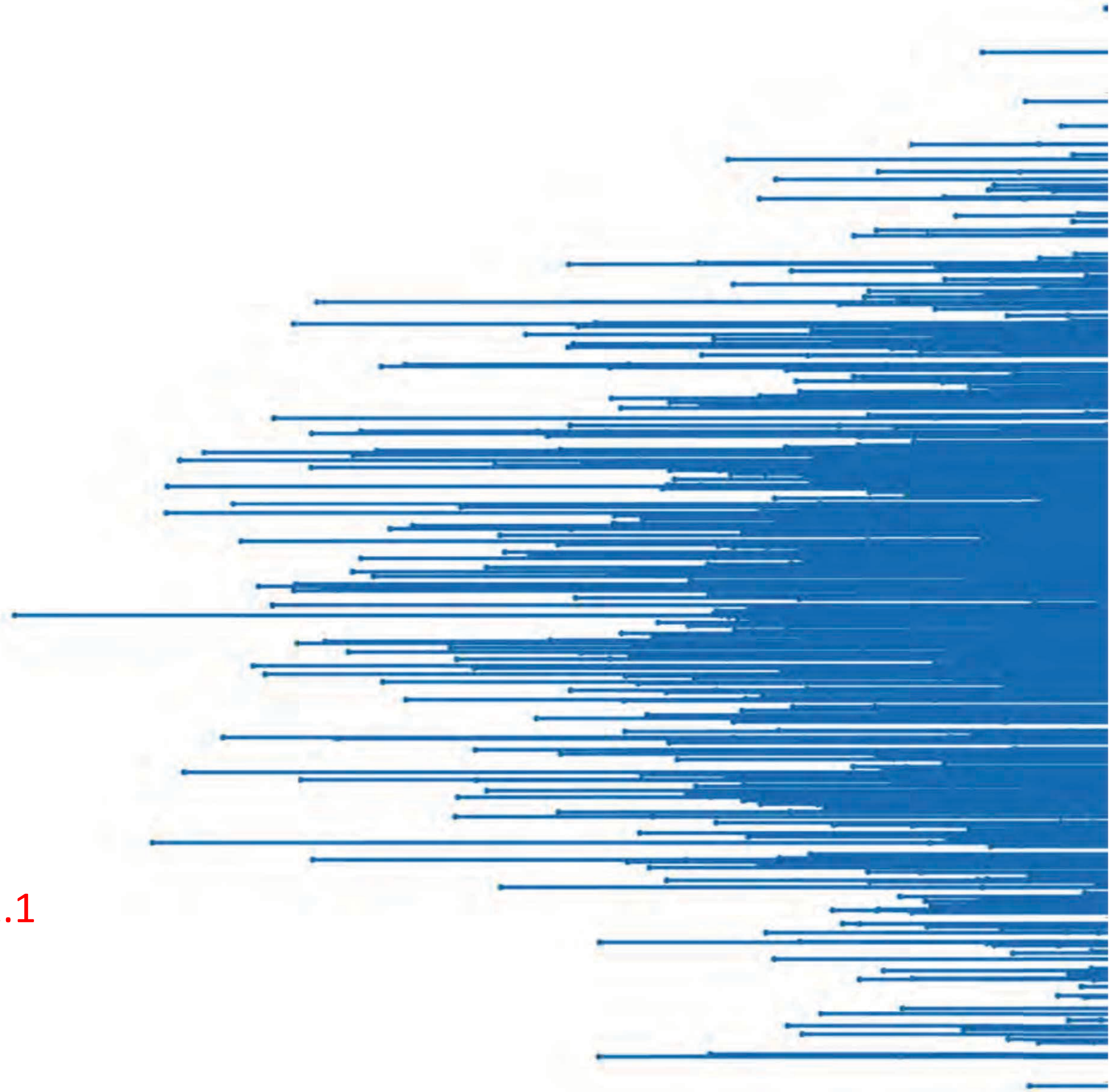
PHYS 2010 (W20)

Classical Mechanics



2020.03.12

Relevant reading:

Knudsen & Hjorth: Ex.1.8, 9.1, 11.1

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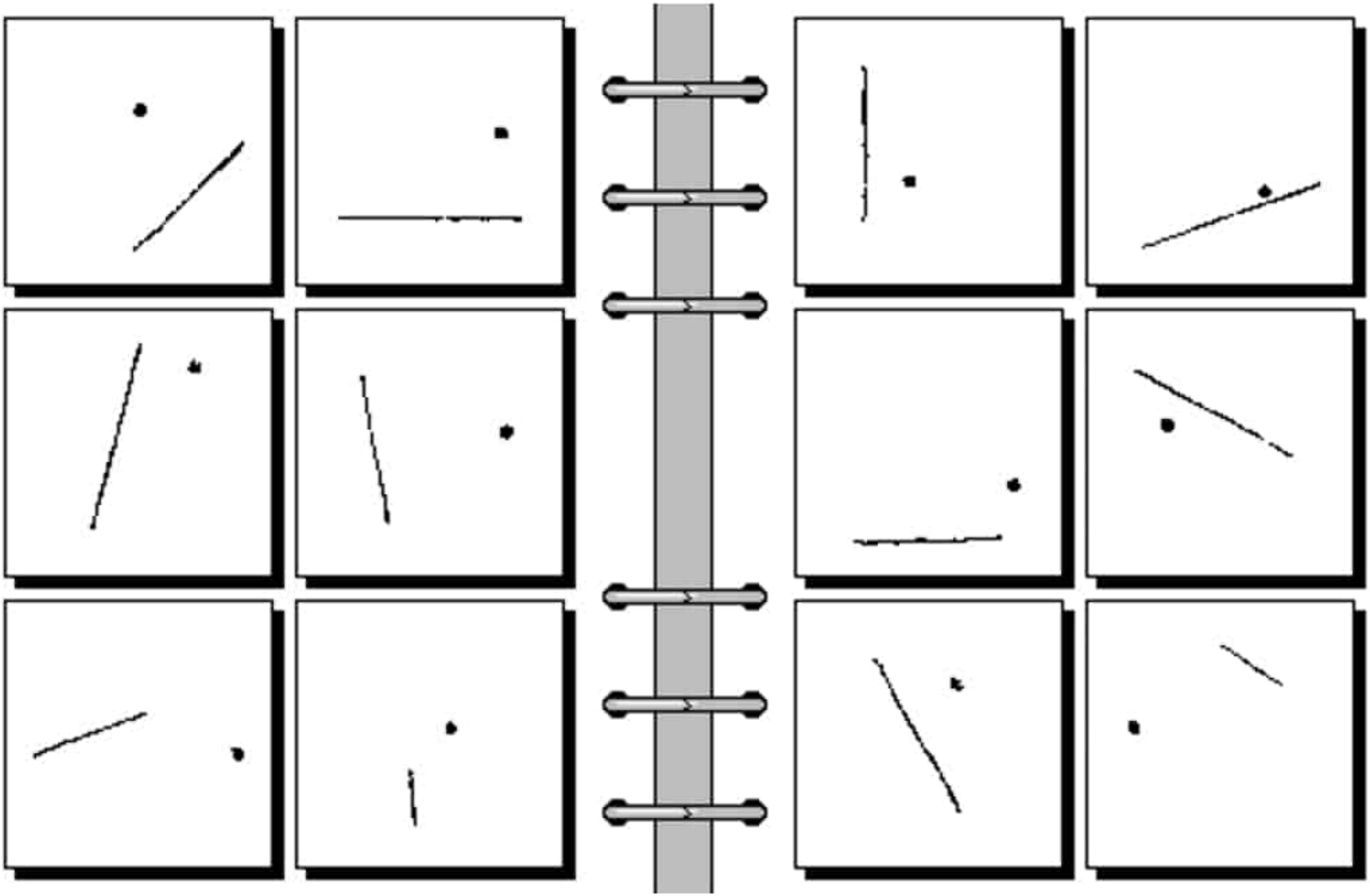
Ref.s:
Knudsen & Hjorth (2000), Fowles & Cassidy (2005)

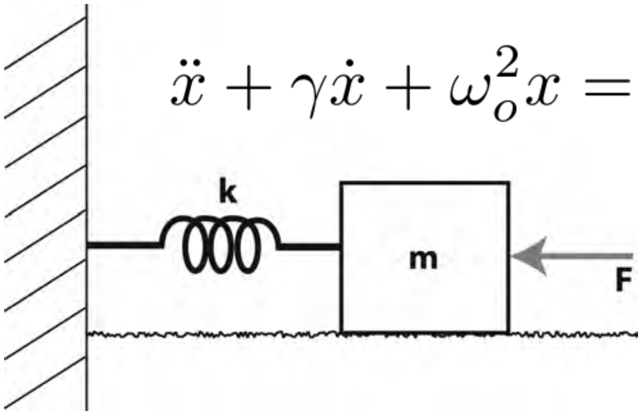


Recall: "Modeling" & Differential equations (DEs)

→ A very common/useful tool in our toolbox....

Harmonic oscillator

$$\ddot{x} + \gamma\dot{x} + \omega_o^2 x = 0$$



Wave equation

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}$$

Diffusion equation

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}$$

Laplace's equation

$$\Delta f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

Maxwell's equations

$$\nabla \cdot \mathbf{E} = \frac{\rho_v}{\varepsilon} \quad \text{(Gauss' Law)}$$

$$\nabla \cdot \mathbf{H} = 0 \quad \text{(Gauss' Law for Magnetism)}$$

$$\nabla \times \mathbf{E} = -\mu\frac{\partial \mathbf{H}}{\partial t} \quad \text{(Faraday's Law)}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \varepsilon\frac{\partial \mathbf{E}}{\partial t} \quad \text{(Ampere's Law)}$$

Logistic eqn. (continuous)

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$$

Aside: SIR model (re Covid-19)

York University
Centre for Disease Modelling

SIR model
('compartmental' model in epidemiology)

S = the number of *susceptibles*, the people who are not yet sick but who could become sick

I = the number of *infecteds*, the people who are currently sick

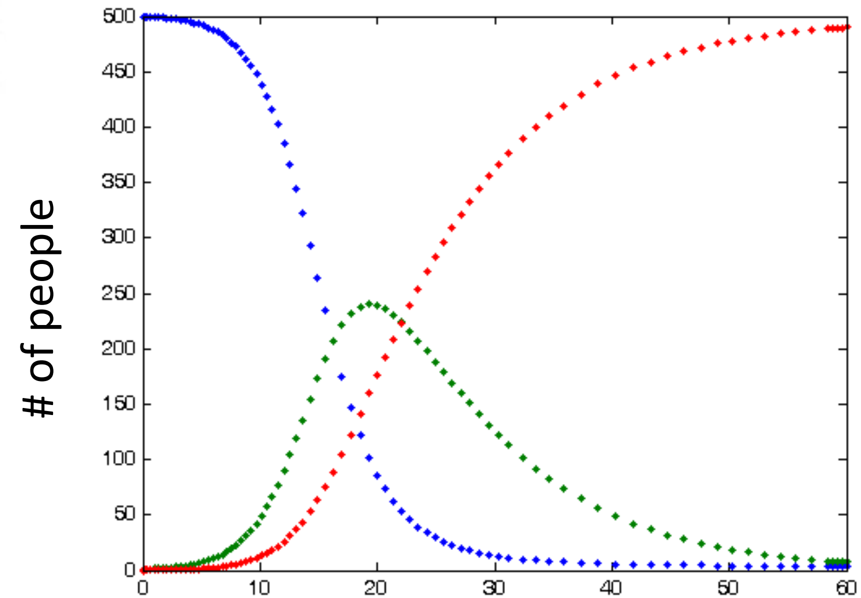
R = the number of *recovered*, or *removed*, the people who have been sick and can no longer infect others or be reinfected.

$$\frac{dS}{dt} = -\beta IS$$

$$\frac{dI}{dt} = \beta IS - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$

Blue=Susceptible
Green=Infected
Red=Recovered

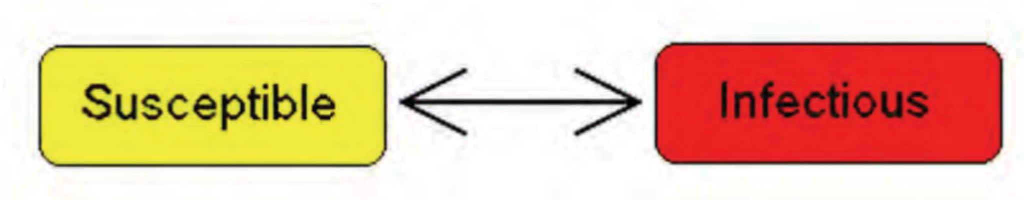


Time

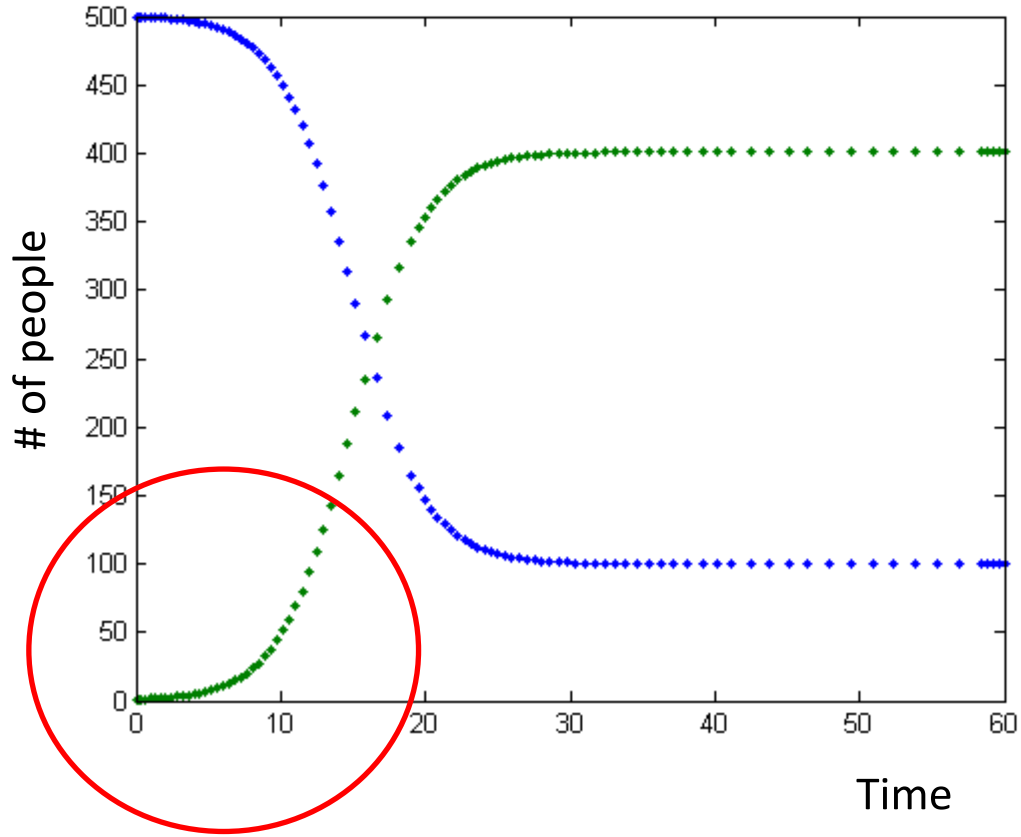
Another term for "recovered"?

wikipedia (re *Compartmental models in epidemiology*)

Aside: SIS model (re Covid-19)



Blue=Susceptible
Green=Infected



Regardless of the details, to first order, initial change patterns are exponential....

wikipedia (re *Compartmental models in epidemiology*)

<u>Aside</u>: SIR model + Diffusion

$$\frac{dS}{dt} = -\beta I S$$

$$\frac{dI}{dt} = \beta I S - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2}$$

Reaction-Diffusion equation

$$\frac{\partial \mathbf{c}}{\partial t} = \mathbf{f}(\mathbf{c}) + D\nabla^2 \mathbf{c},$$

$\mathbf{f}(\mathbf{c})$ describes “reaction kinetics”

→ Pattern formation....

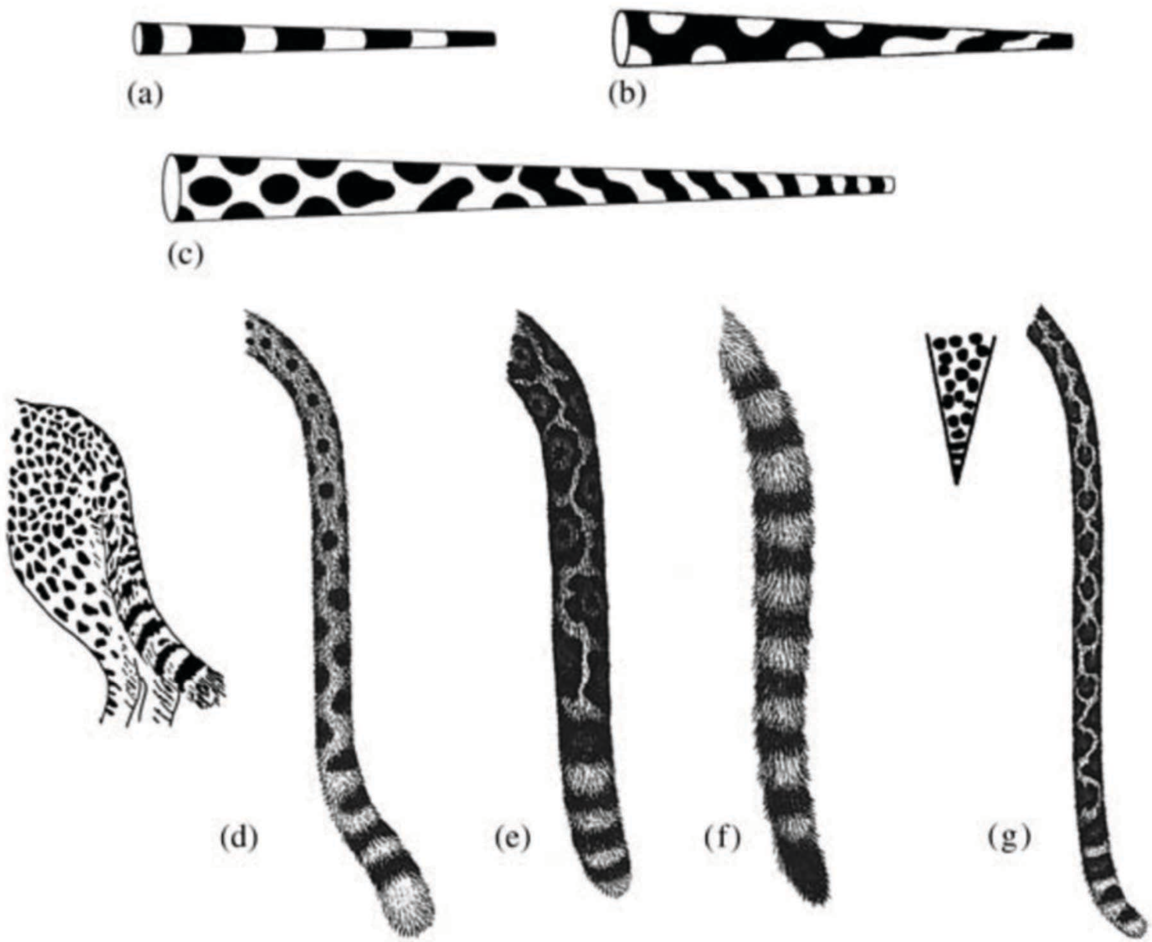
Murray (2003)

Aside: Reaction-Diffusion (or *How the leopard got its spots*)

$$\frac{\partial u}{\partial t} = \gamma f(u, v) + \nabla^2 u, \quad \frac{\partial v}{\partial t} = \gamma g(u, v) + d\nabla^2 v$$

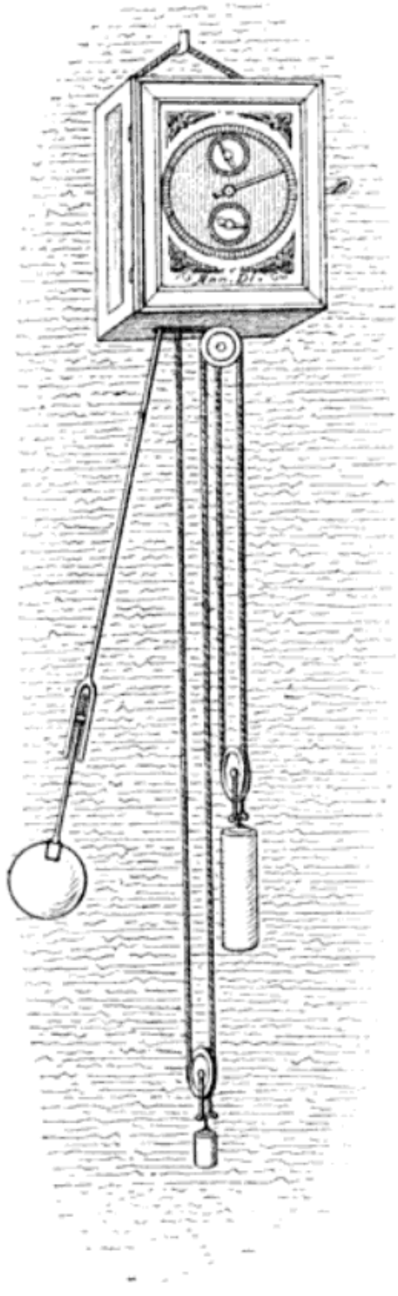
$$f(u, v) = a - u - h(u, v), \quad g(u, v) = \alpha(b - v) - h(u, v)$$

$$h(u, v) = \frac{\rho u v}{1 + u + K u^2}.$$

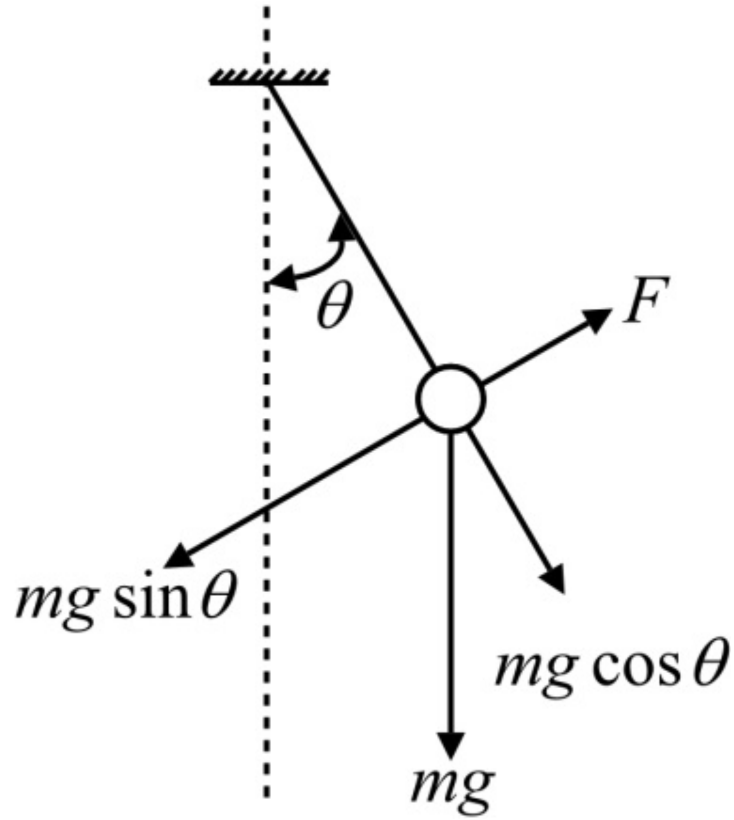


Murray (2003)

<u>Nonlinear oscillations</u>: Pendulum



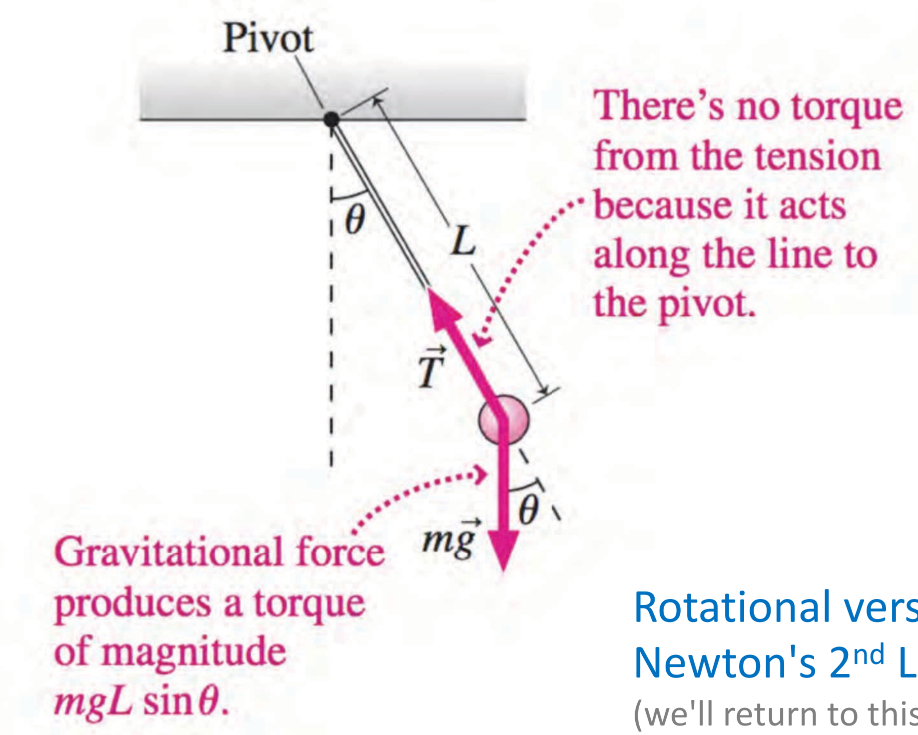
"first pendulum clock" re Christiaan Huygens



$$\frac{d^2\theta}{dt^2} = \ddot{\theta} = -\frac{g}{\ell}\sin(\theta)$$

wikipedia - **Pendulum**

Nonlinear oscillations: Pendulum



Visual depiction of 1st order approx. of Taylor series

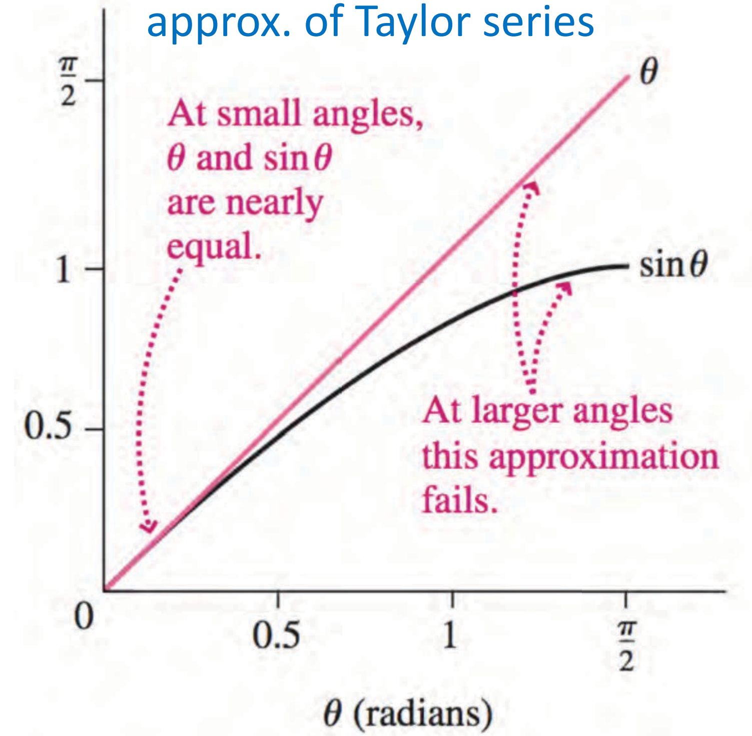


FIGURE 13.11 For θ much less than 1 radian, $\sin\theta$ and θ are nearly equal.

Rotational version of Newton's 2nd Law

(we'll return to this soon...)

$$\tau = I\alpha,$$

Restoring "force" (i.e., torque):

$$\tau = -mgL\sin\theta$$

"Small angles" = SHO

Resulting equation of motion:

$$I\frac{d^2\theta}{dt^2} = -mgL\sin\theta$$

$$I\frac{d^2\theta}{dt^2} = -mgL\theta$$

Wolfson (2007)

<u>Nonlinear oscillations</u>: Pendulum

Moment of inertia
(for a point mass at a distance L
from the axis of rotation)

$$I\frac{d^2\theta}{dt^2} = -mgL\sin\theta$$

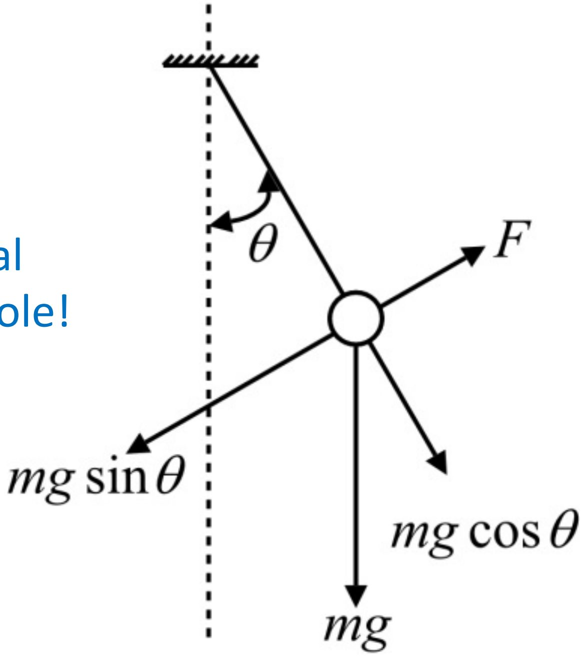
$$I = mL^2$$

<u>Note</u>: We will return soon to rotational motions
(incl. concepts such as moment of inertia)

Pendulum's period of oscillation:

$$\omega = \sqrt{\frac{mgL}{mL^2}} = \sqrt{\frac{g}{L}}$$

→ Remarkably by virtue of being a "rotational problem", mass m does not explicitly play a role!

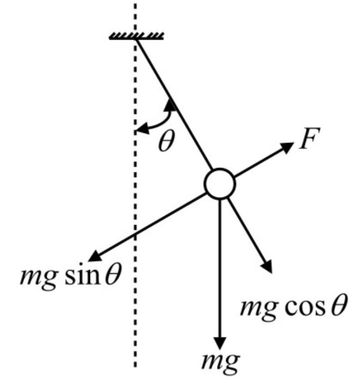


<u>Note</u>: Dimensional analysis would have led to the same conclusion!

$$\frac{d^2\theta}{dt^2} = \ddot{\theta} = -\frac{g}{\ell}\sin(\theta)$$

Nonlinear oscillations: Pendulum

Gravity-driven pendulum



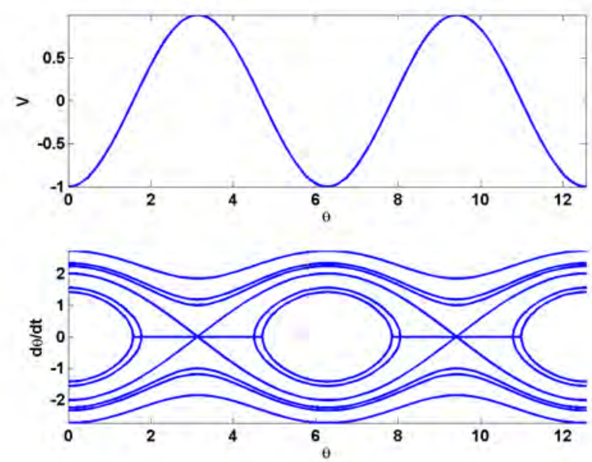
Eqn. of motion

(no damping)

$$\frac{d^2\theta}{dt^2} = \ddot{\theta} = -\frac{g}{\ell}\sin(\theta)$$

Note: Presence of nonlinear term greatly complicates mathematical analysis

Phase space

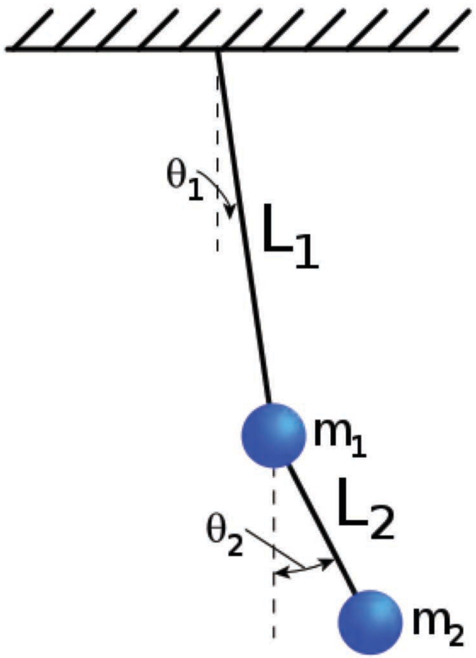


Consider that there are two equilibria w/ differing stability….

wikipedia - **Pendulum (mathematics)**

<u>Nonlinear oscillations</u>: Double pendulum

- Classic example of a relatively simple mechanical system, yet a nonlinear one that exhibits strikingly complex (e.g., chaotic) dynamics



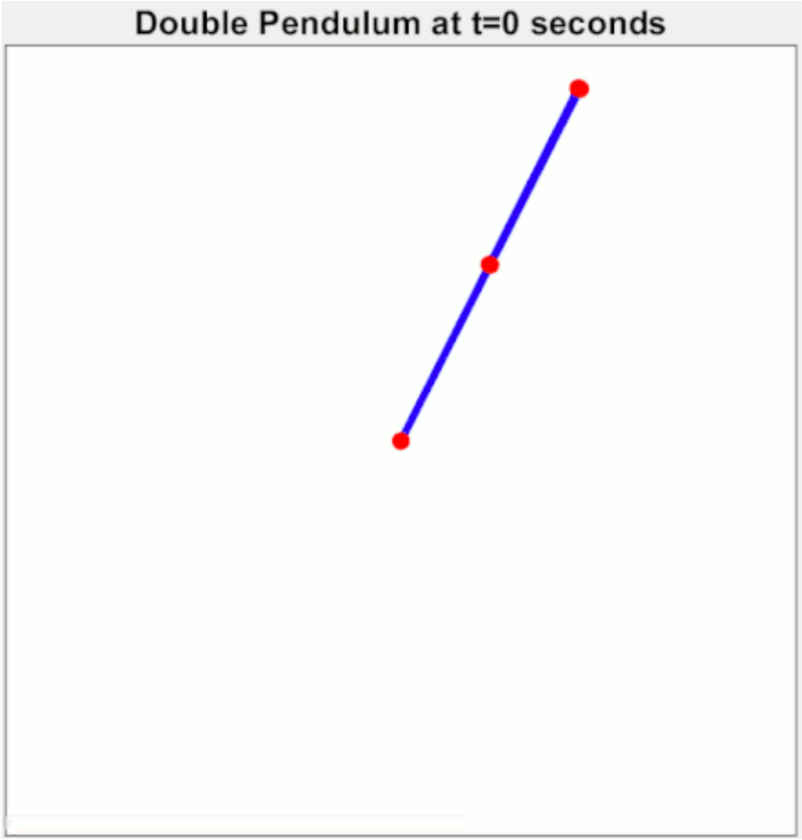
<u>Note</u>: Despite being nonlinear, this system is conservative and thus much more amenable to mathematical analysis via Lagrangian mechanics

wikipedia (double pendulum)

Nonlinear oscillations: Double pendulum

Three double pendulums with near identical initial conditions diverge over time displaying the chaotic nature of the system.

Trajectories of a double pendulum

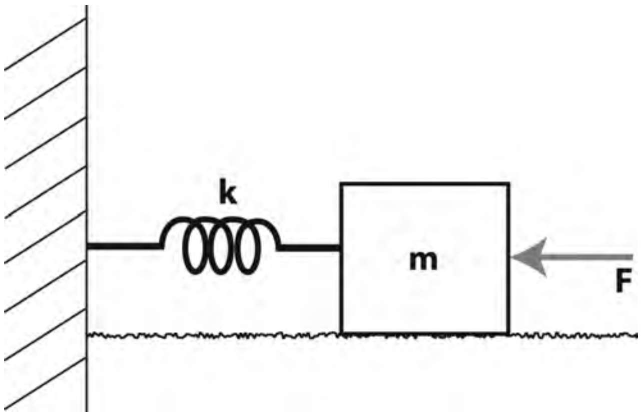


→ Helps motivate the notion of Lyapunov exponents...

wikipedia (double pendulum)

Nonlinear oscillations: van der Pol

- Nonlinear version of a harmonic oscillator



$$\ddot{x} = -x - \varepsilon(x^2 - 1)\dot{x}$$

- Originally proposed to study cardiac dynamics and vacuum tubes

- Nonlinear and exhibits relatively complex behavior, thus has proven a popular model for study in mathematics, physics, and biology

- Physically, how does this differ from a linear damped harmonic oscillator?

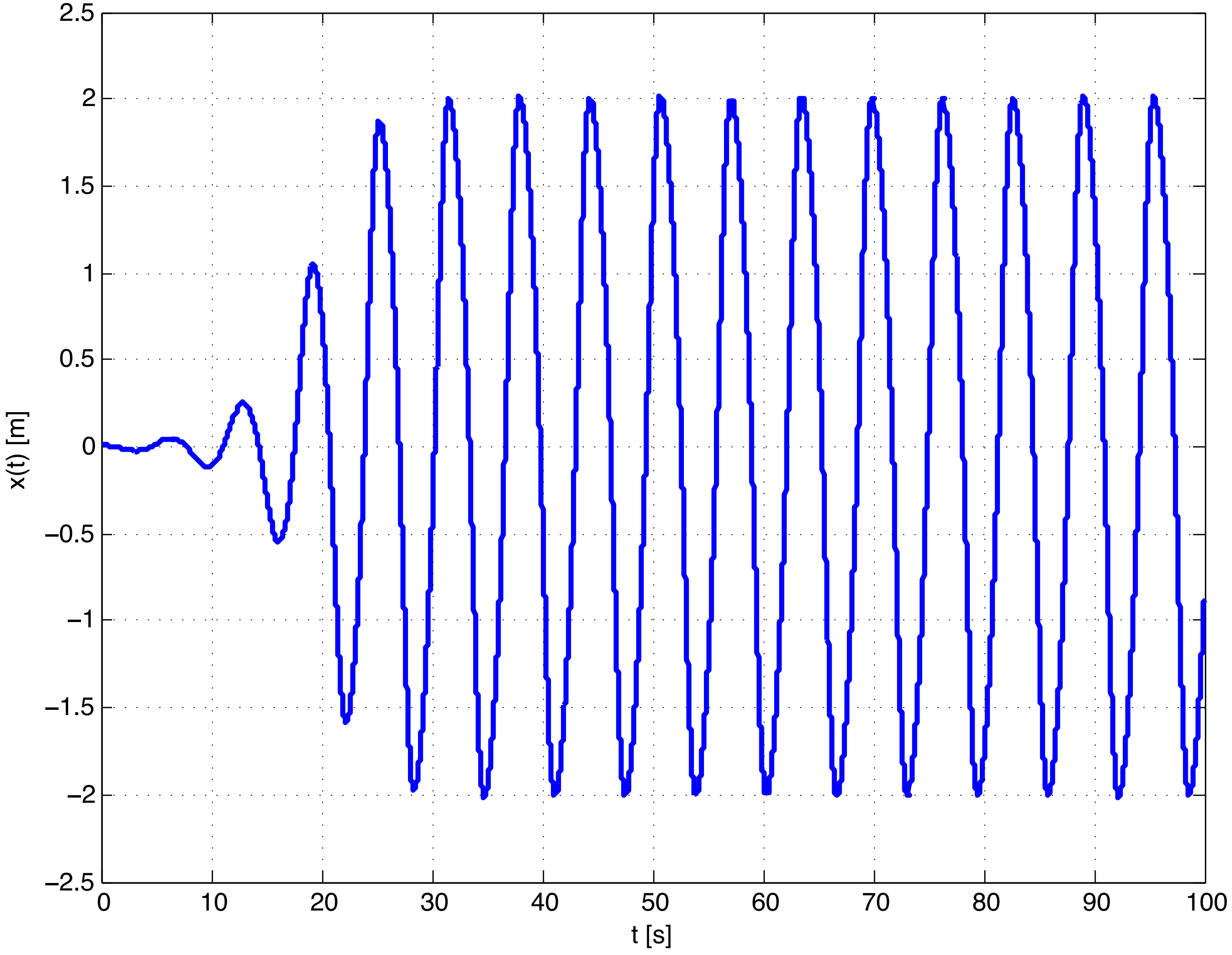
Small displacements → Negative damping
(i.e., non-conservative system)

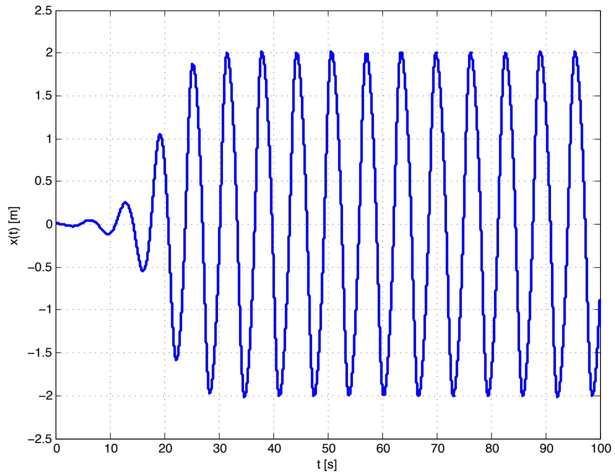
Limit cycles

```
function [out1] = VDPfunction(t,y,flag,P)
% ---------------------------------------------------
%   y(1) ... position x
%   y(2) ... velocity dx/dt
out1(1)= y(2);
out1(2)= (P.mu/P.m)*(1-y(1)^2)*y(2) - (P.k/P.m)*y(1) + (P.A/P.m)*sin(P.wr*t);
out1= out1';     % wants output as a column vector
```

Limit cycles

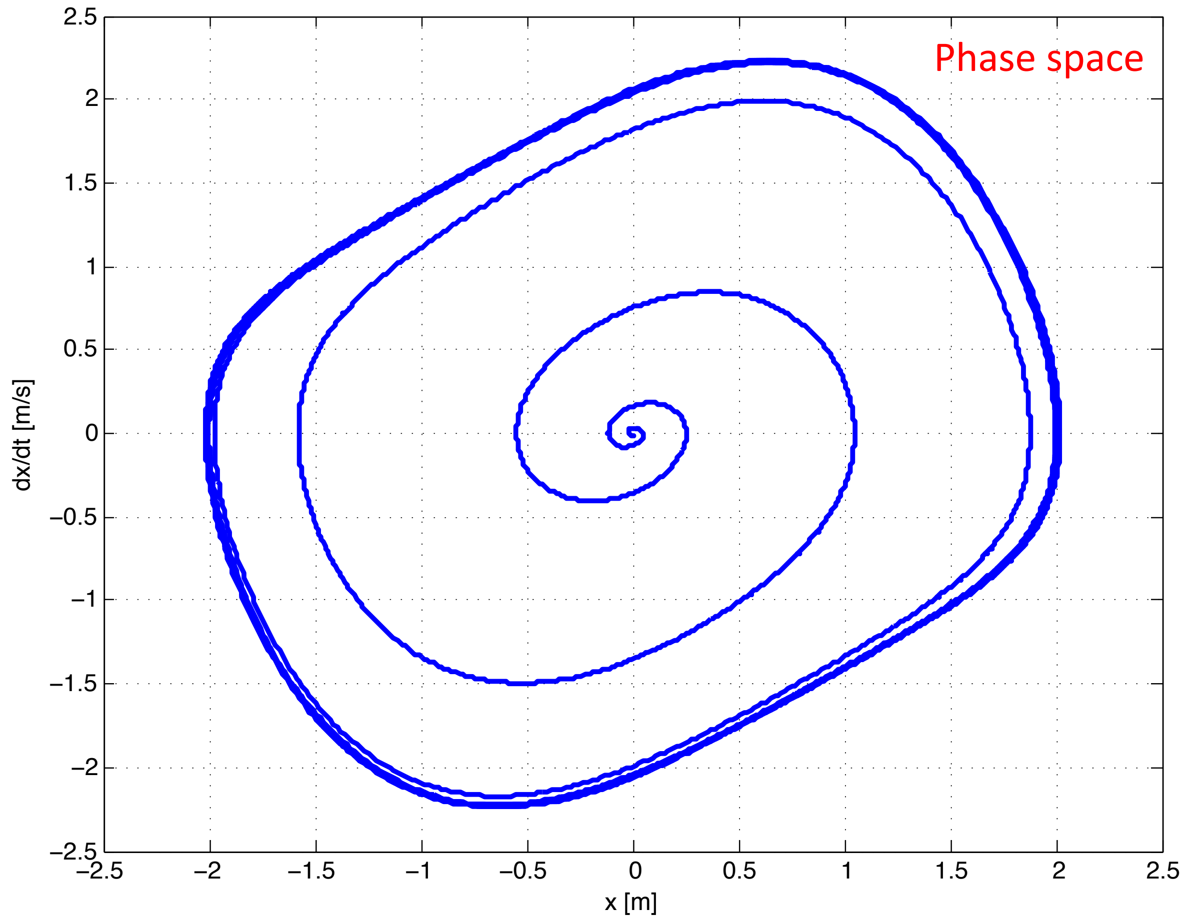
initial non-zero displacement





Limit cycles

→ Even though there is damping, the system oscillates by itself in a stable fashion



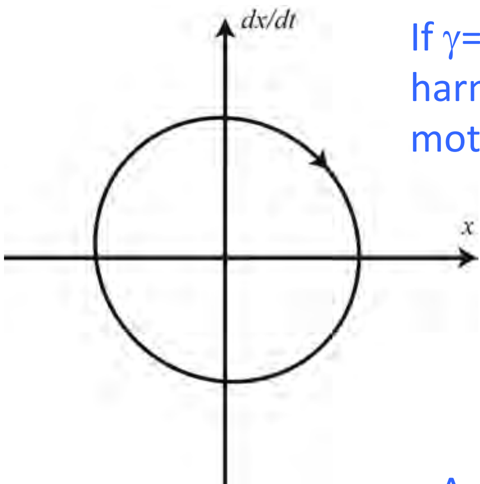
Tangent: Bifurcation analysis

- In the context of dynamical systems, consider bifurcation analysis as a means to assess how the overall ('qualitative') behavior of a system depends upon the parameters (i.e., the 'constants')

→ Consider the damped, undriven harmonic oscillator with a nonzero initial condition

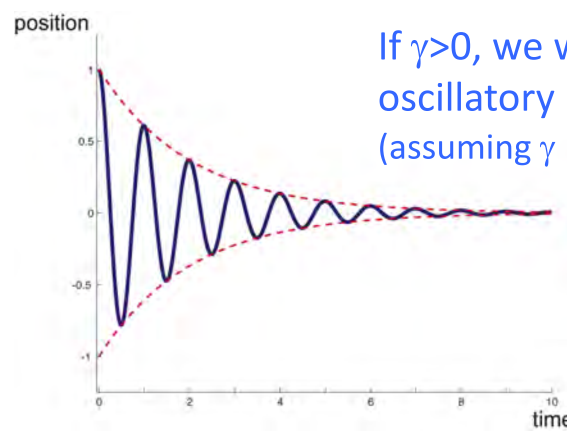
$$\ddot{x} + \gamma \dot{x} + \omega_o^2 x = 0$$

$$\text{e.g.,} \quad x(0) = 0, \quad \dot{x} = 1$$



If γ=0, we will have simple harmonic (i.e., periodic) motion

And if γ<0, then what?



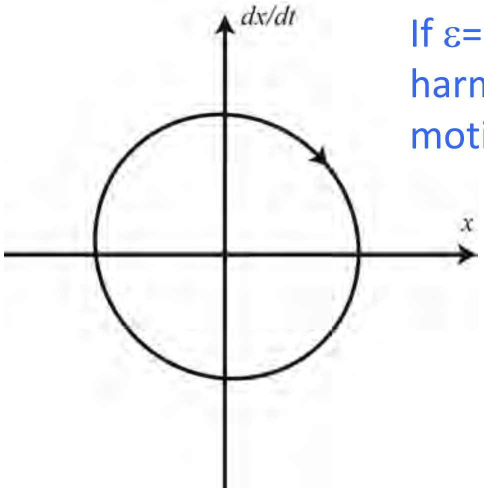
If γ>0, we will have damped oscillatory motion (assuming γ is not too big!)

- Clearly there is a change in the behavior of the system about γ=0

→ We call this a bifurcation

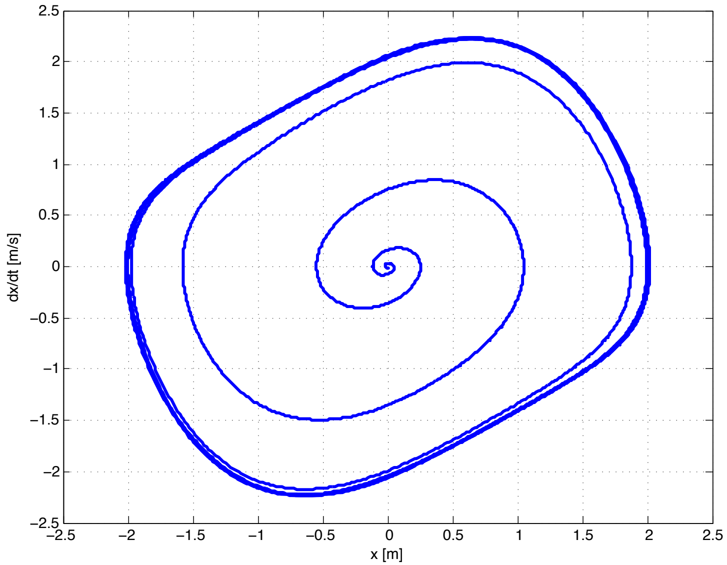
Tangent: Bifurcation analysis

- So what about a nonlinear oscillator? $$\ddot{x} = -x - \varepsilon(x^2 - 1)\dot{x}$$



If ε=0, we will have simple harmonic (i.e., periodic) motion

If ε>0, we will have stable limit cycle



- Physically, a very different type of behavior emerges when ε goes from 0 to a positive value (this is actually called a *supercritical Hopf bifurcation*)

Hopf bifurcation - "... a local bifurcation in which a fixed point of a dynamical system loses stability as a pair of complex conjugate eigenvalues of the linearization around the fixed point cross the imaginary axis of the complex plane. Under reasonably generic assumptions about the dynamical system, we can expect to see a small-amplitude limit cycle branching from the fixed point."

wikipedia (Hopf bifurcation)

Aside: "Bifurcation"?

Aside: "Bifurcation"?

- In the most general sense, a ‘bifurcation’ describes how something ‘splits’

- In dynamical systems theory, bifurcation analysis is a powerful means to study nonlinear systems

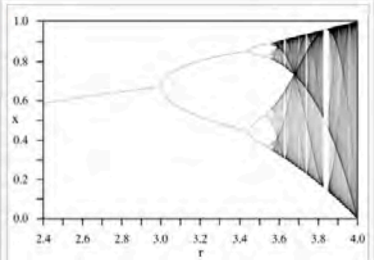
Minimum complexity of a chaotic system [edit]

Discrete chaotic systems, such as the logistic map, can exhibit strange attractors whatever their dimensionality. In contrast, for continuous dynamical systems, the Poincaré–Bendixson theorem shows that a strange attractor can only arise in three or more dimensions. Finite-dimensional linear systems are never chaotic; for a dynamical system to display chaotic behavior, it has to be either nonlinear or infinite-dimensional.

The Poincaré–Bendixson theorem states that a two-dimensional differential equation has very regular behavior. The Lorenz attractor discussed above is generated by a system of three differential equations such as:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma y - \sigma x,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \rho x - xz - y,$$
$$\frac{\mathrm{d}z}{\mathrm{d}t} = xy - \beta z.$$

- Minimum complexity?
- Bifurcation diagram?



Bifurcation diagram of the logistic map $x \to r\,x\,(1 - x)$. Each vertical slice shows the attractor for a specific value of r. The diagram displays period-doubling as r increases, eventually producing chaos.

where x, y, and z make up the system state, t is time, and σ, ρ, β are the system parameters. Five of the terms on the right hand side are linear, while two are quadratic; a total of seven terms. Another well-known chaotic attractor is generated by the Rossler equations which have only one nonlinear term out of seven. Sprott [20] found a three-dimensional system with just five terms, that had only one nonlinear term, which exhibits chaos for certain parameter values. Zhang and Heidel [21][22] showed that, at least for dissipative and conservative quadratic systems, three-dimensional quadratic systems with only three or four terms on the right-hand side cannot exhibit chaotic behavior. The reason is, simply put, that solutions to such systems are asymptotic to a two-dimensional surface and therefore solutions are well behaved.

While the Poincaré–Bendixson theorem shows that a continuous dynamical system on the Euclidean plane cannot be chaotic, two-dimensional continuous systems with non-Euclidean geometry can exhibit chaotic behavior.[23] Perhaps surprisingly, chaos may occur also in linear systems, provided they are infinite dimensional.[24] A theory of linear chaos is being developed in a branch of mathematical analysis known as functional analysis.

wikipedia (~1/3 of the way down for ‘chaos theory’)

Aside: "Bifurcation" & Period doubling

- Consider the "logistic equation" (seemingly simple 1st order nonlinear ODE)

Logistic eqn. (continuous)

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$$

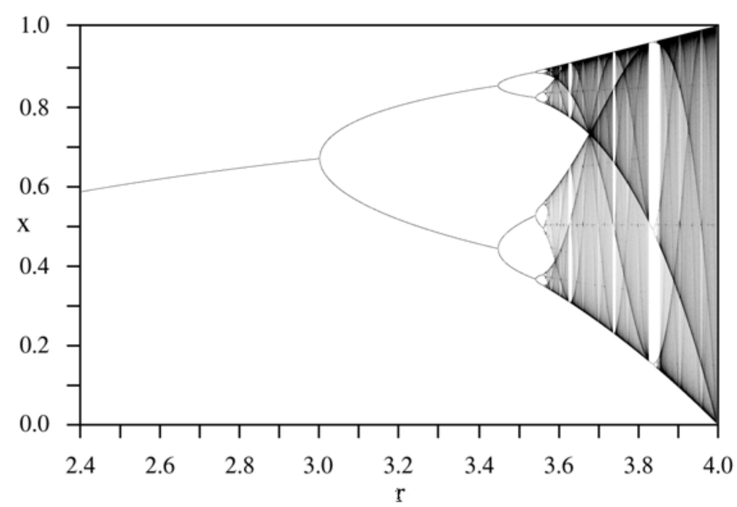
Logistic eqn. (discrete)

$$P_{n+1} = kP_n(1 - P_n/L)$$

- We'll simplify slightly (but keep real-valued): $x_{n+1} = x_n r(1 - x_n)$

- Relatively innocuous equation, right?

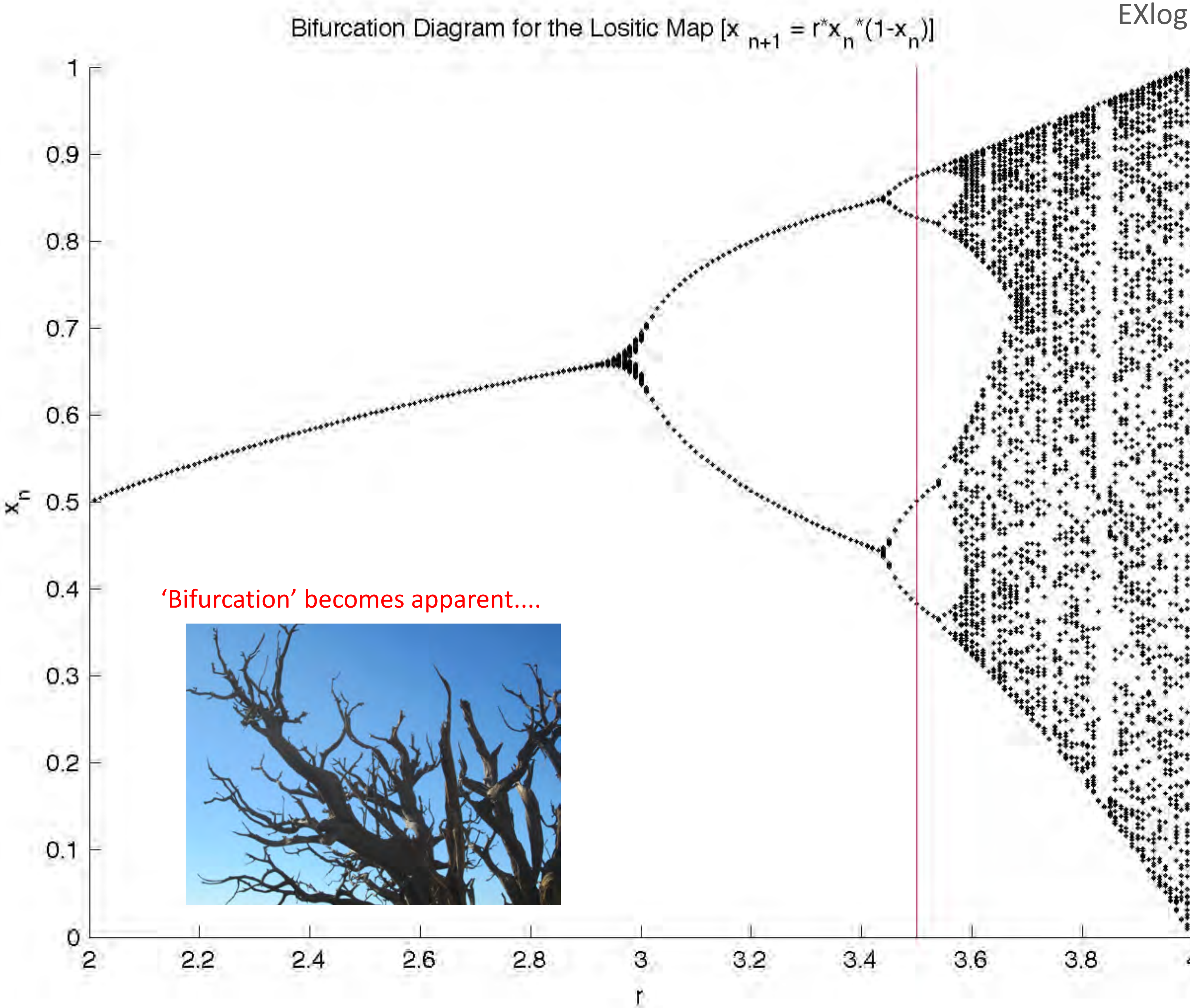
→ Relatively easy to numerically perform a bifurcation analysis (with respect to r) and observe how 'period doubling' emerges, commonly pointed towards as a characteristic of *chaos*



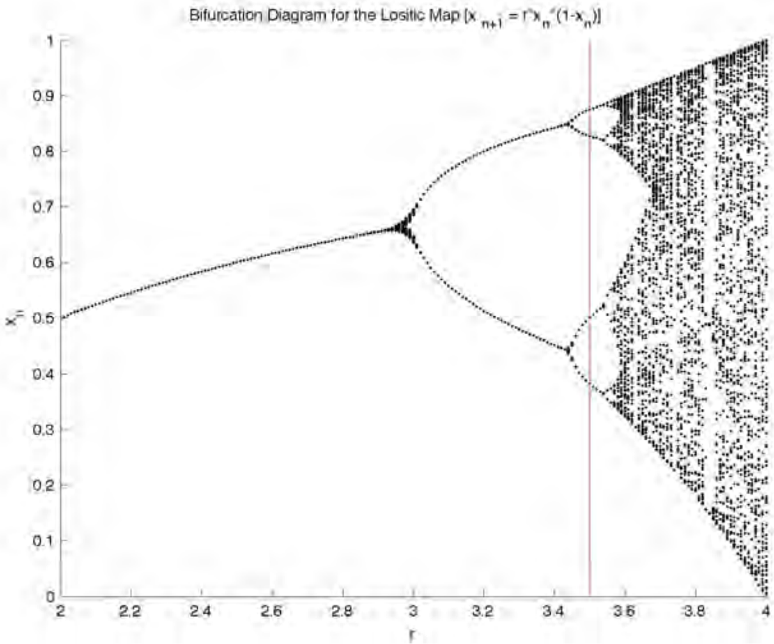
wikipedia (chaos theory)

```
clear;  figure(1); clf; hold on;
% ---------------
% User inputs
range= [2 4];   % min and max values to compute bifurcation diagram over [2 4]
Nr= 200;      % # of steps over range [100]
x0= 0.1;    % starting x value [0.1]
Nsettl= 50; % # of runs allowed for 'settling' [50]
Nit= 100;   % # of iterations to plot for a given value of r [200]
rPlot= 3.5; % for 'timecourse' plot, specify associated r value (must be inside range!)
% ---------------
rmin= range(1); rmax= range(2);
% loop through each r value
for nn=1:Nr
    r(nn)= rmin + nn*(rmax-rmin)/Nr;    % update r
    x= x0;  % reset to IC
    xS(1)= x;   % store first point
    indx=2; % reset indexer (for 2nd iterate)
    for mm=0:Nsettl+Nit     % loop through the iterations of the map
        x= r(nn)*x*(1-x);       % deal with mapping
        xS(nn,indx)= x;        % store values
        indx= indx+1;       % update indexer
    end
    % plot points for a given iteration *past* the settling time
    plot(r(nn)*ones(Nit+1),xS(nn,Nsettl:Nsettl+Nit),'k.')
end
% ----
xlabel('r');    ylabel('x_n')
title('Bifurcation Diagram for the Lositic Map [x_{n+1} = r*x_n*(1-x_n)]')
% ----
% also plot x_n as function of n for relevant r value (as specified)
[junk indxR] = min(abs(r-rPlot));   % search for closest r value to rPlot
n= linspace(0,size(xS,2),size(xS,2));
figure(2); clf;
plot(n,xS(indxR,:),'kd-'); hold on;
xlabel('n');    ylabel('x_n');
stem(Nsettl,max(xS(indxR,:)),'r-','marker','none'); % indicate bound for 'settling'
figure(1); stem(rPlot,max(xS(:)),'r-','marker','none'); % indicate r for which 'time course' is plotted
```

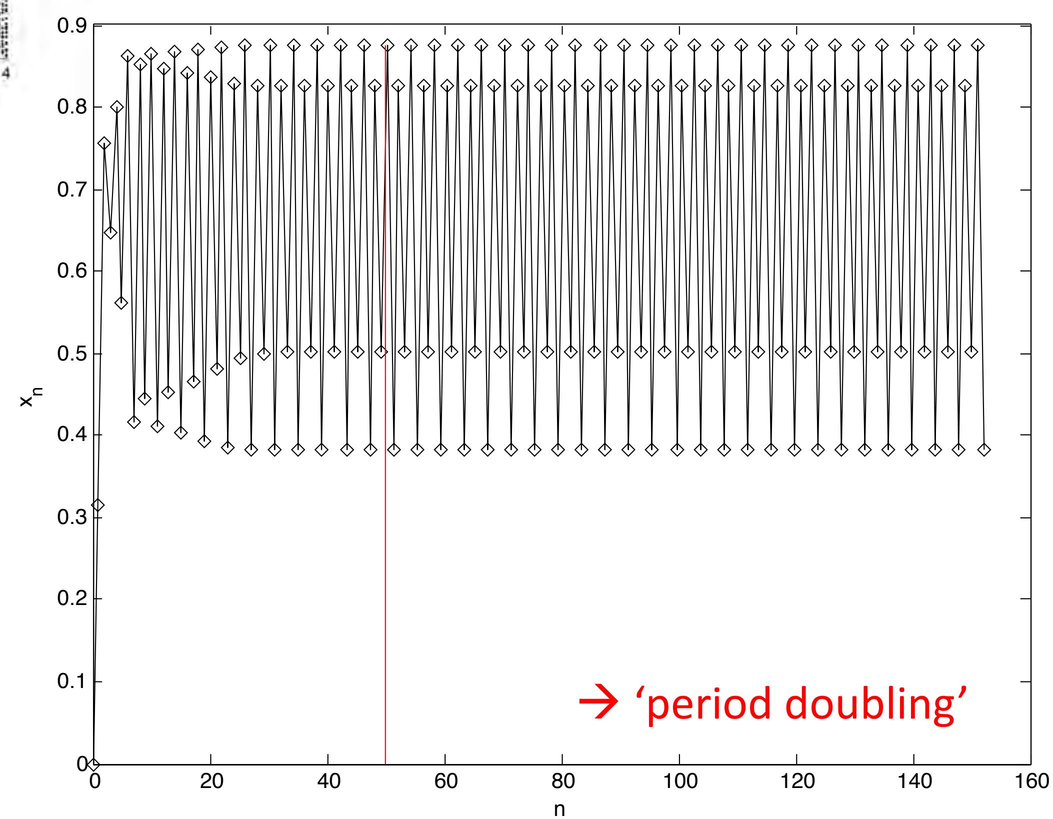
EXlogisticBIF.m



'Bifurcation' becomes apparent....

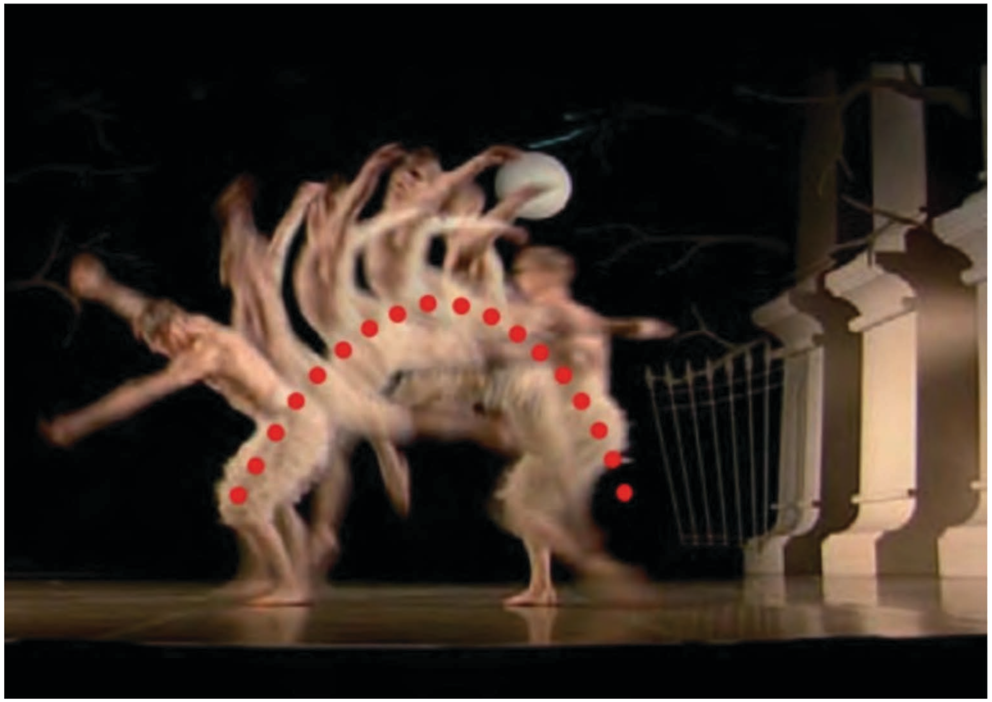


➢ After the settling period, for larger values of *r*, higher and higher orders of periods (i.e., oscillations between different values) emerge



→ Once *r* is large enough (>3.6), there are so many different 'hopping points' that the behavior looks erratic or noisy (even though there is a predictable underlying structure)

Motion of things that are not "point masses"....



Various concepts at play here:

- Projectile motion
- Center of Mass
- Rotation
- Moment of inertia

Finding the center of mass (CM)

Easy case (discrete)

The **center of mass** of a system of n point masses $m_1, m_2, \ldots, m_n$ located at positions $x_1, x_2, \ldots, x_n$ along the x-axis is given by

$$\bar{x} = \frac{\sum x_i m_i}{\sum m_i}.$$

The numerator is the sum of the moments of the masses about the origin; the denominator is the total mass of the system.

- Left-hand term is the vector indicating the center of mass *relative to your chosen coordinate system*

Wolfson notation

$$\vec{r}_{\text{cm}} = \frac{\sum m_i \vec{r}_i}{M}$$

Kesten & Tauck notation

$$x_{\text{CM}} = \frac{1}{M_{\text{tot}}} \sum_{i=1}^{N} m_i x_i$$

✓TIP Choosing the Origin

Choosing the origin at one of the masses here conveniently makes one of the terms in the sum $\sum m_i x_i$ zero. But, as always, the choice of origin is purely for convenience and doesn't influence the actual physical location of the center of mass. Exercise 16 demonstrates this point, repeating Example 9.1 with a different origin.

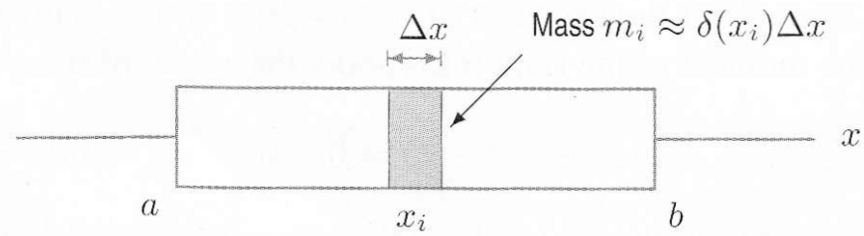
✓TIP Exploit Symmetries

It's no accident that x_{cm} here lies on the vertical line that bisects the triangle; after all, the triangle is symmetric about that line, so its mass is distributed evenly on either side. Exploit symmetry whenever you can; that can save you a lot of computation throughout physics!

Hughes-Hallet et al (2004)
Wolfson

Finding the center of mass

Harder case (1-D continuous mass distribution)



Continuous Mass Density

Instead of discrete masses arranged along the x-axis, suppose we have an object lying on the x-axis between $x = a$ and $x = b$. At point x, suppose the object has mass density (mass per unit length) of $\delta(x)$. To calculate the center of mass of such an object, divide it into n pieces, each of length Δx. On each piece, the density is nearly constant, so the mass of the piece is given by density times length. See Figure 8.51. Thus, if x_i is a point in the i^{th} piece,

$$\text{Mass of the } i^{\text{th}} \text{ piece, } m_i \approx \delta(x_i)\Delta x.$$

Then the formula for the center of mass, $\bar{x} = \sum x_i m_i / \sum m_i$, applied to the n pieces of the object gives

$$\bar{x} = \frac{\sum x_i \delta(x_i)\Delta x}{\sum \delta(x_i)\Delta x}.$$

Interdisciplinary connection: Riemann sums and integrals!

In the limit as $n \to \infty$ we have the following formula:

The **center of mass** $\bar{x}$ of an object lying along the x-axis between $x = a$ and $x = b$ is

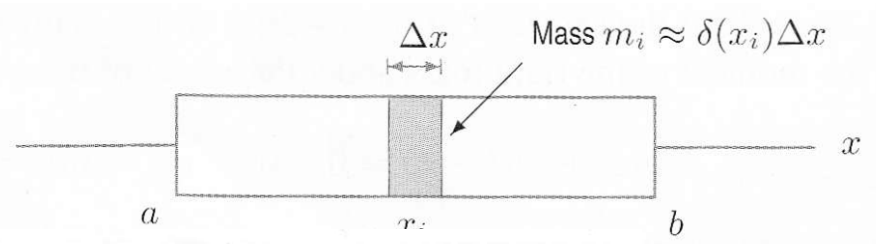
$$\bar{x} = \frac{\int_a^b x\delta(x)\,dx}{\int_a^b \delta(x)\,dx},$$

where $\delta(x)$ is the density (mass per unit length) of the object.

Hughes-Hallet et al (2004)

Finding the center of mass

Harder case (1-D continuous mass distribution)



In the limit as $n \to \infty$ we have the following formula:

The **center of mass** $\overline{x}$ of an object lying along the x-axis between $x = a$ and $x = b$ is

$$\overline{x} = \frac{\int_a^b x\delta(x)\,dx}{\int_a^b \delta(x)\,dx},$$

where $\delta(x)$ is the density (mass per unit length) of the object.

As in the discrete case, the denominator is the total mass of the object.

Wolfson notation

$$\vec{r}_{\text{cm}} = \lim_{\Delta m_i \to 0} \frac{\sum \Delta m_i \vec{r}_i}{M} = \frac{\int \vec{r}\,dm}{M} \quad \left(\begin{array}{c}\text{center of mass,}\\ \text{continuous matter}\end{array}\right)$$

Hughes-Hallet et al (2004)

TACTICS 9.1 Setting Up an Integral

An integral like $\int x\, dm$ can be confusing because you see both x and dm after the integral sign and they don't seem related. But they are, and here's how to proceed:

1. Find a suitable shape for your mass elements, preferably one that exploits any symmetry in the situation. One dimension of the elements should involve an infinitesimal interval in one of the coordinates x, y, or z. In Example 9.3, the mass elements were strips, symmetric about the wing's centerline and with width dx.
2. Find an expression for the infinitesimal area of your mass elements (in a one-dimensional problem it would be the length; in a three-dimensional problem, the volume). In Example 9.3, the infinitesimal area of each mass element was the strip height h multiplied by the width dx.
3. Form ratios that relate the infinitesimal coordinate interval to the physical quantity in the integral—which in Example 9.3 is the mass element dm. Here we formed the ratio of the area of a mass element to the total area, and equated that to the ratio of dm to the total mass M.
4. Solve your ratio statement for the infinitesimal quantity, in this case dm, that appears in your integral. Then you're ready to evaluate the integral.

Sometimes you'll be given a density—mass per volume, per area, or per length—and then in place of steps 3 and 4 you find dm by multiplying the density by the infinitesimal volume, area, or length you identified in step 2.

Although we described this procedure in the context of Example 9.3, it also applies to other integrals you'll encounter in different areas of physics.

Additional slides beyond are for general reference....

<u>Nonlinear systems</u>

- So what can this approach tell us about nonlinear systems (e.g., van der Pol)?

Linearize!

$$\ddot{x} = -x - \varepsilon(x^2 - 1)\dot{x}$$

e.g., Connect pendulum and Taylor series back to simple harmonic oscillator!

1. Find the fixed points of the system

$$\frac{dx}{dt} = y$$

2. For a given fixed point (x_o,y_o), determine the *Jacobian* matrix

$$\frac{dy}{dt} = -x + \epsilon(1 - x^2)y$$

$$\dot{x} = f(x, y)$$

$$\dot{y} = g(x, y)$$

Jacobian

$$J(x_o, y_o) = \left[\begin{array}{cc} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{array} \right] \Bigg|_{x_o, y_o}$$

3. Determine the associated eigenvalues

→ Provides a snapshot in the neighborhood local to the fixed point

Ex. van der Pol

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -x + \epsilon(1 - x^2)y$$

- Fixed point (x_o, y_o) = (0,0)
- Associated Jacobian is then:

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & \epsilon \end{bmatrix}$$

$$J(x_o, y_o) = \left.\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}\right|_{x_o, y_o}$$

- The characteristic equation is then:

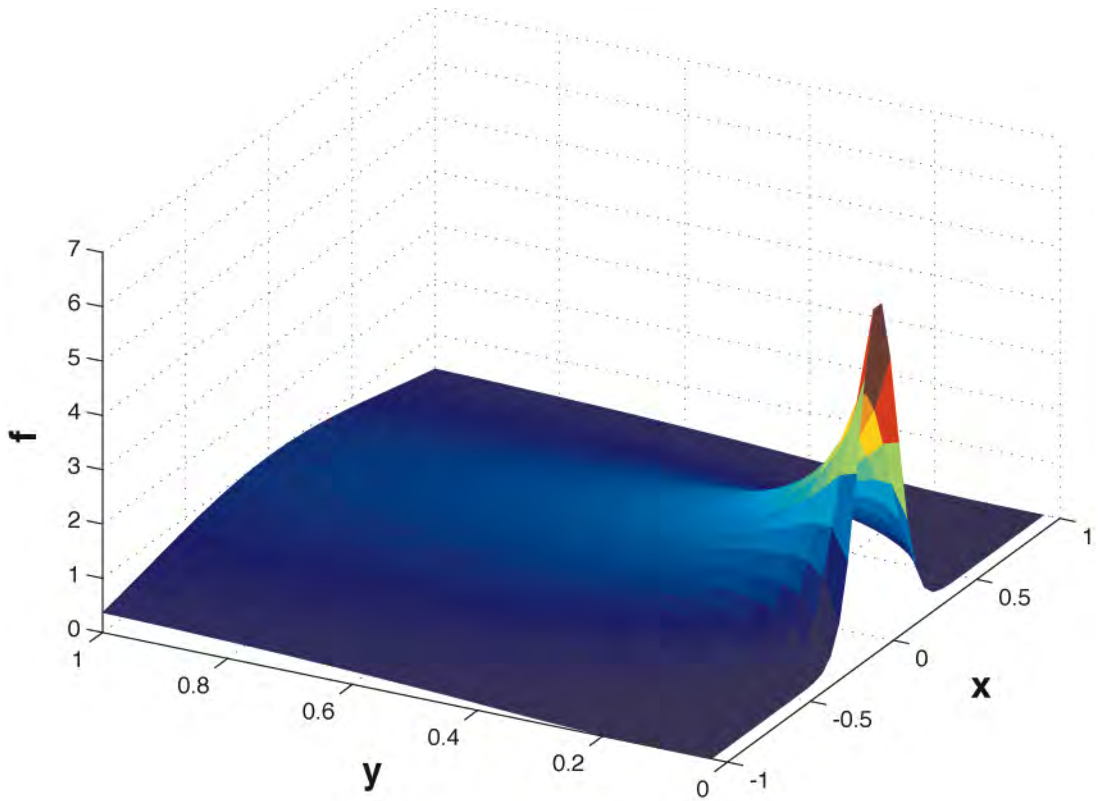
$$\left| \begin{bmatrix} 0 & 1 \\ -1 & \epsilon \end{bmatrix} - \lambda \mathbf{I} \right| = 0$$

- Associated eigenvalues:

$$\lambda = \frac{\epsilon \pm \sqrt{\epsilon^2 - 4}}{2}$$

→ Assuming ε>0, the real parts of both eigenvalues are positive meaning that all solutions will diverge away from the fixed point (i.e., it is unstable)

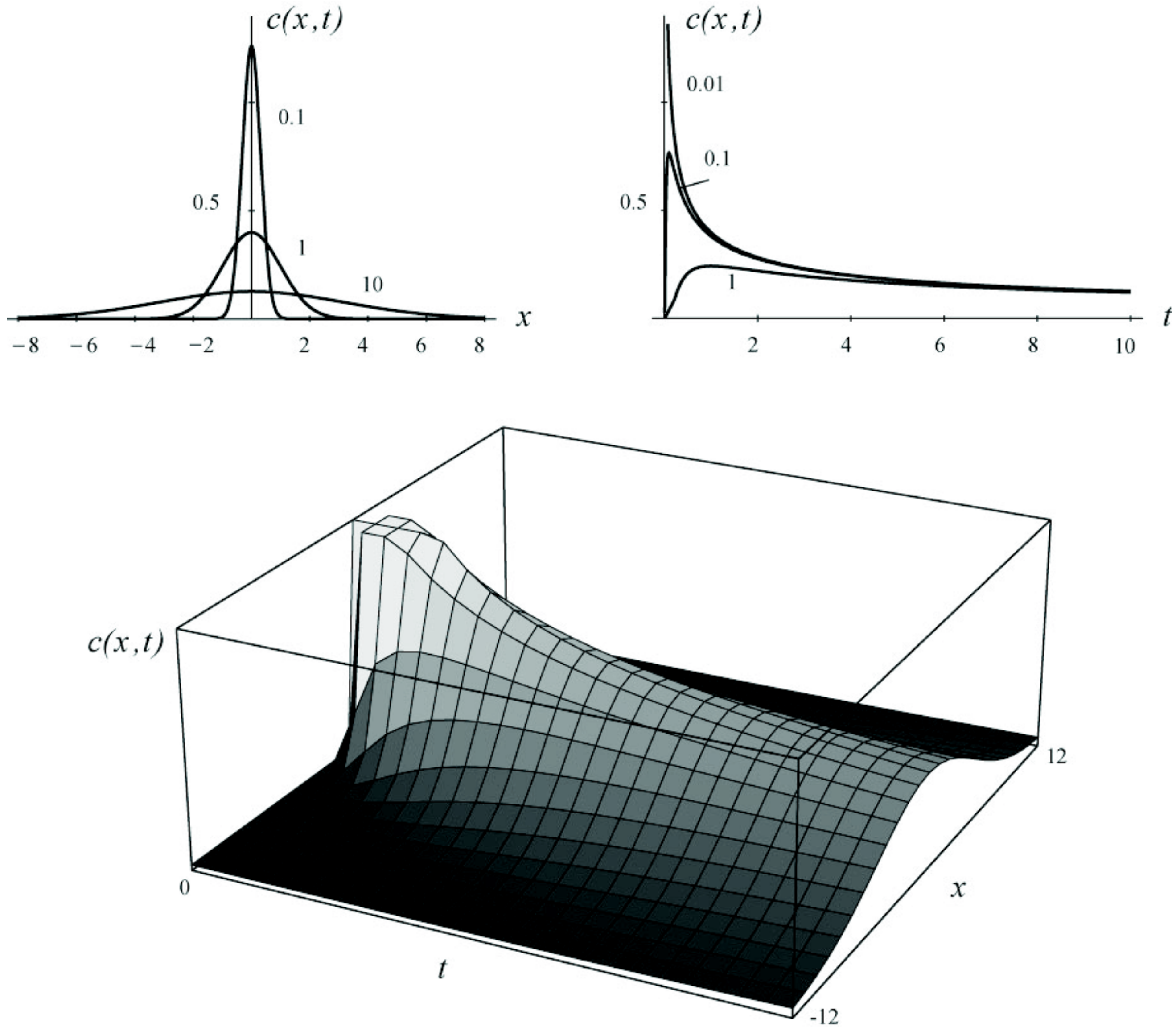
Diffusion processes



$$f(x, y) = \frac{1}{\sqrt{y}} e^{-x^2/y}$$

solution to
diffusion equation!

Diffusion processes



Weiss Fig.3.14 (modified)