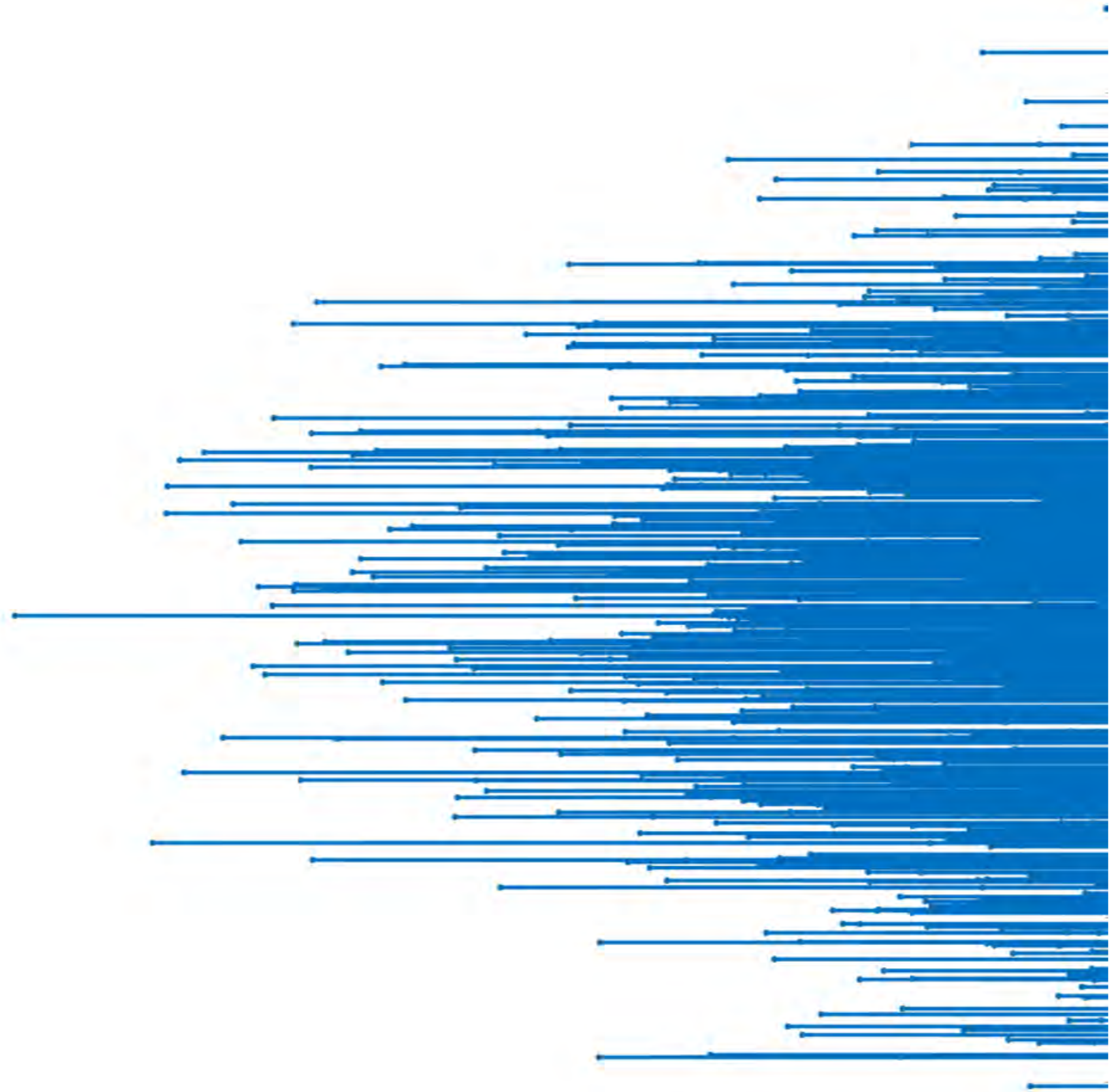
PHYS 2010 (W20)
Classical Mechanics



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Relevant reading:
Knudsen & Hjorth: 15.6

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Ref.s:
Knudsen & Hjorth (2000), Fowles & Cassidy (2005)

This is a word game that's trickier than it looks. I would like you to make the longest word you can using only the six letters illustrated below.

Your answer must be a well-known English word, and you are not allowed to use any letter twice.

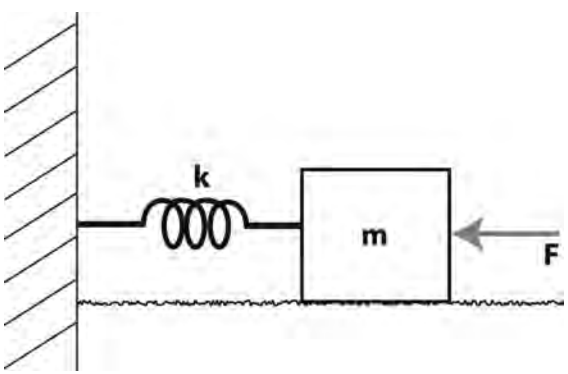
Think you've got a good answer? I can virtually guarantee that my solution will slay your effort.

Looking Ahead.....

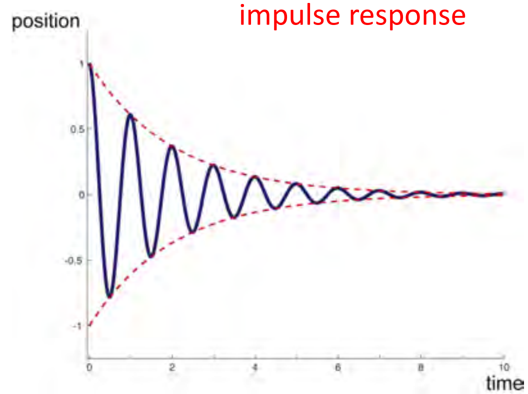
1. Time response of 'system' when subjected to an impulse
   (e.g., striking a bell w/ a hammer)

2. Fourier transform of resulting response
   (e.g., spectrum of bell ringing)

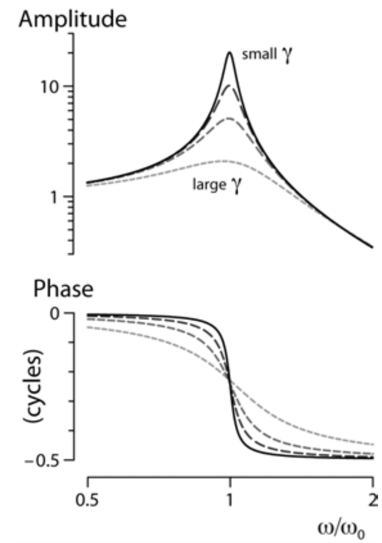
ex. Harmonic oscillator



Temporal
impulse response

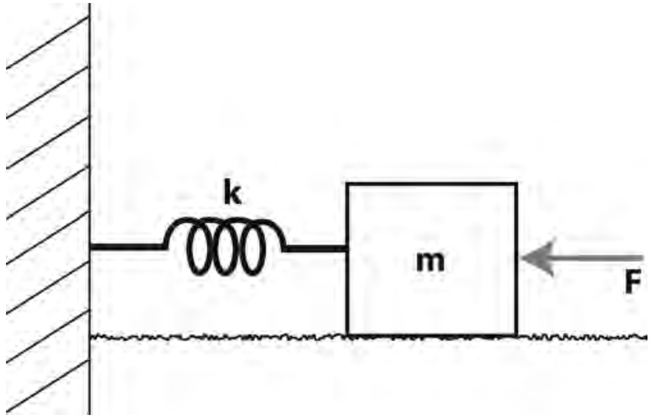


Spectral
impulse response



(Important) Note: The Fourier transform of the impulse response is called the *transfer function*

Case 1: Undamped undriven HO (i.e., SHO)



$$F = ma = m\ddot{x} = -kx$$

Newton's Second Law
Hooke's Law

$$\ddot{x} + \frac{k}{m}x = 0$$

Second order ordinary differential equation
(no need worrying about how to "solve", yet...)

$$x(t) = A\cos(\omega_o t + \phi)$$

$\Rightarrow$ Solution is oscillatory!

$$\omega_o = \sqrt{k/m}$$

System has a *natural* frequency

Case 2: Undamped driven HO

$$\ddot{x} + \frac{k}{m}x = F_o \cos \omega t$$

Sinusoidal driving force at frequency ω

Assumption: Ignore onset behavior and that system oscillates at frequency ω

$$x(t) = B \cos(\omega t + \alpha)$$

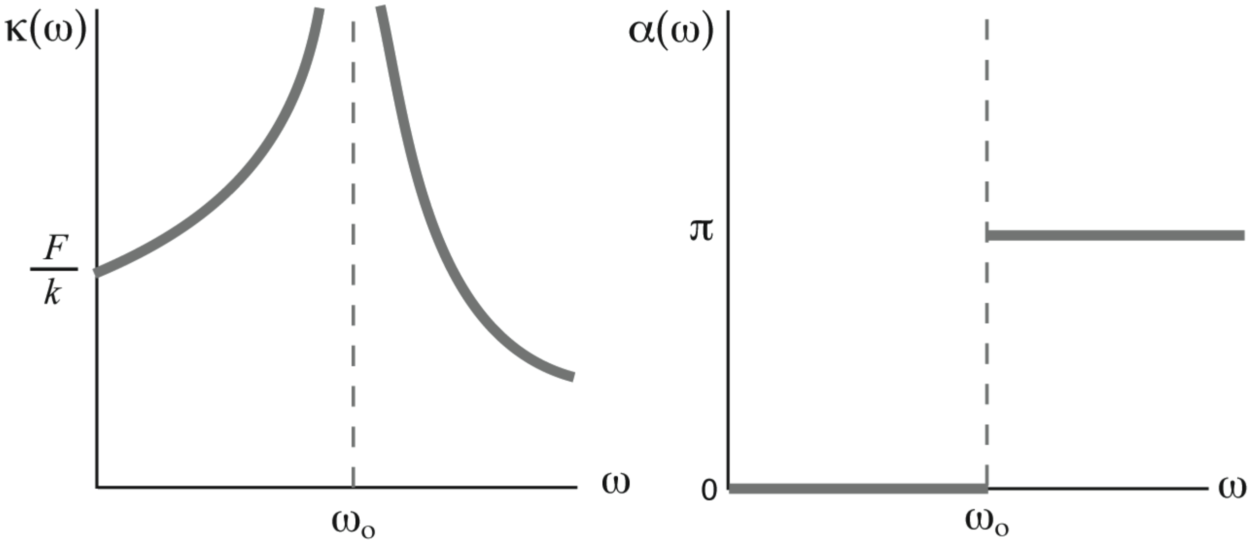
Assumed form of solution

$$-m\omega^2 B \cos \omega t + kB \cos \omega t = F_o \cos \omega t$$

$$\boxed{x(t) = \frac{F_o/m}{\omega_o^2 - \omega^2} \cos(\omega t + \alpha)}$$

Case 2: Undamped driven HO

$$x(t) = \frac{F_o/m}{\omega_o^2 - \omega^2} \cos(\omega t + \alpha) = \kappa(\omega) \cos(\omega t + \alpha)$$



Two Important Concepts Demonstrated Here:

- *Resonance* when system is driven at natural frequency

- Phase shift of 1/2 cycle about resonant frequency

Case 3: Damped undriven HO

$$m\ddot{x} + b\dot{x} + kx = 0$$

Purely sinusoidal solution no longer works!

$$\ddot{x} + \gamma\dot{x} + \omega_o^2 x = 0$$

Change variables

Assumption: Form of solution is a complex exponential

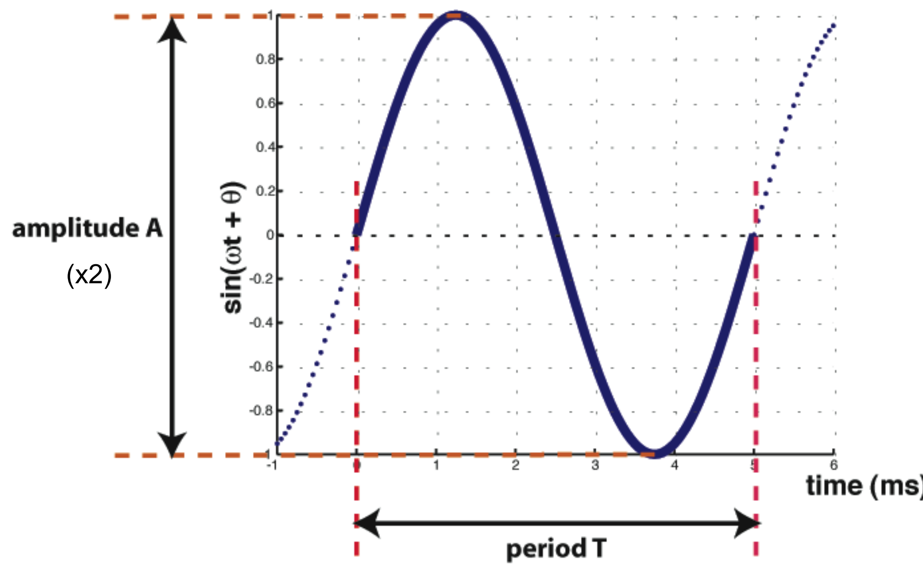
$$x(t) = Ae^{i(\omega t + \delta)}$$

Sinusoid has 3 basic properties:

i. Amplitude - height

ii. Frequency = 1/T [Hz]

iii. Phase - tells you where the peak is (needs a reference)

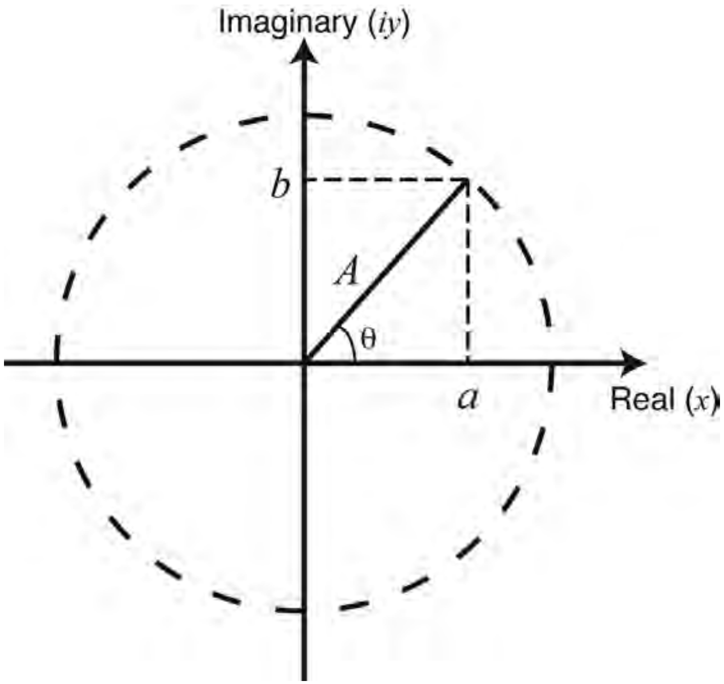


⇒ Phase reveals timing information

Case 3: Damped undriven HO

Motivation for complex solution:

$$a + ib = Ae^{i\theta}$$
$$= A(\cos\theta + i\sin\theta)$$



Cartesian Form

$$a = A\cos(\theta)$$
$$b = A\sin(\theta)$$

$\Leftrightarrow$

Polar Form

$$A = \sqrt{a^2 + b^2}$$
$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$\Rightarrow$ Complex solution contains both magnitude and phase information

Case 3: Damped undriven HO

$$\ddot{x} + \gamma\dot{x} + \omega_o^2 x = 0$$

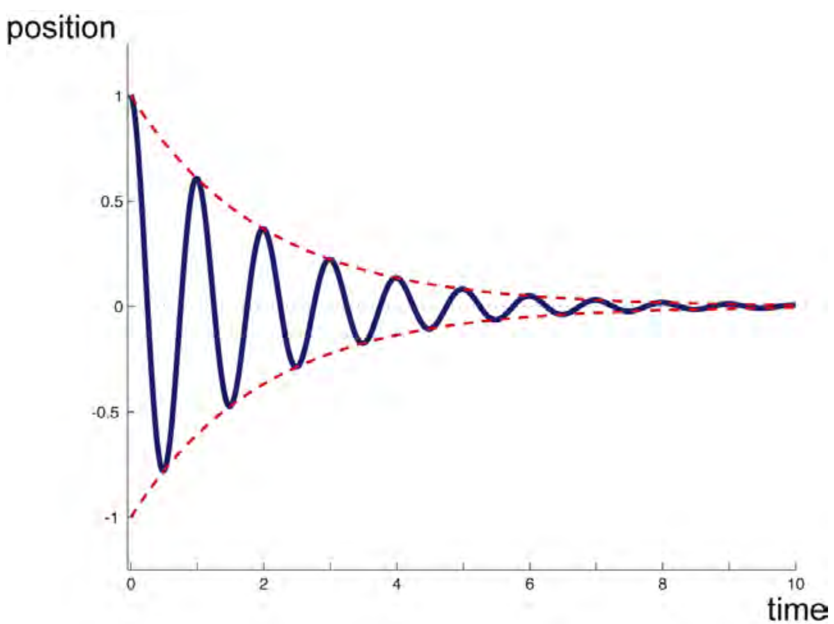
$$x(t) = Ae^{i(\omega t+\delta)}$$

$$x(t) = Ae^{-\gamma t/2}\ e^{i(\omega t+\alpha)}$$

$$\omega^2 = \omega_o^2 - \frac{\gamma^2}{4}$$

(slightly lower frequency of oscillation due to damping)

[A and α are constants of integration, depending upon initial conditions]



$\Rightarrow$ Damping causes energy loss from system

Case 4: Damped driven HO

$$m\ddot{x} = -kx - m\gamma\dot{x} + F_0 \cos\omega t$$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

General (strictly real) solution: $x(t) = x_1(t) + x_2(t)$

$$x(t) = x_0 \exp\left(-\frac{\gamma t}{2}\right)\cos(\omega_{\mathrm{d}} t + \varphi) + A\cos(\omega t - \theta)$$

First part (i.e., x_1) is just a decaying oscillation at the "free" (i.e., undriven) frequency:

$$\omega_{\mathrm{d}} = \omega_0 \left[1 - (\gamma/2\omega_0)^2\right]^{1/2}$$

Second part (i.e., x_2) is the steady-state response and a bit easier to derive via complex exponentials

$$x_2(t) = A\cos\omega t\cos\theta + A\sin\omega t\sin\theta$$

Knudsen & Hjorth (2000)

Case 4: Damped driven HO

$$\ddot{x} + \gamma\dot{x} + \omega_o^2 x = \frac{F_o}{m} e^{i\omega t}$$

Sinusoidal driving force at frequency ω

Assumption: Ignore onset behavior and that system oscillates at frequency ω

$$x(t) = Ae^{-i(\omega t + \delta)}$$

Assumed form of solution

$$A(\omega) = \frac{F_o/m}{[(\omega_o^2 - \omega^2)^2 + (\gamma\omega)^2]^{1/2}}$$

(magnitude)

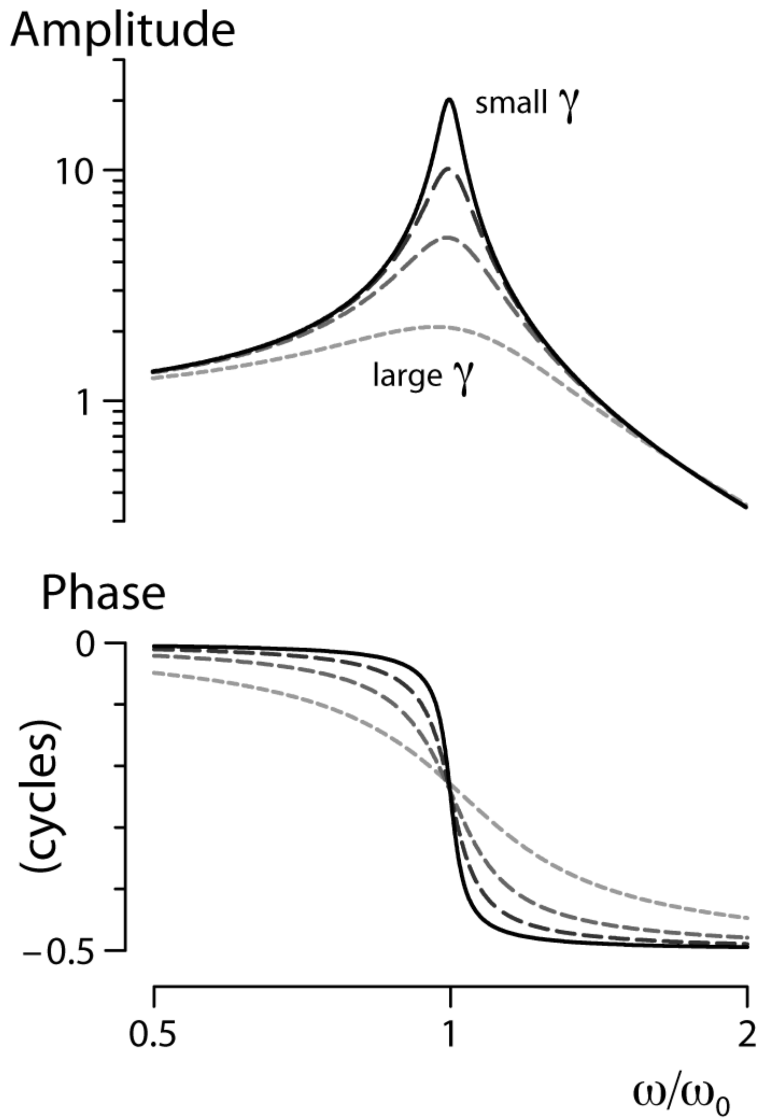
$$\delta(\omega) = \arctan\left(\frac{\gamma\omega}{\omega^2 - \omega_o^2}\right)$$

(phase)

Case 4: Damped driven HO

$$A(\omega) = \frac{F_o/m}{[(\omega_o^2 - \omega^2)^2 + (\gamma\omega)^2]^{1/2}}$$

$$\delta(\omega) = \arctan\left(\frac{\gamma\omega}{\omega^2 - \omega_o^2}\right)$$



⇒ Second-order oscillator behaves as as *band-pass filter*

Case 4: Damped driven HO

Three different key frequencies at play:

1. driving frequency (ω)

$$m\ddot{x} = -kx - m\gamma\dot{x} + F_0 \cos \omega t$$

2. free damped frequency (ω_d)

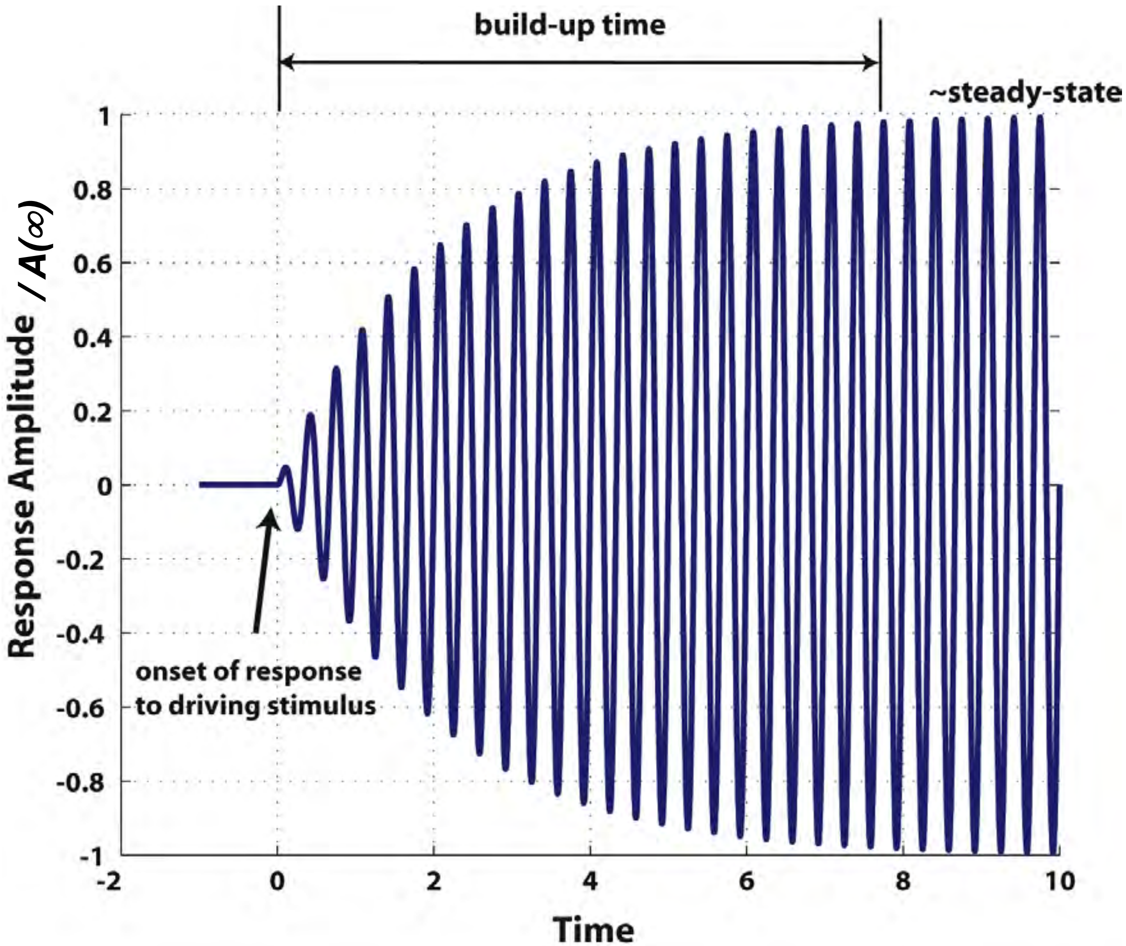
$$\omega_\mathrm{d} = \omega_0 \sqrt{1 - \frac{\gamma^2}{4\omega_0^2}}$$

3. resonant frequency (ω_m)

$$\omega_\mathrm{m} = \omega_0 \sqrt{1 - \frac{\gamma^2}{2\omega_0^2}}$$

Knudsen & Hjorth (2000)

<u>Key Idea</u>: Tuned Responses Take Time



Second Order System
(resonant frequency ω_m)

$\Rightarrow$ External driving force at frequency ω

$$x(t) = A(\infty)\ [1 - e^{(-t/\tau)}]$$

$$\tau = 1/\gamma = Q\ /\ \omega_o$$

Steady-state frequency response

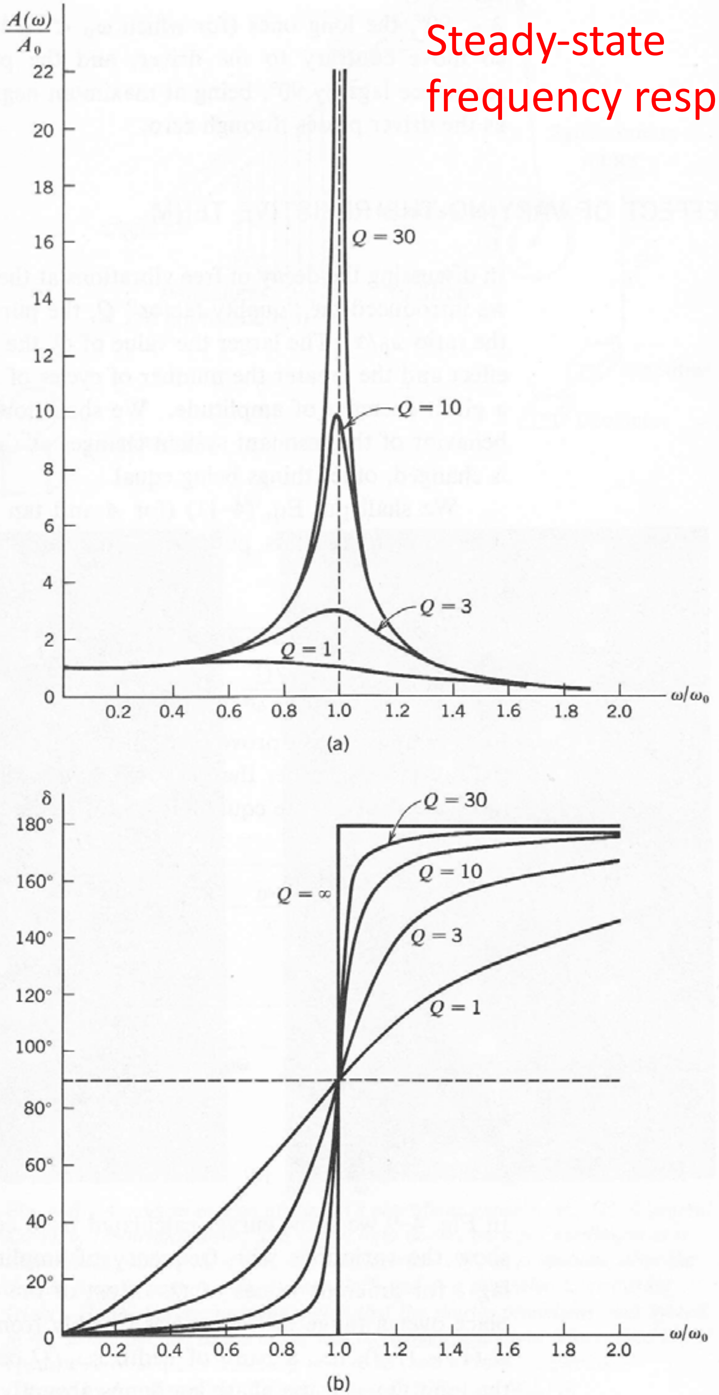


Fig. 4–9 (a) Amplitude as function of driving frequency for different values of Q, assuming driving force of constant magnitude but variable frequency. (b) Phase difference δ as function of driving frequency for different values of Q.

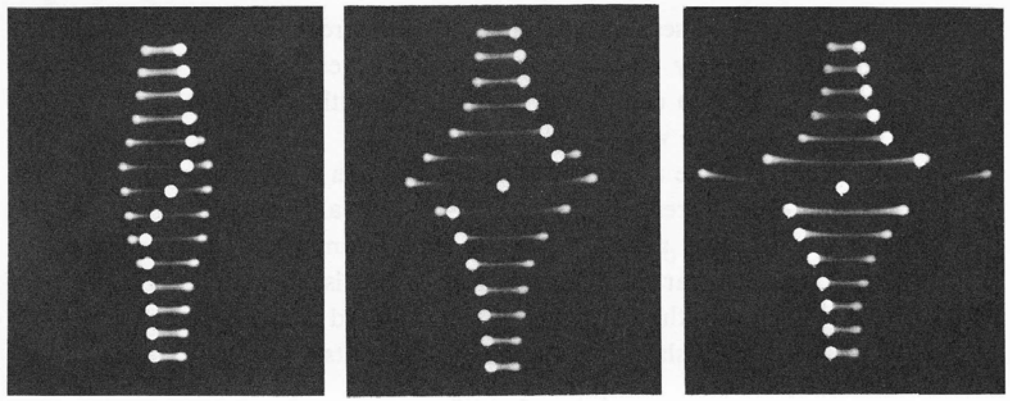
Q is the 'quality factor'

$$\ddot{x} + \gamma \dot{x} + \omega_o^2 x = \frac{F_o}{m} e^{i\omega t}$$

$$Q = \omega_o / \gamma$$

→ Phase information tells us something about the damping

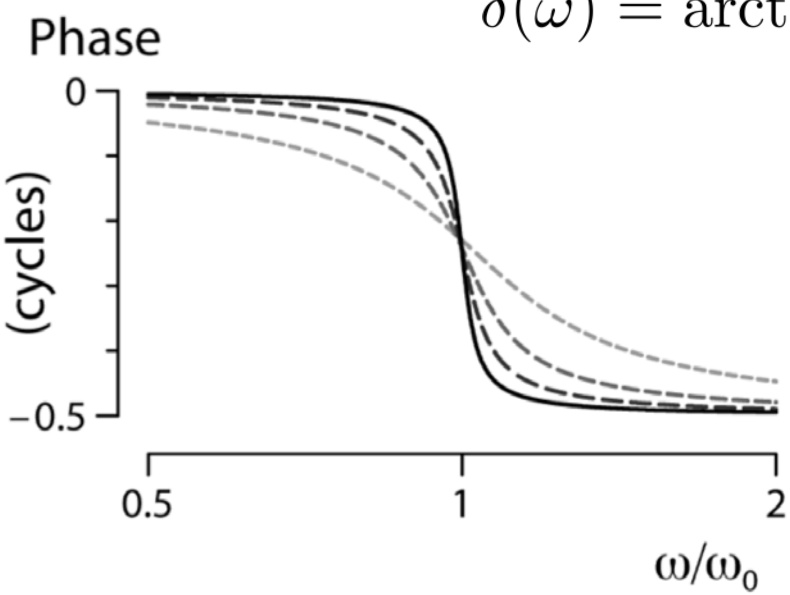
French (1971)



$$\ddot{x} + \gamma\dot{x} + \omega_o^2 x = \frac{F_o}{m} e^{i\omega t}$$

$$\delta(\omega) = \arctan\left(\frac{\gamma\omega}{\omega^2 - \omega_o^2}\right)$$

$N \equiv f_0$ * phase slope
(group delay)



$N \propto 1/\gamma$

⇒ Characterizing phase slope near resonance provides measure of damping

Impulse response

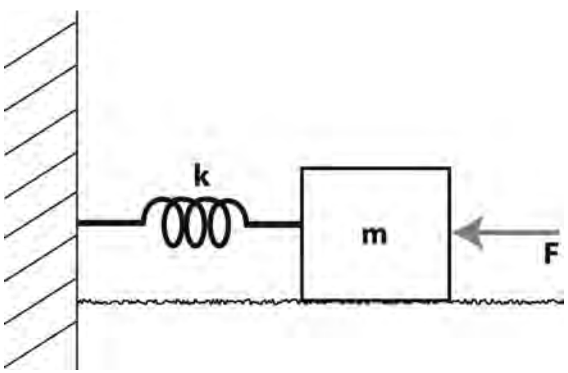
➢ Intuitively defined in two different (but equivalent) ways:

1. Time response of ‘system’ when subjected to an impulse
   (e.g., striking a bell w/ a hammer)

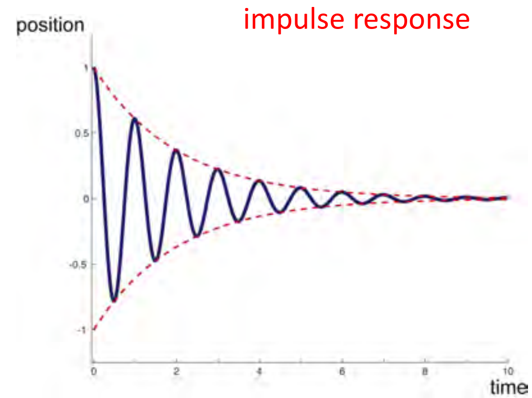
2. Fourier transform of resulting response
   (e.g., spectrum of bell ringing)

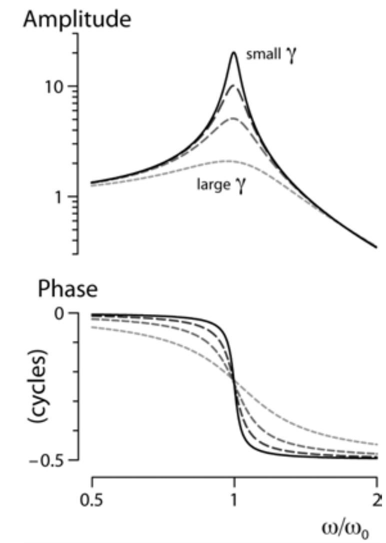
Spectral
impulse response

ex. Harmonic oscillator



Temporal
impulse response





(Important) Note: The Fourier transform of the impulse response is called the *transfer function*