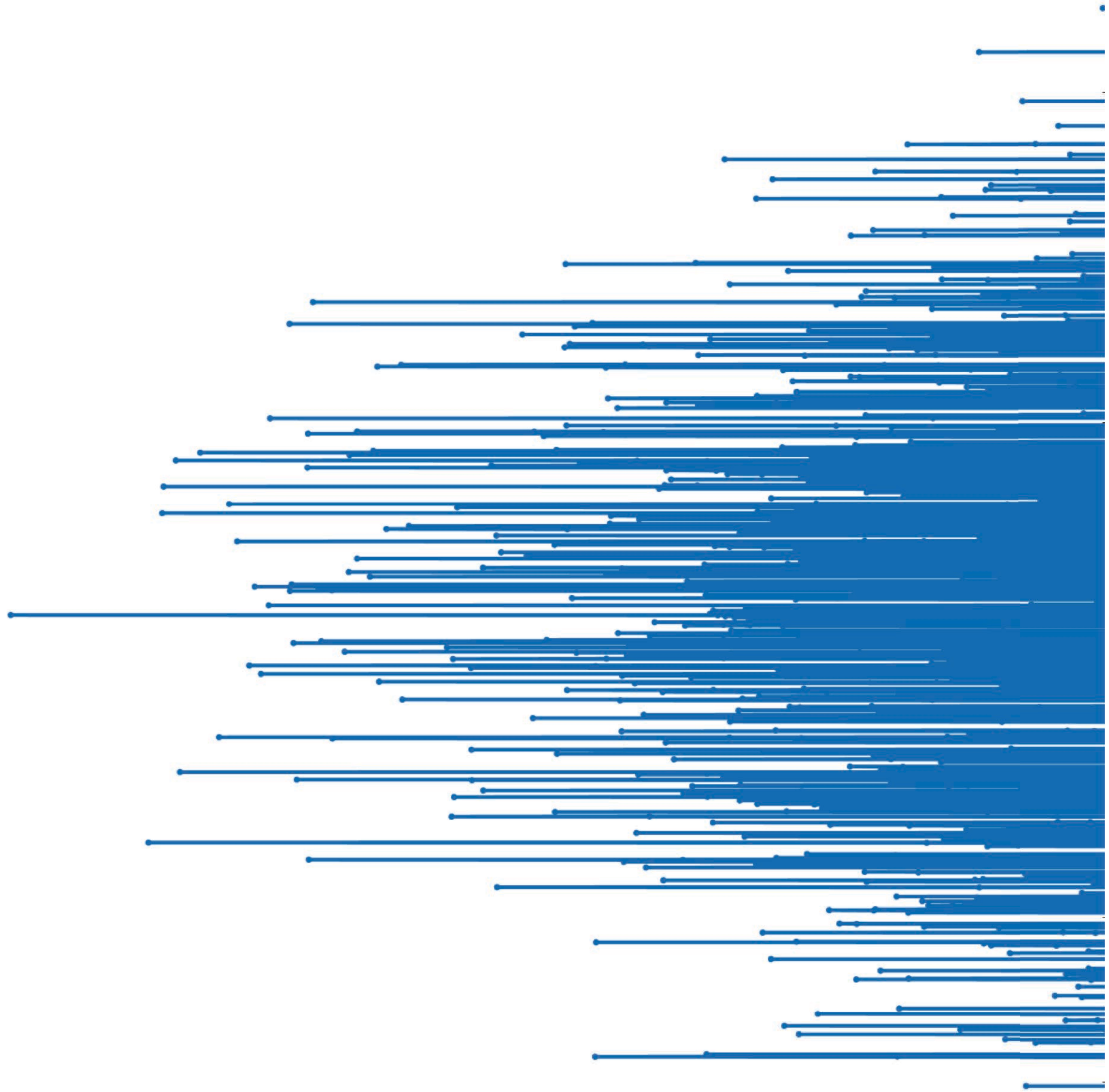
PHYS 2010 (W20)
Classical Mechanics



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Tutorial V

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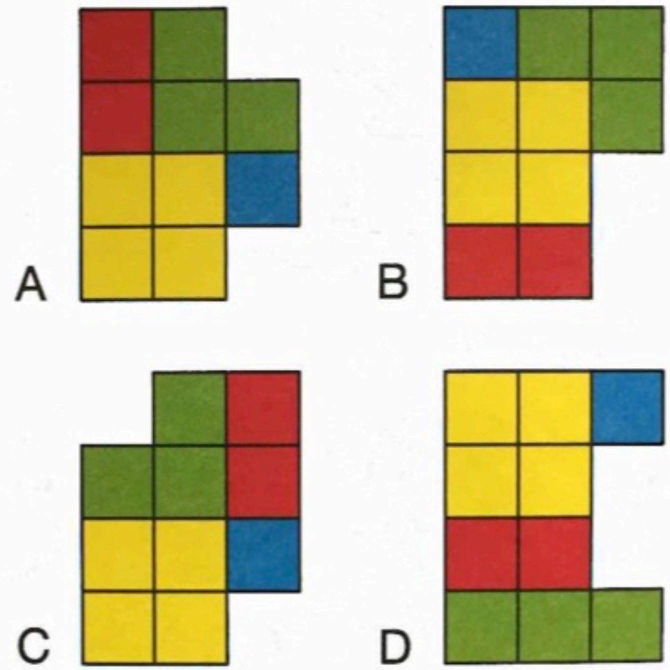
Ref. (re images):
Knudsen & Hjorth (2000), Kesten & Tauck (2012)

16. One-to-Four



Which pattern does not belong?

A B C D

Ex.

A load of mass m lies on a perfectly smooth plane, being pulled in opposite directions by springs 1 and 2, whose coefficients of elasticity are k_1 and k_2 respectively (Fig. 60). If the load be forced out of its state of equilibrium (by being drawn aside), it will begin to oscillate with period T. Will the period of oscillation be altered if the same springs be fastened not at points A_1 and A_2, but at B_1 and B_2? Assume that the springs are subject to Hooke's law for all strains.

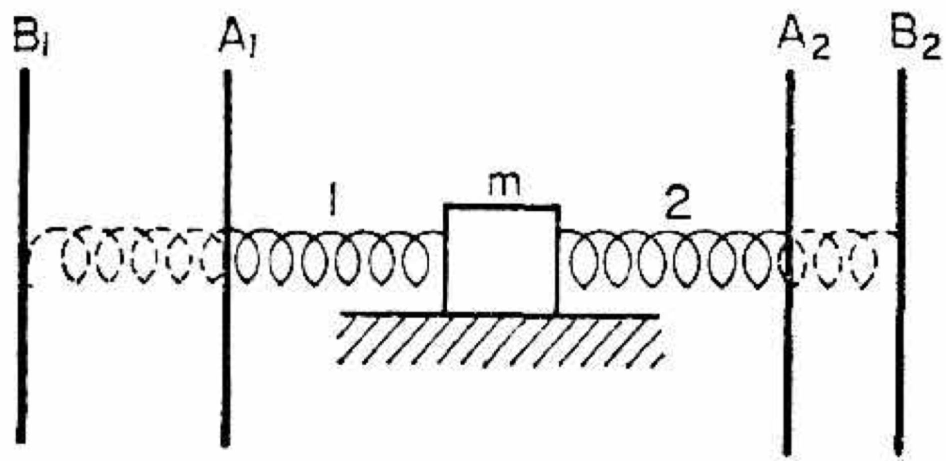


Fig. 60

Shaskol'Skaya & El'Tsin (1963)

Ex. SOL

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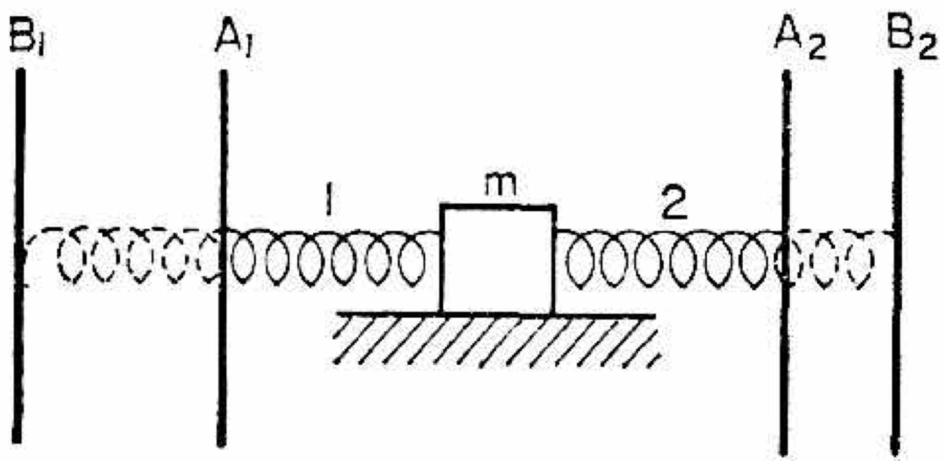


FIG. 60

No

Shaskol'Skaya & El'Tsin (1963)

Problem

(a) If $z = Ae^{j\theta}$, deduce that $dz = jz\, d\theta$, and explain the meaning of this relation in a vector diagram.

(b) Find the magnitudes and directions of the vectors $(2 + j\sqrt{3})$ and $(2 - j\sqrt{3})^2$.

French (1971)

Problem

Find the equivalent electrical circuit for the hanging mass–spring shown in Figure 3-17a and determine the time dependence of the charge q in the system.

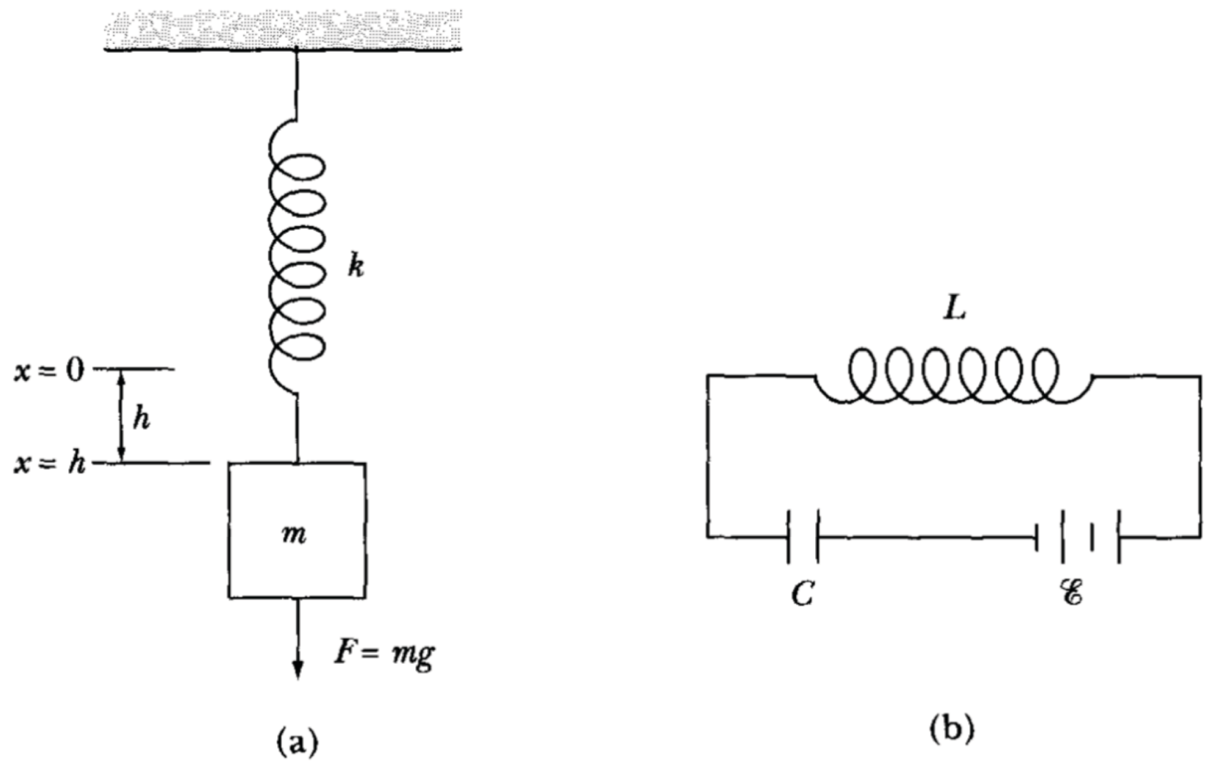
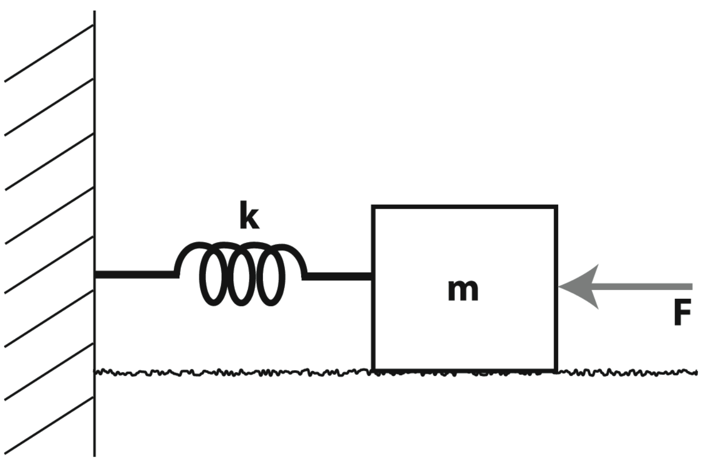


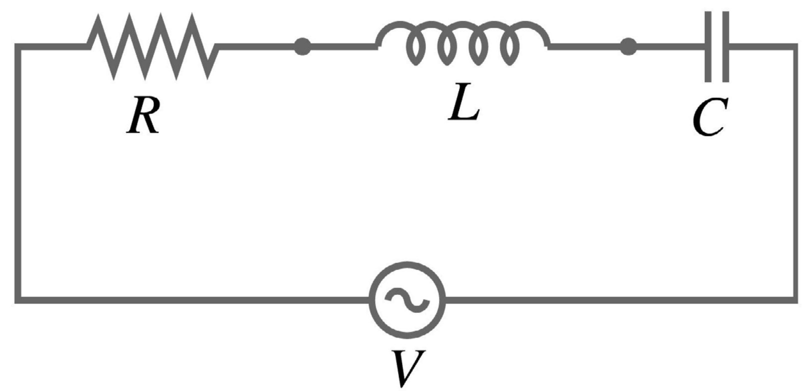
FIGURE 3-17 Example 3.4 (a) hanging mass–spring system; (b) equivalent electrical circuit.

RLC circuit = Damped, Driven Harmonic Oscillator

Mechanical		Electrical
F (force)	←→	V (potential)
v (velocity)	←→	I (current)
x (position)	←→	q (charge)
m (mass)	←→	L (inductance)
b (damping)	←→	R (resistance)
k (spring)	←→	$1/C$ (capacitance)

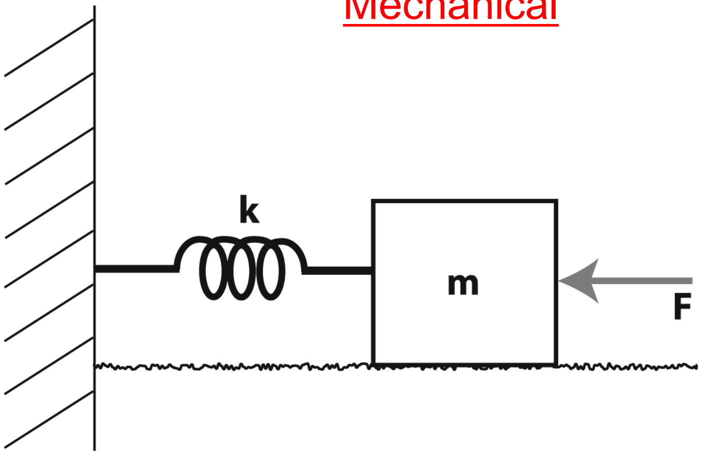
state variables (V, I, q)



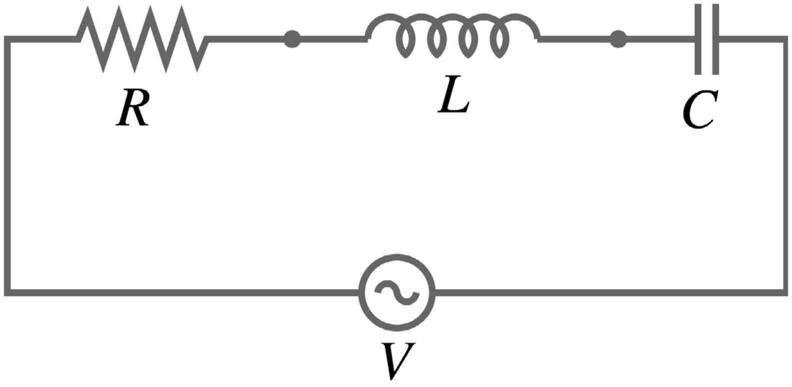


RLC circuit = Damped, Driven Harmonic Oscillator

Mechanical



Electrical



$$m\ddot{x} + b\dot{x} + kx = F_o e^{i\omega t}$$

$$L\ddot{q} + R\dot{q} + \frac{q}{C} = V_o e^{i\omega t}$$

SOL

Solution. Let us first consider the analogous quantities in mechanical and electrical systems. The force F ($= mg$ in the mechanical case) is analogous to the emf $\mathcal{E}$. The damping parameter b has the electrical analog resistance R, which is not present in this case. The displacement x has the electrical analog charge q. We show other quantities in Table 3-1. If we examine Figure 3-17a, we have $1/k \to C$, $m \to L$, $F \to \mathcal{E}$, $x \to q$, and $\dot{x} \to I$. Without the weight of the mass, the equilibrium position would be at $x = 0$; the addition of the gravitational force extends the spring by an amount $h = mg/k$ and displaces the equilibrium position to $x = h$. The equation of motion becomes

$$m\ddot{x} + k(x - h) = 0 \qquad \textbf{(3.73)}$$

or

$$m\ddot{x} + kx = kh$$

with solution

$$x(t) = h + A\cos\omega_0 t \qquad \textbf{(3.74)}$$

where we have chosen the initial conditions $x(t = 0) = h + A$ and $\dot{x}(t = 0) = 0$.

We draw the equivalent electrical circuit in Figure 3-17b. Kirchoff's equation around the circuit becomes

$$L\frac{dI}{dt} + \frac{1}{C}\int I\,dt = \mathcal{E} = \frac{q_1}{C} \qquad \textbf{(3.75)}$$

Thornton & Marion (2004)

SOL

TABLE 3-1 Analogous Mechanical and Electrical Quantities

Mechanical		Electrical	
x	Displacement	q	Charge
$\dot{x}$	Velocity	$\dot{q} = I$	Current
m	Mass	L	Inductance
b	Damping resistance	R	Resistance
$1/k$	Mechanical compliance	C	Capacitance
F	Amplitude of impressed force	$\mathcal{E}$	Amplitude of impressed emf

where q_1 represents the charge that must be applied to C to produce a voltage $\mathcal{E}$. If we use $I = \dot{q}$, we have

$$L\ddot{q} + \frac{q}{C} = \frac{q_1}{C} \tag{3.76}$$

If $q = q_0$ and $I = 0$ at $t = 0$, the solution is

$$q(t) = q_1 + (q_0 - q_1)\cos\omega_0 t \tag{3.77}$$

which is the exact electrical analog of Equation 3.74.

Thornton & Marion (2004)

Problem

Consider the series RLC circuit shown in Figure 3-18 driven by an alternating emf of value $E_0 \sin \omega t$. Find the current, the voltage V_L across the inductor, and the angular frequency ω at which V_L is a maximum.

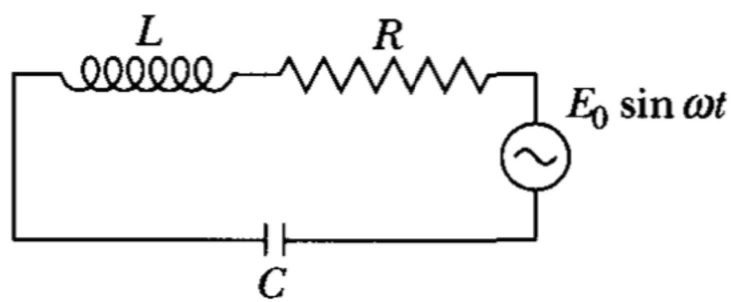


FIGURE 3-18 Example 3.5. RLC circuit with an alternating emf.

SOL

Solution. The voltage across each of the circuit elements in Figure 3-18 are

$$V_L = L\frac{dI}{dt} = L\ddot{q}$$

$$V_R = LI = L\frac{dq}{dt} = L\dot{q}$$

$$V_C = \frac{q}{C}$$

so the voltage drops around the circuit become

$$L\ddot{q} + R\dot{q} + \frac{q}{C} = E_0 \sin \omega t$$

Thornton & Marion (2004)

SOL

We identify this equation as similar to Equation 3.53, which we have already solved. In addition to the relationships in Table 3-1, we also have $\beta = b/2m \rightarrow R/2L$, $\omega_0 = \sqrt{k/m} \rightarrow 1/\sqrt{LC}$, and $A = F_0/m \rightarrow E_0/L$. The solution for the charge q is given by transcribing Equation 3.60, and the equation for the current I is given by transcribing Equation 3.66, which allows us to write

$$I = \frac{-E_0}{\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}} \sin(\omega t - \delta)$$

where δ can be found by transcribing Equation 3.61.

The voltage across the inductor is found from the time derivative of the current.

$$V_L = L\frac{dI}{dt} = \frac{-\omega L E_0}{\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}} \cos(\omega t - \delta)$$
$$= V(\omega)\cos(\omega t - \delta)$$

To find the driving frequency ω_{max}, which makes V_L a maximum, we must take the derivative of V_L with respect to ω and set the result equal to zero. We only need to consider the amplitude $V(\omega)$ and not the time dependence.

$$\frac{dV(\omega)}{d\omega} = \frac{LE_0\left(R^2 - \frac{2L}{C} + \frac{2}{\omega^2 C^2}\right)}{\left[R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2\right]^{3/2}}$$

We have skipped a few intermediate steps to arrive at this result. We determine the value ω_{max} sought by setting the term in parentheses in the numerator equal to zero. By doing so and solving for ω_{max} gives

$$\omega_{max} = \frac{1}{\sqrt{LC - \frac{R^2 C^2}{2}}}$$

which is the result we need. Note the difference between this frequency and those given by the natural frequency, $\omega_0 = 1/\sqrt{LC}$, and the charge resonance frequency (given by transcribing Equation 3.63), $\omega_R = \sqrt{1/LC - 2R^2/L^2}$.

Thornton & Marion (2004)

Problem

A simple harmonic oscillator consists of a 100-g mass attached to a spring whose force constant is 10^4 dyne/cm. The mass is displaced 3 cm and released from rest. Calculate **(a)** the natural frequency ν_0 and the period τ_0, **(b)** the total energy, and **(c)** the maximum speed.

SOL

a) $$\nu_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{10^4\ \text{dyne/cm}}{10^2\ \text{gram}}} = \frac{10}{2\pi}\sqrt{\frac{\frac{\text{gram}\cdot\text{cm}}{\text{sec}^2\cdot\text{cm}}}{\text{gram}}} = \frac{10}{2\pi}\ \text{sec}^{-1}$$

or,

$$\boxed{\nu_0 \cong 1.6\ \text{Hz}}$$

$$\tau_0 = \frac{1}{\nu_0} = \frac{2\pi}{10}\ \text{sec}$$

or,

$$\boxed{\tau_0 \cong 0.63\ \text{sec}}$$

b) $$E = \frac{1}{2}kA^2 = \frac{1}{2}\times 10^4 \times 3^2\ \text{dyne-cm}$$

so that

$$\boxed{E = 4.5\times 10^4\ \text{erg}}$$

c) The maximum velocity is attained when the total energy of the oscillator is equal to the kinetic energy. Therefore,

$$\frac{1}{2}mv_{\text{max}}^2 = 4.5\times 10^4\ \text{erg}$$

$$\text{v}_{\text{max}} = \sqrt{\frac{2\times 4.5\times 10^4}{100}}$$

or,

$$\boxed{v_{\text{max}} = 30\ \text{cm/sec}}$$

Thornton & Marion (2004)

Problem

Allow the motion in the preceding problem to take place in a resisting medium. After oscillating for 10 s, the maximum amplitude decreases to half the initial value. Calculate **(a)** the damping parameter β, **(b)** the frequency ν_1 (compare with the undamped frequency ν_0), and **(c)** the decrement of the motion.

SOL

a) The statement that at a certain time $t = t_1$ the maximum amplitude has decreased to one-half the initial value means that

$$|x_{en}| = A_0 e^{-\beta t_1} = \frac{1}{2} A_0 \tag{1}$$

or,

$$e^{-\beta t_1} = \frac{1}{2} \tag{2}$$

so that

$$\beta = \frac{\ln 2}{t_1} = \frac{0.69}{t_1} \tag{3}$$

Since $t_1 = 10 \text{ sec}$,

$$\boxed{\beta = 6.9 \times 10^{-2} \text{ sec}^{-1}} \tag{4}$$

b) According to Eq. (3.38), the angular frequency is

$$\omega_1 = \sqrt{\omega_0^2 - \beta^2} \tag{5}$$

where, from Problem 3-1, $\omega_0 = 10 \text{ sec}^{-1}$. Therefore,

$$\begin{aligned} \omega_1 &= \sqrt{(10)^2 - \left(6.9 \times 10^{-2}\right)^2} \\ &\cong 10\left[1 - \frac{1}{2}(6.9)^2 \times 10^{-6}\right] \text{ sec}^{-1} \end{aligned} \tag{6}$$

so that

$$\boxed{\nu_1 = \frac{10}{2\pi}(1 - 2.40 \times 10^{-5}) \text{ sec}^{-1}} \tag{7}$$

which can be written as

$$\nu_1 = \nu_0 (1 - \delta) \tag{8}$$

where

$$\delta = 2.40 \times 10^{-5} \tag{9}$$

That is, ν_1 is only slightly different from ν_0.

c) The *decrement* of the motion is defined to be $e^{\beta\tau_1}$ where $\tau_1 = 1/\nu_1$. Then,

$$\boxed{e^{\beta\tau_1} \simeq 1.0445}$$

Thornton & Marion (2004)

Problem

Consider a simple harmonic oscillator. Calculate the *time* averages of the kinetic and potential energies over one cycle, and show that these quantities are equal. Why is this a reasonable result? Next calculate the *space* averages of the kinetic and potential energies. Discuss the results.

Hint: Recall that

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

SOL

a) *Time average:*

The position and velocity for a simple harmonic oscillator are given by

$$x = A \sin \omega_0 t \tag{1}$$

$$\dot{x} = \omega_0 \, A \cos \omega_0 t \tag{2}$$

where $\omega_0 = \sqrt{k/m}$

The time average of the kinetic energy is

$$\langle T \rangle = \frac{1}{\tau} \int_t^{t+\tau} \frac{1}{2} m\dot{x}^2 \, dt \tag{3}$$

where $\tau = \dfrac{2\pi}{\omega_0}$ is the period of oscillation.

By inserting (2) into (3), we obtain

$$\langle T \rangle = \frac{1}{2\tau} mA^2 \omega_0^2 \int_t^{t+\tau} \cos^2 \omega_0 t \, dt \tag{4}$$

or,

$$\boxed{\langle T \rangle = \frac{mA^2 \omega_0^2}{4}} \tag{5}$$

In the same way, the time average of the potential energy is

$$\langle U \rangle = \frac{1}{\tau} \int_t^{t+\tau} \frac{1}{2} kx^2 \, dt$$

$$= \frac{1}{2\tau} kA^2 \int_t^{t+\tau} \sin^2 \omega_0 t \, dt$$

$$= \frac{kA^2}{4} \tag{6}$$

Thornton & Marion (2004)

SOL

and since $\omega_0^2 = k/m$, (6) reduces to

$$\boxed{\langle U \rangle = \frac{mA^2\omega_0^2}{4}} \tag{7}$$

From (5) and (7) we see that

$$\boxed{\langle T \rangle = \langle U \rangle} \tag{8}$$

The result stated in (8) is reasonable to expect from the conservation of the total energy.

$$E = T + U \tag{9}$$

This equality is valid instantaneously, as well as in the average. On the other hand, when T and U are expressed by (1) and (2), we notice that they are described by exactly the same function, displaced by a time $\tau/2$:

$$\left.\begin{aligned} T &= \frac{mA^2\omega_0^2}{2}\cos^2 \omega_0 t \\ U &= \frac{mA^2\omega_0 t}{2}\sin^2 \omega_0 t \end{aligned}\right] \tag{10}$$

Therefore, the time averages of T and U must be equal. Then, by taking time average of (9), we find

$$\langle T \rangle = \langle U \rangle = \frac{E}{2} \tag{11}$$

Thornton & Marion (2004)

SOL

b) *Space average:*

The space averages of the kinetic and potential energies are

$$\bar{T} = \frac{1}{A}\int_0^A \frac{1}{2} m\dot{x}^2 \, dx \tag{12}$$

and

$$\bar{U} = \frac{1}{A}\int_0^A \frac{1}{2} kx^2 \, dx = \frac{m\omega_0^2}{2A}\int_0^A x^2 \, dx \tag{13}$$

(13) is readily integrated to give

$$\boxed{\bar{U} = \frac{m\omega_0^2 A^2}{6}} \tag{14}$$

To integrate (12), we notice that from (1) and (2) we can write

$$\dot{x}^2 = \omega_0^2 A^2 \cos^2 \omega_0 t = \omega_0^2 A^2 \left(1 - \sin^2 \omega_0 t\right)$$

$$= \omega_0^2 \left(A^2 - x^2\right) \tag{15}$$

Thornton & Marion (2004)

SOL

Then, substituting (15) into (12), we find

$$\bar{T} = \frac{m\omega_0^2}{2A}\int_0^A \left[A^2 - x^2\right] dx$$

$$= \frac{m\omega_0^2}{2A}\left[A^3 - \frac{A^3}{3}\right] \tag{16}$$

or,

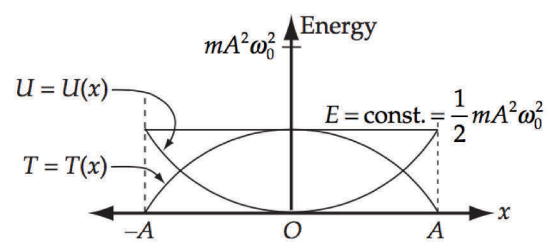
$$\boxed{\bar{T} = 2\,\frac{m\omega_0^2 A^2}{6}} \tag{17}$$

From the comparison of (14) and (17), we see that

$$\boxed{\bar{T} = 2\bar{U}} \tag{18}$$

To see that this result is reasonable, we plot $T = T(x)$ and $U = U(x)$:

$$\left.\begin{aligned} T &= \frac{1}{2}m\omega_0^2 A^2\left[1 - \frac{x^2}{A^2}\right] \\ U &= \frac{1}{2}m\omega_0^2 x^2 \end{aligned}\right] \tag{19}$$



And the area between $T(x)$ and the x-axis is just twice that between $U(x)$ and the x-axis.

Thornton & Marion (2004)