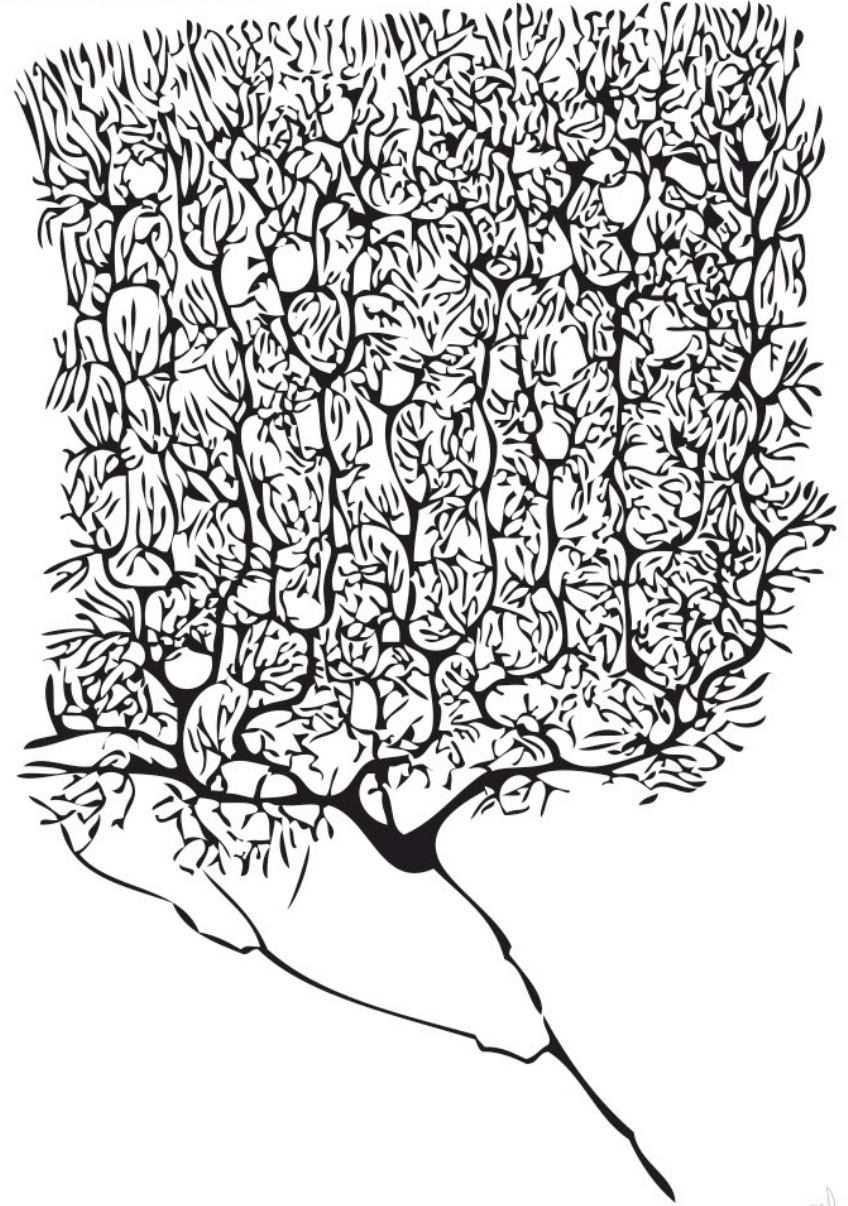


# Cellular Electrodynamics

Santiago Ramón y Cajal (1852-1934)



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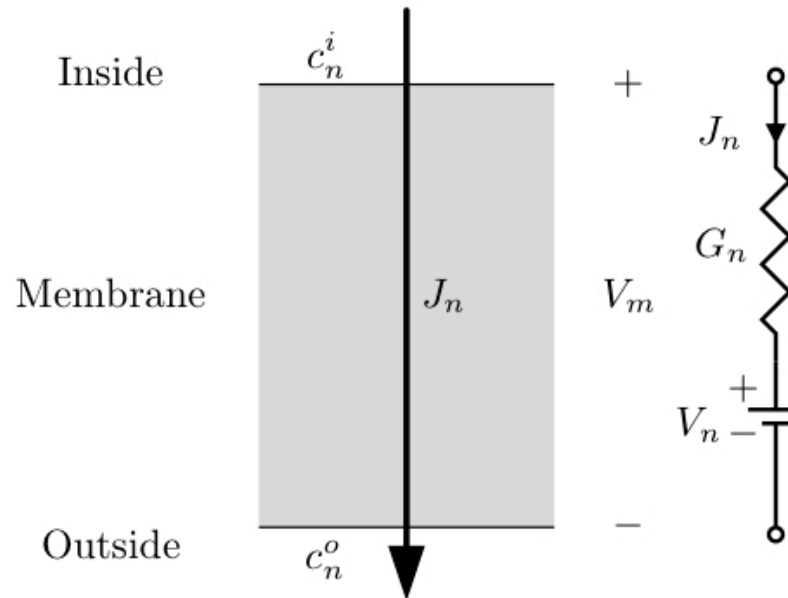
BPHS 4080 Lecture 8

Reference/Acknowledgement:

- TF Weiss (Cellular Biophysics)  
- D Freeman

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# Model of Steady-State Electrodiffusion through Membranes



→ Now we will consider the effect of solutes having charge

## Equations of Electrodifusion

### Nernst-Plank Equation

$$J_n(x, t) = -z_n F D_n \frac{\partial c_n(x, t)}{\partial x} - u_n z_n^2 F^2 c_n(x, t) \frac{\partial \psi(x, t)}{\partial x}$$

### Continuity

$$\frac{\partial J_n(x, t)}{\partial x} = -z_n F \frac{\partial c_n(x, t)}{\partial t}$$

### Poisson's Equation

$$\frac{\partial^2 \psi(x, t)}{\partial x^2} = -\frac{1}{\epsilon} \sum_n z_n F c_n(x, t)$$

## Some new variables

$Z_n$  - charge # (or “valence charge”)  
(e.g., +1, -1, +2, 0, etc...) [re  $1 e = 1.602 \times 10^{-19} \text{ C}$ ]

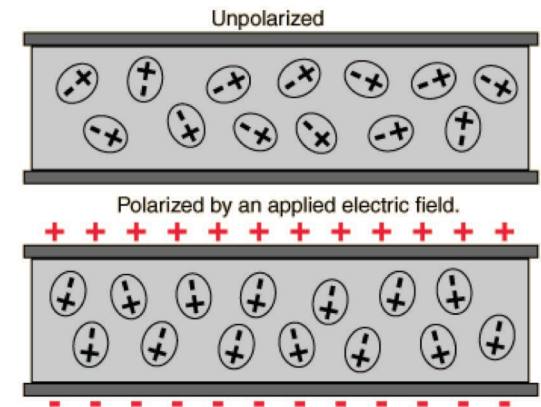
$F$  - Faraday's constant [ $9.65 \times 10^4 \text{ C/mol}$ ]

$J_n$  - current density [ $\text{A/cm}^2$ ]

$\psi$  - electrical potential [V]

$\epsilon$  - permittivity [F/m]

$u_n$  - mechanical mobility [s/kg]



<http://en.wikipedia.org/wiki/Permittivity>

from Einstein  
relation

## Mobility & Stokes-Einstein Relation

$u_n$  - mechanical mobility [s/kg]

from Einstein  
relation

➤ Force ( $f_p$ ) required to move a sphere of radius  $a$  through a viscous medium of viscosity  $\eta$  with a velocity of  $v$  is

$$f_p = 6\pi a\eta v$$

Stoke's Law  
(eqn.3.22)

➤ Particle mobility,  $u_p$ , is defined as the ratio of the particle velocity to the force on the particle

$$u_p \equiv \frac{v}{f_p} = \frac{1}{6\pi a\eta}$$

Similar to  
(reciprocal of)  
impedance

➤ Relating to the diffusion constant (Annus Mirabilis):

$$D = u_p kT = u N_A kT = uRT$$

$$D_n = u_n RT$$

1.  $u_n$  is the molar mechanical mobility of ion  $n$ . In some fields (e.g., solid-state physics), it is customary to use the *molar electrical mobility*,  $\hat{u}_n$ , where  $\hat{u}_n = |z_n|F u_n$ .  $\hat{u}_n$  has units of (cm/s)/(V/cm). In terms of the molar electrical mobility, the Einstein relation is  $D_n = (RT\hat{u}_n)/(|z_n|F)$ .

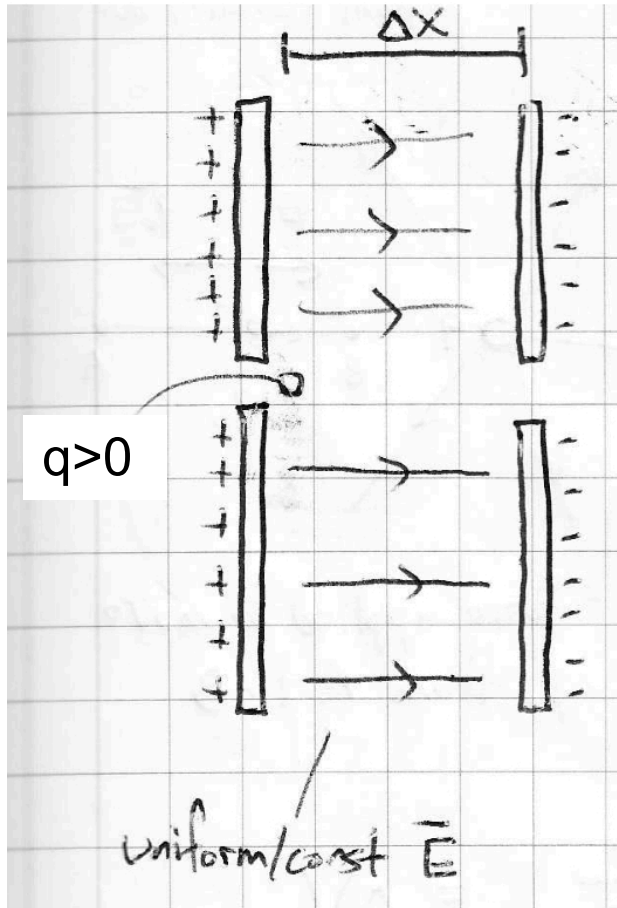
## Nernst-Plank Equation → Electrodiffusion

current  
density

$$J_n(x, t) = \underbrace{-z_n F D_n \frac{\partial c_n(x, t)}{\partial x}}_{\text{diffusion}} - \underbrace{u_n z_n^2 F^2 c_n(x, t) \frac{\partial \psi(x, t)}{\partial x}}_{\text{electric drift}}$$

→ Essentially a charged version of Fick's first law, but now with an additional term due to electric forces (the *drift* term on the right)

## Electric Drift



→ Consider a charge  $q$  placed between two uniformly/oppositely charged plates

- uniform  $E$  field between
- force exerted on charge (Coulomb's law)

$$\mathbf{F} = q\mathbf{E}$$

- $E$  depends upon spatial gradient of the potential

$$E = -\frac{\partial\psi}{\partial x}$$

Think in terms of energy (e.g., where does it come from? conserved?)

$$J_n(x, t) = -z_n F D_n \frac{\partial c_n(x, t)}{\partial x} - u_n z_n^2 F^2 c_n(x, t) \frac{\partial \psi(x, t)}{\partial x}$$

## Continuity Equation

$$\frac{\partial J_n(x, t)}{\partial x} = -z_n F \frac{\partial c_n(x, t)}{\partial t}$$

spatial change in  
current density

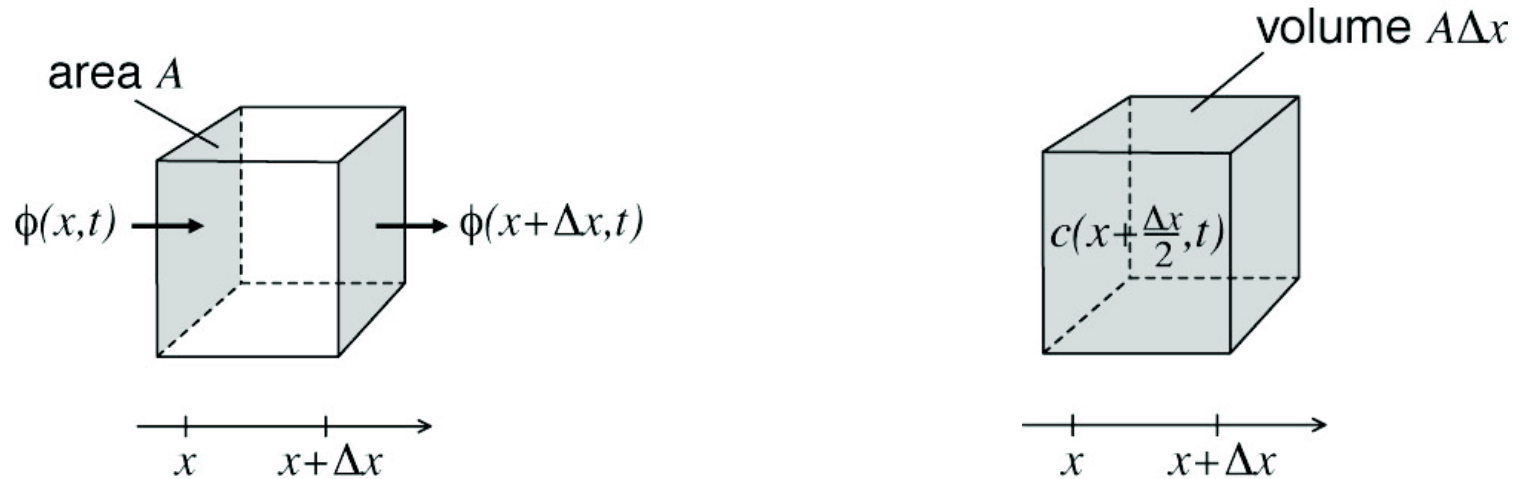
temporal change in  
charge density

→ Just like our derivation for diffusion, this essentially tells us about the conservation of charge



## Review: Continuity Equation (re diffusion)

⇒ imagine a cube (with face area  $A$  and length  $\Delta x$ ) and a time interval  $\Delta t$



solute entering from left - solute exiting from right  
(during time interval  $[t, t + \Delta t]$  )

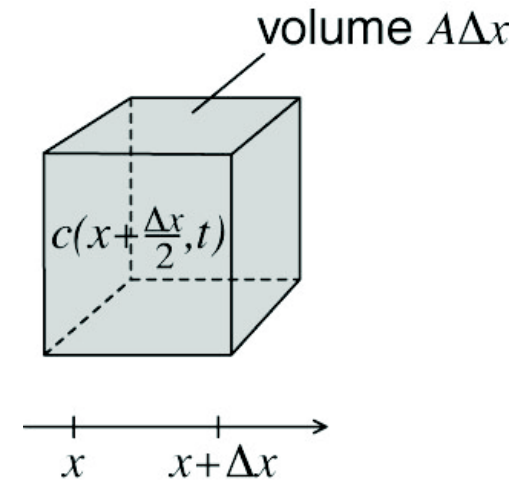
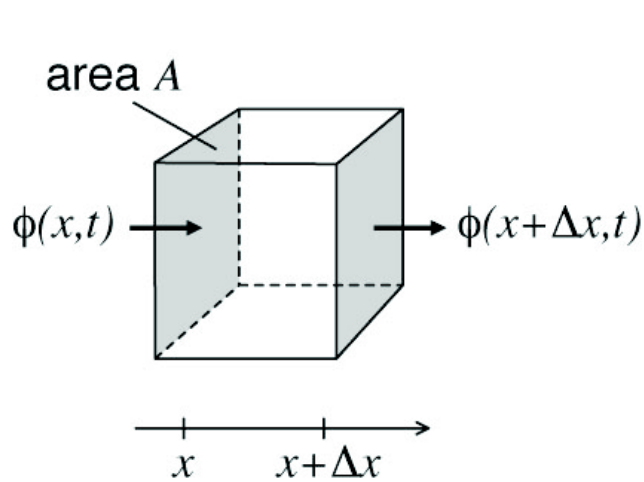
=

change in amount of solute inside cube  
(during time interval  $[t, t + \Delta t]$  )

$$A \Delta t \phi(x, t)$$

$$A \Delta x c(x, t)$$

## Review: Continuity Equation (re diffusion)



solute entering from left - solute exiting from right  
(during time interval  $[t, t + \Delta t]$ )

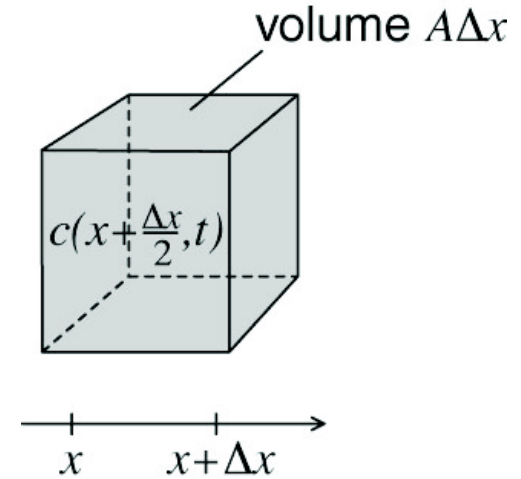
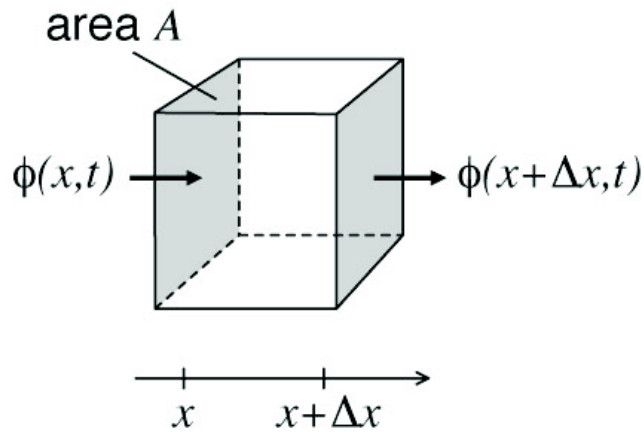
= change in amount of solute inside cube  
(during time interval  $[t, t + \Delta t]$ )

$$A \Delta t \phi(x, t + \Delta t/2) - A \Delta t \phi(x + \Delta x, t + \Delta t/2) = A \Delta x c(x + \Delta x/2, t + \Delta t) - A \Delta x c(x + \Delta x/2, t)$$

$$\frac{\phi(x + \Delta x, t + \Delta t/2) - \phi(x, t + \Delta t/2)}{\Delta x} = \frac{c(x + \Delta x/2, t + \Delta t) - c(x + \Delta x/2, t)}{\Delta t}$$

$$\implies \frac{\partial \phi}{\partial x} = - \frac{\partial c}{\partial t}$$

## Review: Continuity Equation



$$\implies \frac{\partial \phi}{\partial x} = -\frac{\partial c}{\partial t}$$

$$\frac{\partial J_n(x, t)}{\partial x} = -z_n F \frac{\partial c_n(x, t)}{\partial t}$$

Relationship between current density and flux:

$$J_n(x, t) = z_n F \phi_n(x, t)$$

## Poisson's Equation

$$\frac{\partial^2 \psi(x, t)}{\partial x^2} = -\frac{1}{\epsilon} \sum_n z_n F c_n(x, t)$$

→ Stemming from Gauss' Law, relates the charge density and electric potential

charge density [C/m<sup>3</sup>]

$$\rho = \sum_n z_n F c_n(x, t)$$

## Electrostatics [edit]

*Main article: [Electrostatics](#)*

One of the cornerstones of [electrostatics](#) is setting up and solving problems described by the Poisson equation. Solving the Poisson equation amounts to finding the [electric potential](#)  $\phi$  for a given [charge](#) distribution  $\rho_f$ .

The mathematical details behind Poisson's equation in electrostatics are as follows ([SI](#) units are used rather than [Gaussian units](#), which are also frequently used in [electromagnetism](#)).

Starting with [Gauss's law](#) for electricity (also one of [Maxwell's equations](#)) in differential form, we have:

$$\nabla \cdot \mathbf{D} = \rho_f$$

where  $\nabla \cdot$  is the [divergence operator](#),  $\mathbf{D}$  = [electric displacement field](#), and  $\rho_f$  = [free charge density](#) (describing charges brought from outside). Assuming the medium is linear, isotropic, and homogeneous (see [polarization density](#)), we have the [constitutive equation](#):

$$\mathbf{D} = \varepsilon \mathbf{E}$$

where  $\varepsilon$  = [permittivity](#) of the medium and  $\mathbf{E}$  = [electric field](#). Substituting this into Gauss's law and assuming  $\varepsilon$  is spatially constant in the region of interest obtains:

$$\nabla \cdot \mathbf{E} = \frac{\rho_f}{\varepsilon}$$

In the absence of a changing magnetic field,  $\mathbf{B}$ , [Faraday's law of induction](#) gives:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = 0$$

where  $\nabla \times$  is the [curl operator](#) and  $t$  is time. Since the [curl](#) of the electric field is zero, it is defined by a scalar electric potential field,  $\varphi$  (see [Helmholtz decomposition](#)).

$$\mathbf{E} = -\nabla \varphi$$

The derivation of Poisson's equation under these circumstances is straightforward. Substituting the potential gradient for the electric field

$$\nabla \cdot \mathbf{E} = \nabla \cdot (-\nabla \varphi) = -\nabla^2 \varphi = \frac{\rho_f}{\varepsilon},$$

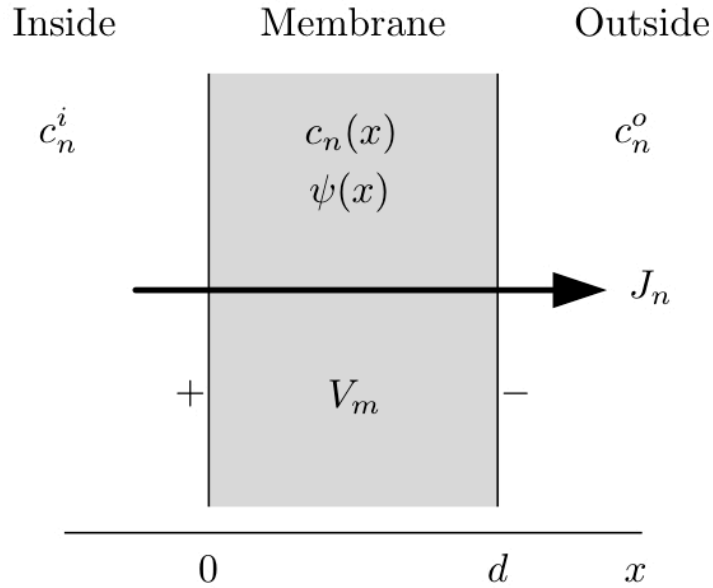
directly obtains **Poisson's equation** for electrostatics, which is:

$$\nabla^2 \varphi = -\frac{\rho_f}{\varepsilon}.$$

Solving Poisson's equation for the potential requires knowing the charge density distribution. If the charge density is zero, then [Laplace's equation](#) results. If the charge density follows a [Boltzmann distribution](#), then the [Poisson-Boltzmann equation](#) results. The Poisson–Boltzmann equation plays a role in the development of the [Debye–Hückel theory of dilute electrolyte solutions](#).

The above discussion assumes that the magnetic field is not varying in time. The same Poisson equation arises even if it does vary in time, as long as the [Coulomb gauge](#) is used. In this more general context, computing  $\phi$  is no longer sufficient to calculate  $\mathbf{E}$ , since  $\mathbf{E}$  also depends on the [magnetic vector potential](#)  $\mathbf{A}$ , which must be independently computed. See [Maxwell's equation in potential formulation](#) for more on  $\phi$  and  $\mathbf{A}$  in Maxwell's equations and how Poisson's equation is obtained in this case.

## Steady-State Electrodiffusion through Membranes



Steady-state

$$\rightarrow \frac{\partial c_n(x, t)}{\partial t} = 0$$

$$\rightarrow \frac{\partial J_n(x, t)}{\partial x} = 0$$

$$\rightarrow J_n = \text{constant}$$

Electrolyte solutions  $\rightarrow$  Electroneutrality

$$\text{if } t \gg \tau_r \text{ and } x \gg \Lambda_D \text{ then } \sum_n z_n F c_n(x, t) = 0$$

$$\frac{\partial^2 \psi(x, t)}{\partial x^2} = -\frac{1}{\epsilon} \sum_n z_n F c_n(x, t)$$

$\rightarrow$  Simplifies Poisson's equation such that  $\psi$  is a linear function across the membrane

## Electrolyte solutions → Electroneutrality

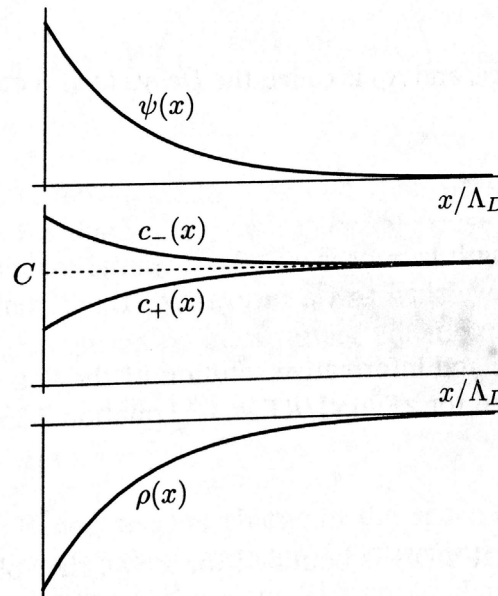
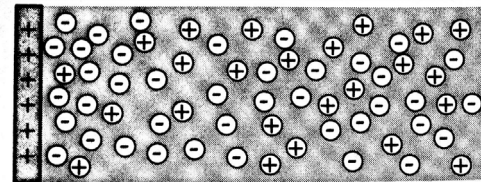
$$\text{if } t \gg \tau_r \text{ and } x \gg \Lambda_D \text{ then } \sum_n z_n F c_n(x, t) = 0$$

- Charge Relaxation Time  $\tau_r$

Measures temporal change in charge density  
(i.e., relaxation time of charge distribution)

- Debye Length  $\Lambda_D$

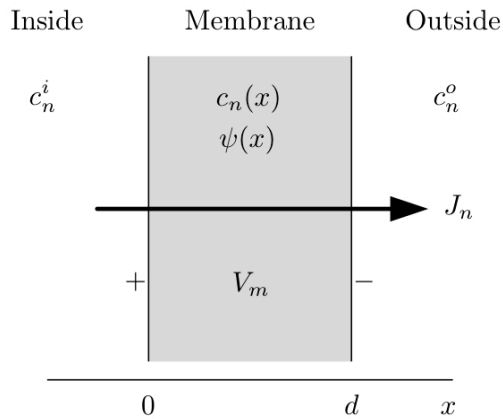
Measures spatial extent of electric potential  
(i.e., distance over which electroneutrality is violated)



**Figure 7.7** The spatial distribution of charge near a plate containing positive fixed charges. The counterions are anions and are in higher concentration near the plate than far from the plate. The cations are at a lower concentration near the plate than far from the plate. The spatial distributions of both mobile ions are exponential, with space constant equal to the Debye length.

→ Both are very small (1 ns and 1 nm respectively; see Weiss v.1 7.2.3), justifying that ionic solutions obey electroneutrality

## Steady-State Electrodiffusion through Membranes



Steady-state

$$\rightarrow \frac{\partial c_n(x, t)}{\partial t} = 0$$

$$\rightarrow \frac{\partial J_n(x, t)}{\partial x} = 0$$

$$\rightarrow J_n = \text{constant}$$

Rearrange Nernst-Planck Equation

$$J_n = -z_n F D_n \frac{dc_n(x)}{dx} - u_n z_n^2 F^2 c_n(x) \frac{d\psi(x)}{dx} = -u_n z_n^2 F^2 c_n(x) \left[ \frac{D_n}{u_n z_n F c_n(x)} \frac{dc_n(x)}{dx} + \frac{d\psi(x)}{dx} \right]$$

Integrate across membrane

$$J_n \underbrace{\int_0^d \frac{dx}{u_n z_n^2 F^2 c_n(x)}}_{\frac{1}{G_n}} = - \int_0^d \frac{d}{dx} \left[ \frac{RT}{z_n F} \ln c_n(x) + \psi(x) \right] dx$$

Rearrange/Rename

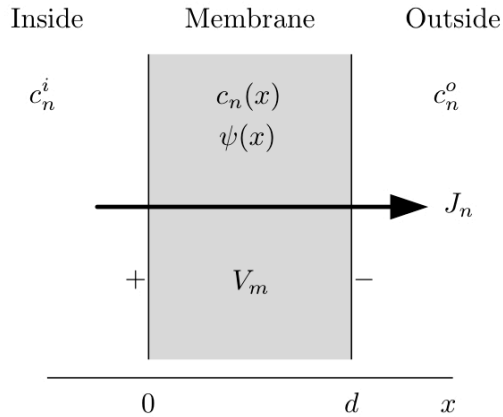
$$J_n \frac{1}{G_n} = - \overbrace{\frac{RT}{z_n F} \ln \frac{c_n(d)}{c_n(0)}}^{V_n} + \overbrace{\psi(0) - \psi(d)}^{V_m}$$



$$J_n = G_n (V_m - V_n)$$



## Steady-State Electrodiffusion through Membranes



Steady-state

$$\rightarrow \frac{\partial c_n(x, t)}{\partial t} = 0$$

$$\rightarrow \frac{\partial J_n(x, t)}{\partial x} = 0$$

$$\rightarrow J_n = \text{constant}$$

### Nernst Equilibrium Potential

$$V_n = \frac{RT}{z_n F} \ln \frac{c_n(d)}{c_n(0)} = \frac{RT}{z_n F} \ln \frac{c_n^o}{c_n^i}$$

$$J_n \frac{1}{G_n} = - \overbrace{\frac{RT}{z_n F} \ln \frac{c_n(d)}{c_n(0)}}^{V_n} + \overbrace{\psi(0) - \psi(d)}^{V_m}$$

$$J_n = G_n (V_m - V_n)$$

$$G_n = \frac{1}{\int_0^d \frac{dx}{u_n z_n^2 F^2 c_n(x)}} \geq 0$$

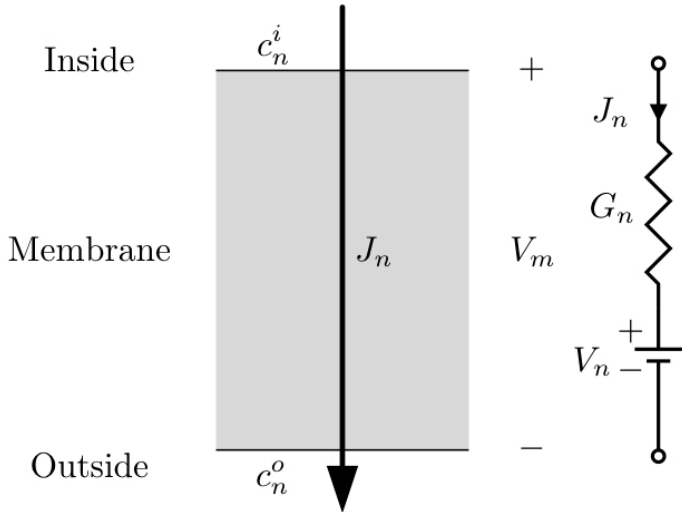
→ Like Ohm's law!

$$\frac{RT}{z_n F \log_{10} e} \sim 59 \text{ mV}$$

(for  $z_n = +1$ , room temp.)

# Model of Steady-State Electrodiffusion through Membranes

Note:  $\ln x = 2.303 \log_{10} x$   
 &  $\log_{10} e = 1/2.303$



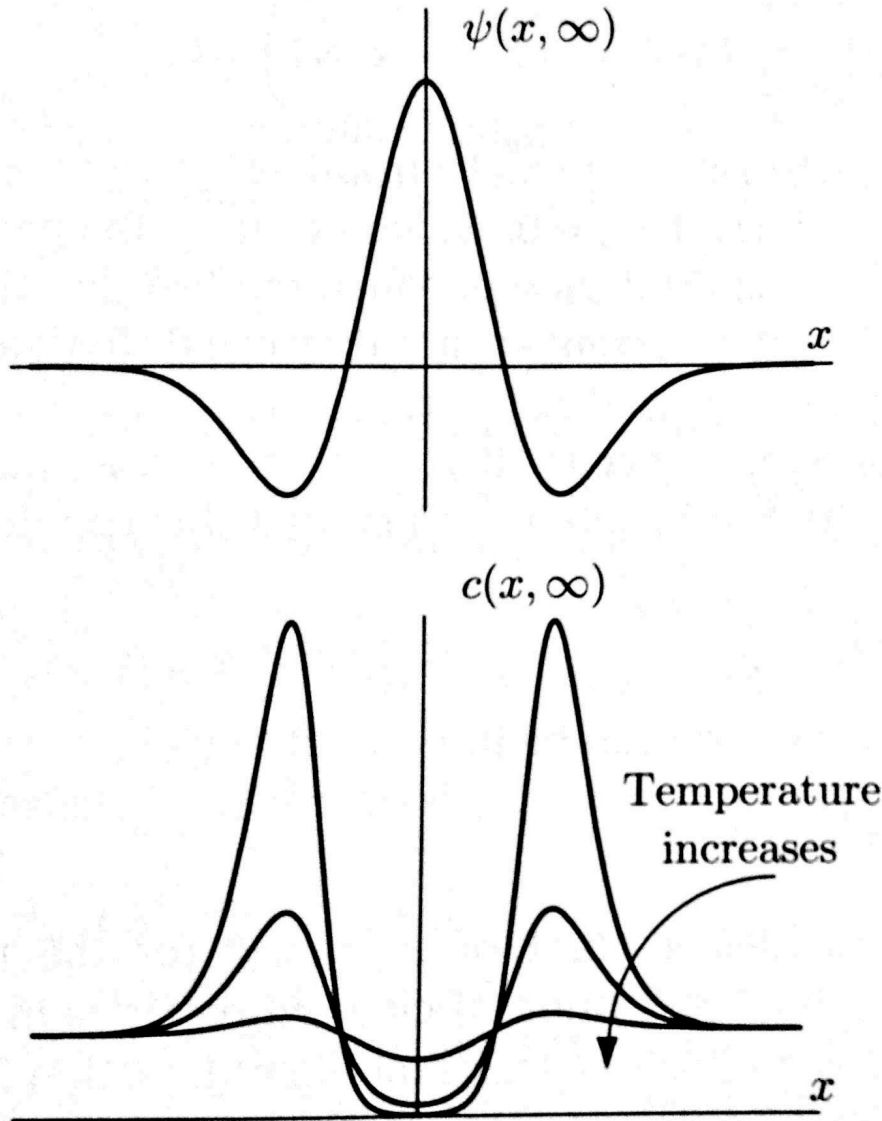
$$\frac{RT}{z_n F \log_{10} e} \sim 59 \text{ mV}$$

(for  $z_n = +1$ , room temp.)

Nernst Equilibrium Potential  $V_n = \frac{RT}{z_n F} \ln \frac{c_n^o}{c_n^i}$

Electrical Conductivity  $G_n = \frac{1}{\int_0^d \frac{dx}{u_n z_n^2 F^2 c_n(x)}} \geq 0$

## Mechanical analog for electrodiffusive equilibrium



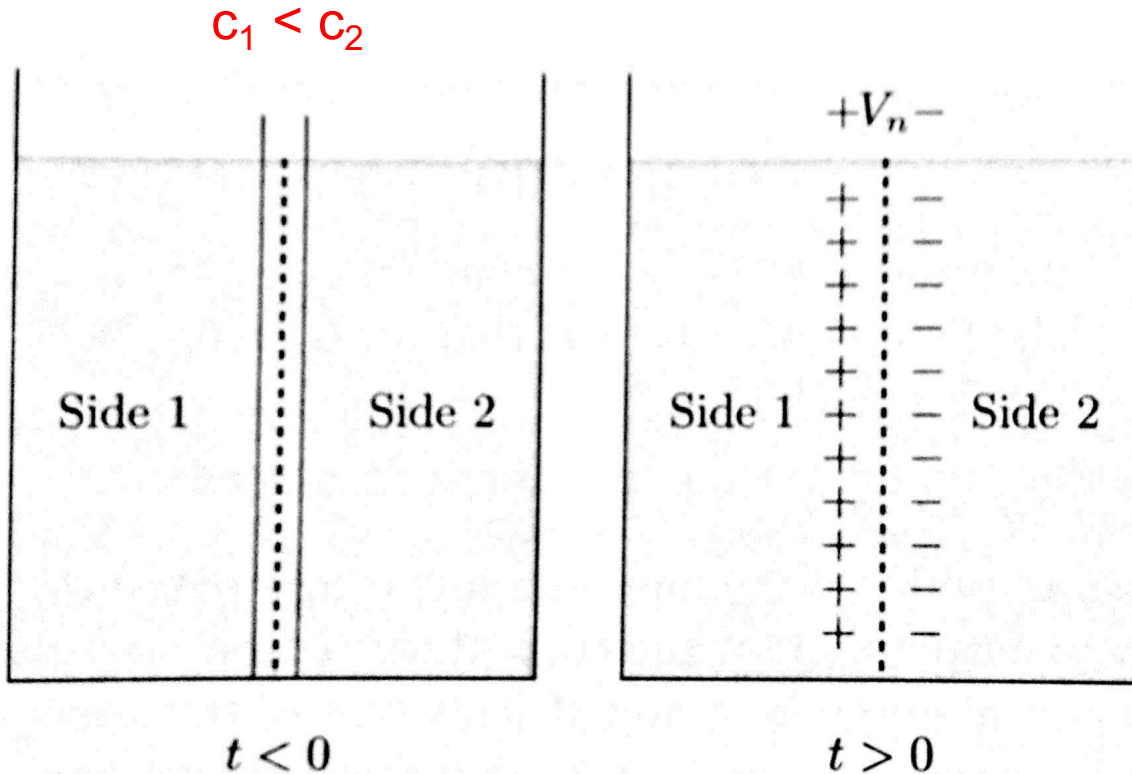
Analog to gravitational potential energy  
(no negative concentrations!)

$$\frac{\partial^2 \psi(x, t)}{\partial x^2} = -\frac{1}{\epsilon} \sum_n z_n F c_n(x, t)$$

**Figure 7.6** The spatial distribution of electric potential and ion concentration at electrodiffusive equilibrium for different temperatures.

## How is the Nernst potential generated?

Assumption: Single permeable ionic species (positively charged)



**Figure 7.16** Illustration of the generation of the Nernst equilibrium potential. A bath is separated into two compartments by a membrane permeable only to ion  $n$ .

→ Note that the creation of a significant  $V_n$  need not require significant concentration changes

# Resting Potential

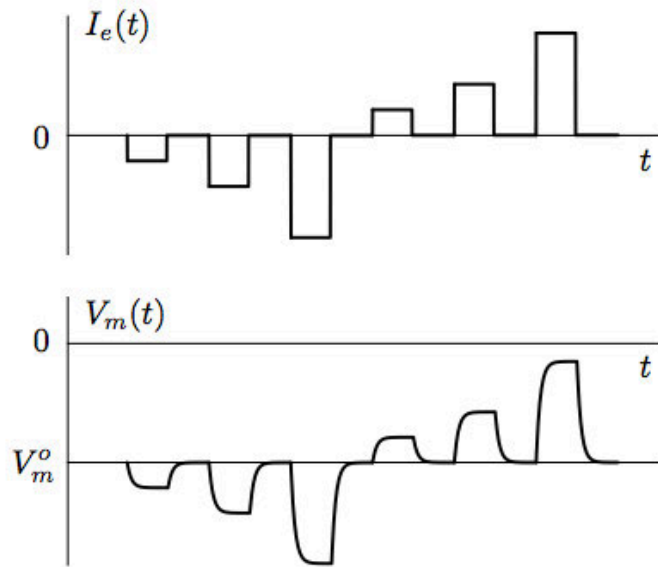
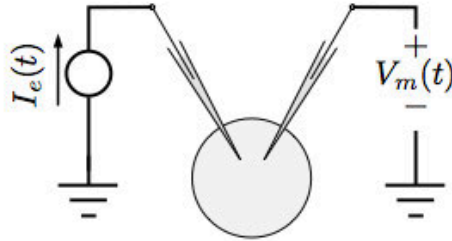


Figure 1.1

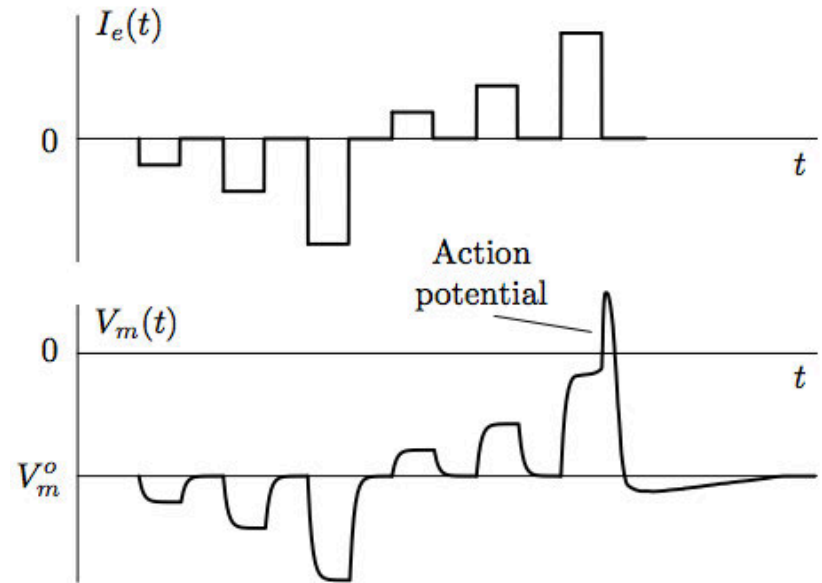
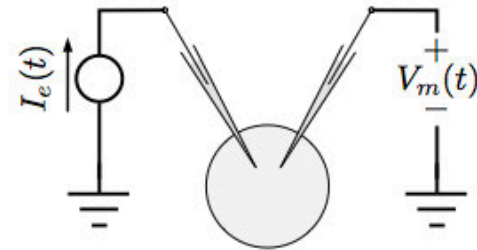
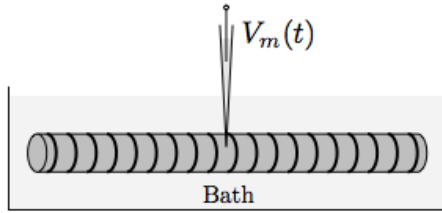


Figure 1.8

→ Independent of whether a cell “fires” an action potential or not, note that there is a baseline trans-membrane potential (“resting potential”)  $V_m^o$

# Resting Potential



Bath composition

Normal solution	Test solution	Normal solution
-----------------	---------------	-----------------

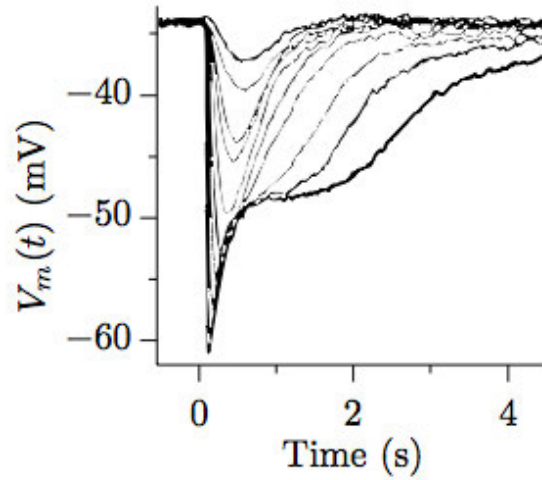
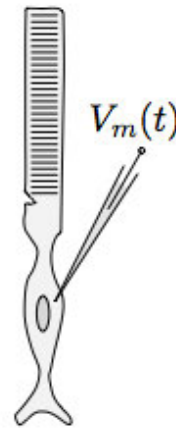


Figure 1.3

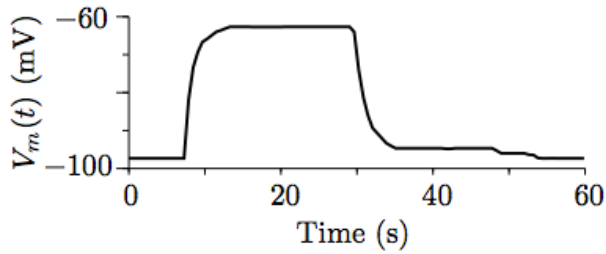


Figure 1.2

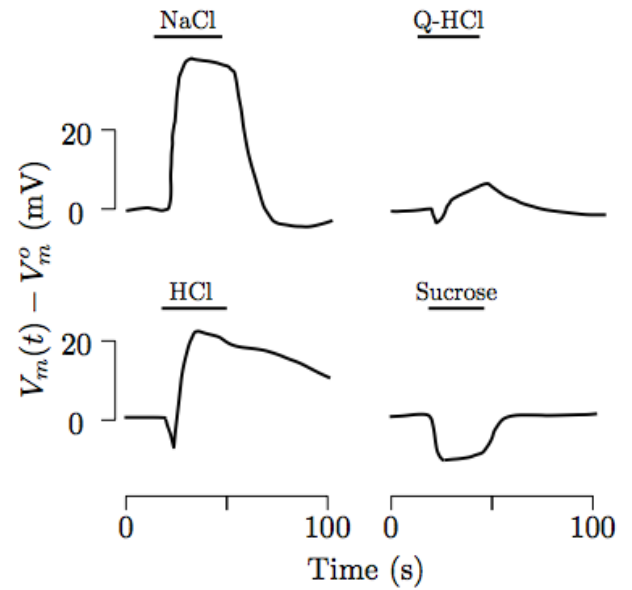


Figure 1.6

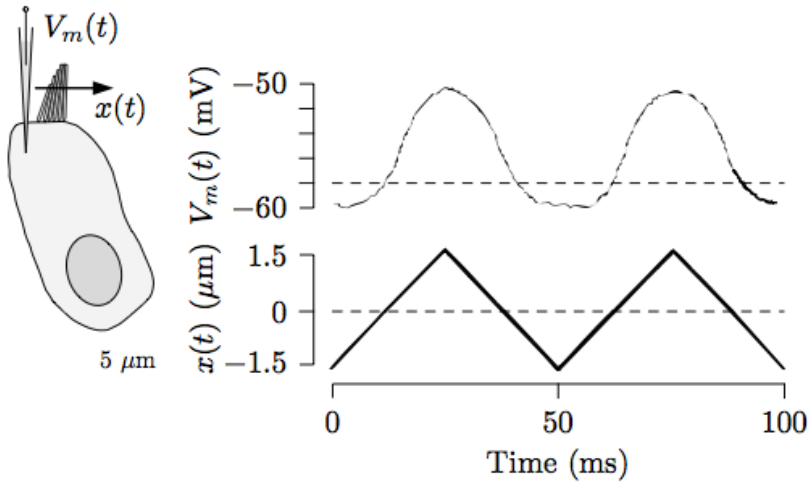
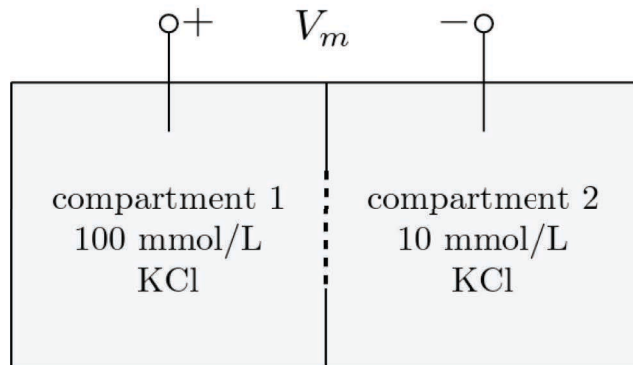


Figure 1.5

→ What is the basis for such a resting potential?

## Problem

- 7.2 As shown in Figure 7.53, compartments 1 and 2 contain well-stirred solutions of potassium chloride and are separated by a membrane that is permeable to only potassium. The potential between compartment 1 and 2 is  $V_m$ . The concentrations of KCl in compartments 1 and 2 are 100 mmol/L and 10 mmol/L, respectively.



**Figure 7.53** Two compartments separated by a membrane that is permeable to potassium only (Exercise 7.2).

- Determine the equilibrium value of  $V_m$ , and give a physical explanation of the sign of the potential.
- A battery is now connected to the solutions, so that  $V_m = -30$  mV. In which direction will current flow through the membrane? Explain.
- Draw an equivalent electrical network for the condition indicated in part b. Label the nodes that represent compartments 1 and 2,  $V_m$ , and label  $I_m$ , defined as the current that flows through the membrane in the direction from compartment 1 to compartment 2.

## Problems SOL

- a. Since only potassium permeates the membrane at equilibrium, the membrane potential must equal the potassium equilibrium potential. Assuming normal room temperature of 25°C,

$$V_m = V_K = 59 \log_{10} \left( \frac{c_K^2}{c_K^1} \right) = 59 \log_{10}(1/10) = -59 \text{ mV.}$$

At equilibrium, the flux due to diffusion (which is from volume 1 to volume 2) is balanced by drift of potassium ions (which must be from volume 2 to volume 1). The drift is in this direction if the potential in volume 1 is less than the potential in volume 2 so that the positive potassium ions flow from a higher to a lower electric potential.

- c. Application of a battery to the two volumes completes an electric circuit consisting of the battery and the membrane as shown in Figure 7.1. The membrane is represented by an equivalent conductance and a battery in series.

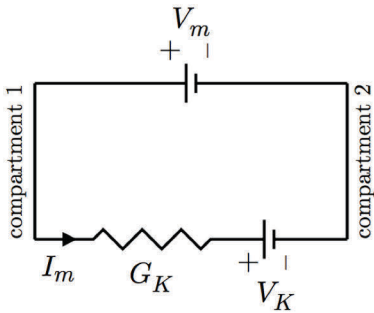


Figure 7.1: Electric network model of membrane and source (Exercise 7.2).

- b. The network shows that  $I_m = G_K(V_m - V_K)$ . When  $V_m > V_K$ ,  $I_m > 0$ , i.e., the current flows from the left to the right compartment. But,  $V_m = -30 \text{ mV}$  and  $V_K = -59 \text{ mV}$ , so that  $V_m - V_K = -30 + 59 = 29 \text{ mV}$  and  $I_m > 0$ .



## Problem

- 7.5 Describe the distinctions between the following terms that refer to ion transport across a cellular membrane: electrodiffusive equilibrium, steady state, resting conditions, and cellular quasi-equilibrium.
- 7.6 The following is a discussion of electroneutrality (Nicholls et al., 1992):

*The intracellular and extracellular solutions must each be electrically neutral. For example, a solution of chloride ions alone cannot exist; their charges must be balanced by an equal number of positive charges on cations such as sodium or potassium (otherwise electrical repulsion would literally blow the solution apart).*

Briefly critique this discussion of electroneutrality.

**Exercise 7.5** In steady state the ionic flux through the membrane, the concentration of ions in the membrane, and the voltage across the membrane are all constant with respect to time. Electrodiffusive equilibrium requires all of the conditions for steady state plus the condition that the ionic flux through the membrane is zero. At equilibrium, the potential across the membrane equals the Nernst equilibrium potentials of each permeant ionic species. Rest requires all of the conditions for steady state plus the condition that the net current through the membrane (total across ionic species) is zero. Quasi-equilibrium requires all of the conditions for steady state plus that the net flux of each ionic species (summed across all of the transport mechanisms for that species) is zero.

As an example, suppose external electrodes pass a constant current through the membrane of a cell. For this case, the membrane could come to a steady-state condition. It could be at electrodiffusive equilibrium if the membrane contains active transport mechanisms to carry all of the current from the external electrodes through the membrane. By definition, the cell is not at rest. Furthermore, the cell could not be in quasi-equilibrium, since the external current must be carried through the membrane by some ionic species.

**Exercise 7.6** The statement is largely correct except for the parenthetical phrase. The solution would not blow up. The excess charges would repel each other and would ultimately reside on the boundaries of the vessel enclosing the solution.

