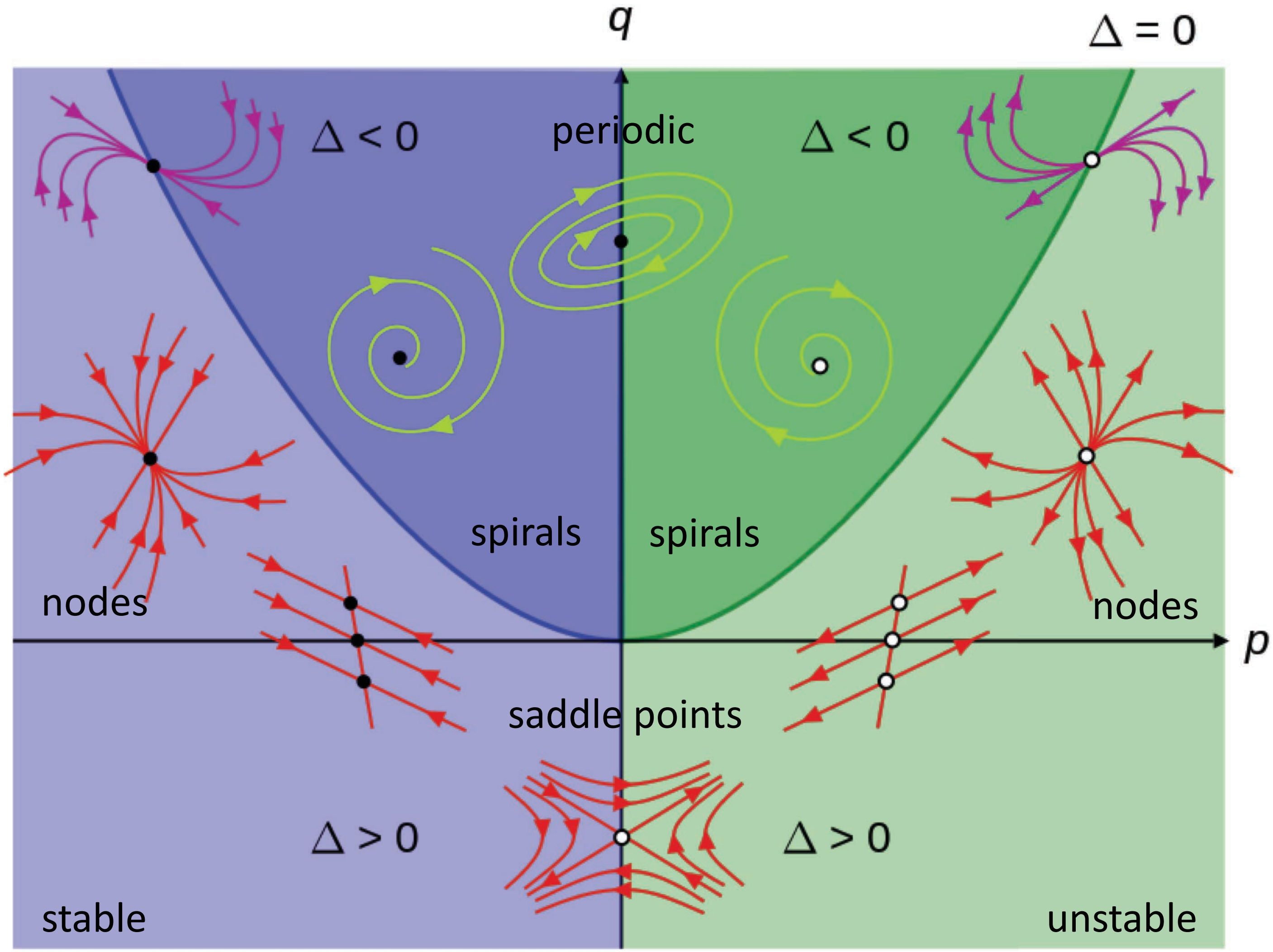
$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

$$x = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} c_1 e^{\lambda_1 t} + \begin{bmatrix} k_3 \\ k_4 \end{bmatrix} c_2 e^{\lambda_2 t}$$

→ Eigenvalues explicitly tell you how the solutions behave!

Reference: Classification of equilibrium points (linear autonomous 2nd order systems)

Orbits



$$\frac{dx}{dt} = Ax + By \qquad p = A + D$$

$$\frac{dy}{dt} = Cx + Dy \qquad q = AD - BC$$

$$\Delta = p^2 - 4q$$

Ex.

$$\frac{dx}{dt} = 5x - 3y$$

$$\frac{dy}{dt} = 2x - 4y$$

→ Only a single equlibrium point exists (at the origin). Stability?

$$\mathbf{A} = \begin{pmatrix} 5 & -3 \\ 2 & -4 \end{pmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

$$p = \mathrm{Tr}(\mathbf{A}) = 5 + (-4) = 1$$

$$q = \det(\mathbf{A}) = 5(-4) - (-3)2 = -14$$

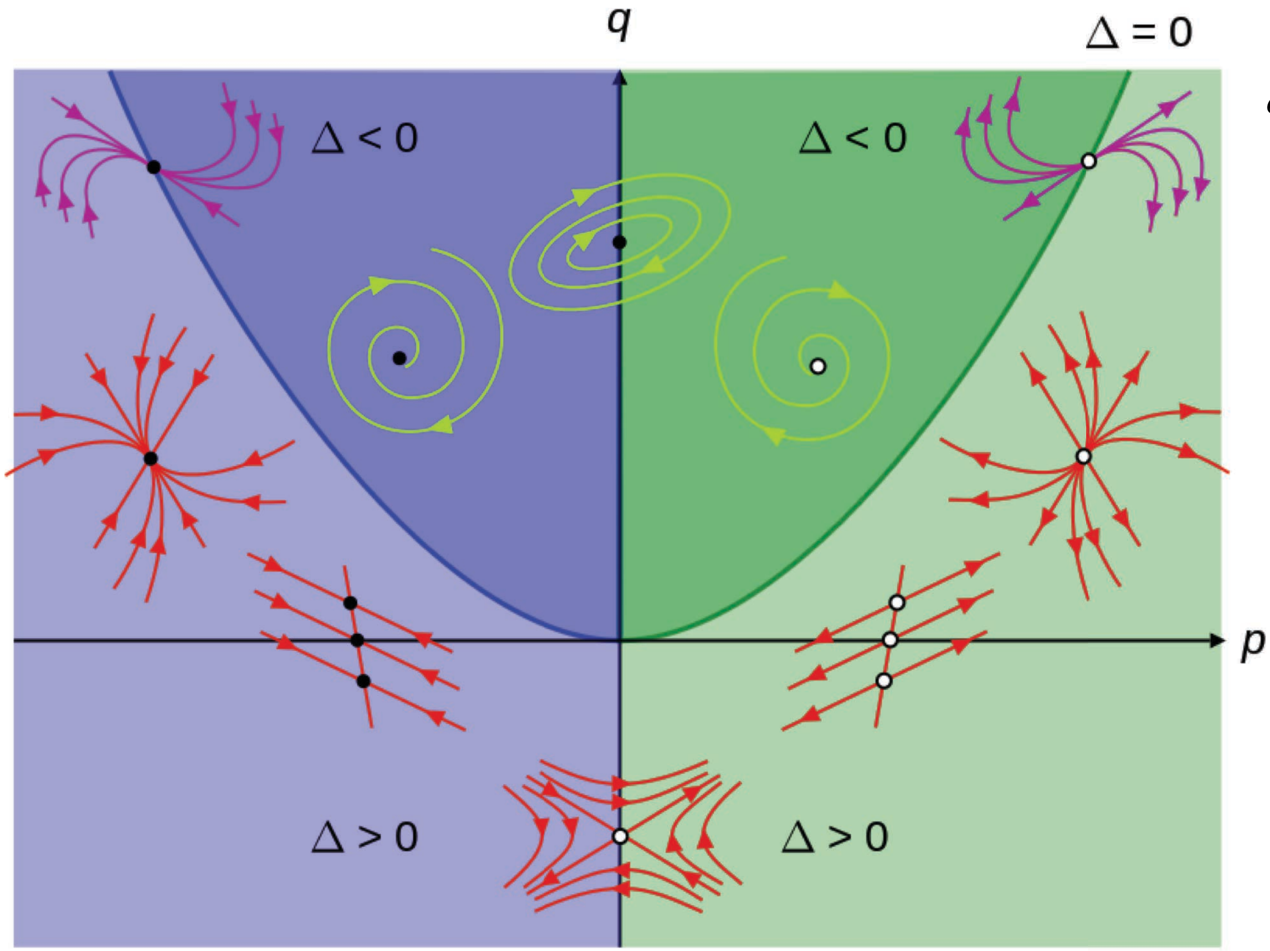
$$\lambda = \frac{1}{2}\left(1 \pm \sqrt{1 + 56}\right)$$

$$x = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} c_1 e^{\lambda_1 t} + \begin{bmatrix} k_3 \\ k_4 \end{bmatrix} c_2 e^{\lambda_2 t}$$

$$\lambda = -3.27, 4.27$$

→ General solution is a linear combination of a (real-valued) exponentials, one converging and one diverging

Ex. (cont.)



$$\frac{dx}{dt} = Ax + By \qquad p = A + D$$

$$\frac{dy}{dt} = Cx + Dy \qquad q = AD - BC$$

$$\Delta = p^2 - 4q$$

$$p = \mathrm{Tr}(\mathbf{A}) = 5 + (-4) = 1$$

$$q = \det(\mathbf{A}) = 5(-4) - (-3)2 = -14$$

$$\lambda = -3.27, 4.27$$

→ Solution curves approach the origin, then diverge away

→ Equilibrium point at origin (where the eigenvectors meet) is said to be a *saddle*

Damped HO (Alternative Approach re complex exponentials)

Rewrite as a system of first order ODEs

$$\ddot{x} + \gamma \dot{x} + \omega_o^2 x = 0$$

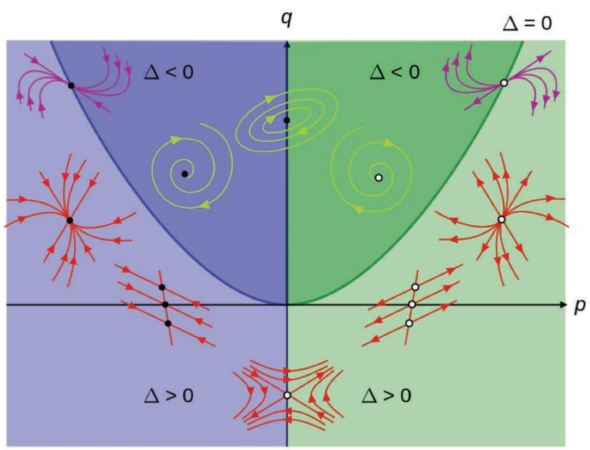
$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -\omega_o^2 x - \gamma y$$

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\omega_o^2 & -\gamma \end{pmatrix}$$

$$\lambda = \frac{1}{2}\left(-\gamma \pm \sqrt{\gamma^2 - 4\omega_o^2}\right)$$

$p = -\gamma$ $\quad$ $q = \omega_o^2$ (>0)



- What if γ is zero? Negative?
- Depending upon the sign and relative values of γ and ω_o, λ can be complex

→ Eigenvalues characterize behavior of all possible solution types!

$$x(t) = Ae^{-\gamma t/2}\; e^{i(\omega t + \alpha)}$$

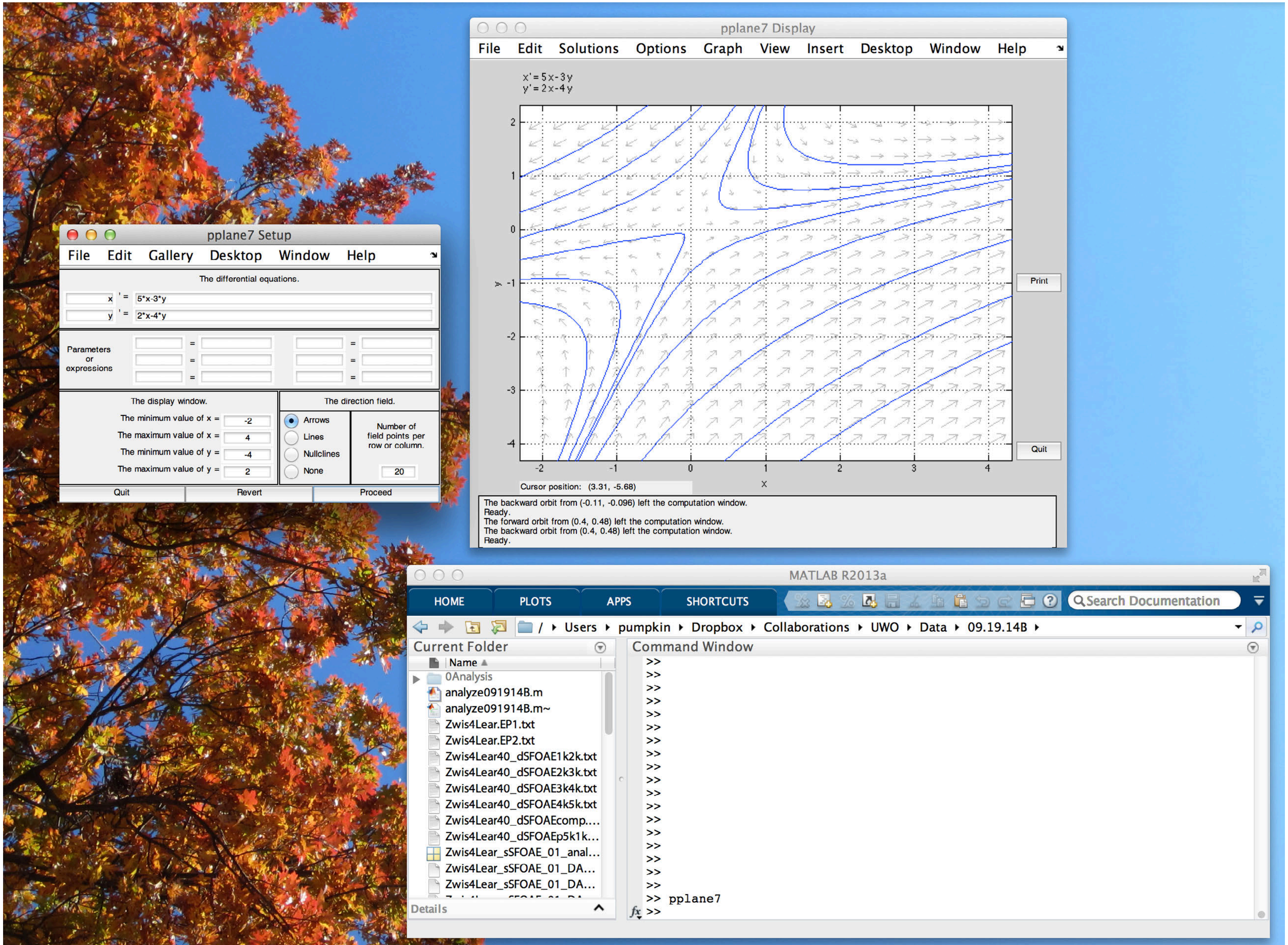
LINode45EX.m
LINfunction.m

```
% ### LINode45EX.m ###       01.26.16
% Numerically integrate a general 2nd order linear autonomous system (w/
% const. coefficients)
% x' = a*x + b*y
% y' = c*x + d*y

clear
% ------------------------------------------------------
% User input (Note: All paramters are stored in a structure)
P.y0(1) = 1.0;   % initial value for x
P.y0(2) = 1;   % initial value for y
P.A= [-3.9 3;      % matrix A to contain coefficients A= [a b
      -2 1];     %                                     c d]

% Integration limits
P.t0 = 0.0;   % Start value
P.tf = 10.0;   % Finish value
P.dt = 0.01;  % time step
% ----------------------------------------------------------------------
% +++
% determine some basic derived quantities
p= P.A(1,1)+ P.A(2,2);  % Tr(A)
q= P.A(1,1)* P.A(2,2)-P.A(1,2)* P.A(2,1); % det(A)
disp(['Tr(A)= ' num2str(p),' and det(A)= ',num2str(q)]);
eigV1= [0.5*(p+sqrt(p^2-4*q)) 0.5*(p-sqrt(p^2-4*q))];     % calc. eigenvalues directly
eigV2= eig(P.A);    % calculate via Matlab's built-in routine
disp(['eigenvalues= ' num2str(eigV1(1)),' and ',num2str(eigV1(2))]);

% +++
% use built-in ode45 to solve
[t y] = ode45('LINfunction', [P.t0:P.dt:P.tf],P.y0,[],P);

% -----------------------------------------------------
% visualize
% NOTE (re variable naming): x=y(1) and y=y(2)
figure(1); clf;
plot(t,y(:,1)); hold on; grid on;
xlabel('t');    ylabel('x(t)')
% Phase plane
figure(2); clf;
plot(y(:,1), y(:,2)); hold on; grid on;
xlabel('x(t)');    ylabel('y(t)')
% "solution space"
figure(3); clf;
plot(p,q,'rx','MarkerSize',9,'LineWidth',3); hold on; grid on;
if (abs(p)<1), pSpan= linspace(-1,1,100);
else    pSpan= linspace(-1.5*p-0.1,1.5*p+0.1,100); end
qSpan= pSpan.^2/4;
plot(pSpan,qSpan,'k-','LineWidth',2); %ylim([-max(qSpan) max(qSpan)])
plot(pSpan,zeros(numel(pSpan),1),'b--','LineWidth',2);
xlabel('Tr(A)');    ylabel('det(A)')
```

```
function [out1] = LINfunction(t,y,flag,P)
% ---------------------------------------------
%   y(1) ... x
%   y(2) ... y
out1(1)= P.A(1,1)*y(1) + P.A(1,2)*y(2);
out1(2)= P.A(2,1)*y(1) + P.A(2,2)*y(2);
out1= out1';
```

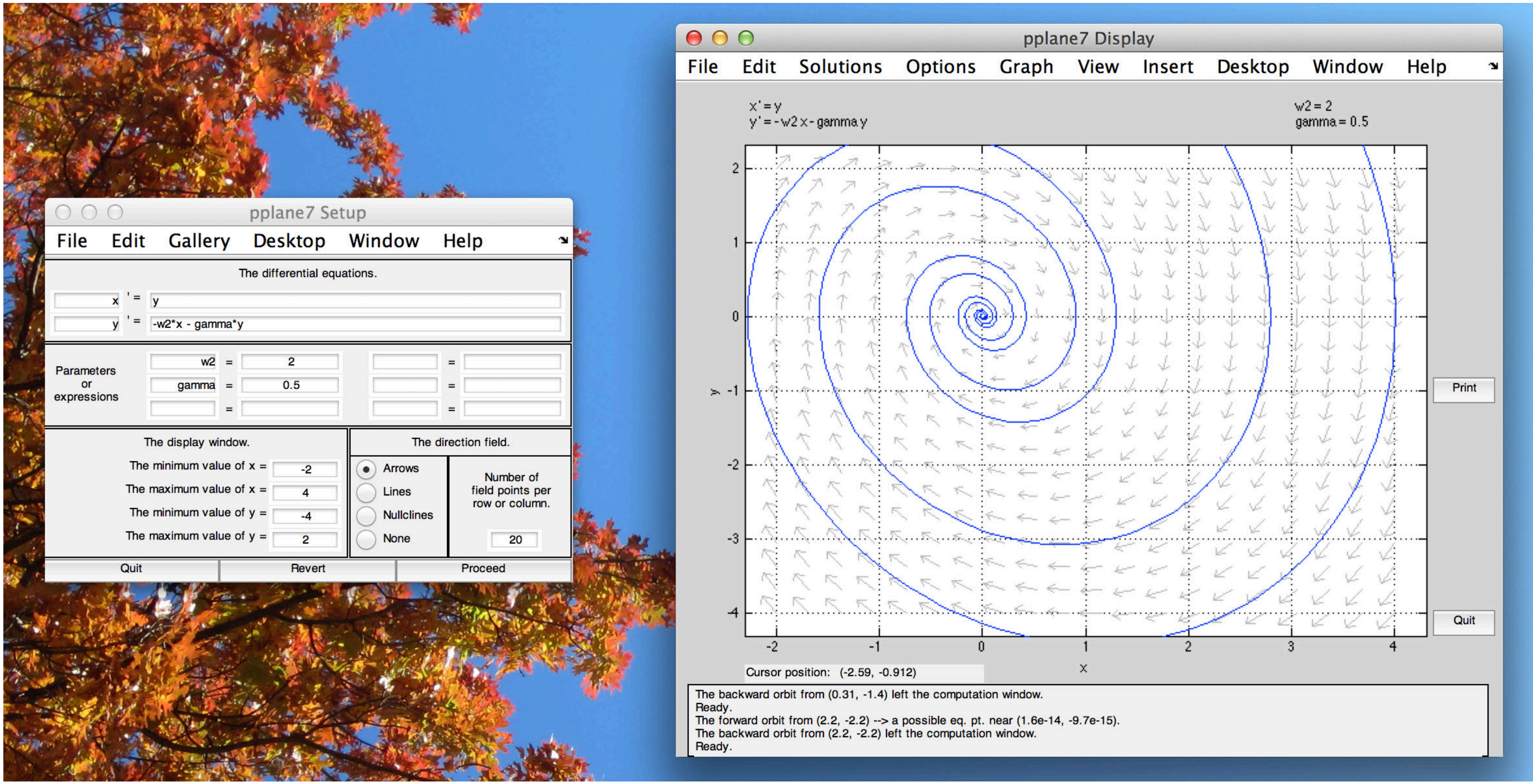
Damped HO (Phase Plane Analysis)

→ Computationally, use our ode45 code or pplane to explore behavior of solution curves



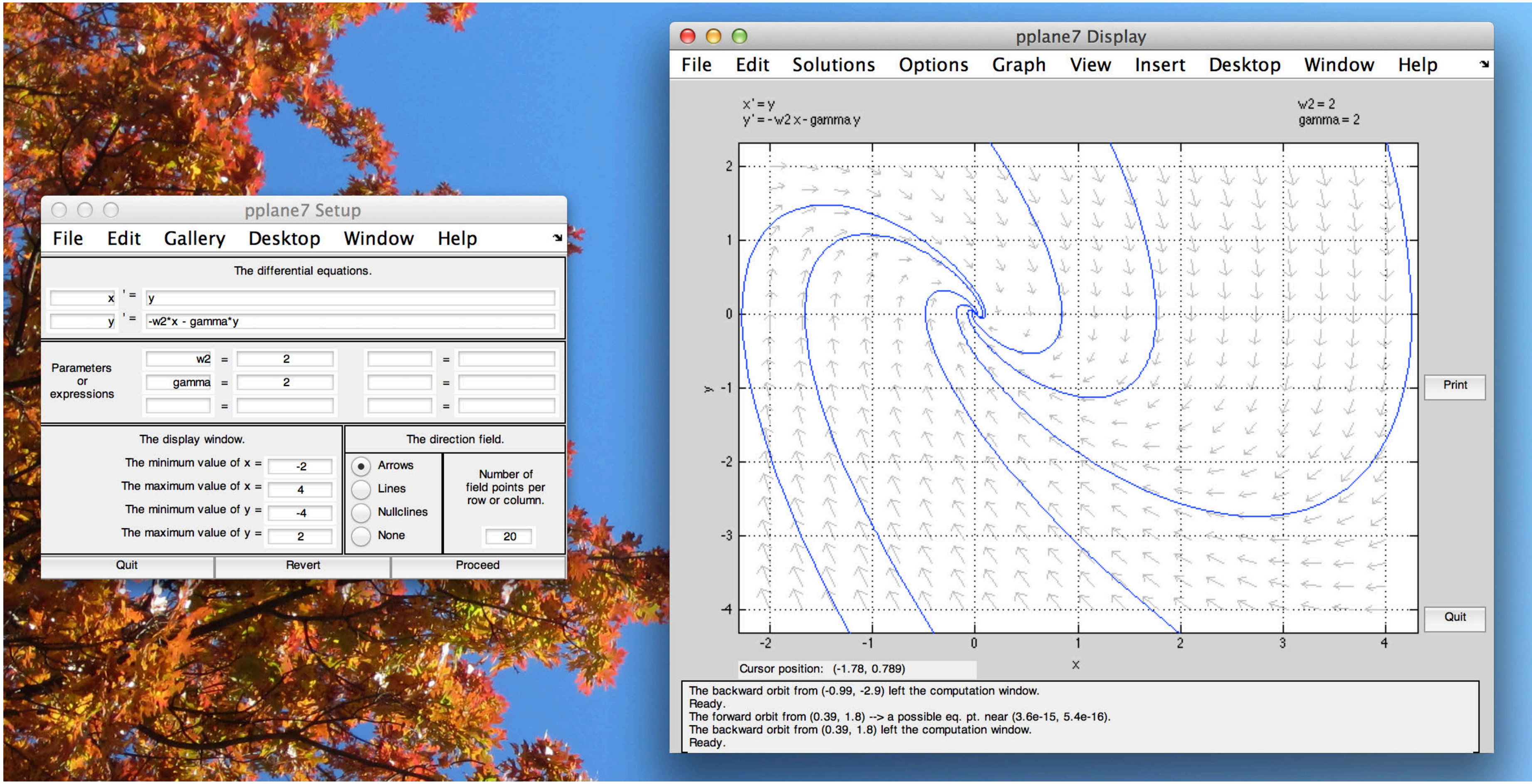
Damped HO (Phase Plane Analysis)

- $\gamma = 0.5$
- $\omega_0^2 = 2$



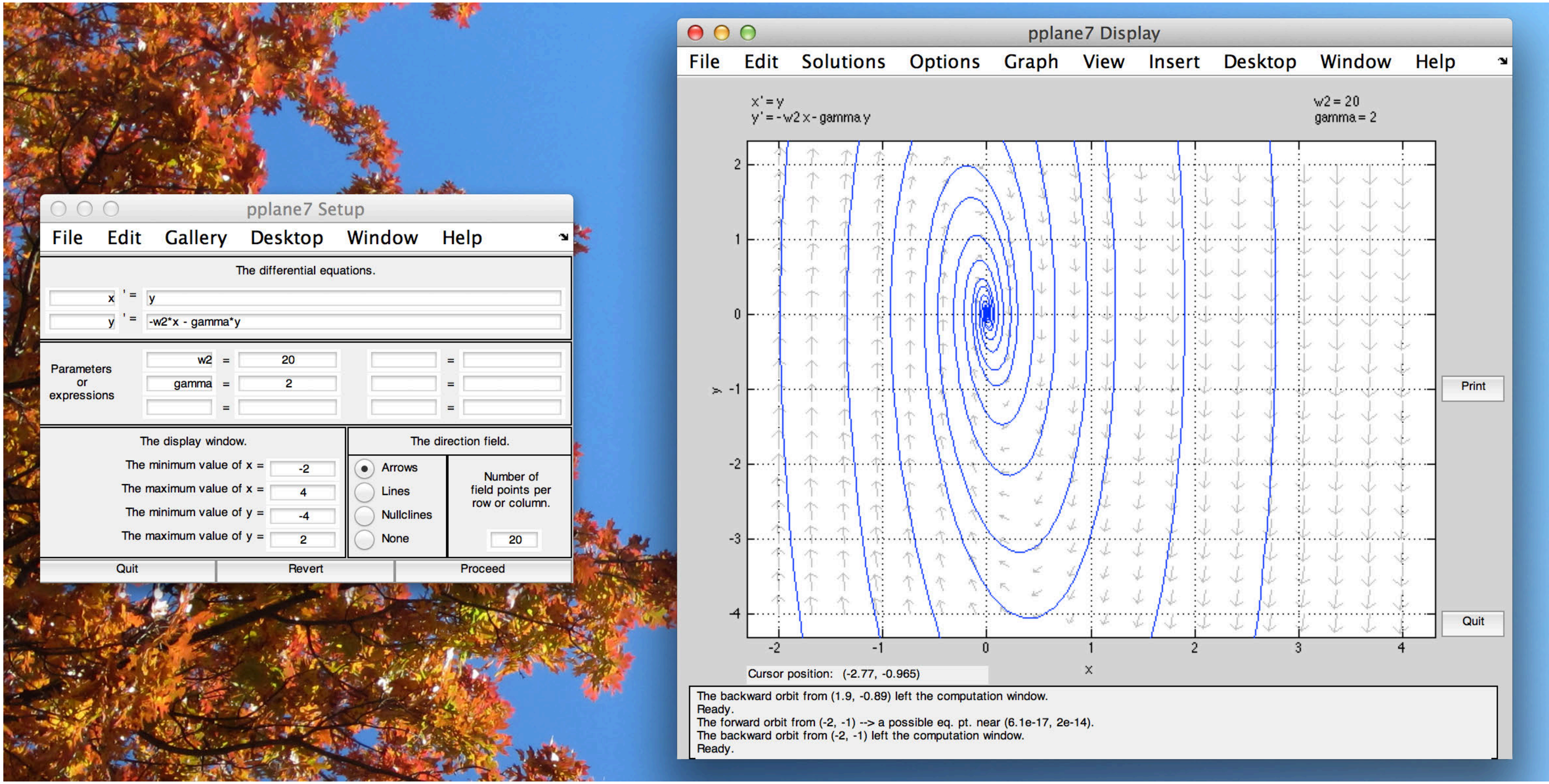
Damped HO (Phase Plane Analysis)

- $\gamma = 2$
- $\omega_0^2 = 2$



Damped HO (Phase Plane Analysis)

- $\gamma = 2$
- $\omega_0^2 = 20$



Damped HO (Phase Plane Analysis)

- $\gamma = 20$
- $\omega_0^2 = 20$