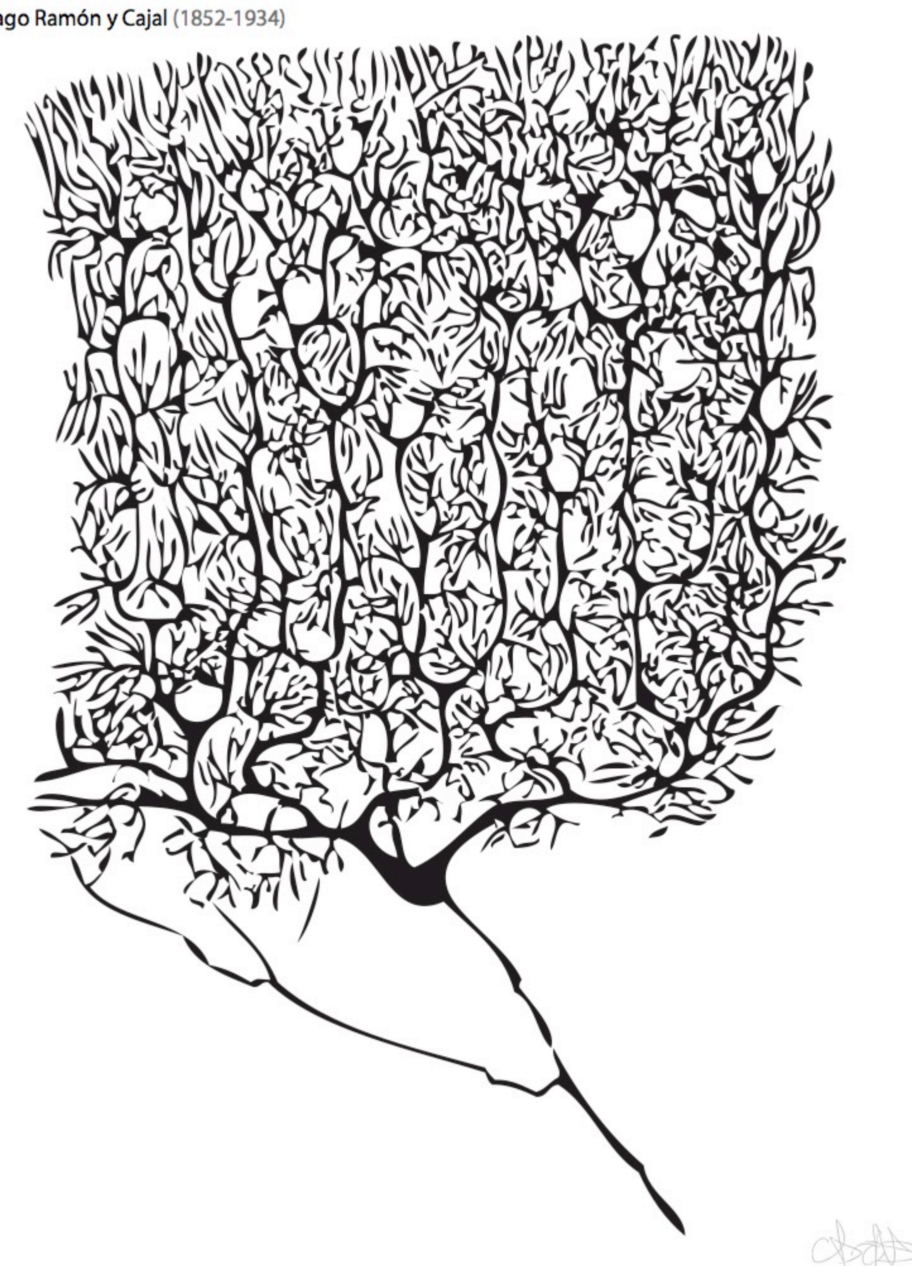
# Cellular Electrodynamics

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York University
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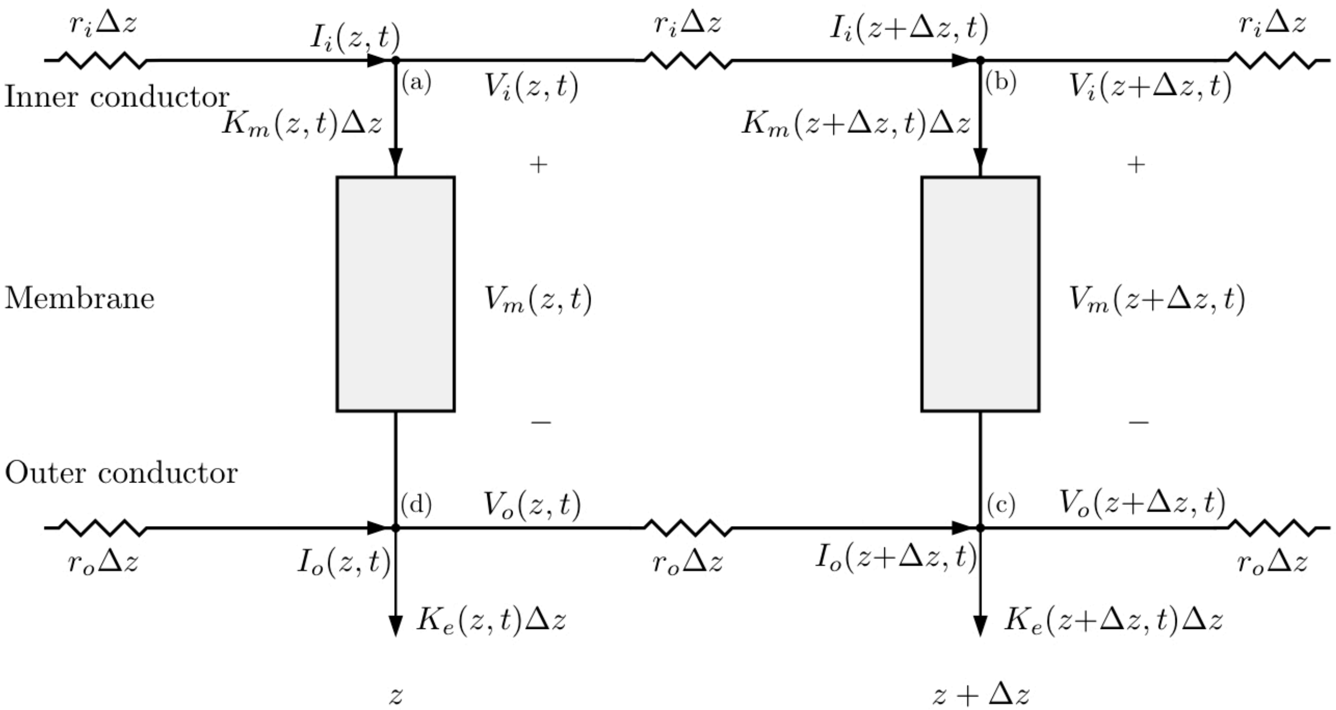
BPHS 4080 Lecture X

Reference/Acknowledgement:
- TF Weiss (Cellular Biophysics)
- D Freeman




## THE Core – Conductor Equation

Assumptions/geometry above, along with Kirchoff's & Ohm's Laws lead us to the...

$$\frac{\partial^2 V_m(z,t)}{\partial z^2} = (r_o + r_i)K_m(z,t) - r_o K_e(z,t)$$

→ Relates spatial change in transmembrane potential to current flowing through the membrane

## Some Implications

Consider no external electrodes
(i.e., $K_e$ = 0):

$$K_m(z,t) = \frac{1}{r_o + r_i} \frac{\partial^2 V_m(z,t)}{\partial z^2}$$

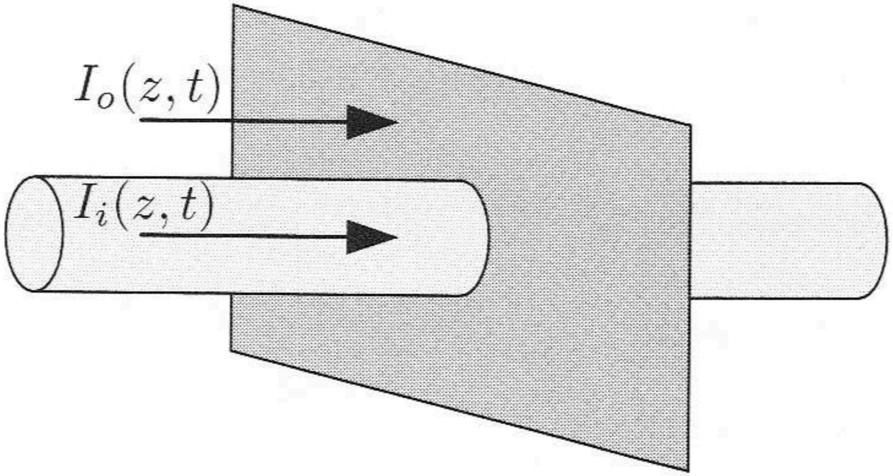



Figure 2.9

Conservation of charge requires: $I_i(z,t) + I_o(z,t) = 0$

**Core – Conductor Equations**

$$\frac{\partial I_i(z,t)}{\partial z} = -K_m(z,t)$$

$$\frac{\partial I_o(z,t)}{\partial z} = K_m(z,t) - K_e(z,t)$$

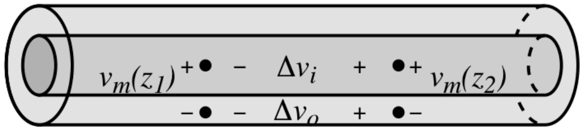
$$\frac{\partial V_i(z,t)}{\partial z} = -r_i I_i(z,t)$$

$$\frac{\partial V_o(z,t)}{\partial z} = -r_o I_o(z,t)$$

$$I_o(z,t) = \frac{1}{r_o + r_i} \frac{\partial V_m(z,t)}{\partial z}$$

# Relation between extracellular and intracellular potentials

Some useful interrelationships...




KVL:

$$v_m(z_2) + \Delta v_o - v_m(z_1) - \Delta v_i = 0$$

$$\Delta v_o - \Delta v_i = v_m(z_1) - v_m(z_2)$$

$$\Delta v_i = -\int_{z_1}^{z_2} r_i I_i dz = -r_i \int_{z_1}^{z_2} I_i dz$$

$$\Delta v_o = -\int_{z_1}^{z_2} r_o I_o dz = -r_o \int_{z_1}^{z_2} I_o dz$$

$$I_o = -I_i$$

$$\Delta v_i = -r_i \int_{z_1}^{z_2} I_i dz = -r_i \int_{z_1}^{z_2} (-I_o) dz = -\frac{r_i}{r_o} \Delta v_o$$

$$\Delta v_o - \Delta v_i = \Delta v_o + \frac{r_i}{r_o} \Delta v_o = v_m(z_1) - v_m(z_2)$$

$$\Delta v_o = \frac{r_o}{r_o + r_i} \left(v_m(z_1) - v_m(z_2)\right)$$

let $z_1 \to -\infty$: then $v_o(z_1) \to 0$ and $v_m(z_1) \to V_m^o$

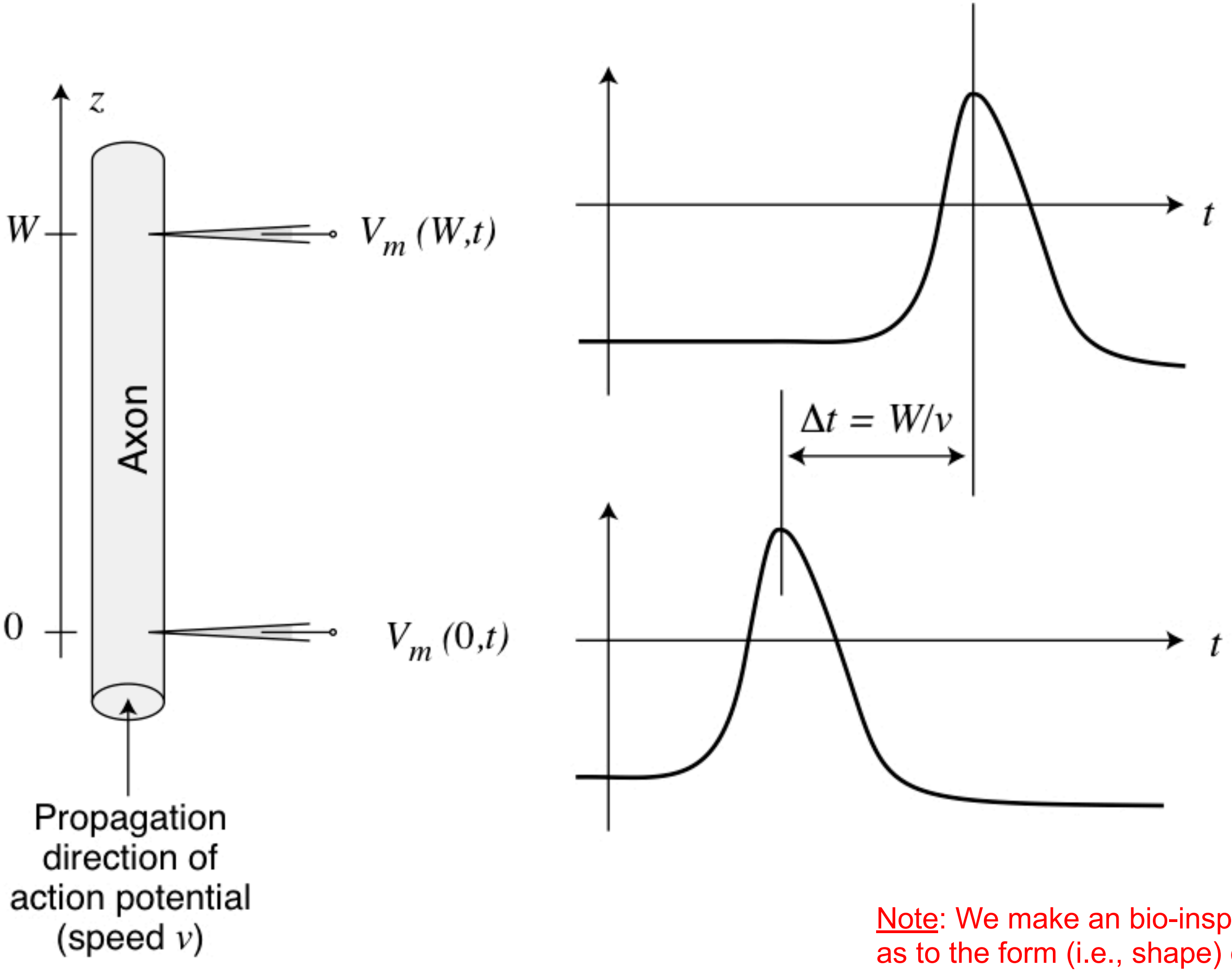
$$\Delta v_o = v_o(z_2) - v_o(z_1) = v_o(z_2) = \frac{r_o}{r_o + r_i} \left(V_m^o - v_m(z_2)\right)$$

$$v_o(z_2) = -\frac{r_o}{r_o + r_i} \left(v_m(z_2) - V_m^o\right)$$

# Propagation at *Constant Velocity*

Assumption: Membrane potential behaves in a ***wave-like*** fashion (const. velocity)

$$V_m(z,t) = f\left(t \pm \frac{z}{v}\right)$$

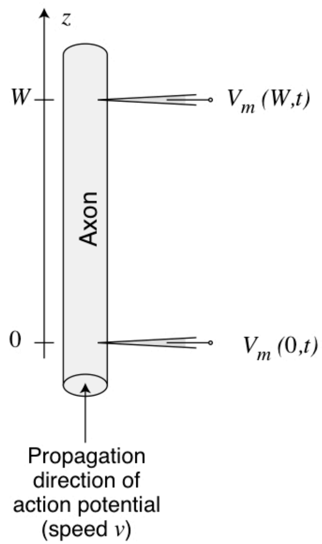



Note: We make an bio-inspired assumption as to the form (i.e., shape) of $f$ here

# Propagation at *Constant Velocity*

$$V_m(z,t) = f\left(t \pm \frac{z}{v}\right)$$

“Snapshot” at one specific time
→ ***Spatial dependence***




“Snapshots” at two different spatial locations
→ ***Temporal dependence***

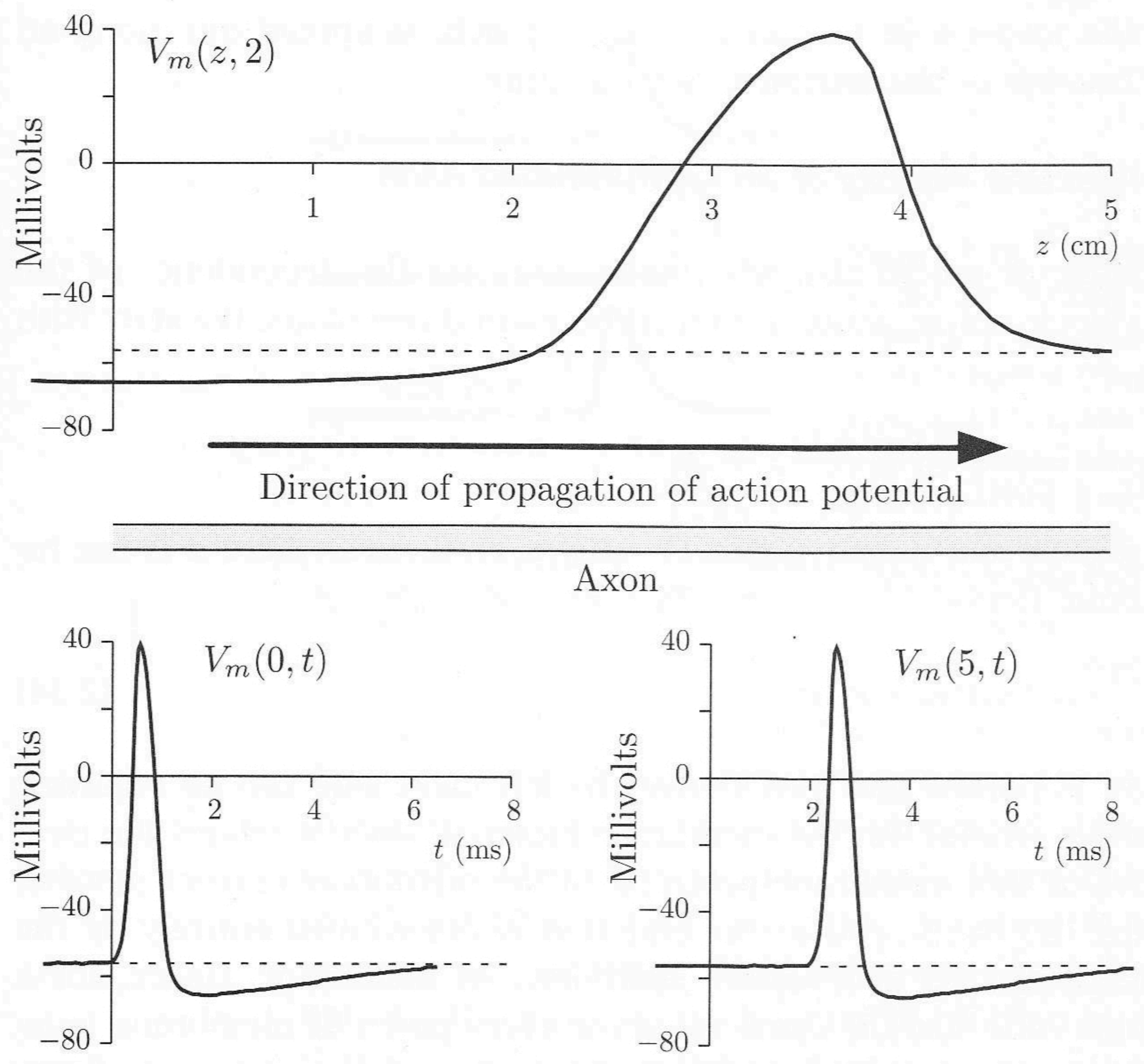



Figure 2.13

# Propagation at *Constant Velocity*

$$V_m(z,t) = f\left(t \pm \frac{z}{v}\right)$$

Think carefully about what the diacritical dot means here!

$$\frac{\partial V_m(z,t)}{\partial z} = \pm\frac{1}{v}\dot{f}\left(t \pm \frac{z}{v}\right) \quad \text{and} \quad \frac{\partial V_m(z,t)}{\partial t} = \dot{f}\left(t \pm \frac{z}{v}\right)$$

$$\frac{\partial^2 V_m(z,t)}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 V_m}{\partial t^2}$$

Wave equation
(differential form)

$$I_o(z,t) = \pm\frac{1}{(r_o + r_i)v}\frac{\partial V_m(z,t)}{\partial t}$$

and

$$K_m(z,t) = \frac{1}{(r_o + r_i)\,v^2}\frac{\partial^2 V_m(z,t)}{\partial t^2}$$

→ So when we assume a wave propagating at constant velocity, the core conductor model yields explicit time relationships as well

$$I_o(z,t) = \frac{1}{r_o + r_i}\frac{\partial V_m(z,t)}{\partial z}$$

## Connecting action potential-type waves to the core conductor model

$$I_o(z,t) = \pm \frac{1}{(r_o + r_i)\nu} \frac{\partial V_m(z,t)}{\partial t}$$

and

$$K_m(z,t) = \frac{1}{(r_o + r_i)\,\nu^2} \frac{\partial^2 V_m(z,t)}{\partial t^2}$$

$$I_i(z,t) + I_o(z,t) = 0$$

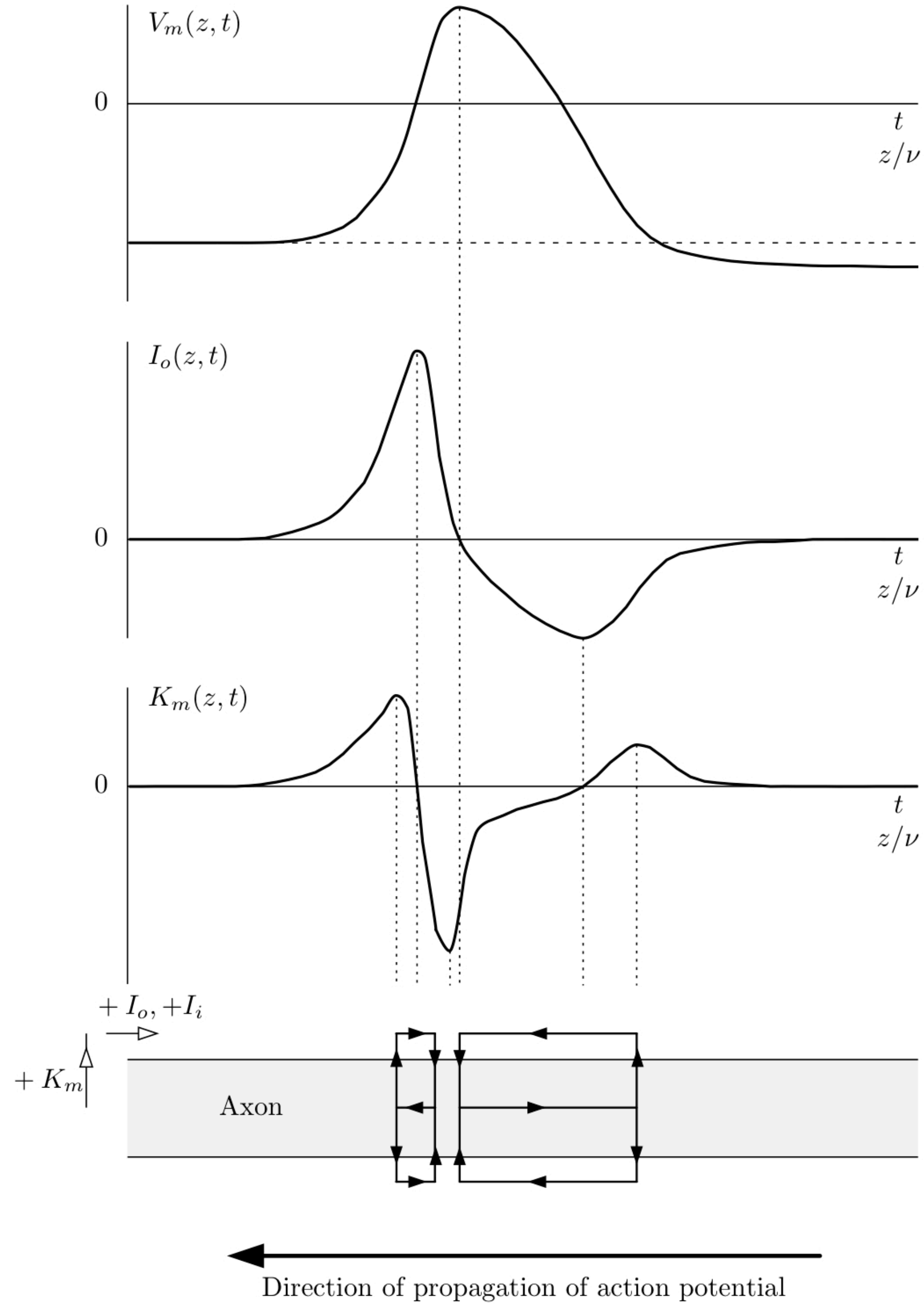



Figure 2.12

Conduction Velocity *(unmyelinated axon)*

$$K_m(z,t) = \frac{1}{(r_o + r_i)v^2}\frac{\partial^2 V_m(z,t)}{\partial t^2}$$

$$J_m(z,t) = K_m(z,t)/(2\pi a)$$

$$\frac{\partial^2 V_m(z,t)/\partial t^2}{J_m(z,t)} = 2\pi a(r_o + r_i)v^2$$

Left-side: constant, only depends upon electrical properties of membrane per unit area

Right-side: constant, velocity depends only upon axon diameter and fluid resistances

Conduction Velocity *(unmyelinated axon)*

Change $r_o$

$$\frac{\partial^2 V_m(z,t)/\partial t^2}{J_m(z,t)} = 2\pi a(r_o + r_i)\nu^2$$

Changing resistivity affects conduction velocity

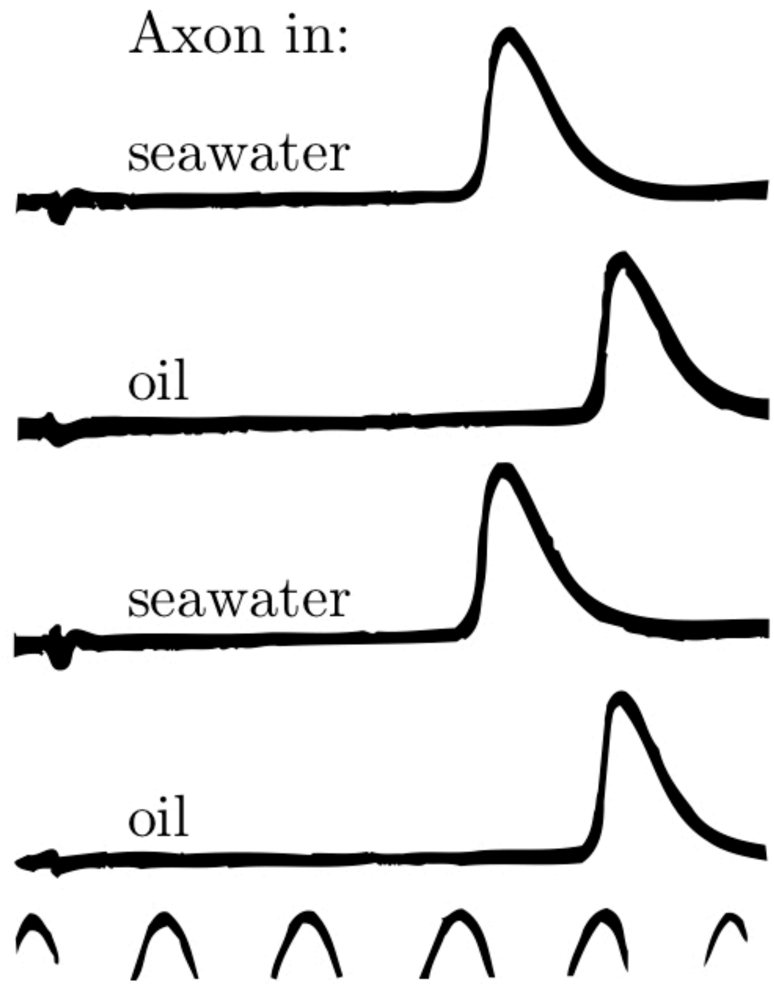



Figure 2.14

# Conduction Velocity *(unmyelinated axon)*

Change $r_i$

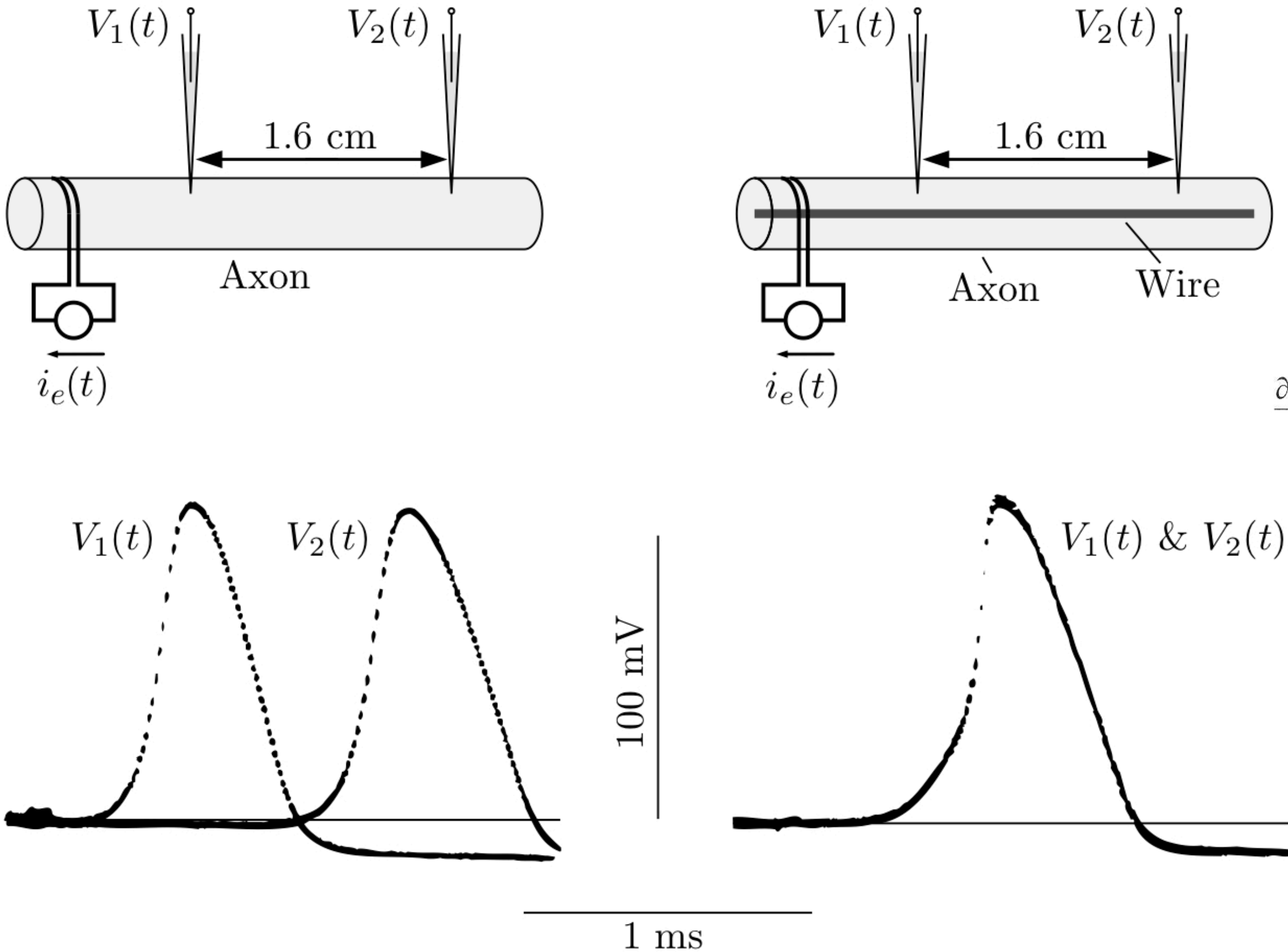



Figure 2.15

Inner wire essentially acts like a short

$$\frac{\partial^2 V_m(z,t)/\partial t^2}{J_m(z,t)} = 2\pi a (r_o + r_i) v^2$$

→ 'Space clamp'

Conduction Velocity *(unmyelinated axon)*

Assume $r_i >> r_o$

$$2\pi a r_i v^2 = K_m$$

$$r_i = \frac{\rho_i}{\pi a^2}$$

$$v = \sqrt{\frac{K_m a}{2\rho_i}}$$

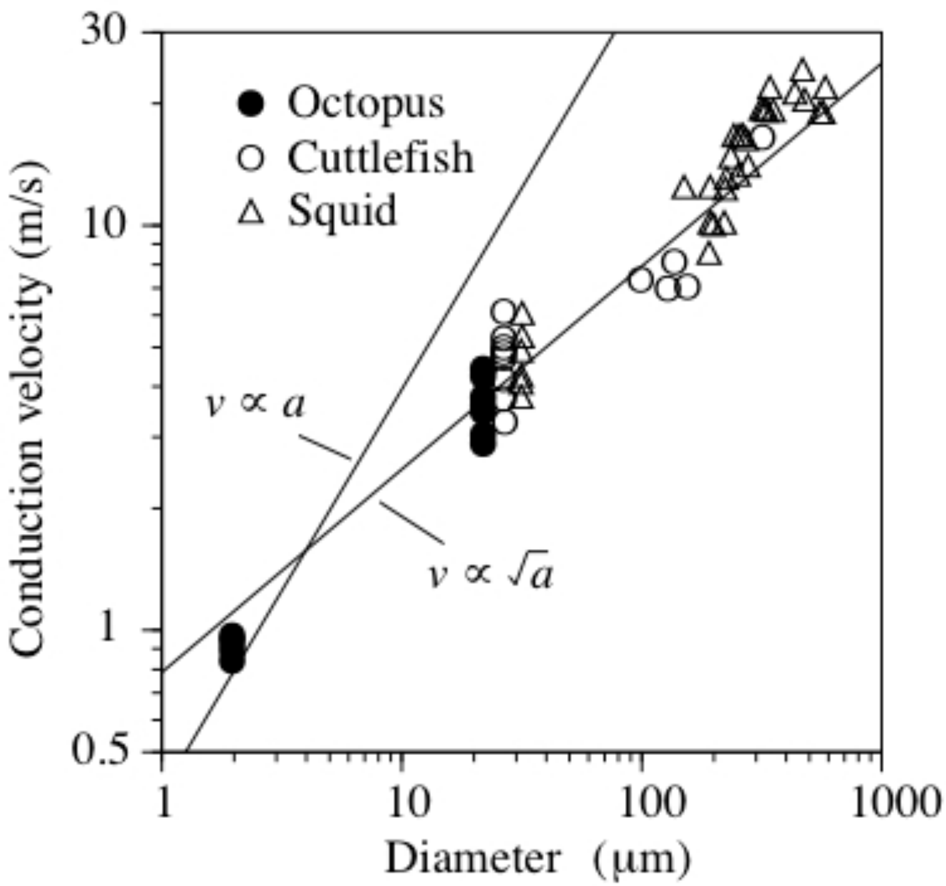


Figure 2.16

→ thicker axons = faster propagation

## Core Conductor Model

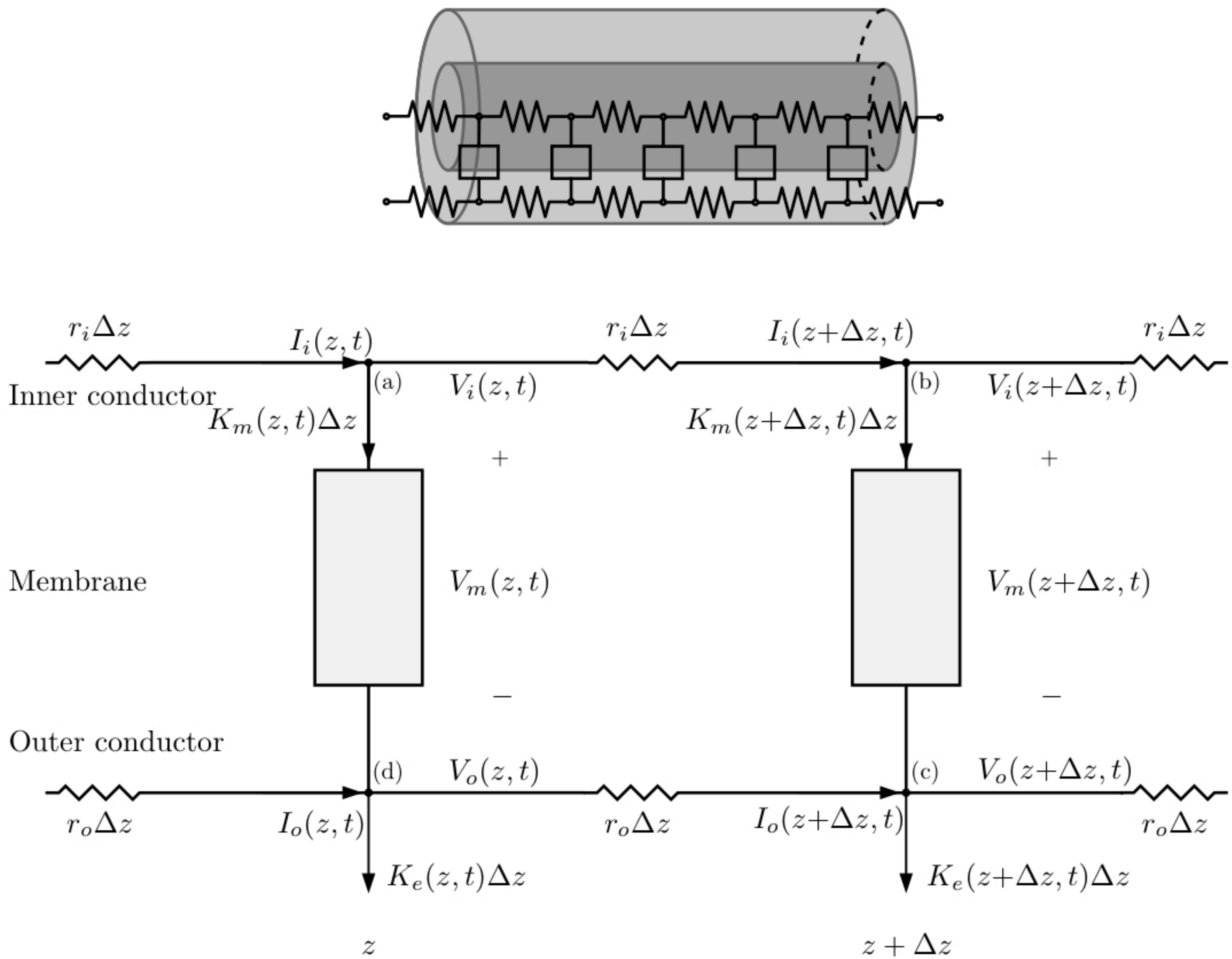
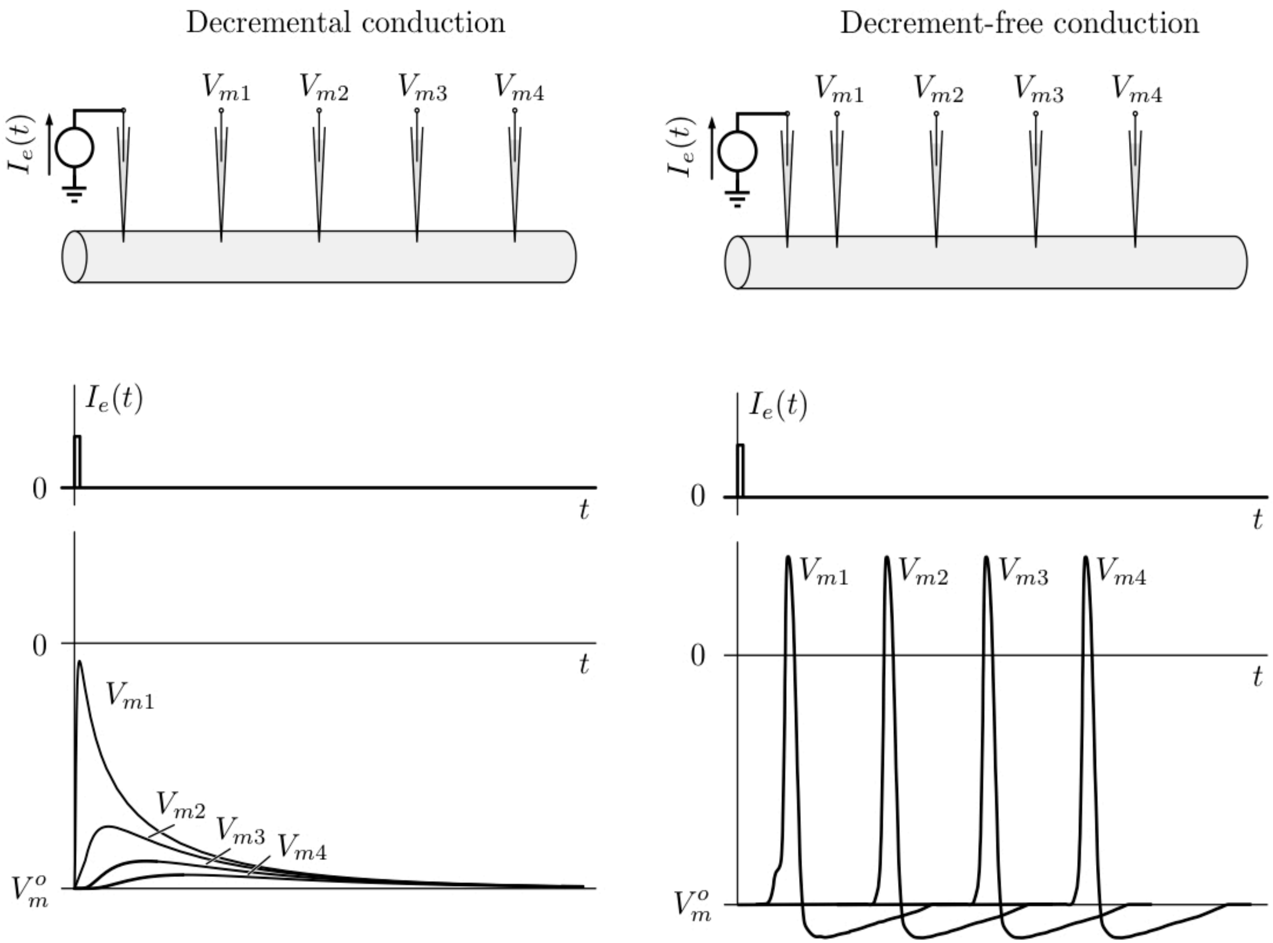
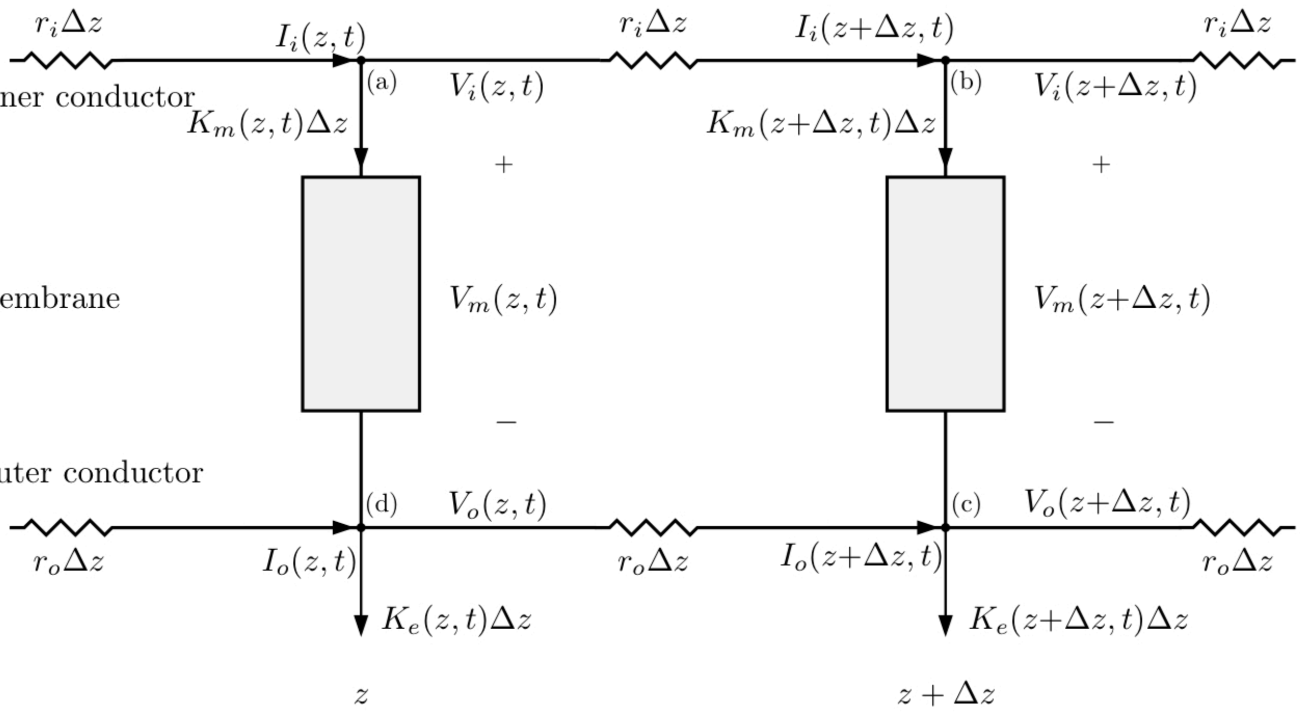



Figure 2.7



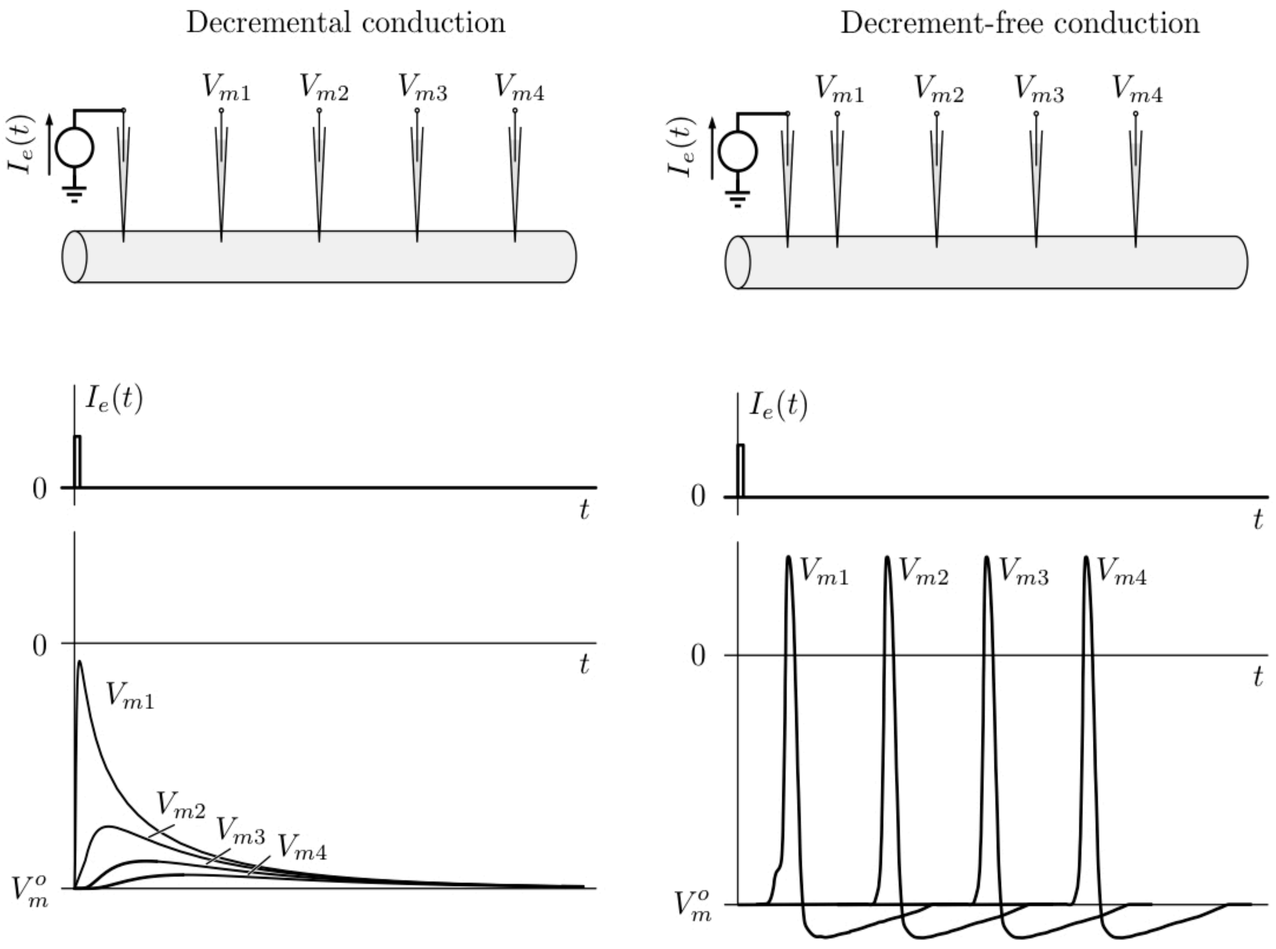

Core-Conductor Model (starting point) → Model for electrically large cells




No assumptions made about membrane!

# THE Core – Conductor Equation

$$\frac{\partial^2 V_m(z,t)}{\partial z^2} = (r_o + r_i)K_m(z,t) - r_o K_e(z,t)$$




Note dynamics of response....

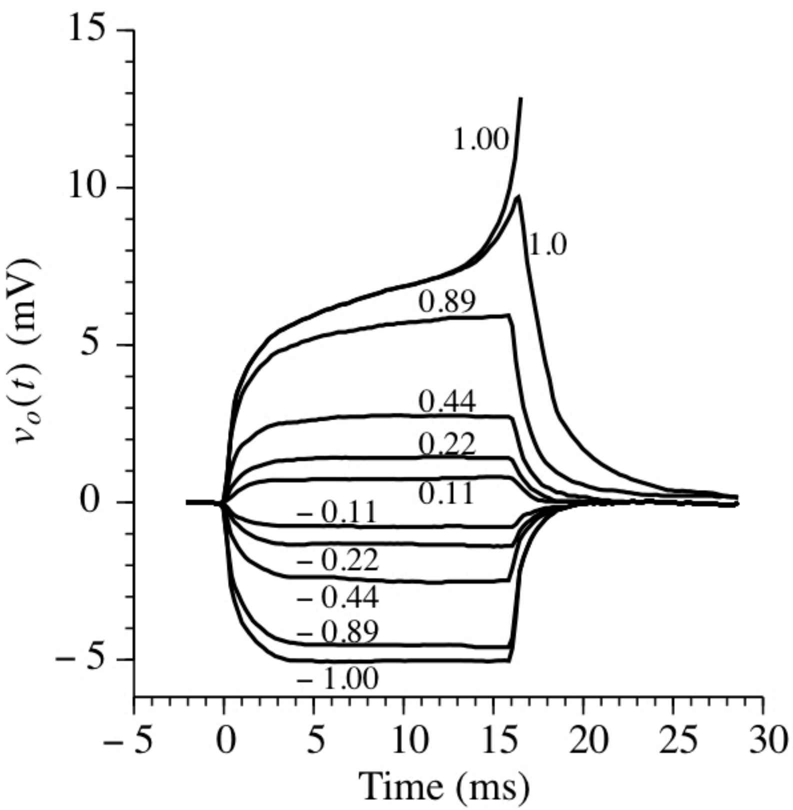



Figure 3.1

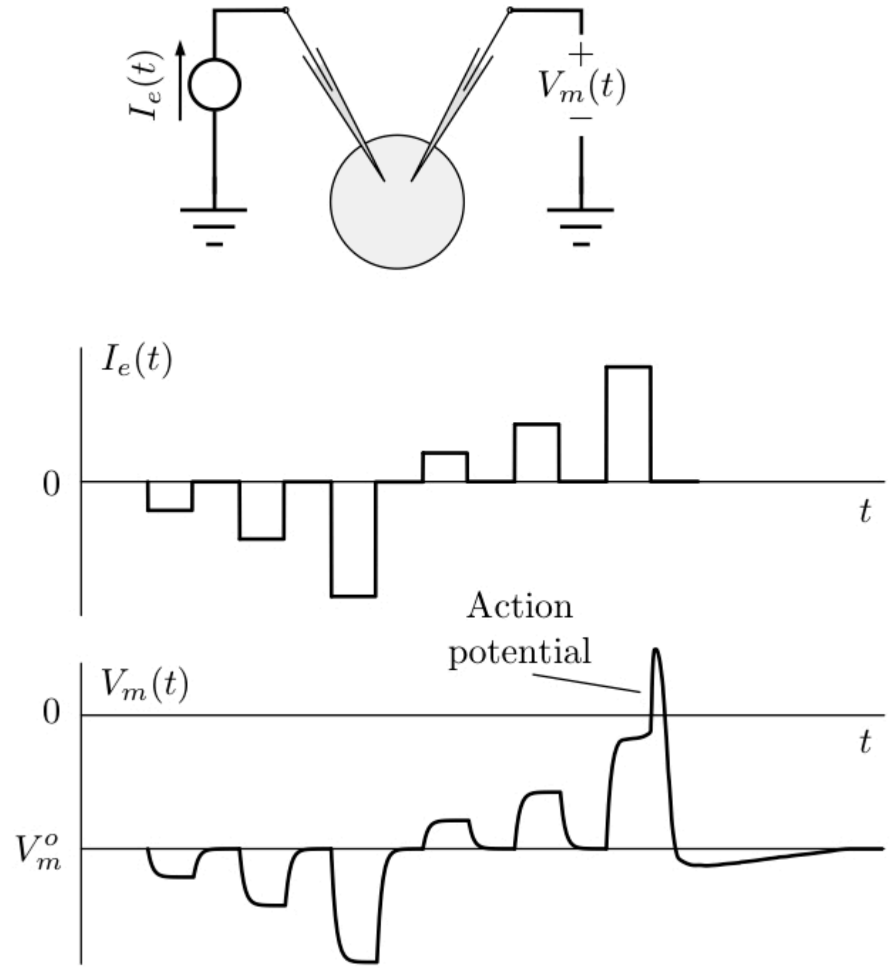



Figure 1.8

1. Linear (to a point)

2. Delay apparent

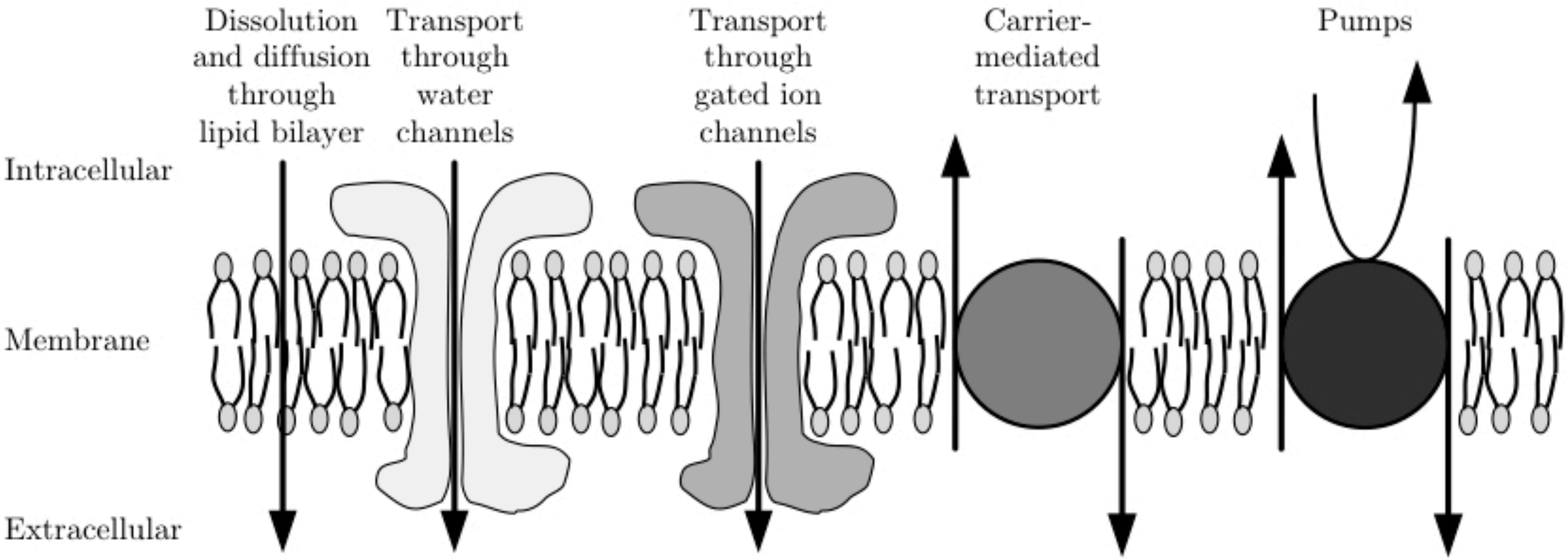
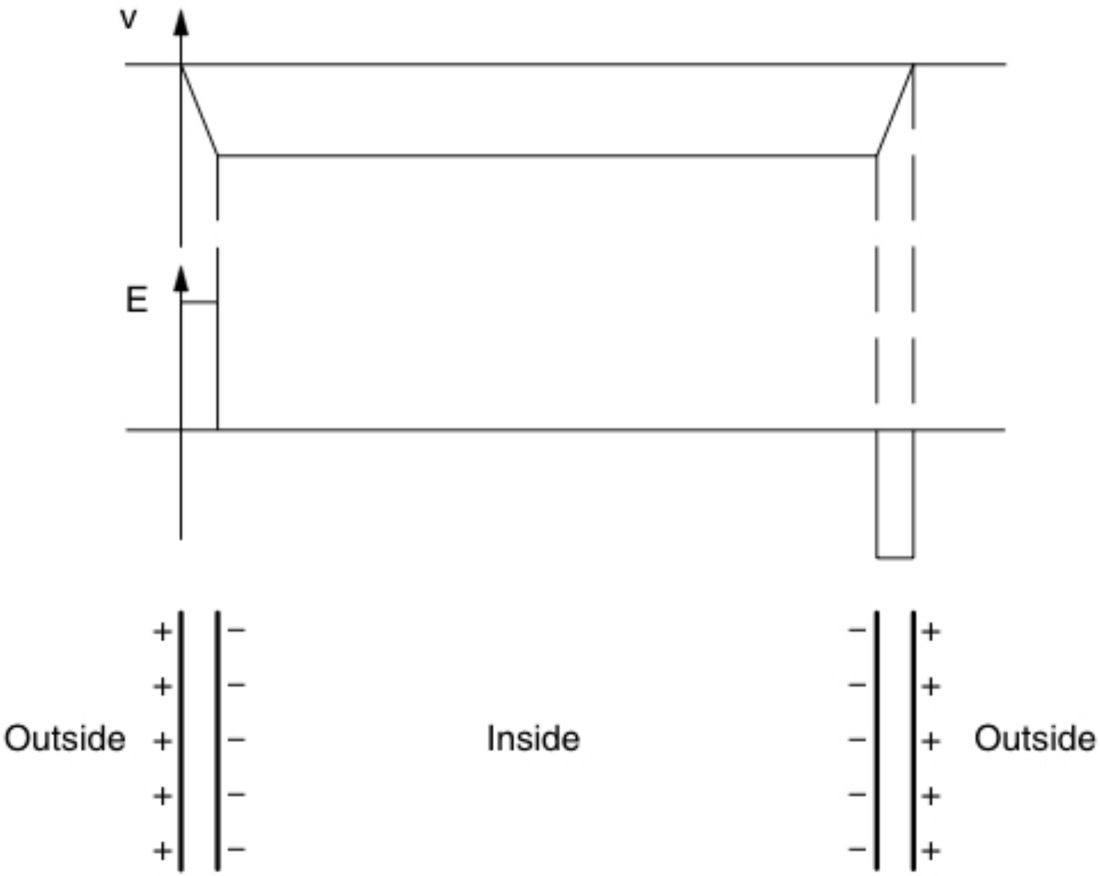



Figure 2.19

Idea: Membrane not only allows for charge transport, but also charge separation

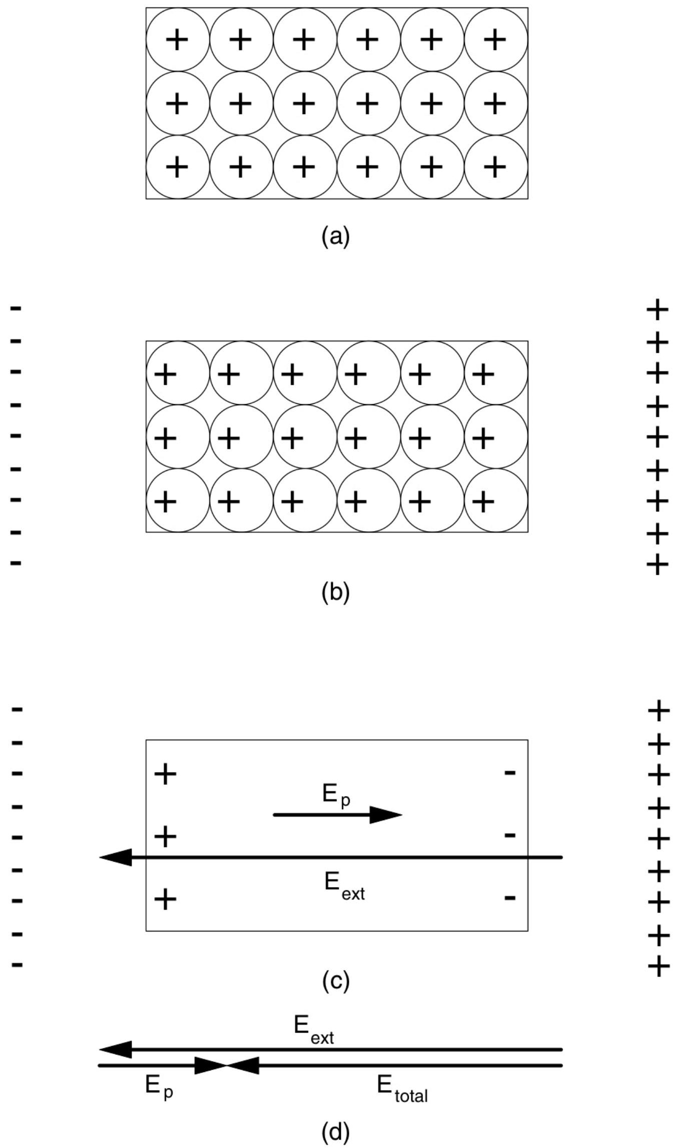
# Cell Membrane = Capacitor




- Steady-state electrodiffusion cause charge buildup on both sides of membrane

- Charge separation acts like parallel-plate capacitor
  ($C$ ~ 1 μF/cm²)

# Lipid Bilayer = Dielectric

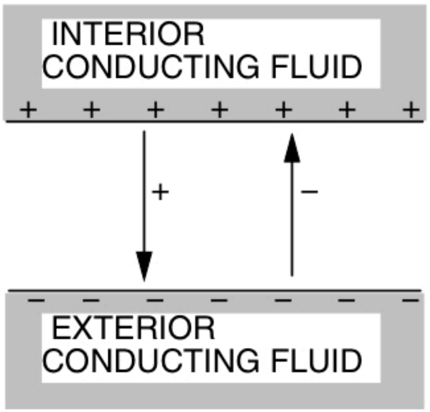


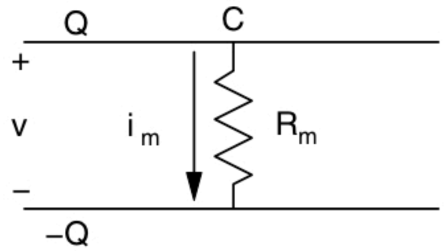

- Lipid bilayer is an insulator
(i.e., acts as a dielectric w/ const. $\kappa$)

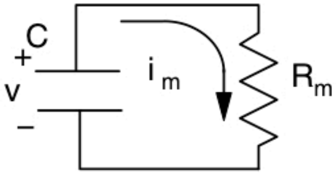
- $\kappa$~3-7, meaning more charge separation can occur (higher capacitance)

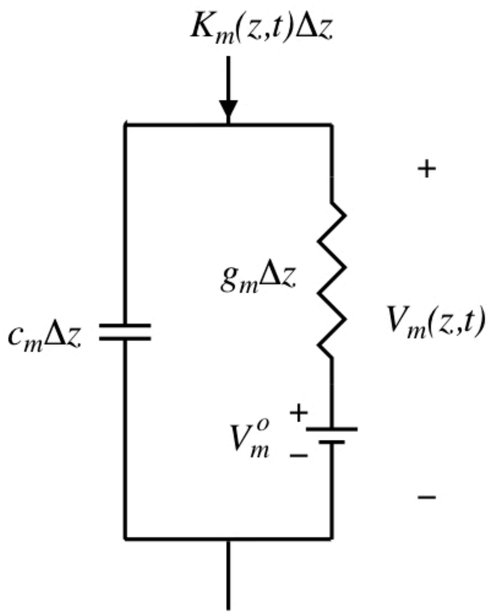
Hobbie & Roth

# Circuit Representation



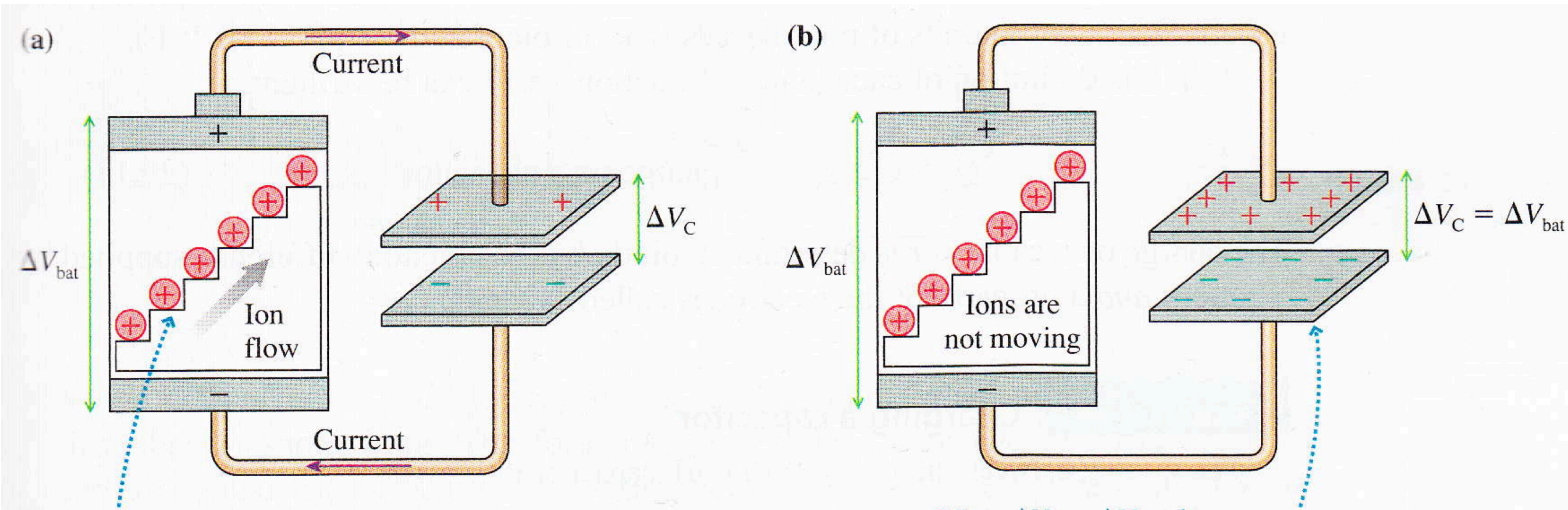

Resistor and capacitor in series
→ RC time constant

Figure 3.6

Hobbie & Roth
Weiss

<u>Review</u>: Capacitance

- Charging a parallel-plate capacitor




The charge escalator moves charge from one plate to the other. $\Delta V_C$ increases as the charge separation increases.

When $\Delta V_C = \Delta V_{bat}$, the current stops and the capacitor is fully charged.

$$Q = C\,\Delta V_C \qquad \text{(charge on a capacitor)}$$

→ Stored charge is proportional to potential difference. Constant of proportionality is characterizes the “capacitance”

Knight

# Review: RC Circuits

KVL (combined w/ Ohm's law):

$$\Delta V_{\text{cap}} + \Delta V_{\text{res}} = \frac{Q}{C} - IR = 0$$

$$I = -\frac{dQ}{dt}$$

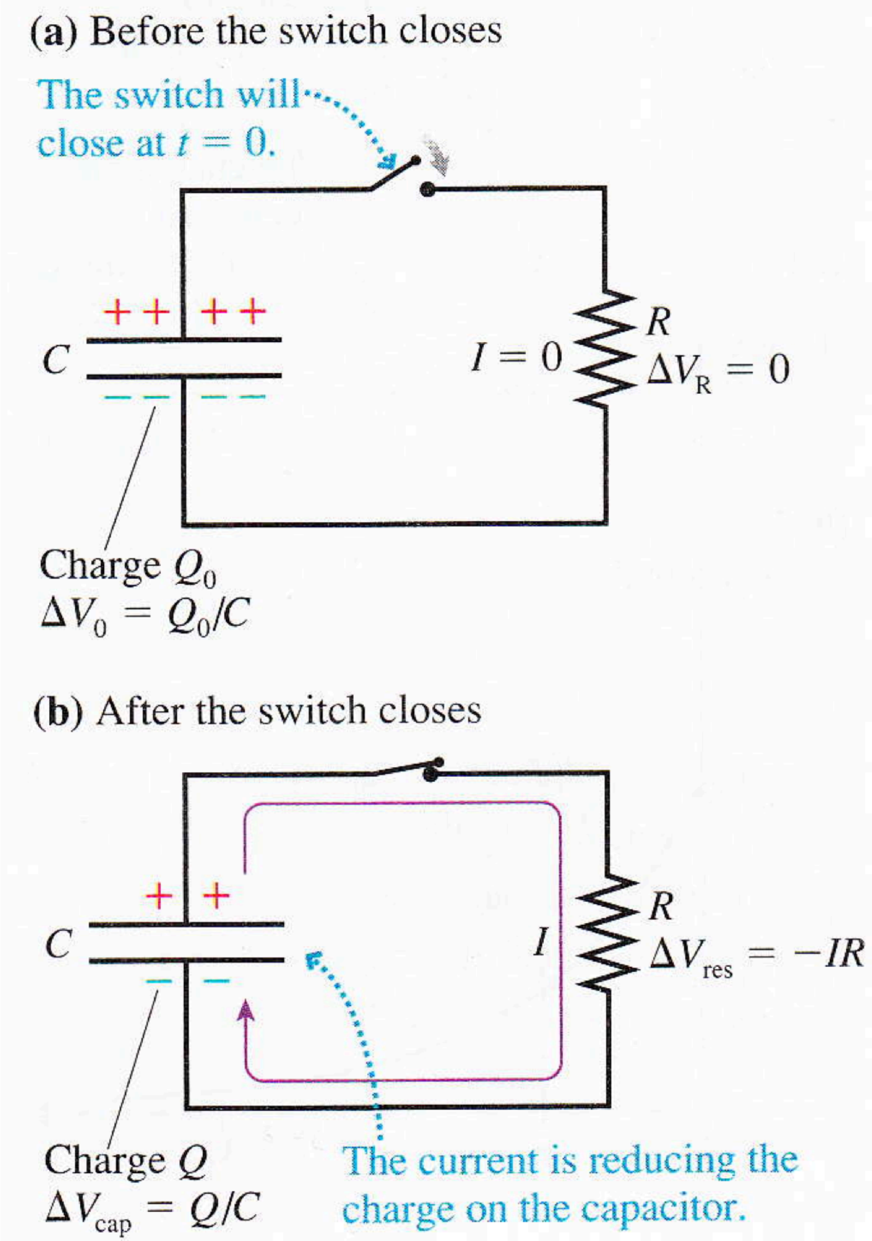
Negative because resistor current removes charge from capacitor

$$\frac{dQ}{dt} + \frac{Q}{RC} = 0 \qquad \frac{dQ}{Q} = -\frac{1}{RC}dt$$

$$\int_{Q_0}^{Q} \frac{dQ}{Q} = -\frac{1}{RC}\int_0^t dt$$

$$\ln Q\Big|_{Q_0}^{Q} = \ln Q - \ln Q_0 = \ln\left(\frac{Q}{Q_0}\right) = -\frac{t}{RC}$$

$$Q = Q_0 e^{-t/RC}$$




Knight

# Review: RC Circuits

$$Q = Q_0 e^{-t/RC}$$

$$Q = C\,\Delta V_C$$

$$\tau = RC$$

"RC time constant"

$$\Delta V_C = \Delta V_0 e^{-t/\tau}$$

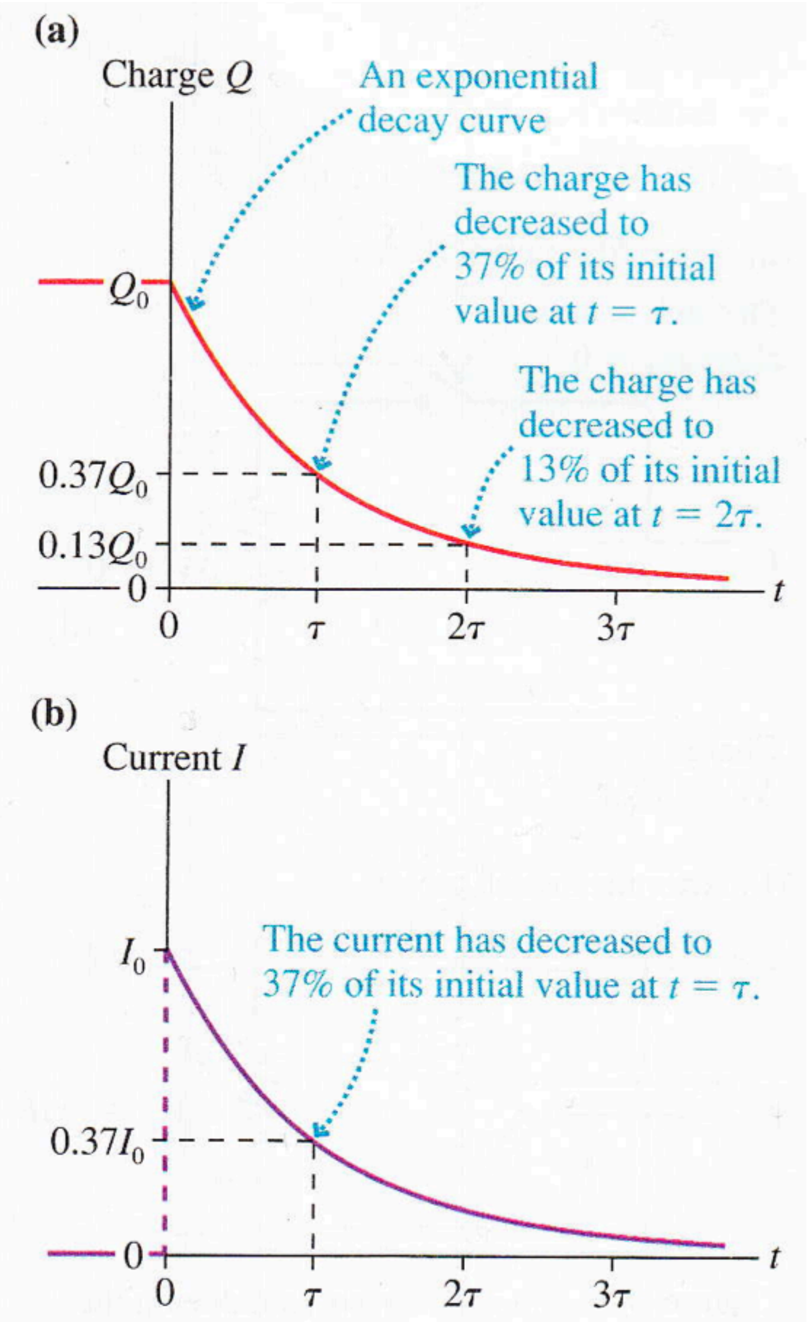
→ Resistor dissipates energy stored in the capacitor

Current through the capacitor?

$$I = \frac{dQ}{dt}$$

$$I_C = C\frac{dV_C}{dt}$$

$$Q = C\,V_C$$




Knight

# Review: RC Circuits

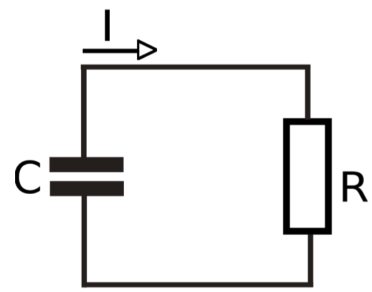
**DC** (some energy initially stored via charged capacitor) → KCL

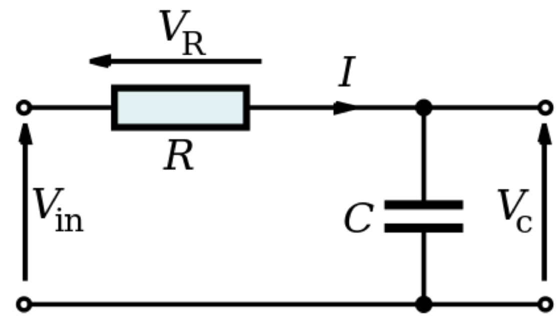
$$C\frac{dV}{dt} + \frac{V}{R} = 0$$

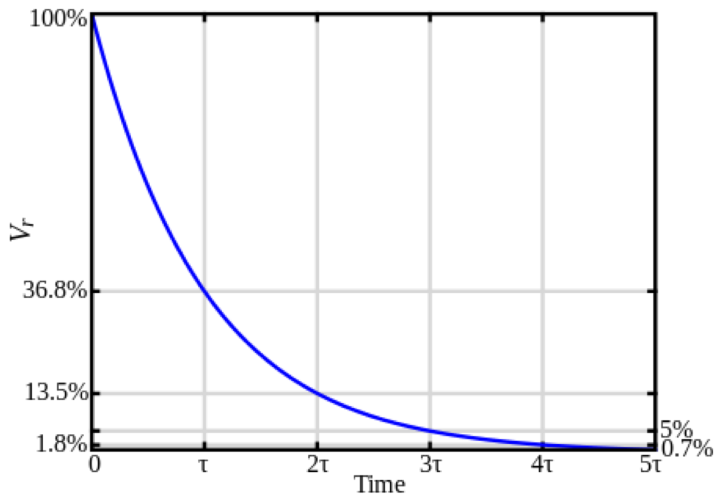
$$V(t) = V_o e^{-\frac{t}{RC}}$$

$$\tau = RC$$

"RC time constant"



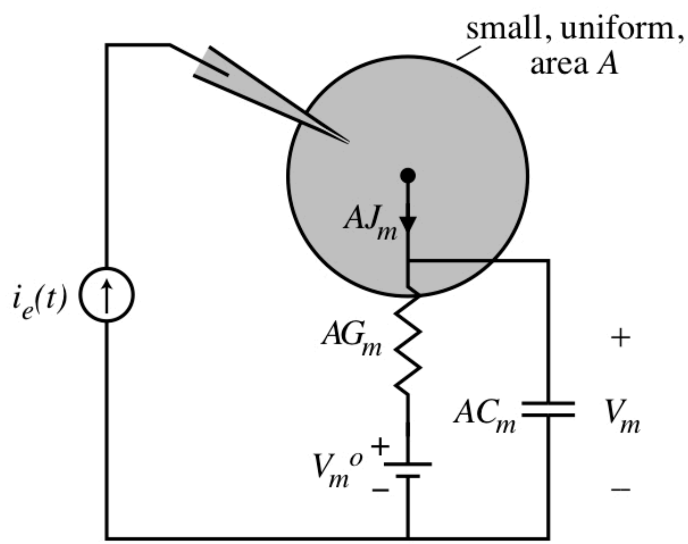

**AC** (sinusoidally-driven at $\omega$, steady-state) → KVL

Complex impedance

$$Z = R - \frac{i}{\omega C}$$

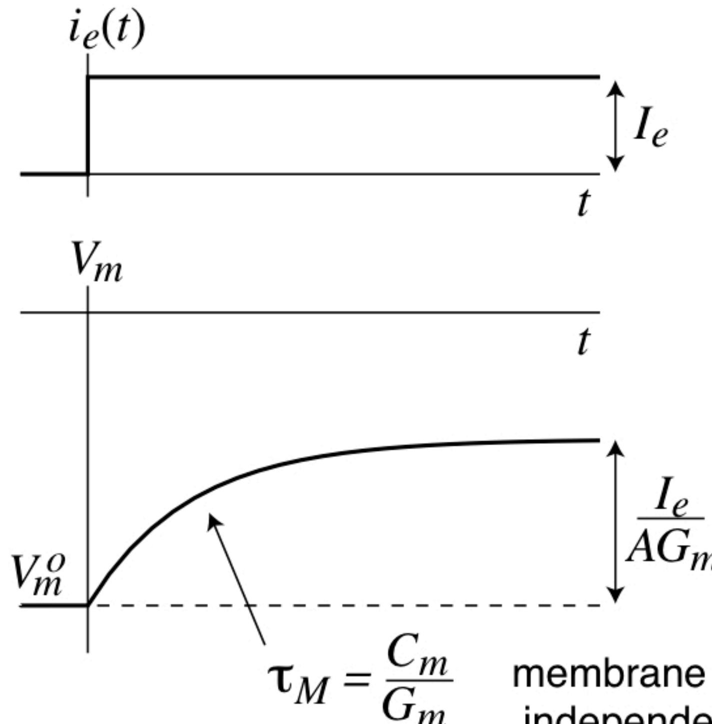
Think: RLC without the inductor

Wikipedia (RC circuit)



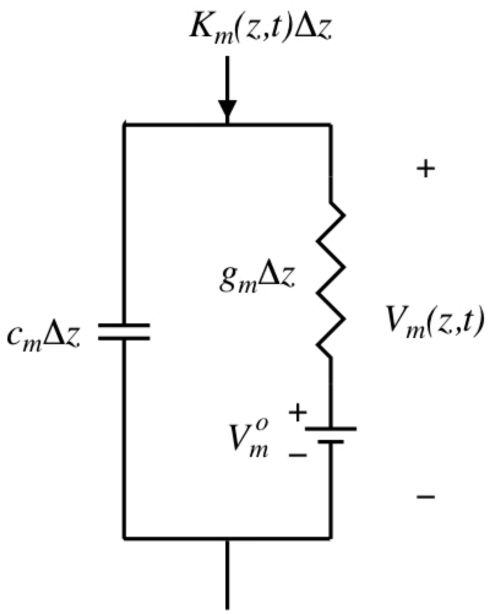

$$i_e(t) = AJ_m = AC_m \frac{dV_m}{dt} + AG_m(V_m - V_m^o)$$

$$\frac{AC_m}{AG_m}\frac{dV_m}{dt} + V_m = V_m^o + \frac{i_e(t)}{AG_m}$$




membrane time constant
independent of cell size

see Weiss Figure 3.8



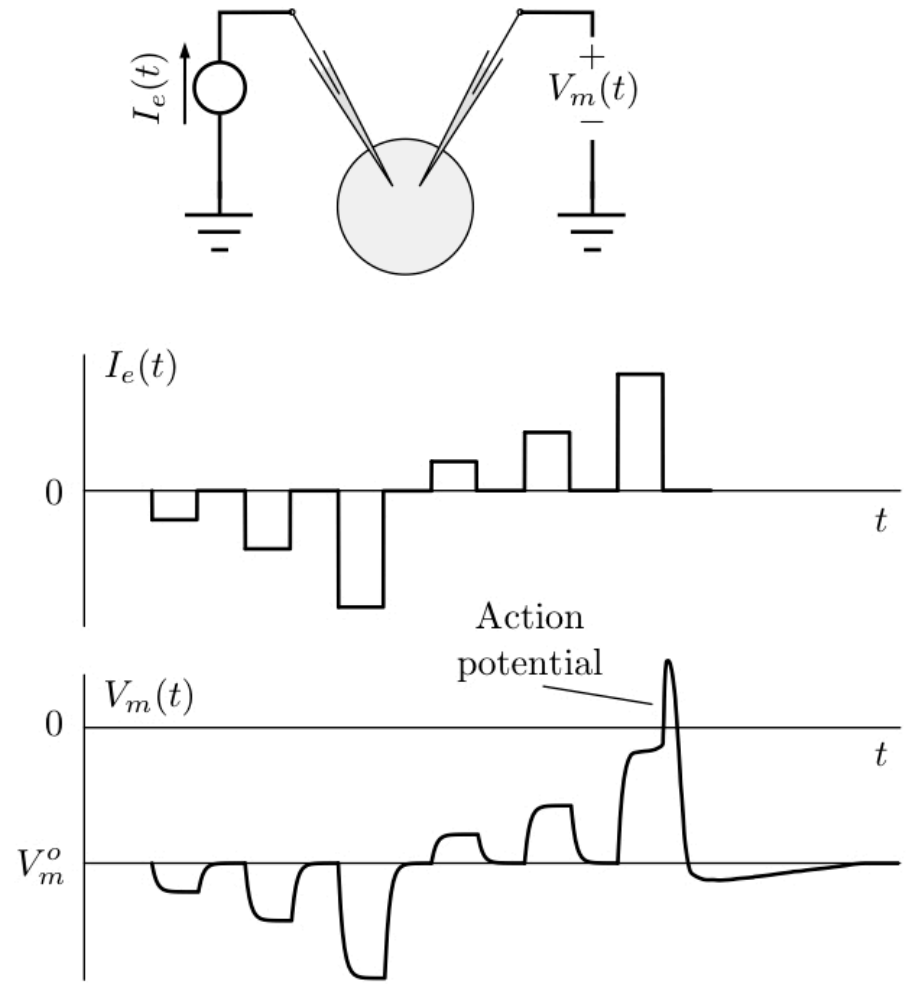




Figure 1.8

→ Delay explained

(but not action potentials yet!)

# Cable Model - History

- First solved by William Thomson (aka Lord Kelvin) in ~1855

- Motivated by Atlantic submarine cable for intercontinental telegraphy

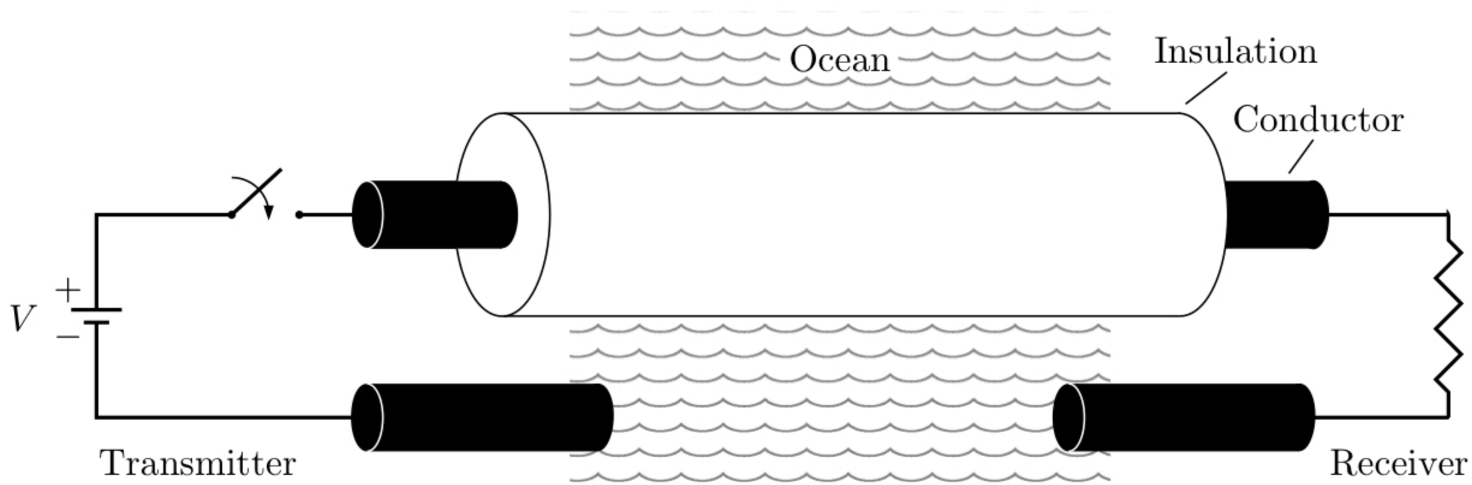



Figure 3.8

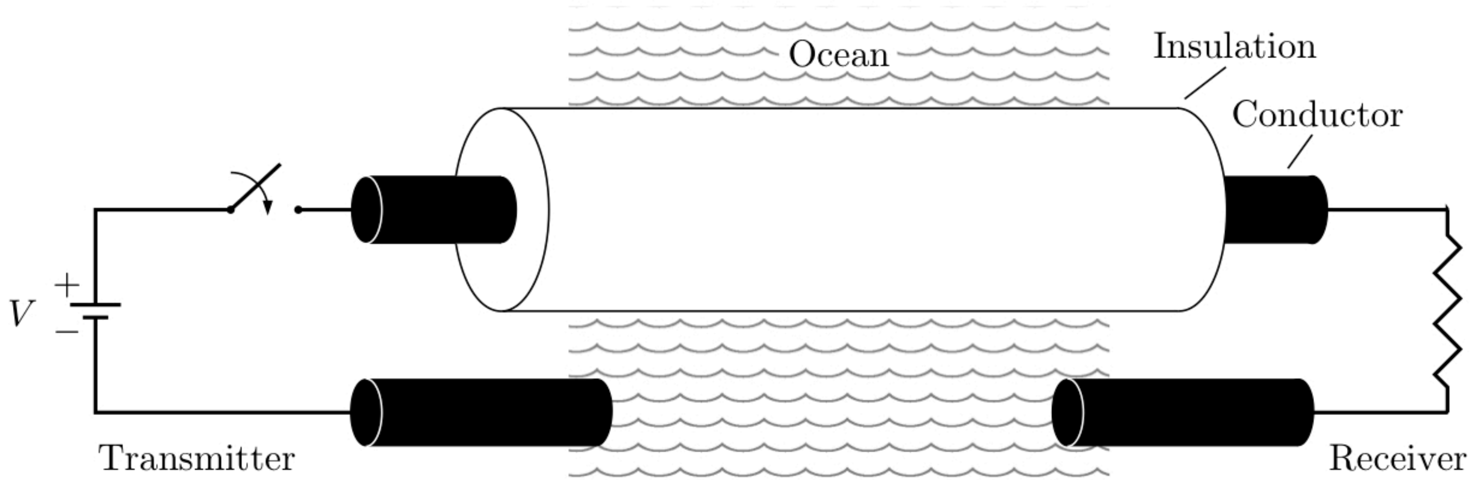


Figure 3.8

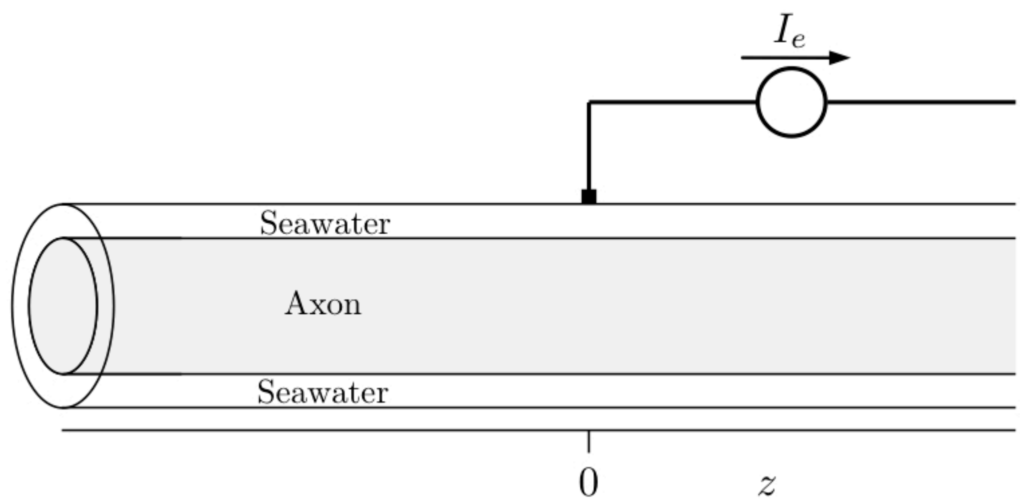


Figure 3.9

# Cable Model - Overview

- Uses the Core Conductor model as underlying basis

- Assumes membrane that it can be described as a parallel capacitance and conductance

- Linear

**Core Conductor Model**