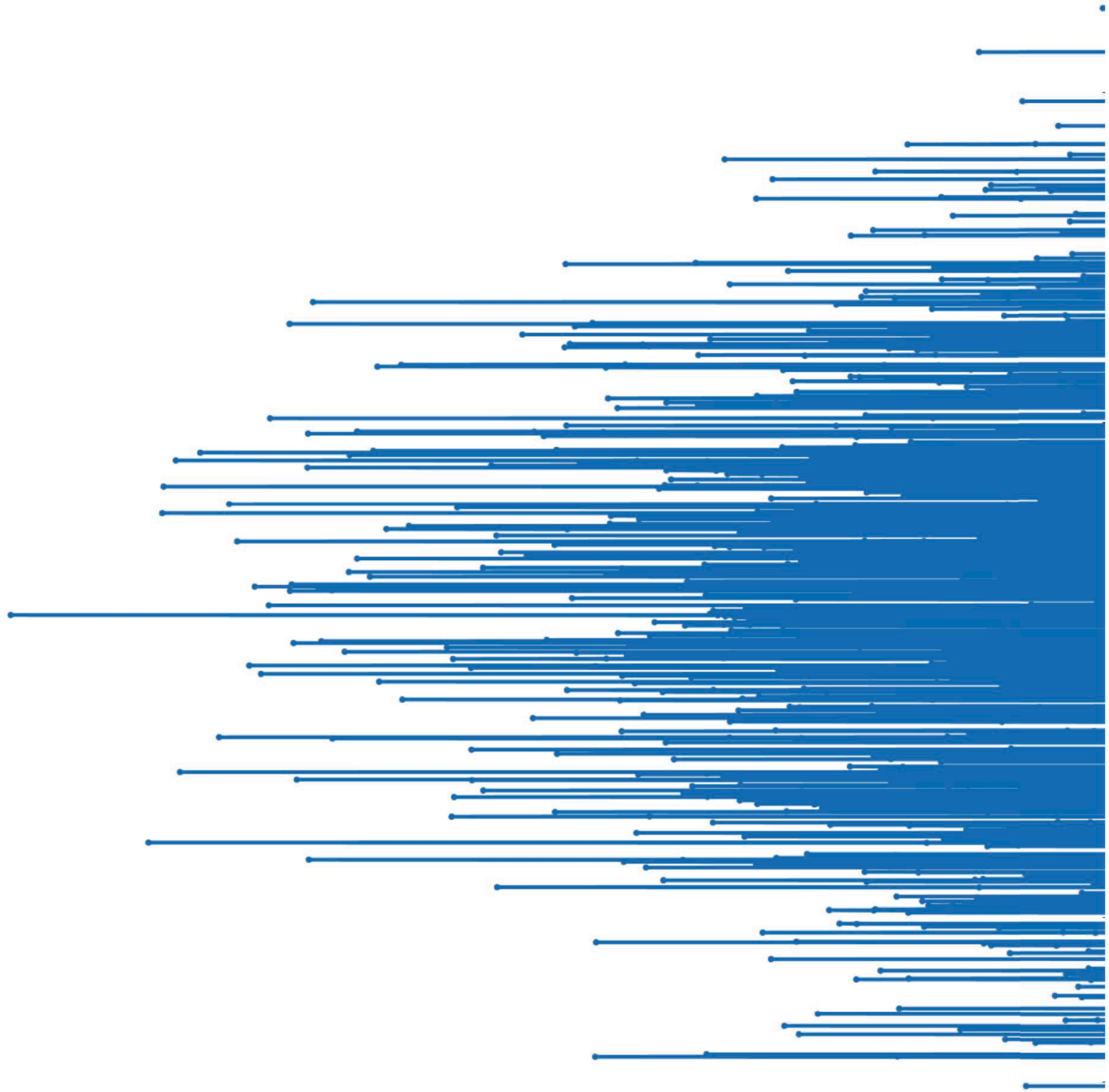
PHYS 2010 (W20)
Classical Mechanics



2020.03.10

<u>Relevant reading</u>:
Knudsen & Hjorth: X

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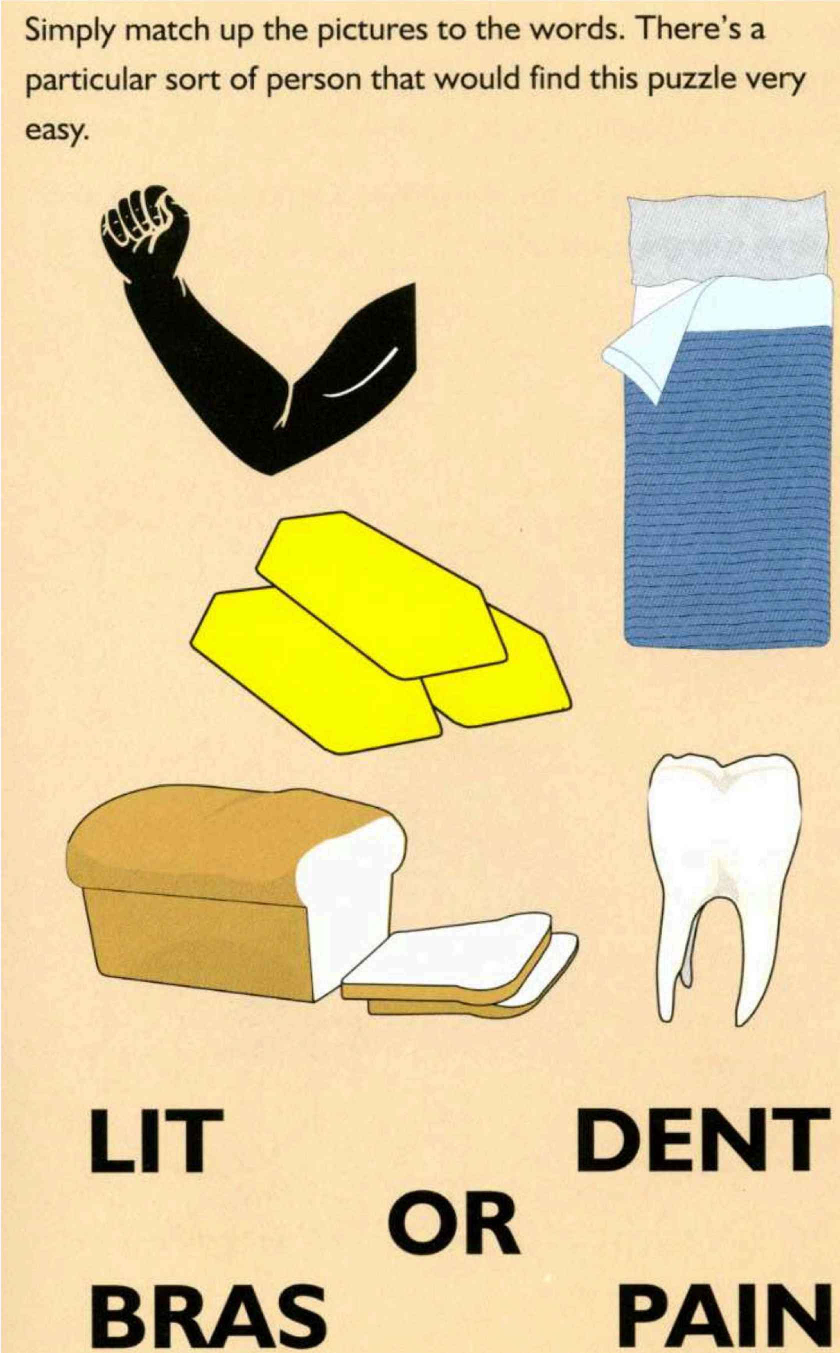
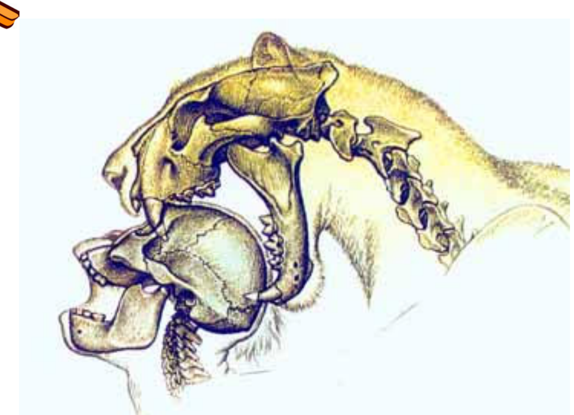
<u>Ref.s</u>:
Knudsen & Hjorth (2000), Fowles & Cassidy (2005)





Bed

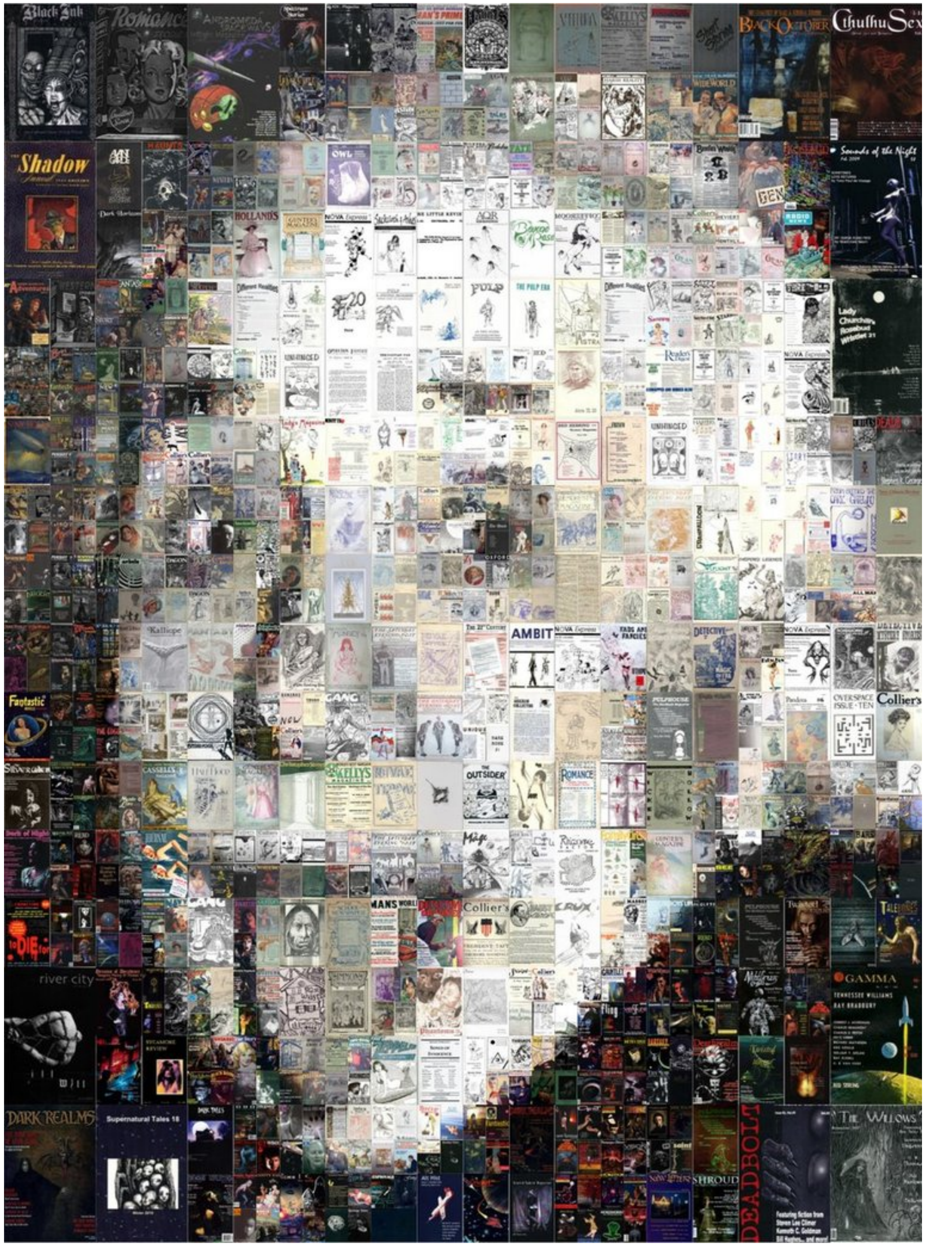
Tooth

Gold

Arm

Bread


PULP
THE PULP ERA
UNHINGED
AMBIT
NOVA Express



- ‘Photo mosaics’ use images as an underlying set of ‘basis functions’
- Note that we could just as easily choose a different set of basis images....

... and it’s not too hard to imagine that some choices might be better than others!

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt.$$

→ Similar idea underlies the notion of Fourier analysis, the choice of basis functions being sinusoids

http://cornejo-sanchez.deviantart.com/art/Einstein-Mosaic-227306670

Fourier analysis

- Deep history throughout mathematics, physics, engineering, biology, ...

- Backbone of modern signal processing and linear systems theory

- Lays at foundation of many modern methodologies in medical imaging (e.g., MRI, CT scans)

- Builds off the basic idea of a *Taylor series* (which posits we can describe a function as an infinite series of polynomials)

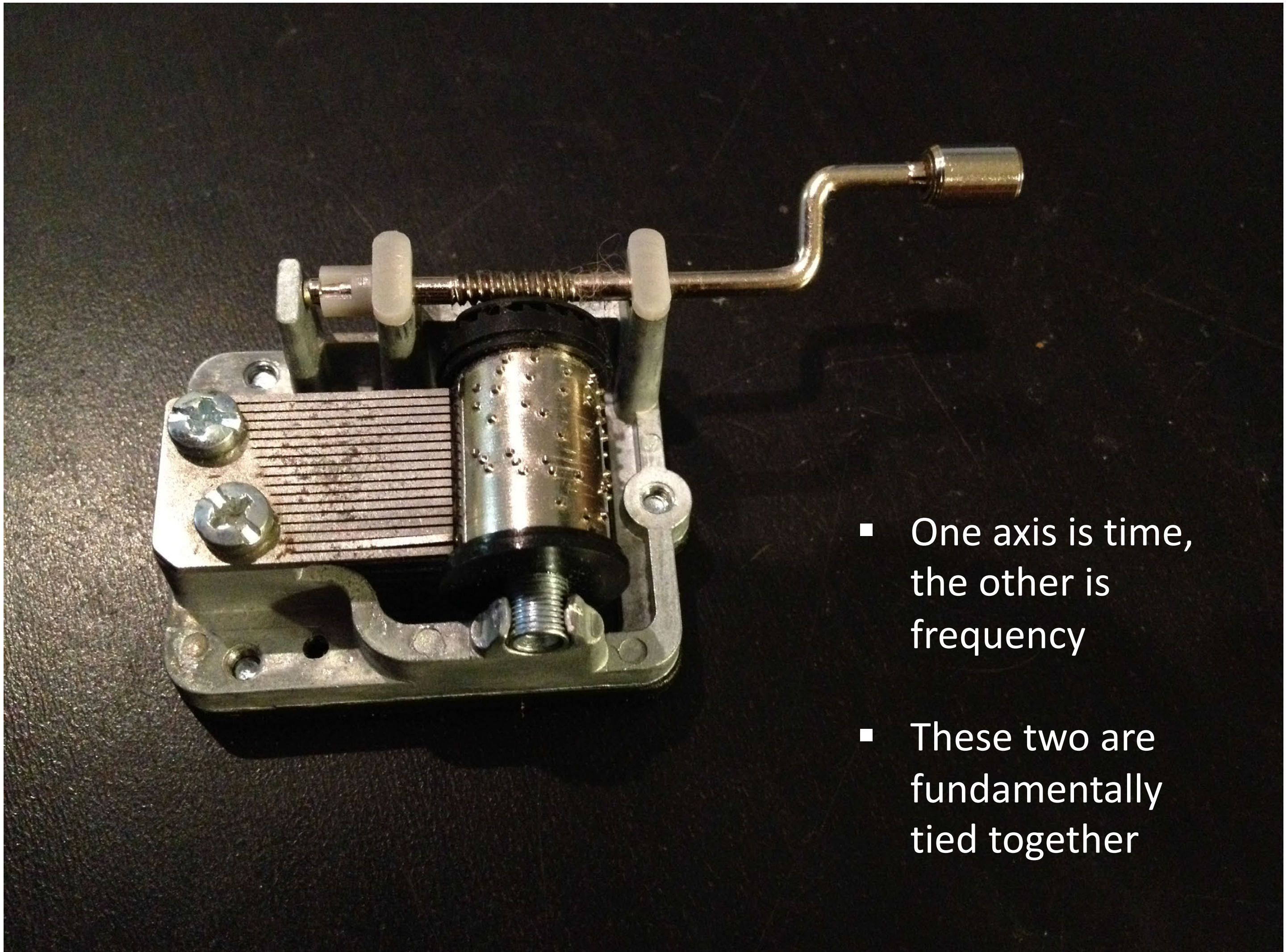
Joseph Fourier (1768-1830)

Basic idea: Represent 'signal' as a sum of sinusoids

Note: We focus on 1-D here for clarity, but these ideas generalize to higher dimensions (e.g., 2-D for images)

<u>Key idea:</u> Fourier transform

- Allows one to go from a time domain description (e.g., recorded signal) to a spectral description (i.e., what frequency components make up that signal)



Fourier series

Intuitive connection back to Taylor series:

$$y(x_1 + \Delta x) \approx y(x_1) + \sum_{n=1}^{N} \frac{1}{n!} \left. \frac{d^n y}{dx^n} \right|_{x_1} (\Delta x)^n. \qquad \text{(D.2)}$$

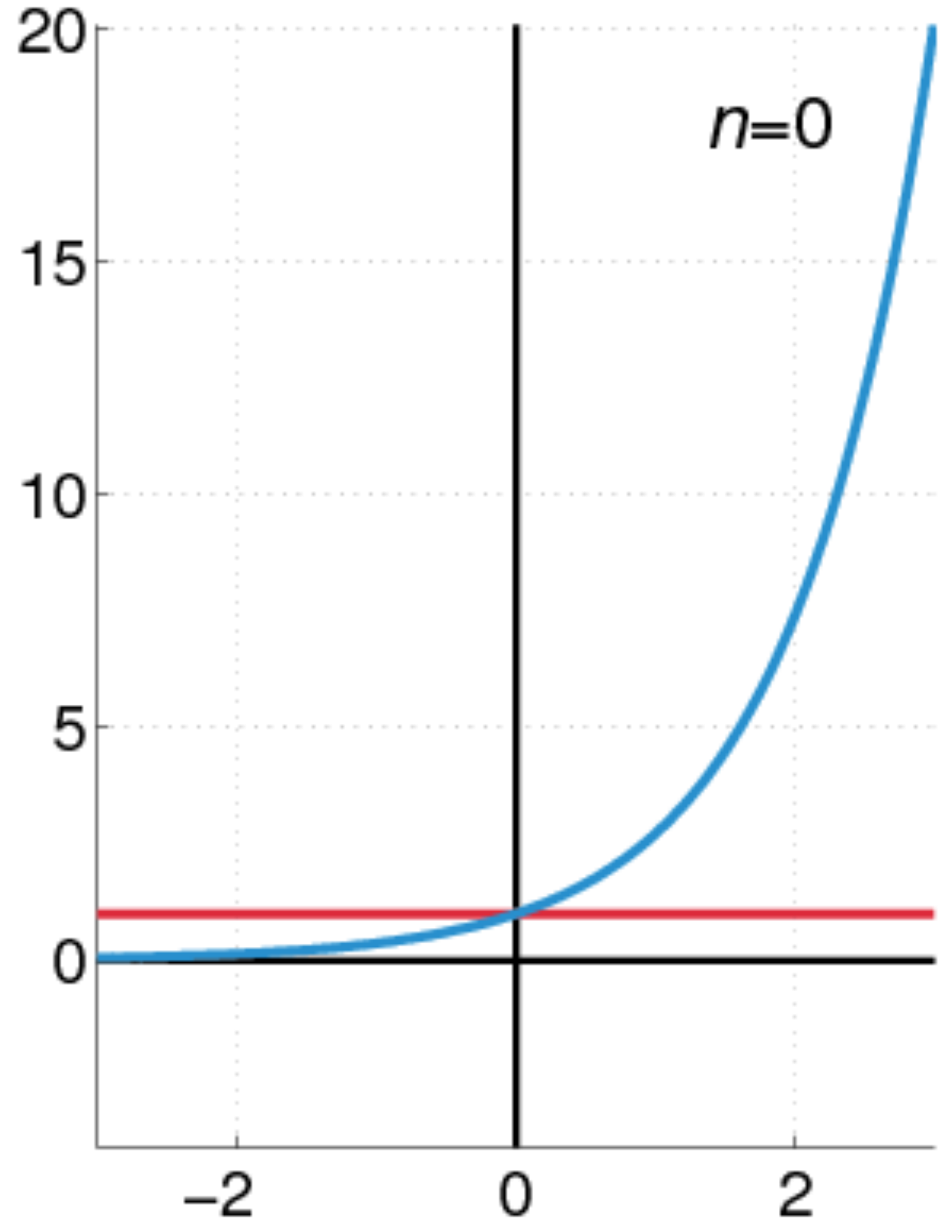
$$f(x) = f(x_o) + f'(x_o)(x - x_o) + \frac{f''(x_o)}{2!}(x - x_o)^2 + \cdots + \frac{f^{(n)}(x_o)}{n!}(x - x_o)^n + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_o)}{n!}(x - x_o)^n$$

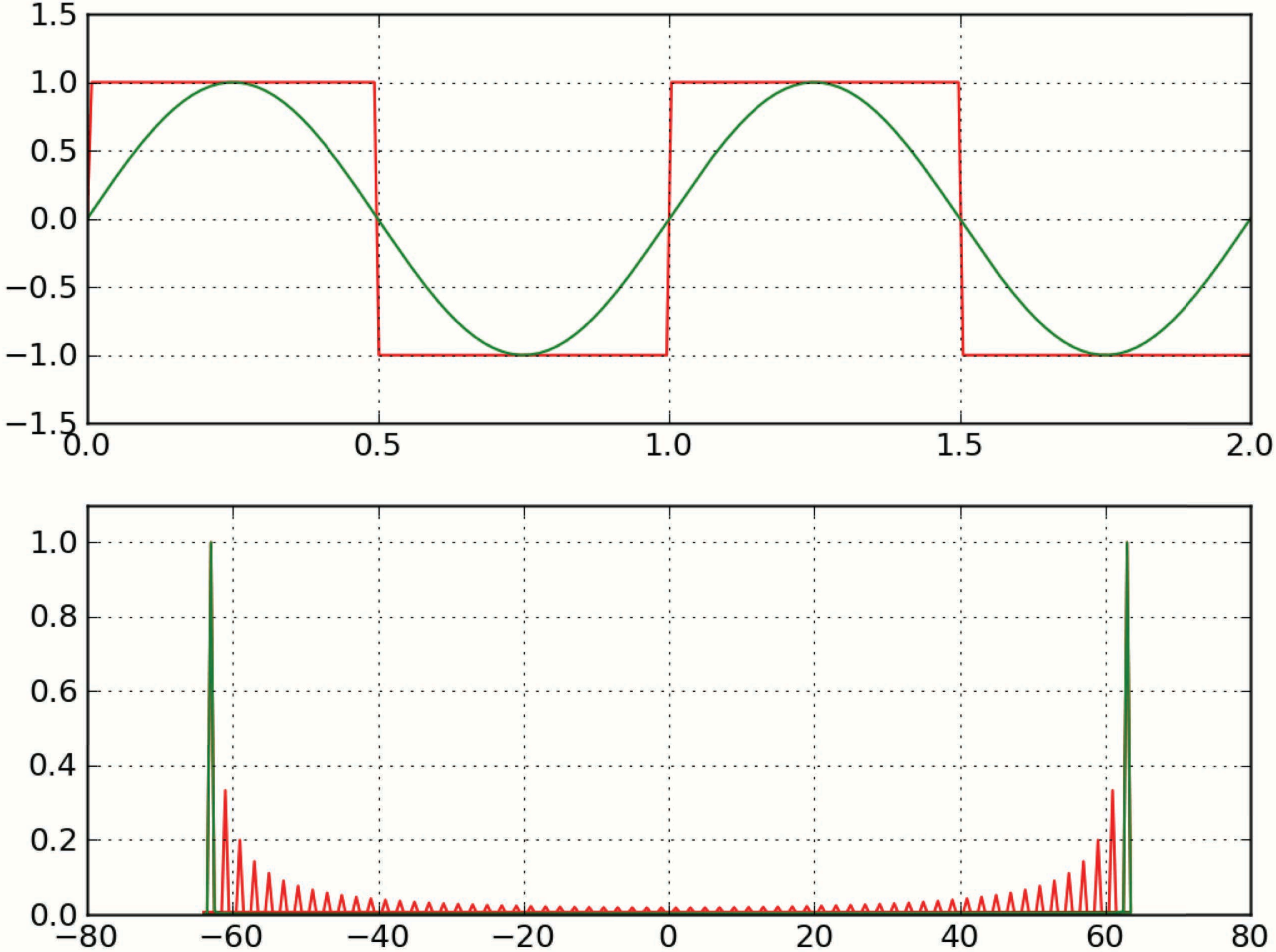
Taylor series → Expand as a (infinite) sum of polynomials

Different Idea: Fourier series → Expand as a (infinite) sum of sinusoids

“The exponential function e^x (in blue), and the sum of the first n+1 terms of its Taylor series at 0 (in red).”



wikipedia (Taylor series)



wikipedia (square wave)

Fourier series

$$f(t) = a_o + a_1 \sin(\omega t) + b_1 \cos(\omega t) + \\ + a_2 \sin(2\omega t) + b_2 \cos(2\omega t) + \\ + a_3 \sin(3\omega t) + b_3 \cos(3\omega t) + \cdots$$

$$= A_0 + A_1 \sin(\omega t + \phi_1) \\ + A_2 \sin(2\omega t + \phi_2) \\ + A_3 \sin(3\omega t + \phi_3) + \cdots$$

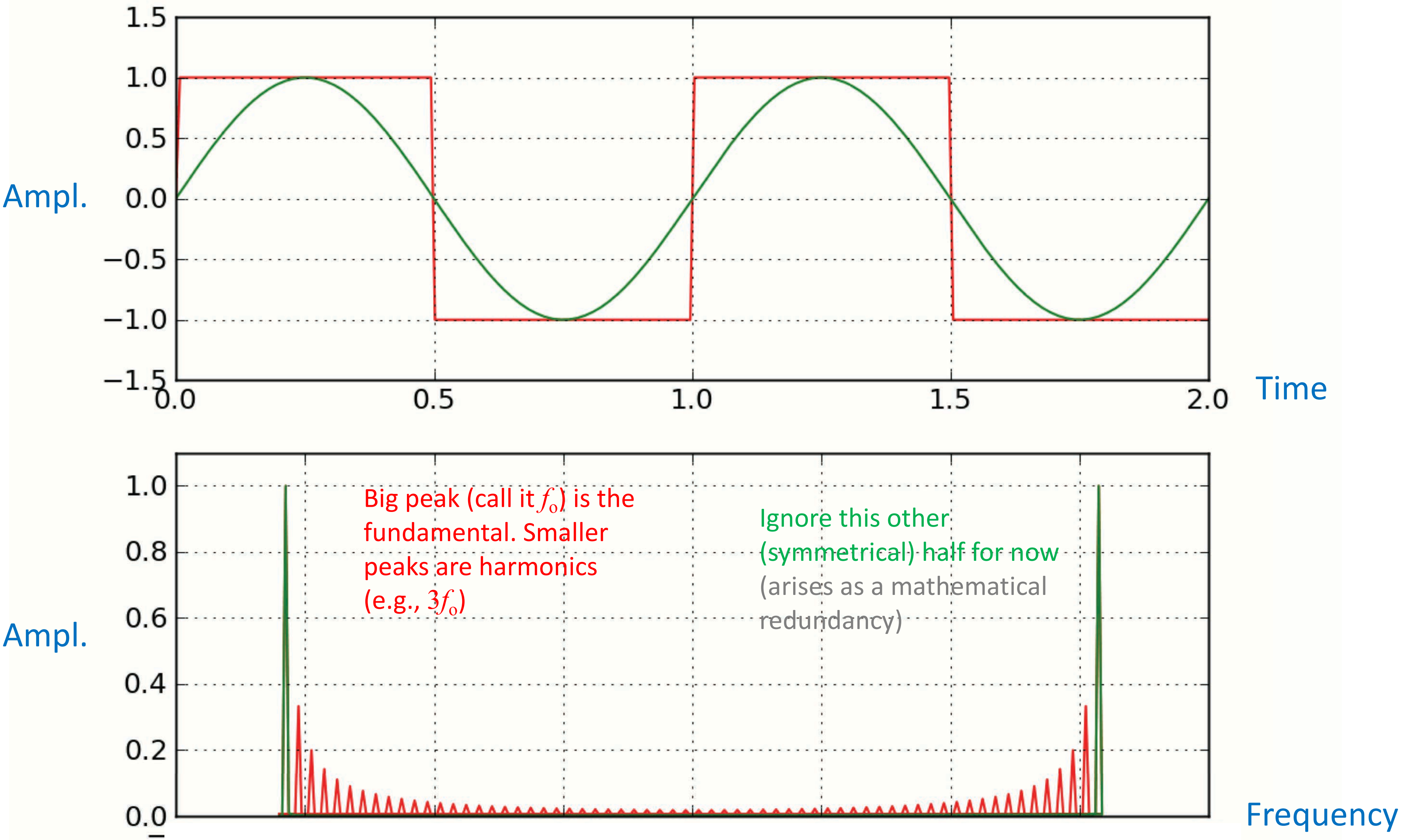
$$= \sum_{n=0}^{\infty} A_n \sin(n\omega t + \phi_n)$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)\,dt \qquad n = 0, 1, 2, \ldots$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t)\,dt \qquad n = 1, 2, \ldots$$

$$= \sum_{n=0}^{\infty} B_n e^{in\omega t} \quad \text{where } B_n \in \mathbb{C},\ i = \sqrt{-1}$$

Complex #s are much more compact and easier to deal with e.g., Square "Waves"

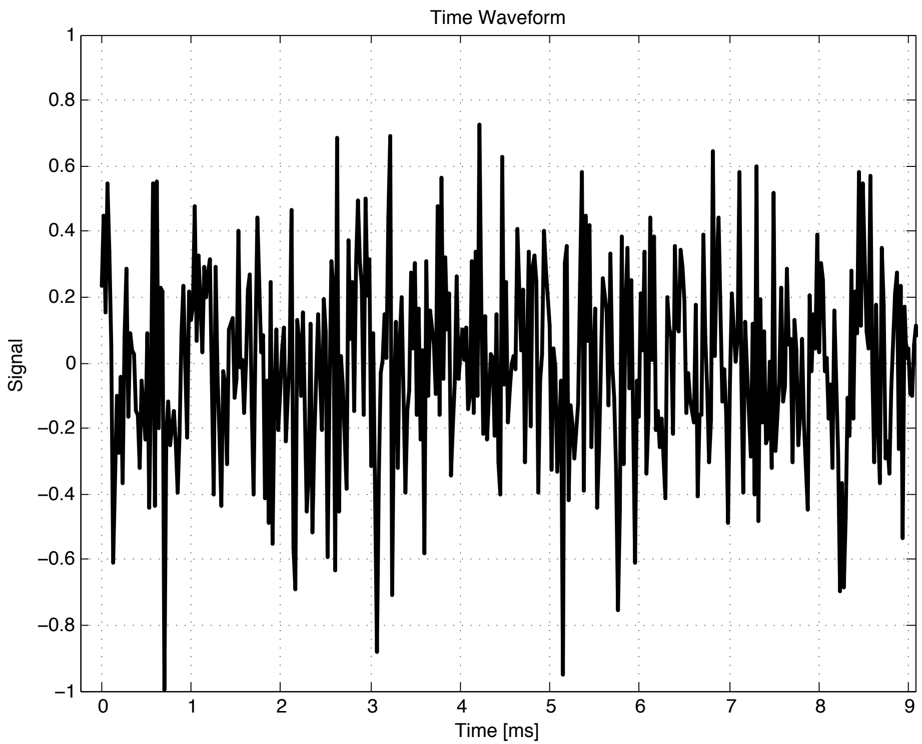


wikipedia (square wave)

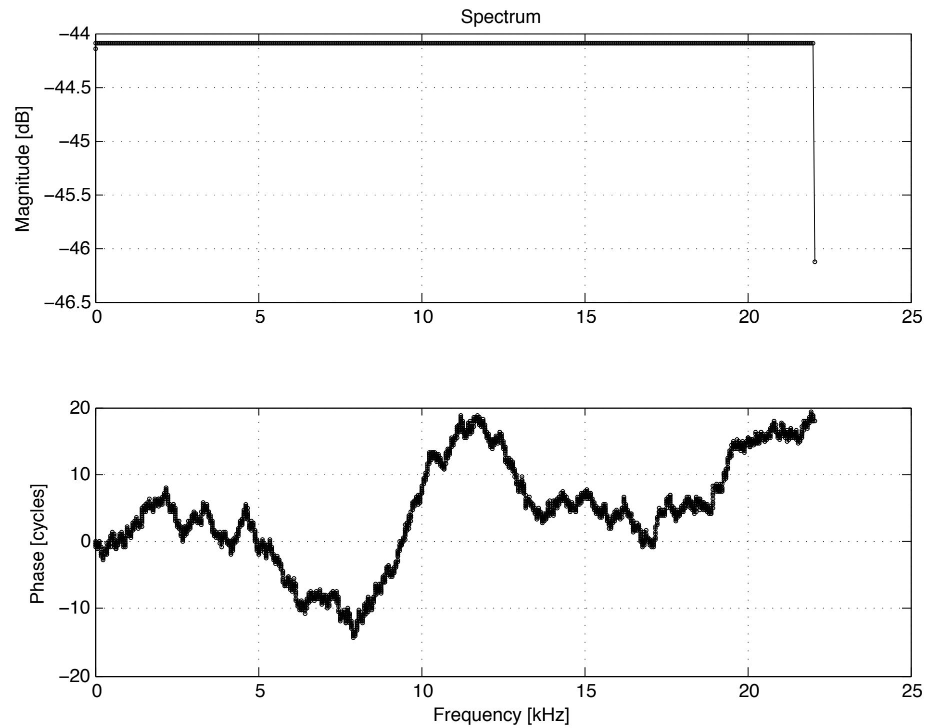
Fourier transforms of basic (1-D) waveforms

`stimT= 7 - noise (Gaussian distribution)`

Time domain



Spectral domain



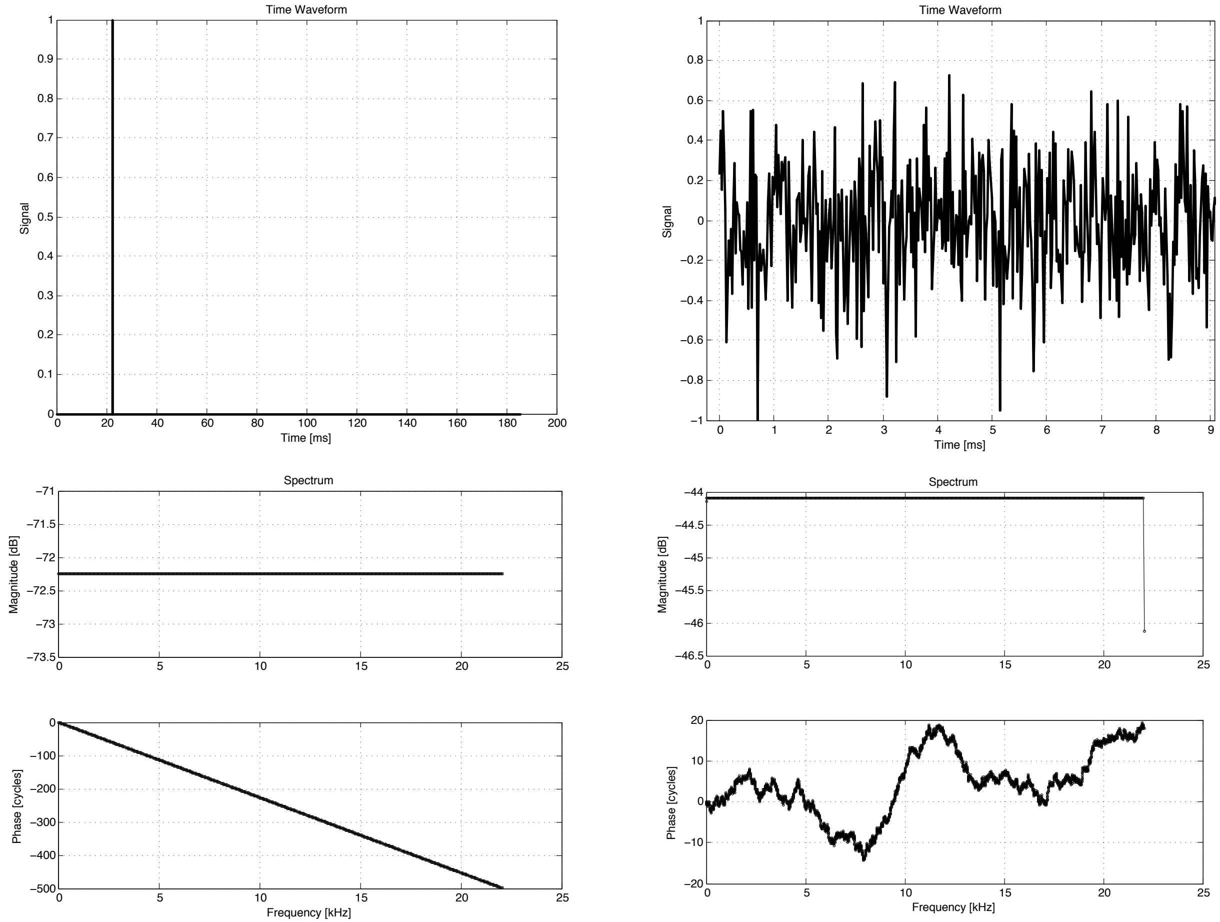
- Magnitude is flat just like an impulse (i.e., flat), but the phase is random

Fourier transforms of basic (1-D) waveforms

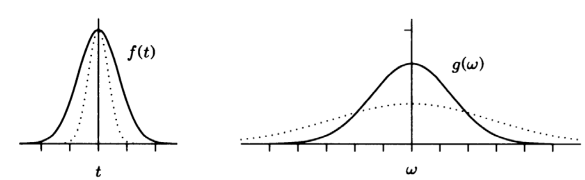
Impulse | Noise

Time domain

Spectral domain



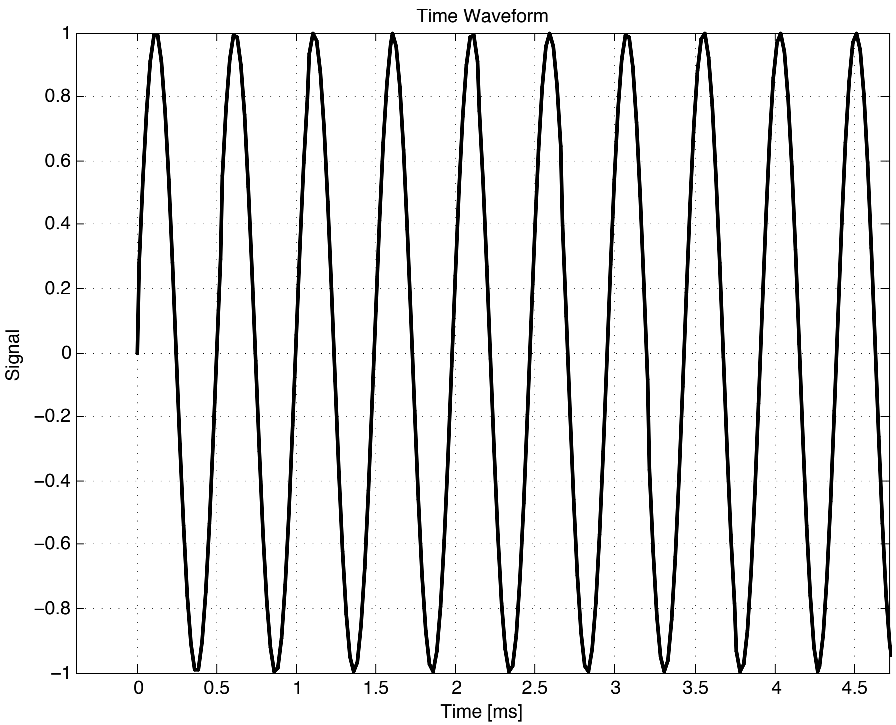
→ Remarkable that the magnitudes are identical (more or less) between two signals with such different properties. The key difference here is the phase: *Timing is a critical piece of the puzzle!*



Fourier transforms of basic (1-D) waveforms

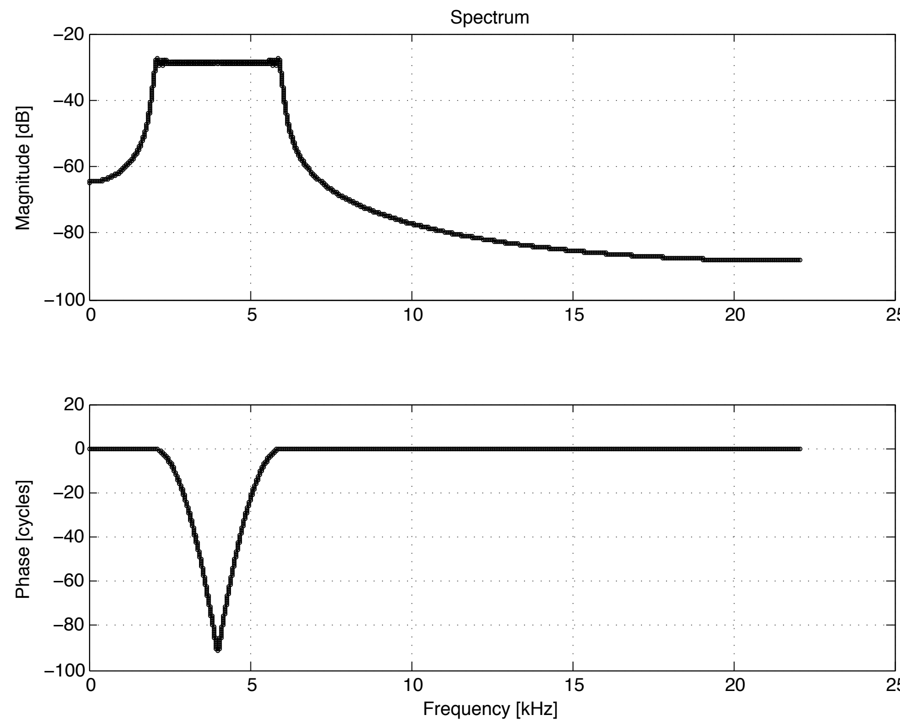

```
stimT= 6 - chirp (flat mag.)
```

Time domain



Hard to see on this timescale, but frequency is changing (increasing) with time

Spectral domain

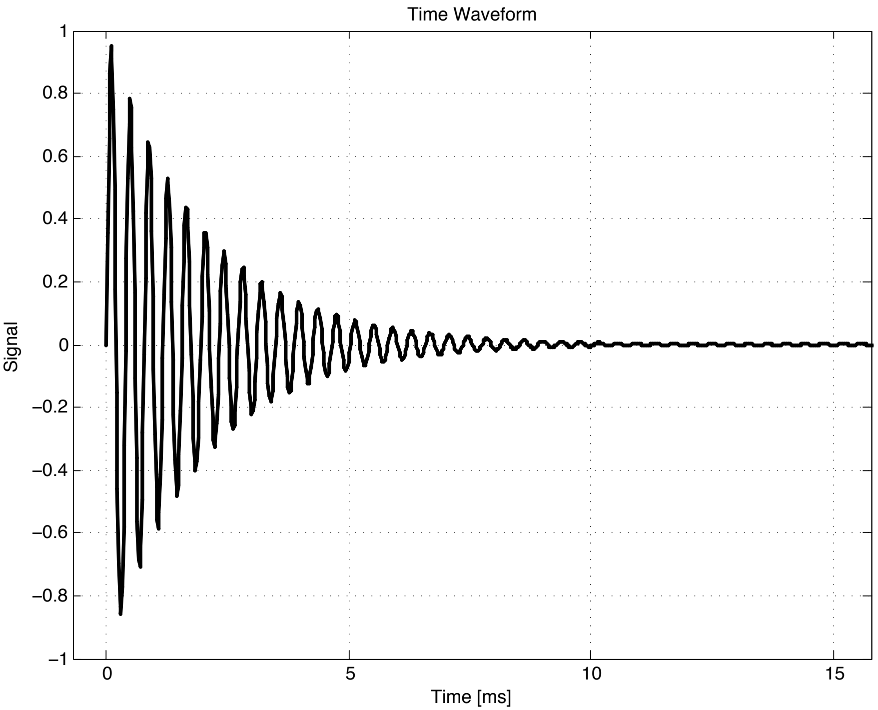


Fourier transforms of basic (1-D) waveforms

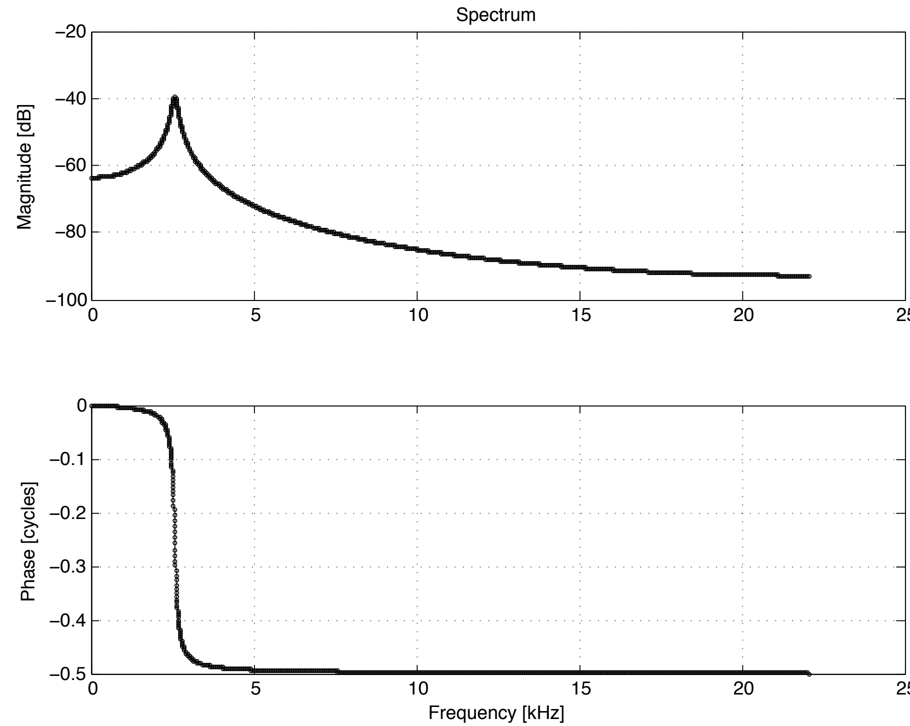
EXspecREP3.m

```
stimT= 8 - exponentially decaying sinusoid
```

Time domain



Spectral domain



- This seems to look familiar....

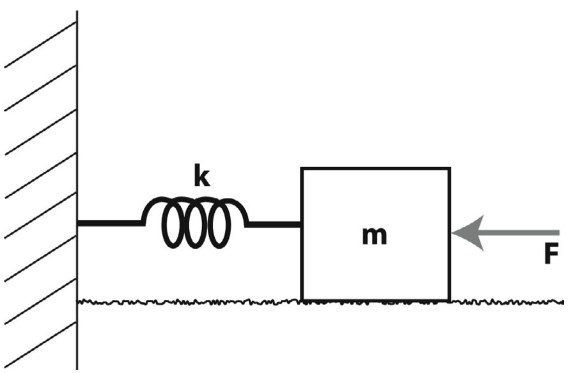
Impulse response

➢ Intuitively defined in two different (but equivalent) ways:

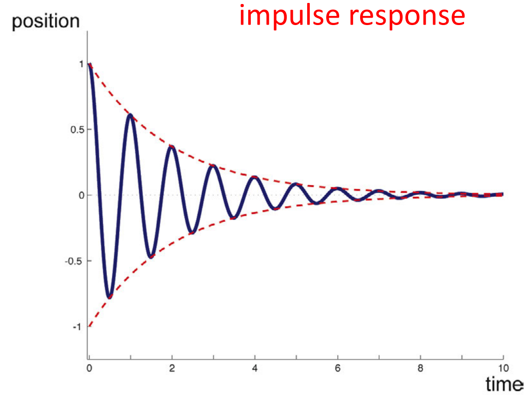
1. Time response of ‘system’ when subjected to an impulse
   (e.g., striking a bell w/ a hammer)

2. Fourier transform of resulting response
   (e.g., spectrum of bell ringing)

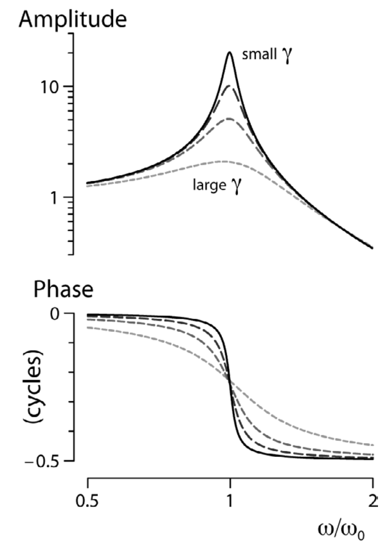
ex. Harmonic oscillator



Temporal impulse response

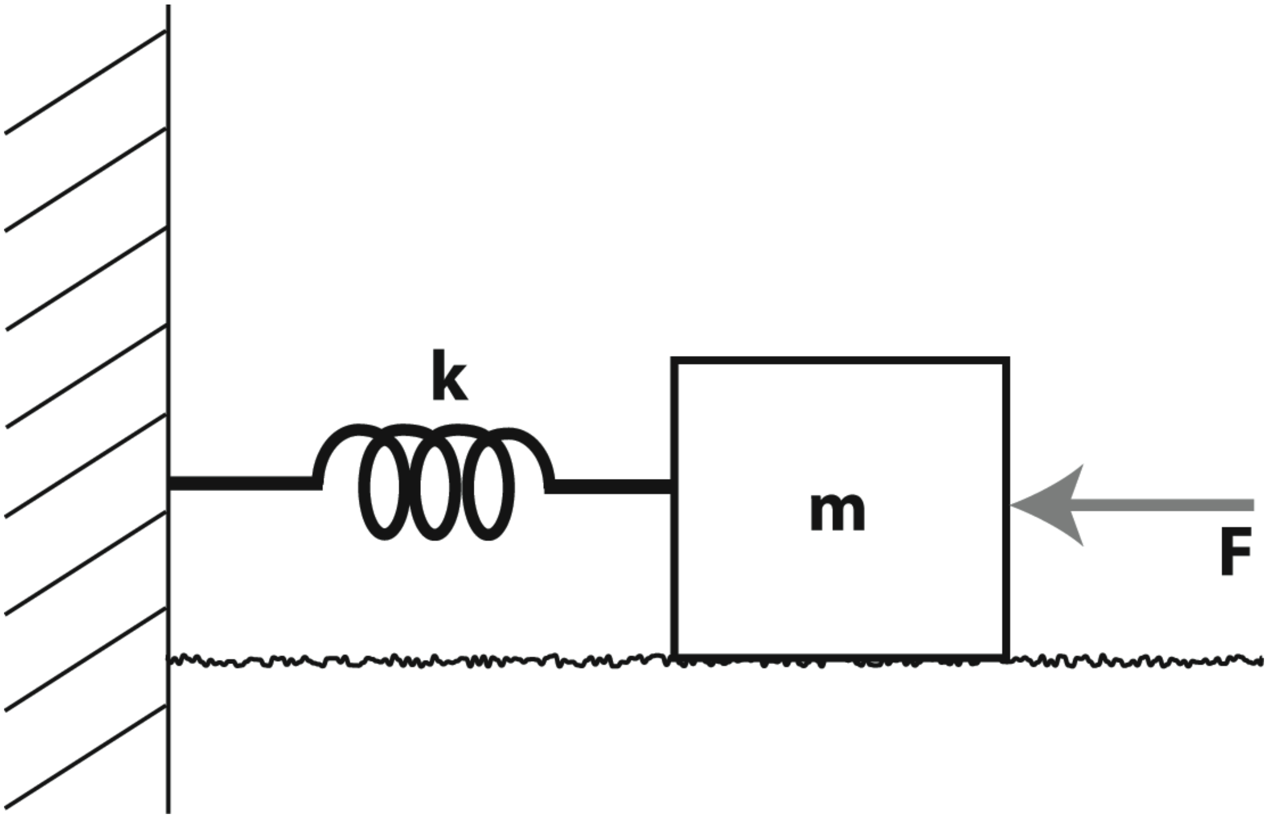


Spectral impulse response



(Important) Note: The Fourier transform of the impulse response is called the *transfer function*

Superposition & Linearity



→ When dealing with linear oscillators (or linear systems in general), superposition takes a domineering position in how we approach analysis and modeling

<u>Tangent</u>: Superposition & Linearity (beyond the HO)

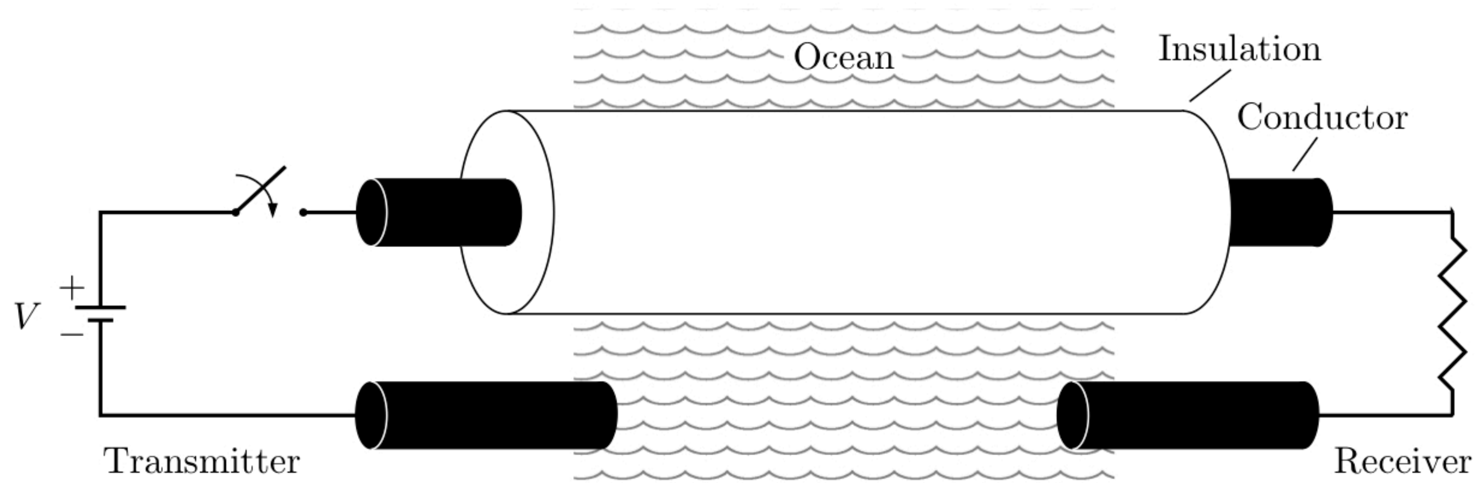


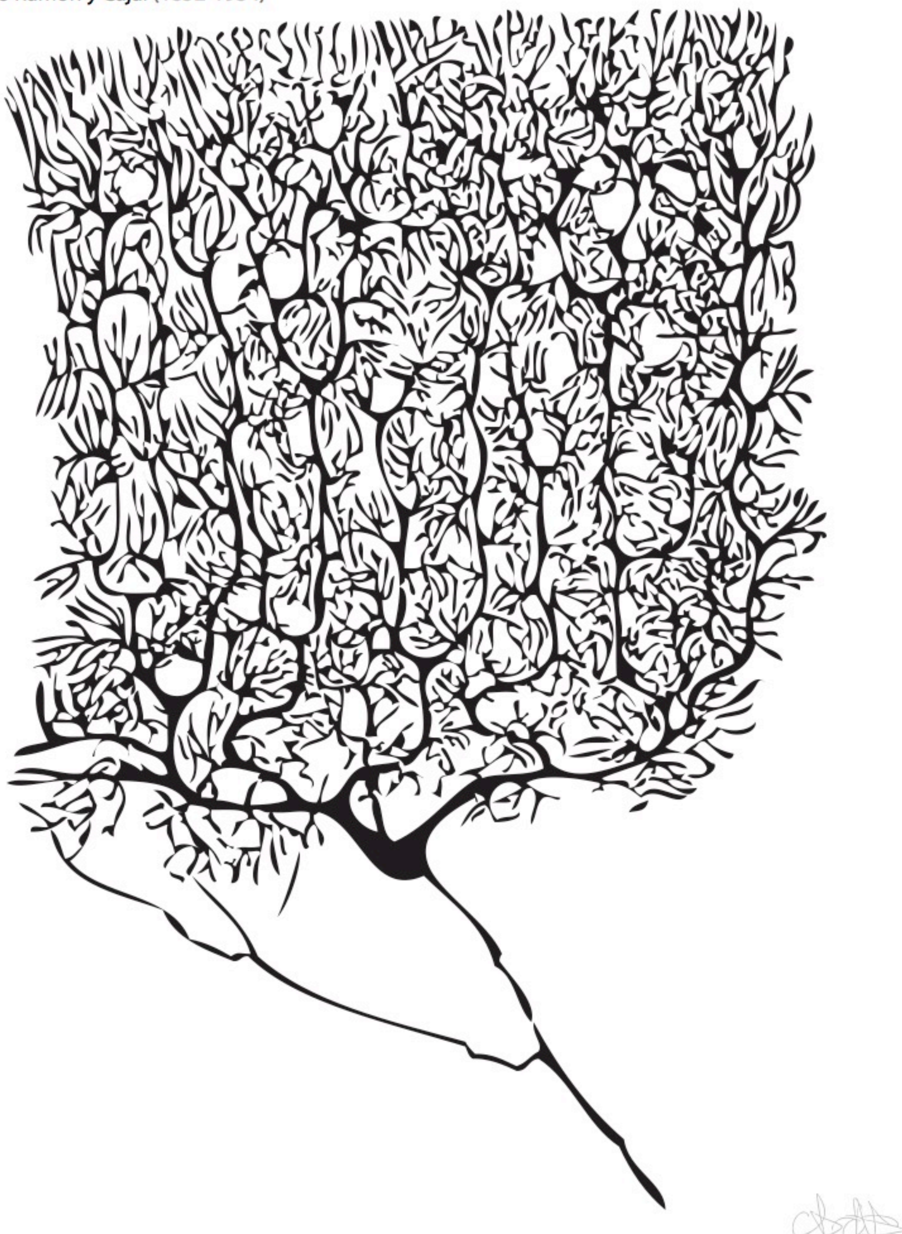
Figure 3.8

"cable model"

- First solved by William Thomson (aka Lord Kelvin) in ~1855
- Motivated by Atlantic submarine cable for intercontinental telegraphy
- Directly applicable to transmission lines and BNC cables

Weiss (1996)

Santiago Ramón y Cajal (1852-1934)



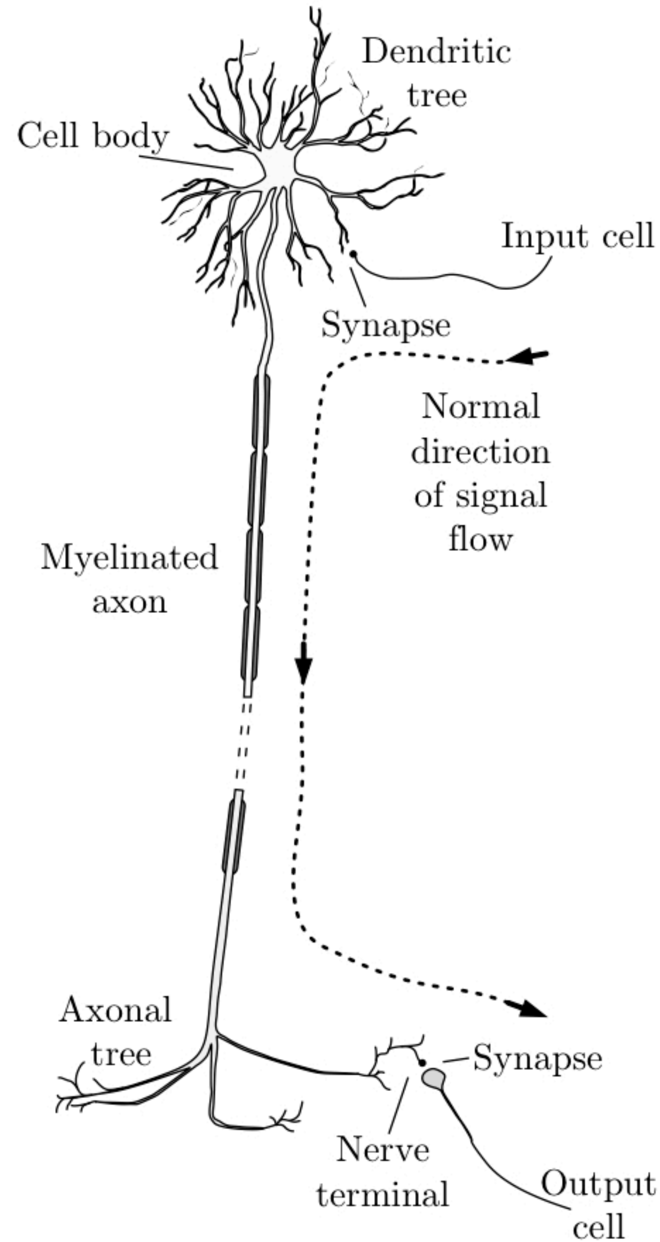


Figure 1.22

Weiss (1996)

Tangent: Superposition & Linearity (beyond the HO)

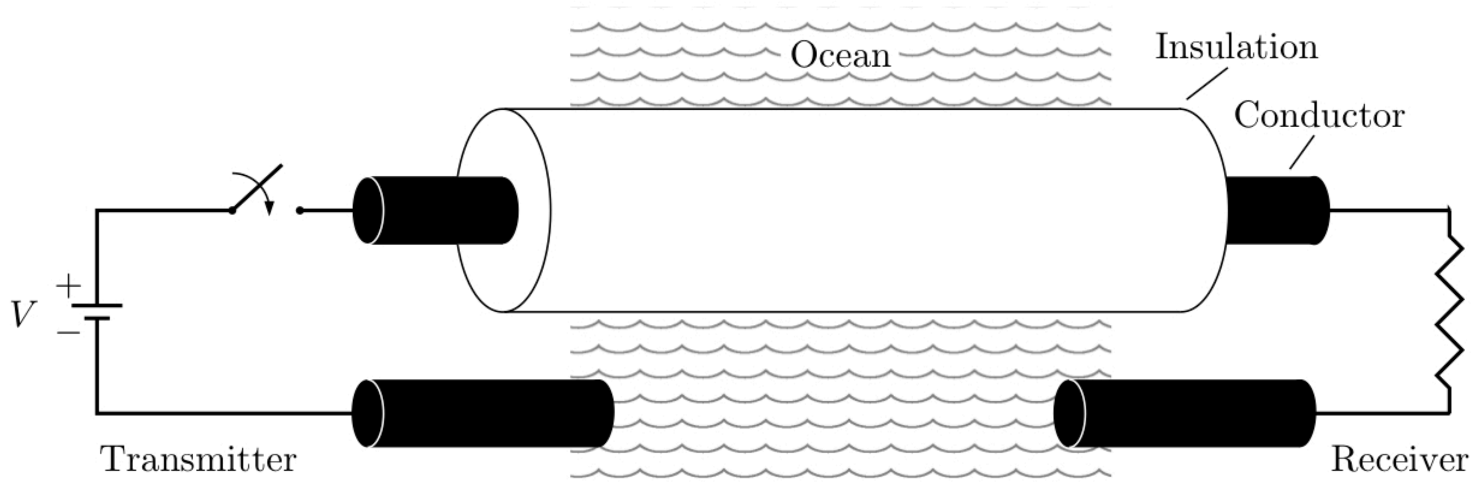


Figure 3.8

→ Axon behaves in fashion similar to a leaky submarine cable

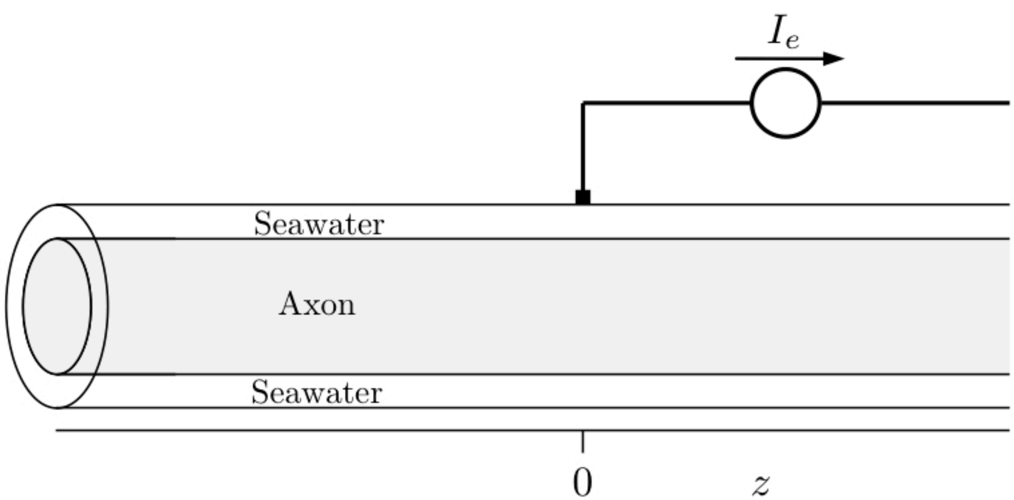


Figure 3.9

Tangent: Superposition & Linearity (beyond the HO)

Cable Equation

Linear PDE

Let $v_m(z,t) = V_m(z,t) - V_m^o$ and $|v_m(z,t)| << |V_m^o|$:

$$\boxed{v_m(z,t) + \tau_M \frac{\partial v_m(z,t)}{\partial t} - \lambda_C^2 \frac{\partial^2 v_m(z,t)}{\partial z^2} = r_o \lambda_C^2 K_e(z,t)}$$

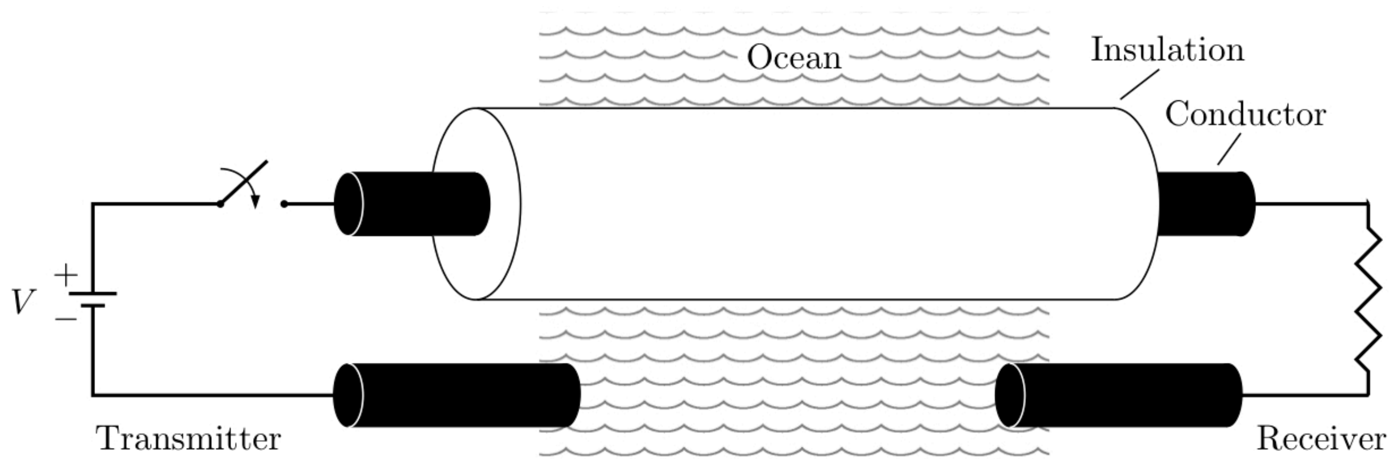
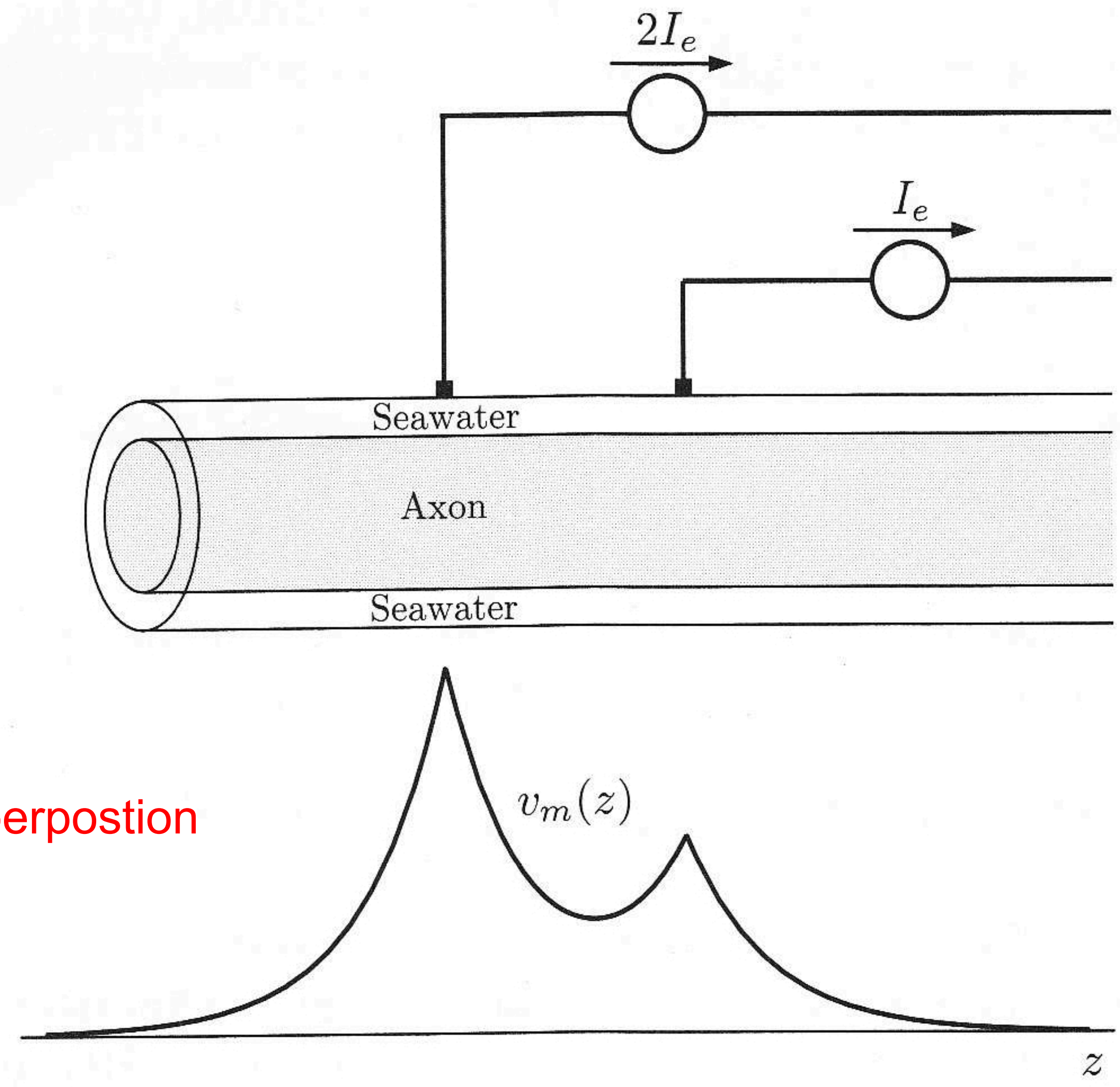


Figure 3.8

Tangent: Superposition & Linearity (beyond the HO)



Linearity → Superpostion

Figure 3.19

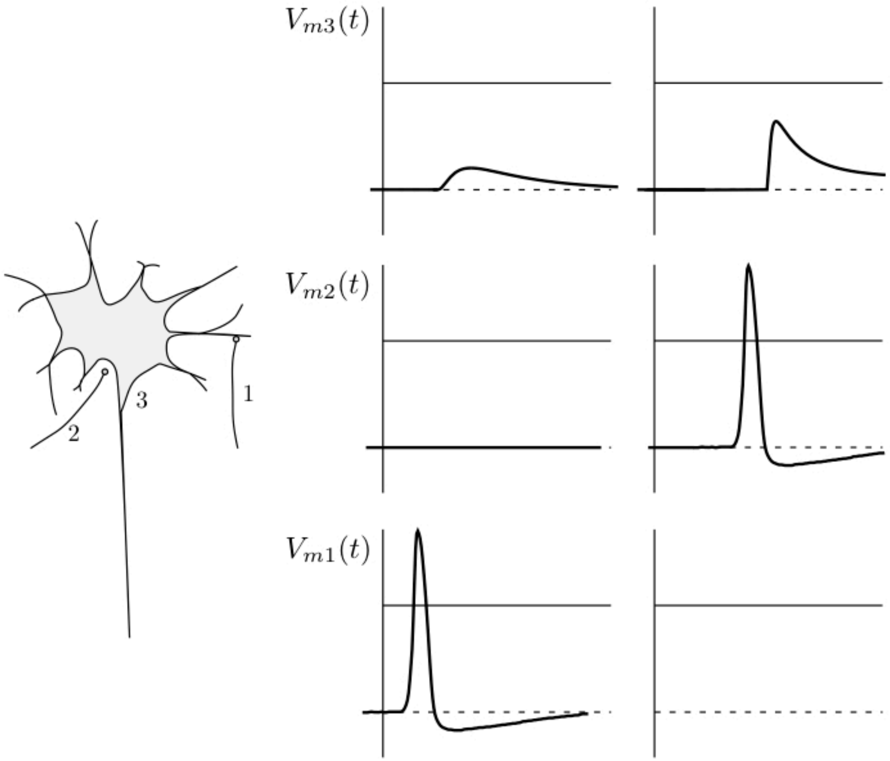


Figure 3.34

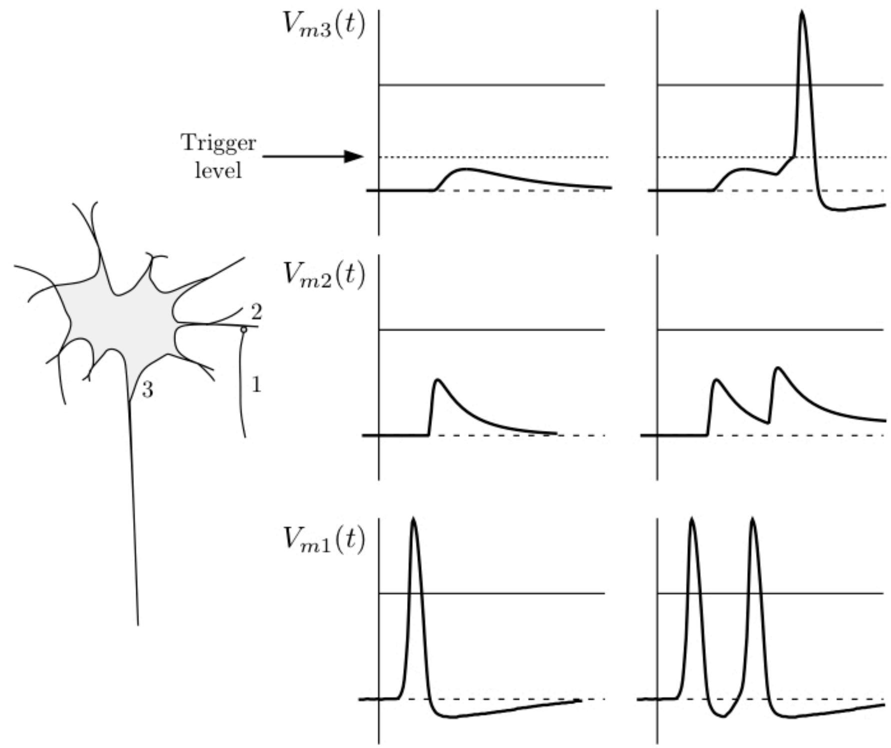


Figure 3.35

"Electronic distance"

"Temporal integration"

→ Key considerations with regard to synapses (i.e., inter-neuron communication)

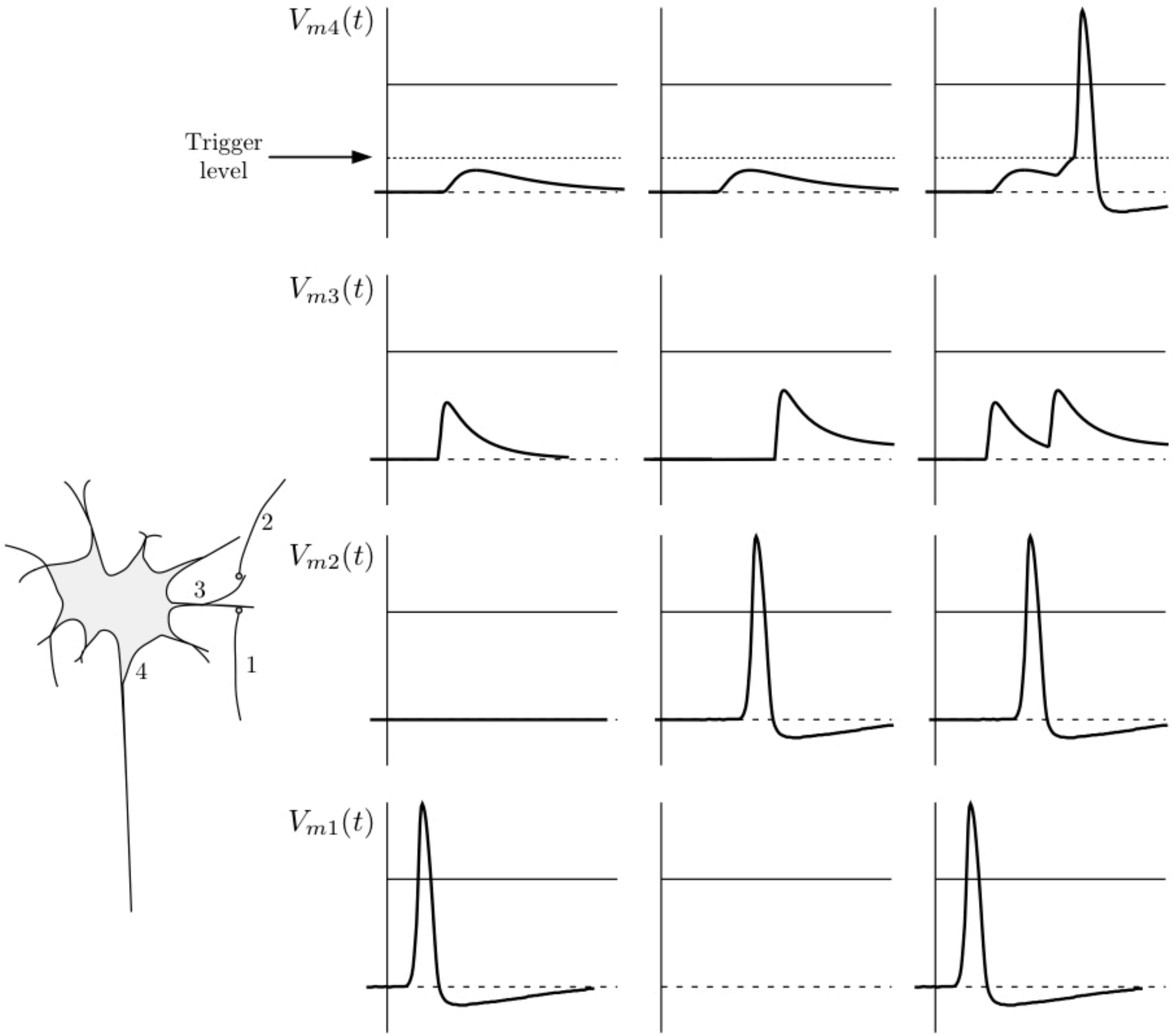
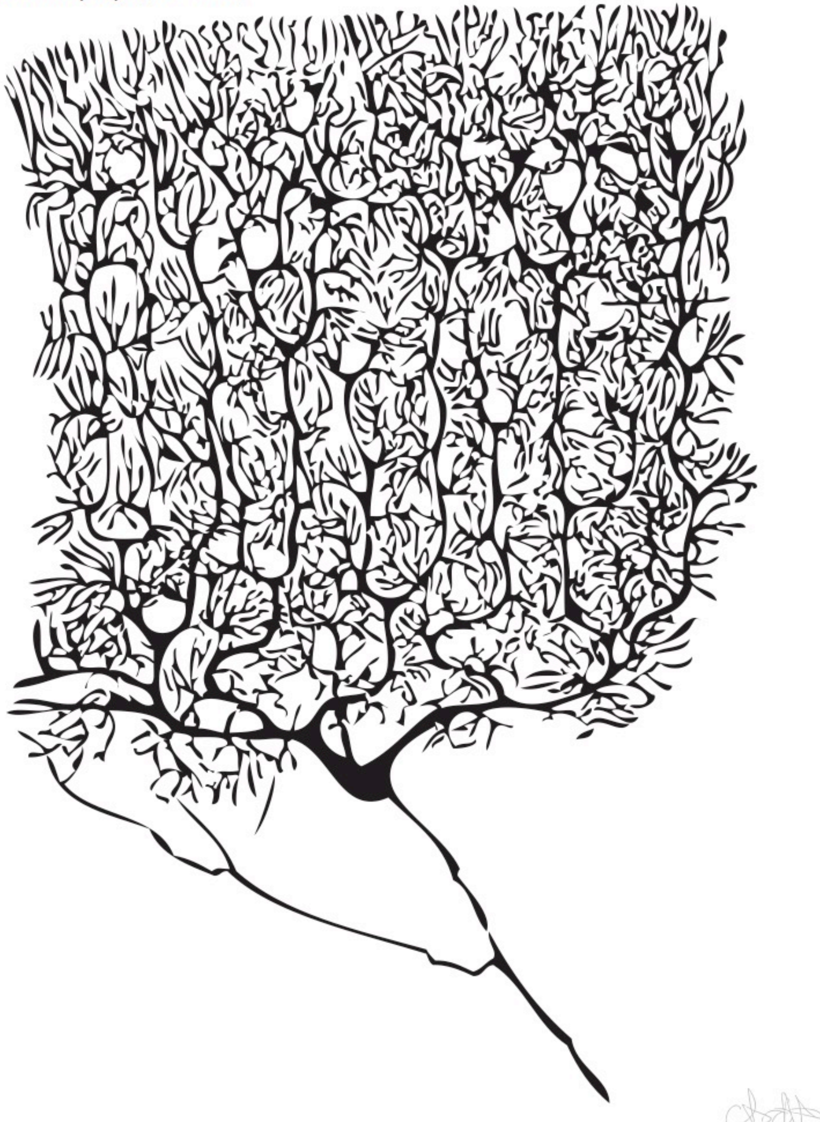


Figure 3.36

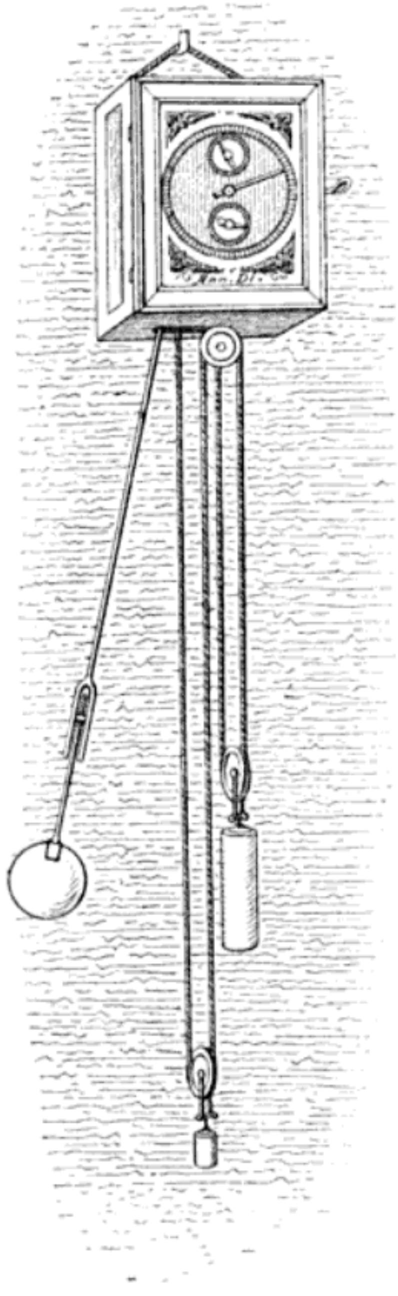
Spatial integration

Santiago Ramón y Cajal (1852-1934)

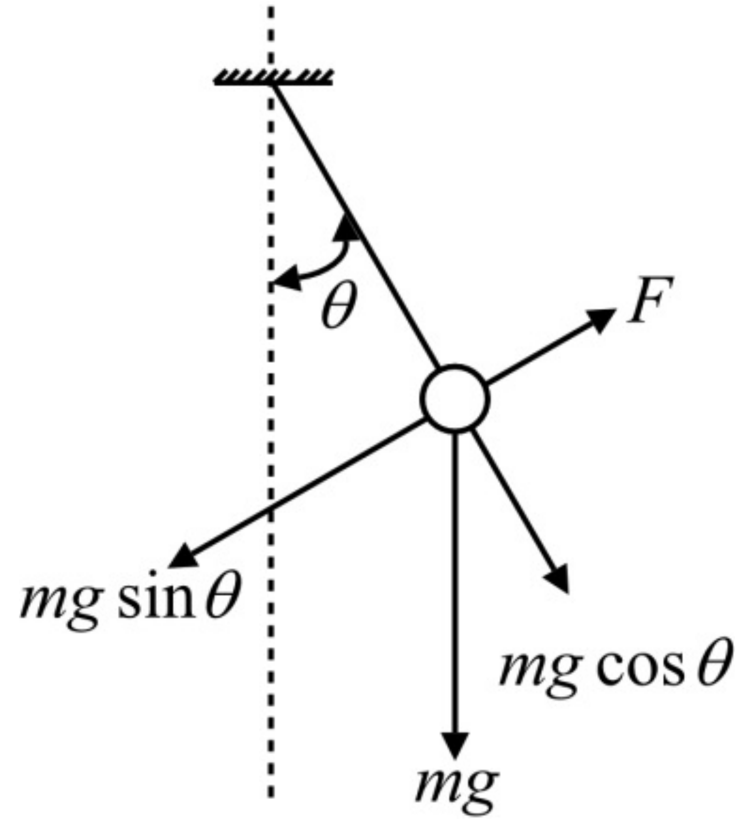


In Short: Superposition plays a key role in how your brain works as a "network" to process information

Nonlinear oscillations: Pendulum



"first pendulum clock" re Christiaan Huygens



$$\frac{d^2\theta}{dt^2} = \ddot{\theta} = -\frac{g}{\ell}\sin(\theta)$$

wikipedia - **Pendulum**

Additional slides beyond are for general reference....

General properties of Fourier transforms

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{i\omega t} d\omega$$

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$$

Devries (1994)

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} F(k) dk\,.$$

Kutz (2012)

- Don't be confused by different notations (there are a lot out there!)
- Keep in mind that the Fourier transform is defined over the interval [-∞, ∞], though most 'signals' we deal with computationally are finite (e.g., [-L,L])

(the resolution here is an implicit assumption of a 'periodic boundary condition'; we'll come back to this)

- Connection back to previous topics covered:

Further, the kernel of the transform, $\exp(\pm ikx)$, describes oscillatory behavior. Thus the Fourier transform is essentially an eigenfunction expansion over all continuous wavenumbers k. And once we are on a finite domain $x \in [-L, L]$, the continuous eigenfunction expansion becomes a discrete sum of eigenfunctions and associated wavenumbers (eigenvalues).

General properties of Fourier transforms

- The Fourier transform is a *linear process*

That is, if $f_1(t)$ and $f_2(t)$ are two functions having Fourier transforms $g_1(\omega)$ and $g_2(\omega)$, then the Fourier transform of $f_1(t) + f_2(t)$ is

$$\begin{aligned} g(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [f_1(t) + f_2(t)]\, e^{-i\omega t}\, dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(t) e^{-i\omega t}\, dt + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_2(t) e^{-i\omega t}\, dt \\ &= g_1(\omega) + g_2(\omega). \end{aligned}$$

→ Linearity is a seemingly innocuous property that has vast implications... (we'll see some in later lectures)

- Another key property is that of scaling:

$$\mathcal{F}[f(\alpha t)] = \frac{1}{|\alpha|} g\left(\frac{\omega}{\alpha}\right)$$

$$\mathcal{F}^{-1}[g(\beta\omega)] = \frac{1}{|\beta|} f\left(\frac{t}{\beta}\right)$$

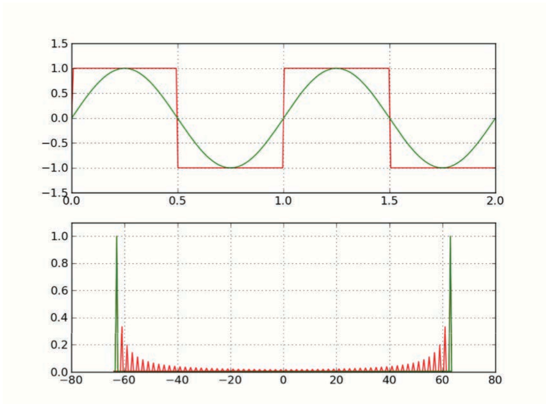
Devries (1994)

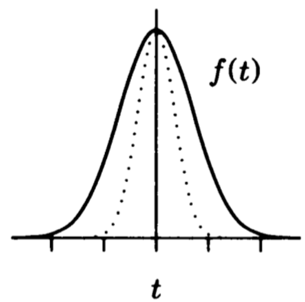
General properties of Fourier transforms

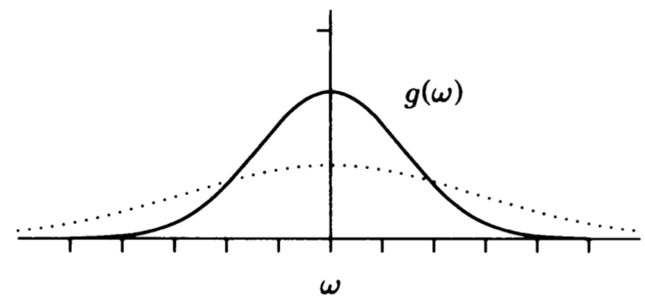
$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{i\omega t} d\omega \qquad \mathcal{F}[f(\alpha t)] = \frac{1}{|\alpha|} g(\frac{\omega}{\alpha})$$

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \qquad \mathcal{F}^{-1}[g(\beta\omega)] = \frac{1}{|\beta|} f(\frac{t}{\beta})$$

- Due to this scaling, as $f(t)$ gets narrower (i.e., more localized in time), $g(\omega)$ gets broader (i.e., more spread out across frequency)







in large part, the role of the additional functions is to cancel the oscillations. Thus, the more localized a function is in time, the more delocalized it is in frequency.

This is more than a casual observation — it's a fundamental property of Fourier transforms, and has direct physical consequences. Usually stated in terms of position and momentum rather than time and frequency, the statement is that the product of the width of the function, Δx, and the width of the transform of the function, Δp, is always greater than or equal to a specific *nonzero* value, $\hbar$ — Heisenberg's uncertainty principle.

Devries (1994)

General properties of Fourier transforms

- A handful of other properties arise (e.g., shifting, time reversal), which give rise to numerous symmetries that can be summarized as follows:

If $f(t)$ is real,	then $\Re g(\omega)$ is even and $\Im g(\omega)$ is odd;
if $f(t)$ is imaginary,	then $\Re g(\omega)$ is odd and $\Im g(\omega)$ is even;
if $f(t)$ is even,	then $g(\omega)$ is even;
if $f(t)$ is odd,	then $g(\omega)$ is odd;
if $f(t)$ is real and even,	then $g(\omega)$ is real and even;
if $f(t)$ is real and odd,	then $g(\omega)$ is imaginary and odd;
if $f(t)$ is imaginary and even,	then $g(\omega)$ is imaginary and even;
if $f(t)$ is imaginary and odd,	then $g(\omega)$ is real and odd.

- Another key feature is that of *derivatives*:

$$\mathcal{F}[f'(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(t) e^{-i\omega t}\, dt$$

Integrating by parts has:

$$\mathcal{F}[f'(t)] = \frac{e^{-i\omega t}}{\sqrt{2\pi}} f(t)\Big|_{-\infty}^{\infty} + \frac{i\omega}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$$

$$\boxed{\mathcal{F}[f'(t)] = i\omega g(\omega)}$$

→ The first term on the right-side must be zero

Devries (1994)

Aside: Using Fourier transforms to solve linear differential equations

Note: This topic is a bit beyond the scope of 2030, but is worth pointing out here for future reference

$$\widehat{f^{(n)}} = (ik)^n \widehat{f}.$$

[Kutz notation]
Basic idea is generalizable to higher order derivatives

$$\mathcal{F}[f'(t)] = i\omega g(\omega)$$

Consider the ***linear***, non-autonomous ODE:

$$y'' - \omega^2 y = -f(x) \quad x \in [-\infty, \infty]$$

Take Fourier transform of both sides and work through:

$$\widehat{y''} - \omega^2 \widehat{y} = -\widehat{f}$$
$$-k^2 \widehat{y} - \omega^2 \widehat{y} = -\widehat{f}$$
$$(k^2 + \omega^2)\widehat{y} = \widehat{f}$$

$$\widehat{y} = \frac{\widehat{f}}{k^2 + \omega^2}$$

We end up with an integral solution that can than either be evaluated analytically or numerically

$$y(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \frac{\widehat{f}}{k^2 + \omega^2} dk.$$

→ Other avenues deal with PDEs and integrals, with many practical implications in physics (e.g., delta functions in quantum mechanics, Parseval's identity), though such is beyond our scope here

Kutz (2012)