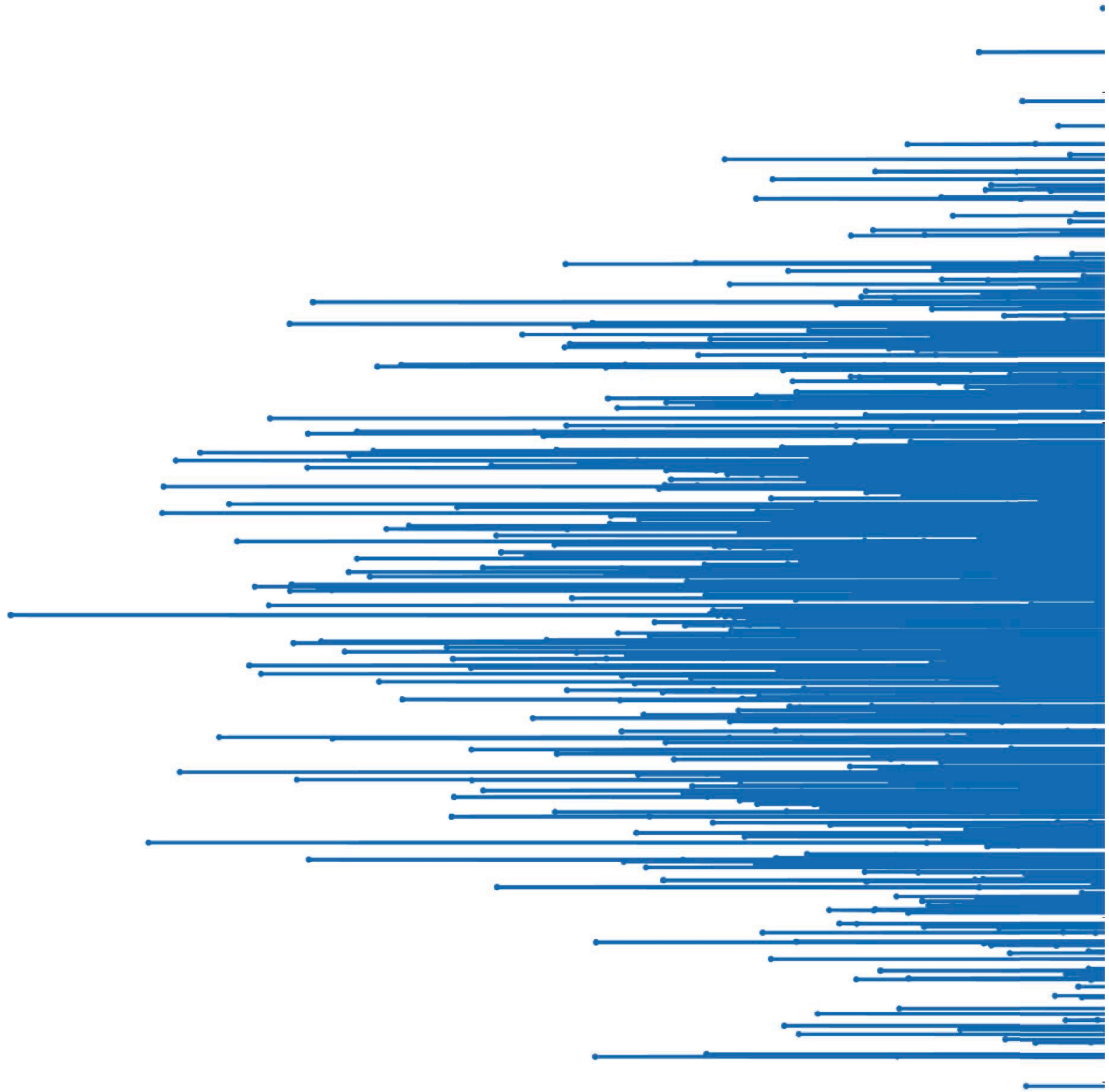
PHYS 2010 (W20)
Classical Mechanics



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Tutorial III

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Ref. (re images):
Knudsen & Hjorth (2000), Kesten & Tauck (2012)

A mass M slides without friction on the roller coaster track shown in Fig. 1.4. The curved sections of the track have radius of curvature R. The mass begins its descent from the height h. At some value of h, the mass will begin to lose contact with the track. Indicate on the diagram where the mass loses contact with the track and calculate the minimum value of h for which this happens.

(*Wisconsin*)

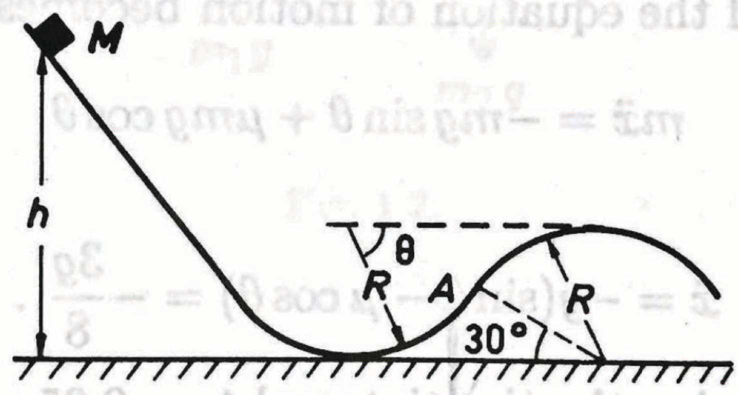


Fig. 1.4.

Solution:

Before the inflection point A of the track, the normal reaction of the track on the mass, N, is

$$N = \frac{mv^2}{R} + mg\sin\theta \ ,$$

where v is the velocity of the mass. After the inflection point,

$$N + \frac{mv^2}{R} = mg\sin\theta \ ,$$

for which $\sin\theta = \frac{R}{2R}$, or $\theta = 30°$.

The mass loses contact with the track if $N \leq 0$. This can only happen for the second part of the track and only if

$$\frac{mv^2}{R} \geq mg\sin\theta \ .$$

The conservation of mechanical energy

$$mg[h - (R - R\sin\theta)] = \frac{1}{2}mv^2$$

then requires

$$h - R + R\sin\theta \geq \frac{R\sin\theta}{2} \ ,$$

or

$$h \geq R - \frac{R\sin\theta}{2} \ .$$

The earliest the mass can start to lose contact with the track is at A for which $\theta = 30°$. Hence the minimum h required is $\frac{3R}{4}$.

Yung-Kuo et al

Show that $\vec{F} = 2xy\vec{i} + xy\vec{j}$ cannot be a gradient vector field.

We have $F_1 = 2xy$ and $F_2 = xy$. Since $\partial F_1/\partial y = 2x$ and $\partial F_2/\partial x = y$, in this case

$$\partial F_2/\partial x - \partial F_1/\partial y \neq 0$$

so $\vec{F}$ cannot be a gradient field.

Hughes-Hallet et al (2013)

The gravitational field, $\vec{F}$, of an object of mass M is given by

$$\vec{F} = -\frac{GM}{r^3}\vec{r}.$$

Show that $\vec{F}$ is a gradient field by finding a potential function for $\vec{F}$.

Solution The vector $\vec{F}$ points directly in toward the origin. If $\vec{F} = \operatorname{grad} f$, then $\vec{F}$ must be perpendicular to the level surfaces of f, so the level surfaces of f must be spheres. Also, if $\operatorname{grad} f = \vec{F}$, then $\|\operatorname{grad} f\| = \|\vec{F}\| = GM/r^2$ is the rate of change of f in the direction toward the origin. Now, differentiating with respect to r gives the rate of change in a radially outward direction. Thus, if $w = f(x, y, z)$ we have

$$\frac{dw}{dr} = -\frac{GM}{r^2} = GM\left(-\frac{1}{r^2}\right) = GM\frac{d}{dr}\left(\frac{1}{r}\right).$$

So let's try

$$w = \frac{GM}{r} \quad \text{or} \quad f(x, y, z) = \frac{GM}{\sqrt{x^2 + y^2 + z^2}}.$$

We calculate

$$f_x = \frac{\partial}{\partial x}\frac{GM}{\sqrt{x^2 + y^2 + z^2}} = \frac{-GMx}{(x^2 + y^2 + z^2)^{3/2}},$$
$$f_y = \frac{\partial}{\partial y}\frac{GM}{\sqrt{x^2 + y^2 + z^2}} = \frac{-GMy}{(x^2 + y^2 + z^2)^{3/2}},$$
$$f_z = \frac{\partial}{\partial z}\frac{GM}{\sqrt{x^2 + y^2 + z^2}} = \frac{-GMz}{(x^2 + y^2 + z^2)^{3/2}}.$$

So

$$\operatorname{grad} f = f_x\vec{i} + f_y\vec{j} + f_z\vec{k} = \frac{-GM}{(x^2 + y^2 + z^2)^{3/2}}(x\vec{i} + y\vec{j} + z\vec{k}) = \frac{-GM}{r^3}\vec{r} = \vec{F}.$$

Our computations show that $\vec{F}$ is a gradient field and that $f = GM/r$ is a potential function for $\vec{F}$.

Hughes-Hallet et al (2013)

For what values of the constants a, b, and c is the force $\mathbf{F} = \mathbf{i}(ax + by^2) + \mathbf{j}cxy$ conservative?

Taking the curl, we have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ax + by^2 & cxy & 0 \end{vmatrix} = \mathbf{k}(c - 2b)y$$

This shows that the force is conservative, provided $c = 2b$. The value of a is immaterial.

Let $\vec{F}$ be the vector field given by $\vec{F}(x, y) = \dfrac{-y\vec{i} + x\vec{j}}{x^2 + y^2}$.

(a) Calculate $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}$. Does the curl test imply that $\vec{F}$ is path-independent?

(b) Calculate $\displaystyle\int_C \vec{F} \cdot d\vec{r}$, where C is the unit circle centered at the origin and oriented counterclockwise. Is $\vec{F}$ a path-independent vector field?

(c) Explain why the answers to parts (a) and (b) do not contradict Green's Theorem.

Solution

(a) Taking partial derivatives, we have

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{1}{x^2+y^2} - \frac{x \cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

Similarly,

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{-1}{x^2+y^2} + \frac{y \cdot 2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

Thus,

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0.$$

Since $\vec{F}$ is undefined at the origin, the domain of $\vec{F}$ contains a hole. Therefore, the curl test does not apply.

(b) On the unit circle, $\vec{F}$ is tangent to the circle and $||\vec{F}|| = 1$. Thus,[3]

$$\int_C \vec{F} \cdot d\vec{r} = ||\vec{F}|| \cdot \text{Length of curve} = 1 \cdot 2\pi = 2\pi.$$

Since the line integral around the closed curve C is nonzero, $\vec{F}$ is not path-independent. We observe that $\vec{F} = \operatorname{grad}(\arctan(y/x))$ and $\arctan(y/x)$ is θ from polar coordinates, for $-\pi/2 < \theta < \pi/2$. The fact that θ increases by 2π each time we wind once around the origin counterclockwise explains why $\vec{F}$ is not path-independent.

(c) The domain of $\vec{F}$ is the "punctured plane," as shown in Figure 18.48. Since $\vec{F}$ is not defined at the origin, which is inside C, Green's Theorem does not apply. In this case

$$2\pi = \int_C \vec{F} \cdot d\vec{r} \neq \int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx dy = 0.$$

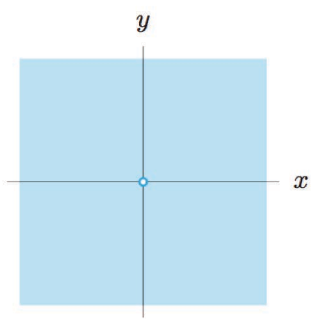


Figure 18.48: The domain of $\vec{F}(x,y) = \frac{-y\vec{i}+x\vec{j}}{x^2+y^2}$ is the plane minus the origin

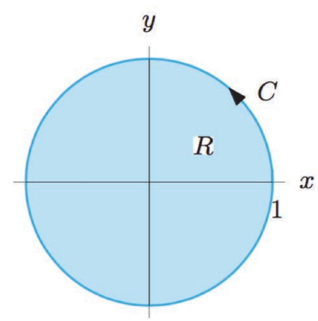


Figure 18.49: The region R is *not* contained in the domain of $\vec{F}(x,y) = \frac{-y\vec{i}+x\vec{j}}{x^2+y^2}$

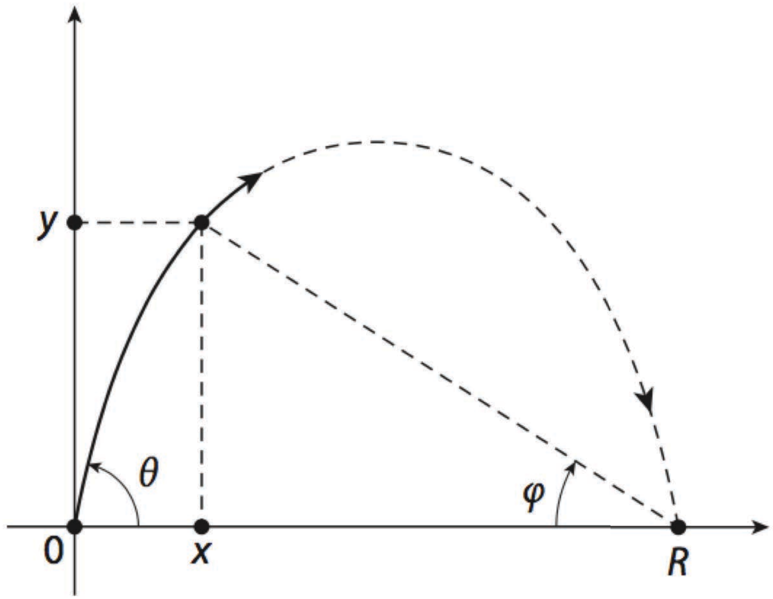
Hughes-Hallet et al (2013)

“Willie Mays, at the crack of the bat, will take a brief look at the flight of the ball, run without looking back, be at exactly the right spot at the right time, and take the ball over his shoulder with a basket catch. How he does it no one knows, certainly not Willie Mays.”

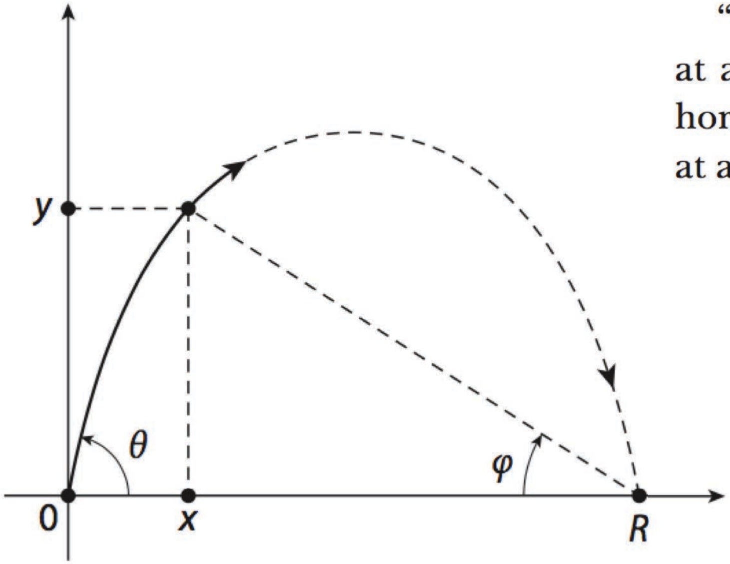
Vannevar Bush (1890–1974)

→ So how does an outfielder know where to go to catch a baseball?

May help to neglect air resistance....



Hint – May be useful to dig up:
Chapman, S. “Catching a Baseball,” *Am. J. Phys.*, Oct. 1968, pp. 868–870.



"Let the ball leave the bat (the origin) with an initial speed of V at an angle θ with the ground. As is well known … the vertical and horizontal displacements [that is, the x- and y-coordinates of the ball] at any time t [$t = 0$ is the instant the batter hits the ball] are

$$y = V\sin(\theta)t - \frac{1}{2}gt^2,$$

$$x = V\cos(\theta)t,$$

Never changes due to (assumed) lack of drag

Assume batter is at origin and fielder is (luckily) right where ball will land (R)

Thus, the fielder does not actually see the arc of the ball's trajectory, but, instead, the ball appears to him to be simply first rising and then falling in a vertical plane that passes through the fielder and the batter. What visual cue to the fielder can there be in this situation—the toughest one that a fielder can face—that tells him that the ball is coming right to him? This is the question that Chapman thought he answered.

Nahin (2016)

To start, we define $t = T$ to be the time when the ball returns to Earth (that is, when the fielder catches the ball). Then, as $y(T) = 0$, we have

$$V\sin(\theta)T - \frac{1}{2}gT^2 = 0,$$

and solving for $T > 0$, we get

$$T = \frac{2\,V\sin(\theta)}{g}.$$

Substituting this result into the equation for x, and since $x(T) = R$, we have

$$R = \frac{2V^2\sin(\theta)\cos(\theta)}{g}.$$

From the geometry of Figure 17.1 we can immediately write, for every instant of time $0 < t < T$,

$$\tan(\phi) = \frac{y}{R-x} = \frac{V\sin(\theta)t - \frac{1}{2}gt^2}{\frac{2V^2\sin(\theta)\cos(\theta)}{g} - V\cos(\theta)t} = \frac{t\left[V\sin(\theta) - \frac{1}{2}gt\right]}{V\cos(\theta)\left[\frac{2\,V\sin(\theta)}{g} - t\right]}$$

$$= \frac{t[2V\sin(\theta) - gt]\frac{1}{2}}{V\cos(\theta)\frac{1}{g}[2V\sin(\theta) - gt]},$$

$$= \frac{g}{2V\cos(\theta)}t,$$

and so we arrive at the simple result

$$\tan(\phi) = (\text{constant})t.$$

That is, for a fielder standing right where the ball will land, the tangent of his line-of-sight elevation angle to the ball's instantaneous location increases linearly with time.

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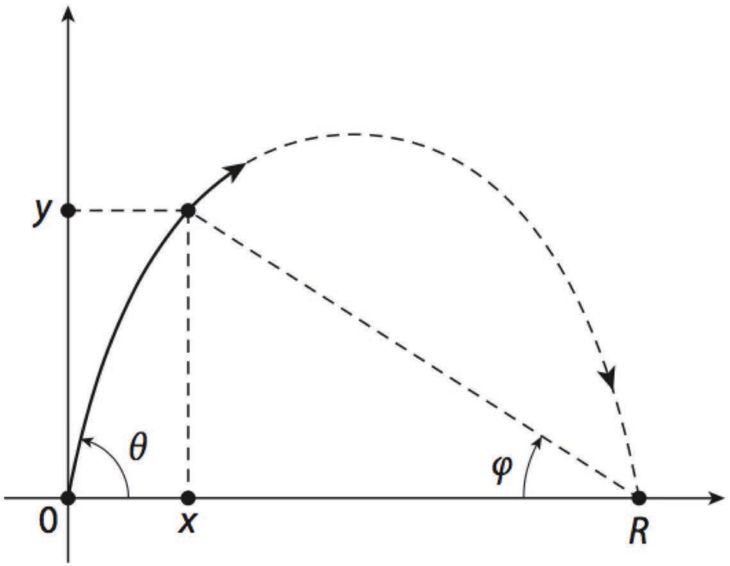
Assume the fielder is at some distance +/- s relative to R



Suppose that τ is the fielder's reaction time and that once he decides he has to move, the fielder runs at the constant speed v that just gets him to $x = R$ at time $t = T$, that is,

$$s = v(T - \tau).$$

Now relax the assumption that the fielder is at R...

The fielder's coordinate along the horizontal axis at time $t \geq \tau$ is $(R - s) + v(t - \tau)$, and so now we can write

$$\tan(\phi) = \frac{y}{(R - s) + v(t - \tau) - x}.$$

Since

$$s = vT - v\tau,$$

then

$$\tau = \frac{vT - s}{v} = T - \frac{s}{v},$$

Nahin (2016)

and so

$$\tan(\phi) = \frac{V\sin(\theta)t - \frac{1}{2}gt^2}{R - s + v\left(t - T + \frac{s}{v}\right) - V\cos(\theta)t}$$

$$= \frac{t\left[V\sin(\theta) - \frac{1}{2}gt\right]}{\frac{2V^2\sin(\theta)\cos(\theta)}{g} - s + v(t - T) + s - V\cos(\theta)t}$$

$$= \frac{t[2V\sin(\theta) - gt]\frac{1}{2}}{\frac{2V^2\sin(\theta)\cos(\theta)}{g} + v\left[t - \frac{2V\sin(\theta)}{g}\right] - V\cos(\theta)t}$$

$$= \frac{\frac{1}{2}gt[2V\sin(\theta) - gt]}{2V^2\sin(\theta)\cos(\theta) + v[gt - 2V\sin(\theta)] - Vg\cos(\theta)t}$$

$$= \frac{\frac{1}{2}gt\,[2V\sin(\theta) - gt]}{2V^2\sin(\theta)\cos(\theta) - v[2V\sin(\theta) - gt] - Vg\cos(\theta)t}$$

$$= \frac{\frac{1}{2}gt[2V\sin(\theta) - gt]}{V\cos(\theta)[2V\sin(\theta) - gt] - v[2V\sin(\theta) - gt]}$$

$$= \frac{\frac{1}{2}gt[2V\sin(\theta) - gt]}{[2V\sin(\theta) - gt][V\cos(\theta) - v]}$$

$$= \frac{gt}{2[V\cos(\theta) - v]},$$

Nahin (2016)

or, once again,

$$\tan(\phi) = (\text{constant})t.$$

So, just as before, even with the added complications of the two new variables s and τ, the tangent of the fielder's line-of-sight elevation angle to the instantaneous location of the ball increases linearly with time. Amazing!

Nahin (2016)

and so

$$\tan(\phi) = \frac{V\sin(\theta)t - \frac{1}{2}gt^2}{R - s + v\left(t - T + \frac{s}{v}\right) - V\cos(\theta)t}$$

$$= \frac{t\left[V\sin(\theta) - \frac{1}{2}gt\right]}{\frac{2V^2\sin(\theta)\cos(\theta)}{g} - s + v(t - T) + s - V\cos(\theta)t}$$

$$= \frac{t[2V\sin(\theta) - gt]\frac{1}{2}}{\frac{2V^2\sin(\theta)\cos(\theta)}{g} + v\left[t - \frac{2V\sin(\theta)}{g}\right] - V\cos(\theta)t}$$

$$= \frac{\frac{1}{2}gt[2V\sin(\theta) - gt]}{2V^2\sin(\theta)\cos(\theta) + v[gt - 2V\sin(\theta)] - Vg\cos(\theta)t}$$

$$= \frac{\frac{1}{2}gt\,[2V\sin(\theta) - gt]}{2V^2\sin(\theta)\cos(\theta) - v[2V\sin(\theta) - gt] - Vg\cos(\theta)t}$$

$$= \frac{\frac{1}{2}gt[2V\sin(\theta) - gt]}{V\cos(\theta)[2V\sin(\theta) - gt] - v[2V\sin(\theta) - gt]}$$

$$= \frac{\frac{1}{2}gt[2V\sin(\theta) - gt]}{[2V\sin(\theta) - gt][V\cos(\theta) - v]}$$

$$= \frac{gt}{2[V\cos(\theta) - v]},$$

or, once again,

$$\tan(\phi) = (\text{constant})t.$$

Physical Laws should have mathematical beauty.

—written on a Moscow blackboard in 1955 by the 1933 Nobel Prize in Physics winner Paul Dirac

Nahin (2016)

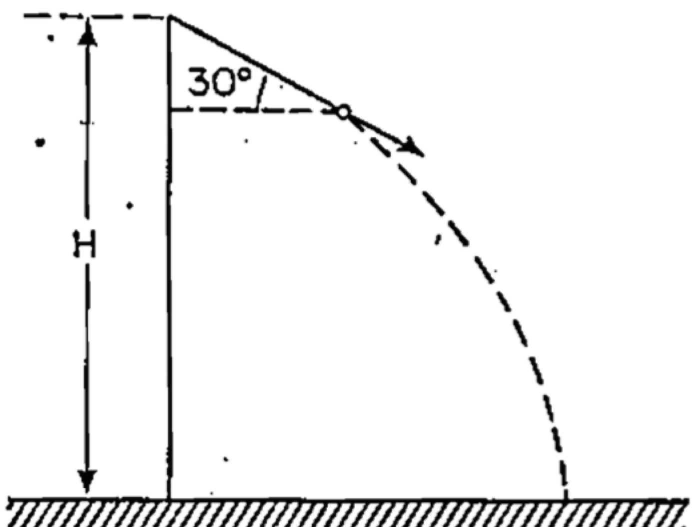


FIG. 43

81. Starting from a height H, a ball slips without friction, down a smooth plane inclined at an angle of 30° to the horizontal (Fig. 43). The length of the plane is $H/3$. The ball then falls on to a horizontal surface with an impact that may be taken as perfectly elastic. How high does the ball rise after striking the horizontal plane?

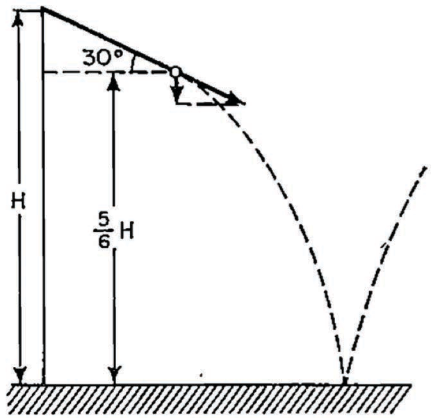


FIG. 192

81. The ball slips from the plane (see Fig. 192) with a velocity o. $\sqrt{gH/3}$ at an inclination of 30° to the horizontal. Then the ball describes a parabola and falls on to the horizontal plane with a velocity inclined at some unknown angle to the horizontal. But the height to which the ball will rise after an absolutely elastic impact on the plane depends on only the vertical component of this velocity. The value of this component can be found by calculating the speed with which the ball will fall from a height of $\frac{5}{6}H$ with an initial velocity of $\frac{1}{2}\sqrt{gH/3}$. From the equation

$$\frac{5}{6}H = \frac{1}{2}\sqrt{\frac{gH}{3}}\,t + \frac{gt^2}{2}$$

we find that the time of fall of the ball

$$t = \frac{\sqrt{21}-1}{2}\sqrt{\frac{H}{3g}}.$$

Therefore its velocity at the end of the fall will be

$$v = v_0 + gt = \frac{\sqrt{21}}{2}\sqrt{\frac{gH}{3}}.$$

Therefore the height to which the ball will rise after its elastic impact on the plane will equal

$$\frac{v^2}{2g} = \frac{7H}{8}.$$

Shaskol'Skaya & El'Tsin (1963)

82. A bullet of mass m hits a wooden block of mass M, which is suspended from a thread of length l (a ballistic pendulum), and is embedded in it. Find through what angle the block will swing if the bullet's velocity is v (Fig. 44).

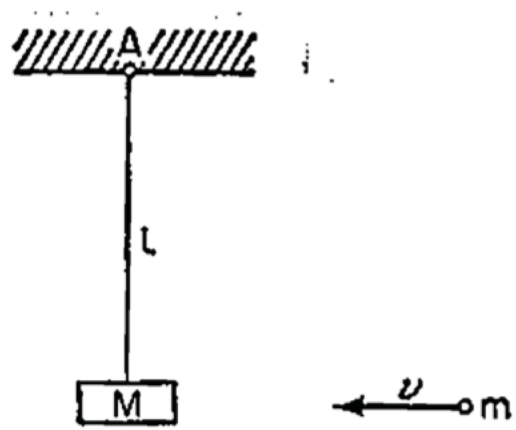


FIG. 44

82. A bullet of mass m, travelling with a velocity v, has momentum mv. After the bullet embeds itself in the block, the block plus the bullet will have exactly the same momentum (the impact is completely inelastic). Therefore the velocity v_1, which the block acquires immediately upon the bullet's hitting it, will be determined from the law of conservation of momentum: $mv = (M + m)v_1$. Also the kinetic energy of block and bullet will be

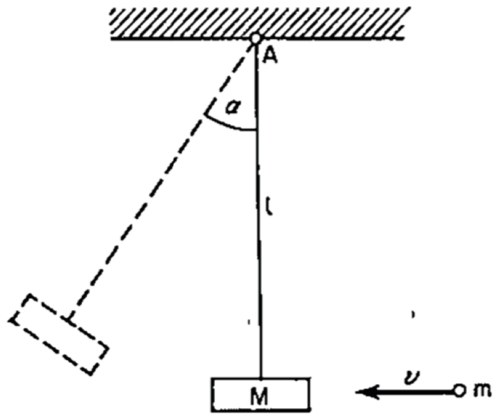


FIG. 193

$$\frac{(M+m)v_1^2}{2} = \frac{m}{M+m} \cdot \frac{mv^2}{2}.$$

Then the block will rise, and this kinetic energy will be changed into potential energy. Since the whole mass $(M + m)$ is virtually at a distance of l from the point of suspension A (Fig. 193), its centre of gravity will rise, in consequence of a swing through an angle α on the part of the pendulum, through height $\Delta h = l\,(1 - \cos\alpha)$. At the farthest point of its swing, through an angle α_0, the potential energy must equal the initial kinetic energy, i.e.

$$\frac{m}{M+M} \cdot \frac{mv^2}{2} = (M+m)gl\,(1-\cos\alpha_0).$$

Hence the angle through which the pendulum swings is given by the relationship

$$\sin^2\frac{\alpha_0}{2} = \frac{m^2v^2}{4(M+m)^2gl}.$$

Shaskol'Skaya & El'Tsin (1963)