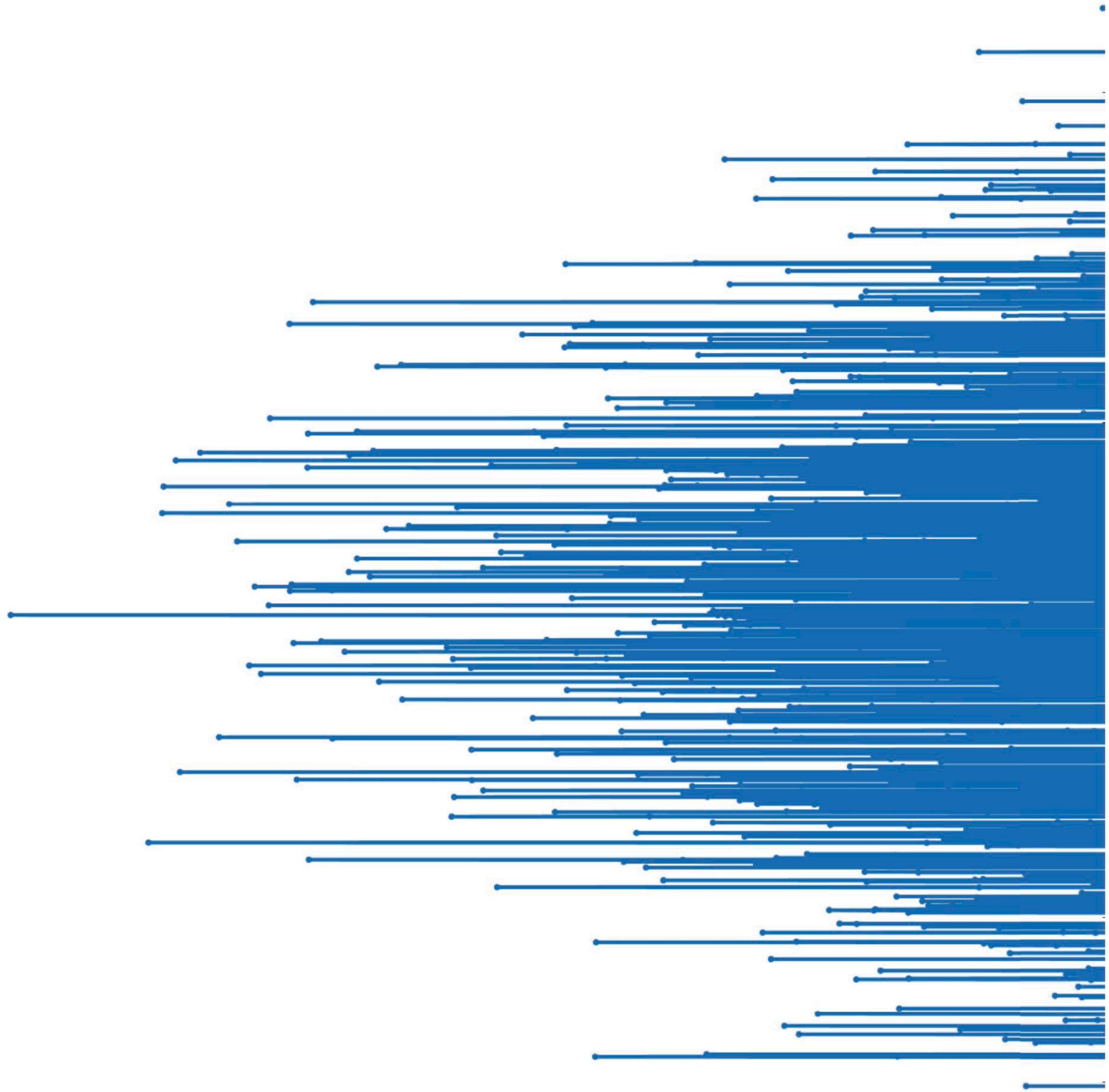
PHYS 2010 (W20)
Classical Mechanics



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Tutorial IV

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Ref. (re images):
Knudsen & Hjorth (2000), Kesten & Tauck (2012)

1. THE RETURNING EXPLORER

AN OLD RIDDLE runs as follows. An explorer walks one mile due south, turns and walks one mile due east, turns again and walks one mile due north. He finds himself back where he started. He shoots a bear. What color is the bear? The time-honored answer is: "White," because the explorer must have started at the North Pole. But not long ago someone made the discovery that the North Pole is not the only starting point that satisfies the given conditions! Can you think of any other spot on the globe from which one could walk a mile south, a mile east, a mile north and find himself back at his original location?

Problem

Calculate $\nabla \cdot \mathbf{r}$:

Here $\mathbf{r}$ is just the "coordinate vector" (i.e., $\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$)

SOL

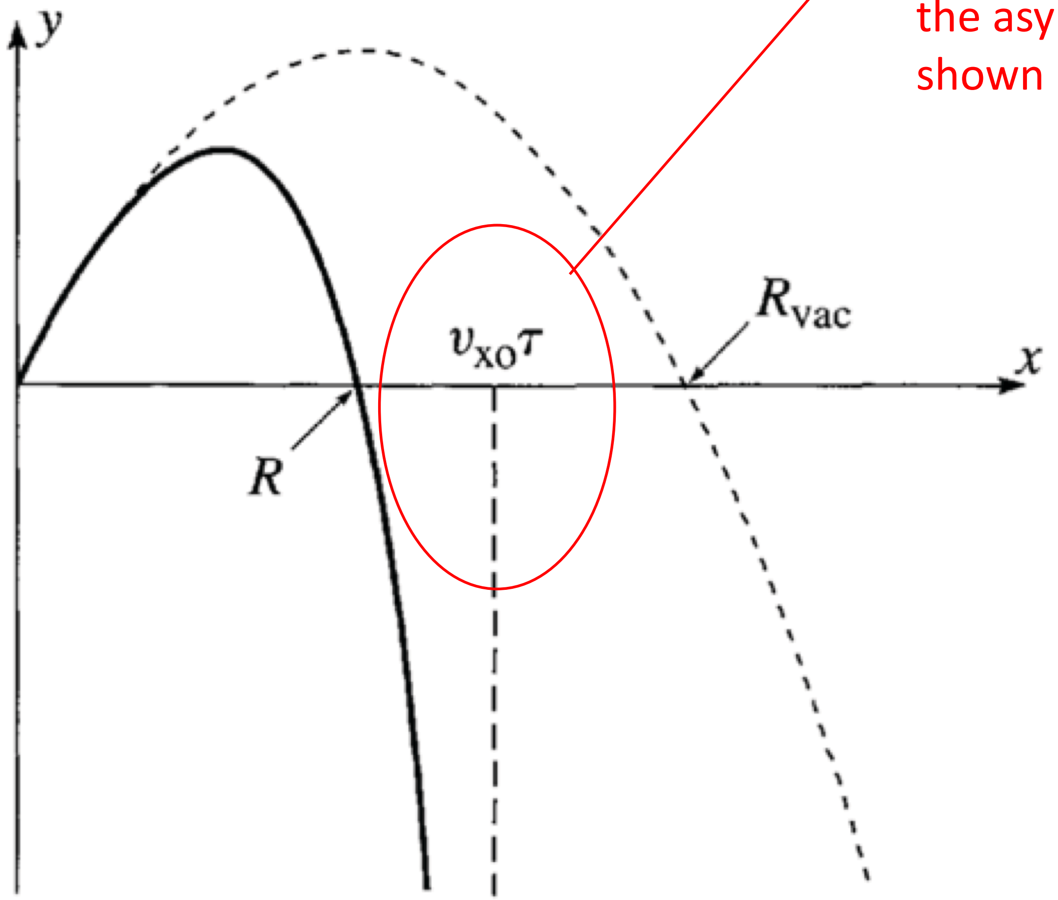
$$\nabla \cdot \mathbf{r} = \left(\hat{\mathbf{e}}_x \frac{\partial}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \right) \cdot (\hat{\mathbf{e}}_x x + \hat{\mathbf{e}}_y y + \hat{\mathbf{e}}_z z)$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z},$$

which reduces to $\nabla \cdot \mathbf{r} = 3$.

Arfken & Weber (2013)

Problem



For a projectile experiencing linear drag, determine an expression for the asymptotic horizontal limit as shown here.

SOL

From Fowles & Cassidy (2005)

$$m\frac{d^2\mathbf{r}}{dt^2} = -m\gamma\mathbf{v} - \mathbf{k}mg$$

$$\ddot{x} = -\gamma\dot{x} \qquad x = \frac{\dot{x}_0}{\gamma}(1-e^{-\gamma t})$$

$$\ddot{y} = -\gamma\dot{y}$$

$$\ddot{z} = -\gamma\dot{z} - g \qquad z = \left(\frac{\dot{z}_0}{\gamma} + \frac{g}{\gamma^2}\right)(1-e^{-\gamma t}) - \frac{g}{\gamma}t$$

From Taylor (2005)

$$m\dot{\mathbf{v}} = m\mathbf{g} - b\mathbf{v},$$

$$\tau = 1/k = m/b \qquad \text{[for linear drag].}$$

$$x(t) = v_{xo}\tau\left(1 - e^{-t/\tau}\right)$$

$$y(t) = (v_{yo} + v_{ter})\tau\left(1 - e^{-t/\tau}\right) - v_{ter}t.$$

→ So be careful re the units of the constant used in the drag force!

Problem

A particle moving in 1-D experiencing quadratic drag only can be described by the following ODE. Solve this via separation of variables to determine $v(t)$ and $x(t)$. Sketch them as well, noting any relevant time constants.

$$m\frac{dv}{dt} = -cv^2.$$

SOL

$$m\int_{v_o}^{v}\frac{dv'}{v'^2} = -c\int_0^t dt'$$

$$x(t) = x_o + \int_0^t v(t')\,dt'$$
$$= v_o\tau\ln(1+t/\tau),$$

$$m\left(\frac{1}{v_o} - \frac{1}{v}\right) = -ct$$

$$v(t) = \frac{v_o}{1 + cv_o t/m} = \frac{v_o}{1+t/\tau}$$

$$\tau = \frac{m}{cv_o} \qquad \text{[for quadratic drag]}$$

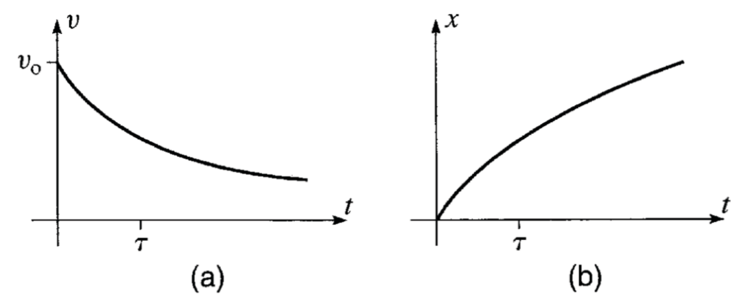


Figure 2.8 The motion of a body, such as a bicycle, coasting horizontally and subject to a quadratic air resistance. **(a)** The velocity is given by (2.49) and goes to zero like $1/t$ as $t \to \infty$. **(b)** The position is given by (2.51) and goes to infinity as $t \to \infty$.

Taylor (2005)

Problem

Find the scalar potential for the gravitational force on a unit mass m_1,

$$\mathbf{F}_G = -\frac{Gm_1m_2\hat{\mathbf{r}}}{r^2} = -\frac{k\hat{\mathbf{r}}}{r^2},$$

SOL

Need to integrate: $\mathbf{F} = -\nabla\varphi$

$$\varphi_G(r) - \varphi_G(\infty) = -\int_{\infty}^{\mathbf{r}} \mathbf{F}_G \cdot d\mathbf{r} = +\int_{\mathbf{r}}^{\infty} \mathbf{F}_G \cdot d\mathbf{r}.$$

By use of $\mathbf{F}_G = -\mathbf{F}_{\text{applied}}$, a comparison with Eq. (1.95a) shows that the potential is the work done in bringing the unit mass in from infinity. (We can define only potential difference. Here we arbitrarily assign infinity to be a zero of potential.) The integral on the right-hand side of Eq. (1.132) is negative, meaning that $\varphi_G(r)$ is negative. Since $\mathbf{F}_G$ is radial, we obtain a contribution to φ only when $d\mathbf{r}$ is radial, or

$$\varphi_G(r) = -\int_r^{\infty} \frac{k\,dr}{r^2} = -\frac{k}{r} = -\frac{Gm_1m_2}{r}.$$

The final negative sign is a consequence of the attractive force of gravity. ■

Arfken & Weber

Problem

A baseball is dropped from a tall tower. Assume only gravity and air resistance proportional to the balls velocity squared (i.e., quadratic drag) act on the ball. Provide a numerical estimate of the associated terminal velocity and make a rough time sketch of the associated timecourse of the speed.

SOL

$$\mathbf{f} = -f(v)\hat{\mathbf{v}}, \tag{2.1}$$

where $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$ denotes the unit vector in the direction of $\mathbf{v}$, and $f(v)$ is the magnitude of $\mathbf{f}$.

The function $f(v)$ that gives the magnitude of the air resistance varies with v in a complicated way, especially as the object's speed approaches the speed of sound. However, at lower speeds it is often a good approximation to write[1]

$$f(v) = bv + cv^2 = f_{\text{lin}} + f_{\text{quad}} \tag{2.2}$$

where f_{lin} and f_{quad} stand for the linear and quadratic terms respectively,

$$f_{\text{lin}} = bv \quad \text{and} \quad f_{\text{quad}} = cv^2. \tag{2.3}$$

The physical origins of these two terms are quite different: The linear term, f_{lin}, arises from the viscous drag of the medium and is generally proportional to the viscosity of the medium and the linear size of the projectile (Problem 2.2). The quadratic term, f_{quad}, arises from the projectile's having to accelerate the mass of air with which it is continually colliding; f_{quad} is proportional to the density of the medium and the cross-sectional area of the projectile (Problem 2.4). In particular, for a spherical projectile (a cannonball, a baseball, or a drop of rain), the coefficients b and c in (2.2) have the form

$$b = \beta D \quad \text{and} \quad c = \gamma D^2 \tag{2.4}$$

where D denotes the diameter of the sphere and the coefficients β and γ depend on the nature of the medium. For a spherical projectile in air at STP, they have the approximate values

$$\beta = 1.6 \times 10^{-4} \text{ N·s/m}^2 \tag{2.5}$$

and

$$\gamma = 0.25 \text{ N·s}^2\text{/m}^4. \tag{2.6}$$

Taylor (2005)

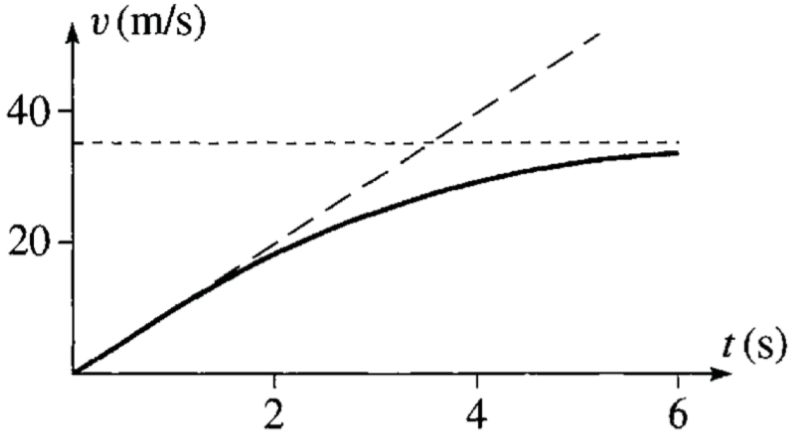
SOL

$$m\dot{v} = mg - cv^2$$

$$v_{\text{ter}} = \sqrt{\frac{mg}{c}}$$

$$c = \gamma D^2 \text{ where } \gamma = 0.25\ \text{N}\cdot\text{s}^2/\text{m}^4$$

$$v_{\text{ter}} = \sqrt{\frac{mg}{\gamma D^2}} = \sqrt{\frac{(0.15\,\text{kg}) \times (9.8\,\text{m/s}^2)}{(0.25\,\text{N}\cdot\text{s}^2/\text{m}^4) \times (0.07\,\text{m})^2}} = 35\ \text{m/s}$$



Taylor (2005)

SOL

$$\frac{f_{\text{quad}}}{f_{\text{lin}}} = \frac{cv^2}{bv} = \frac{\gamma D}{\beta} v = \left(1.6 \times 10^3 \, \frac{\text{s}}{\text{m}^2}\right) Dv$$

$$\frac{f_{\text{quad}}}{f_{\text{lin}}} \approx 600 \qquad \text{[baseball]}.$$

$$\frac{f_{\text{quad}}}{f_{\text{lin}}} \approx 1 \qquad \text{[raindrop]}.$$

$$\frac{f_{\text{quad}}}{f_{\text{lin}}} \approx 10^{-7} \qquad \text{[Millikan oildrop]}.$$

Taylor (2005)

Problem

Calculate the scalar potential for the **centrifugal** force per unit mass, $\mathbf{F}_C = \omega^2 r \hat{\mathbf{r}}$, radially **outward**. Physically, you might feel this on a large horizontal spinning disk at an amusement park.

SOL

integrating from the origin outward and taking $\varphi_C(0) = 0$, we have

$$\varphi_C(r) = -\int_0^r \mathbf{F}_C \cdot d\mathbf{r} = -\frac{\omega^2 r^2}{2}.$$

If we reverse signs, taking $\mathbf{F}_{SHO} = -k\mathbf{r}$, we obtain $\varphi_{SHO} = \frac{1}{2}kr^2$, the simple harmonic oscillator potential.

The gravitational, centrifugal, and simple harmonic oscillator potentials are shown in Fig. 1.34. Clearly, the simple harmonic oscillator yields stability and describes a restoring force. The centrifugal potential describes an unstable situation. ■

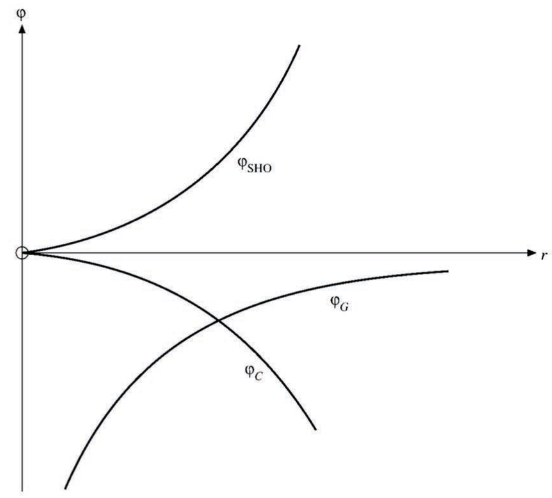


FIGURE 1.34 Potential energy versus distance (gravitational, centrifugal, and simple harmonic oscillator).

Arfken & Weber