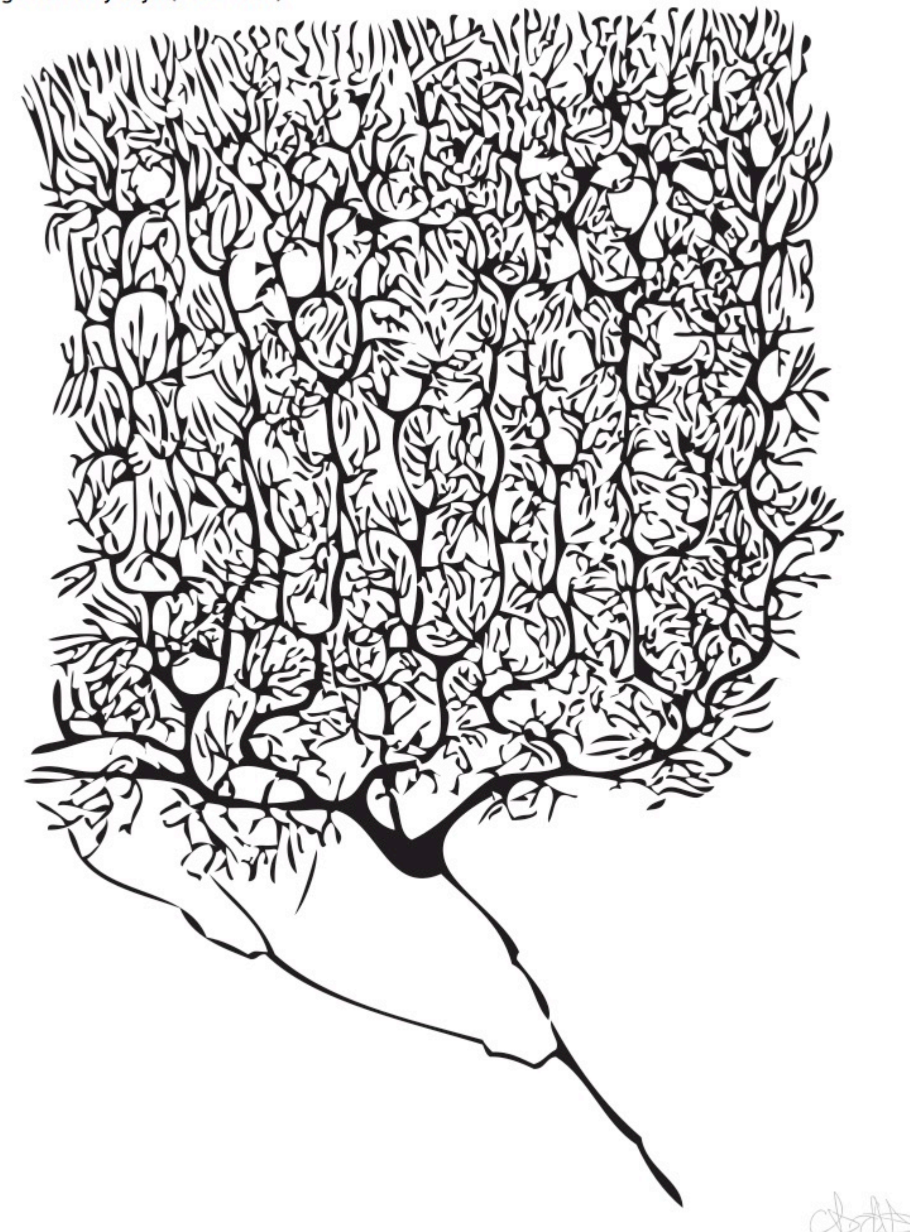
Cellular Electrodynamics

Santiago Ramón y Cajal (1852-1934)



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York University

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BPHS 4080 Lecture 15

Reference/Acknowledgement:

- TF Weiss (Cellular Biophysics)
- D Freeman

239. Tangled Stars

A

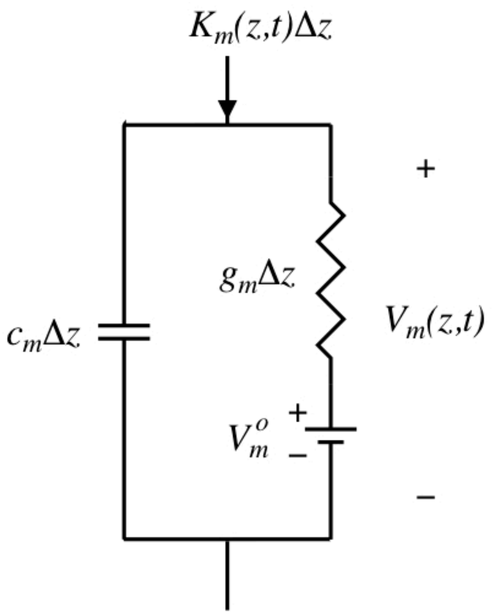
B

C

D

Three of the four tangled stars are identical.
Which one is different?

A B C D



→ Cell membrane acts like an RC filter

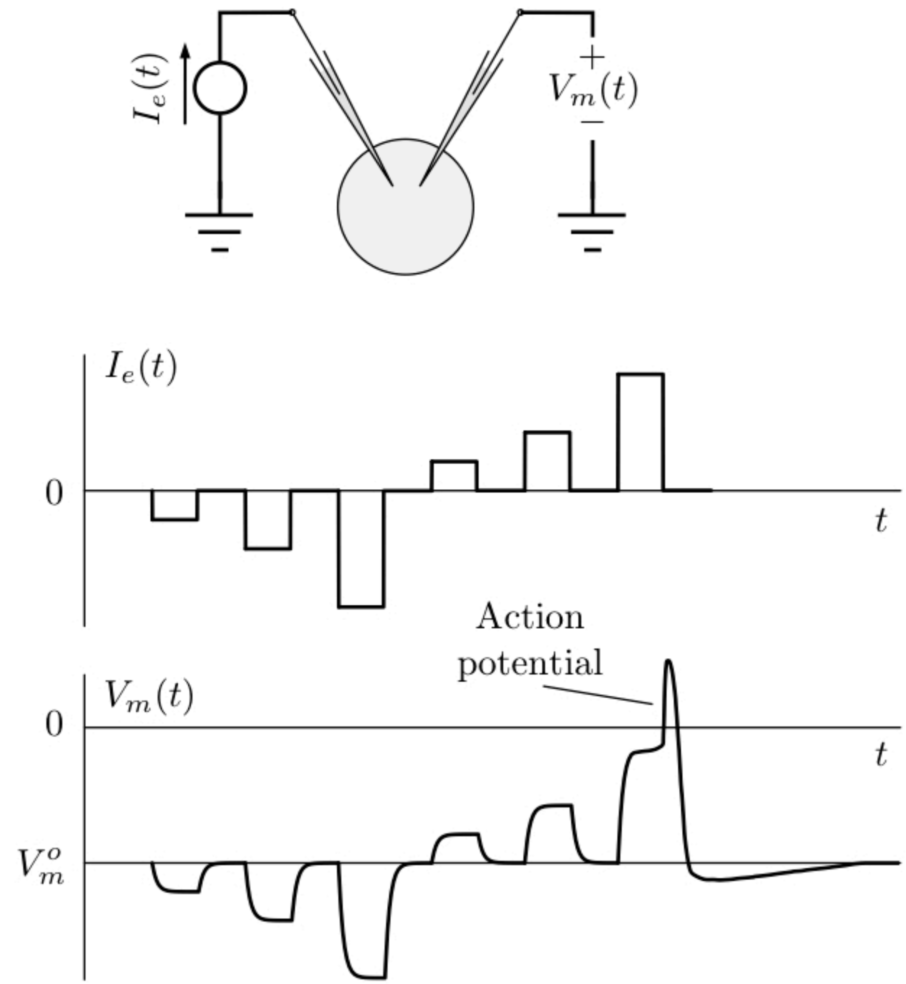
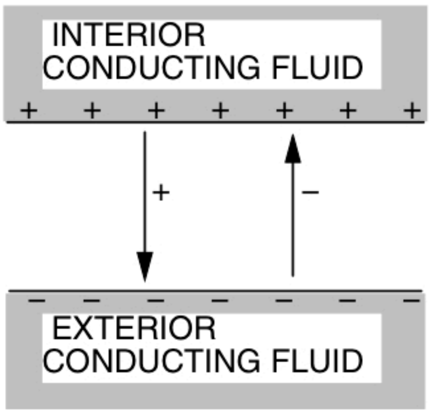
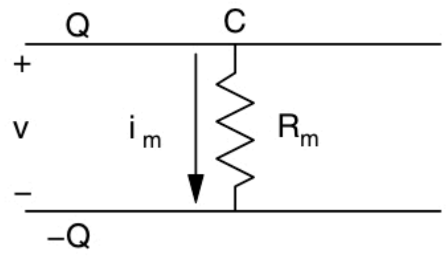
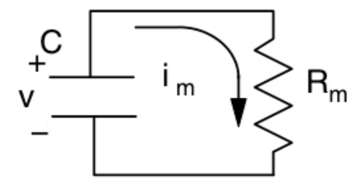


Figure 1.8

Circuit Representation







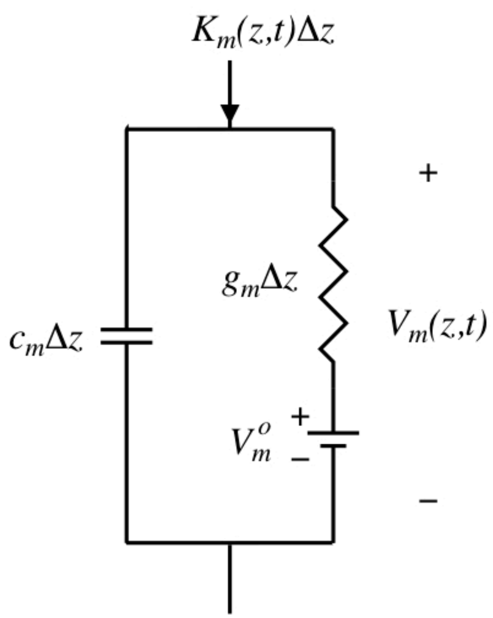


Figure 3.6

Resistor and capacitor in series
→ RC time constant

Hobbie & Roth
Weiss

<u>Cable Model</u> - History

- First solved by William Thomson (aka Lord Kelvin) in ~1855

- Motivated by Atlantic submarine cable for intercontinental telegraphy

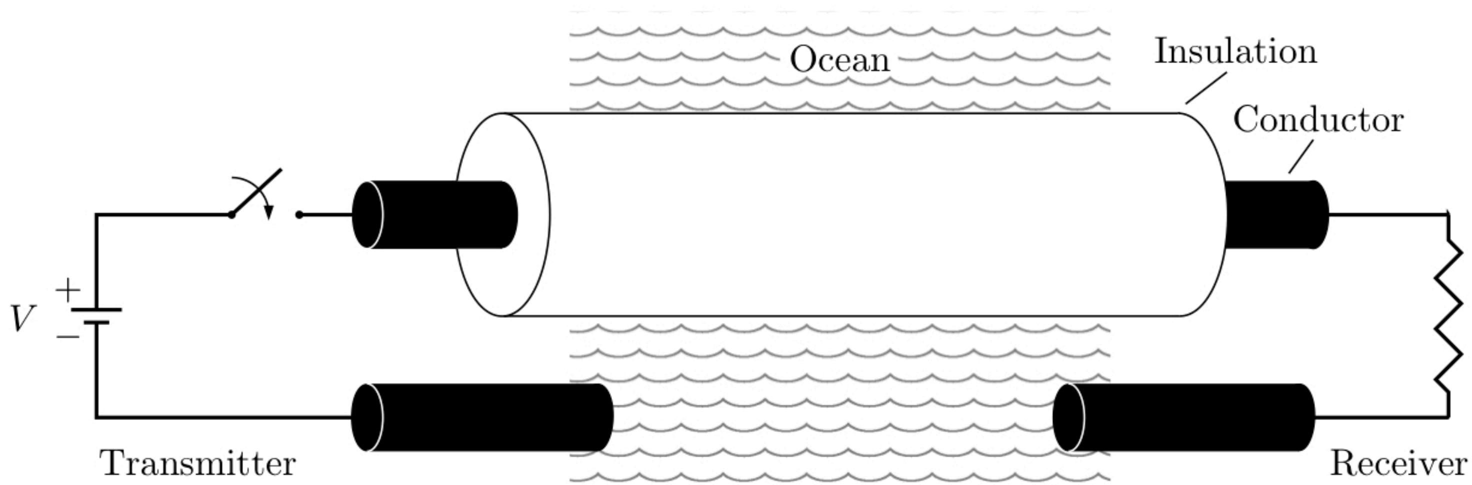


Figure 3.8

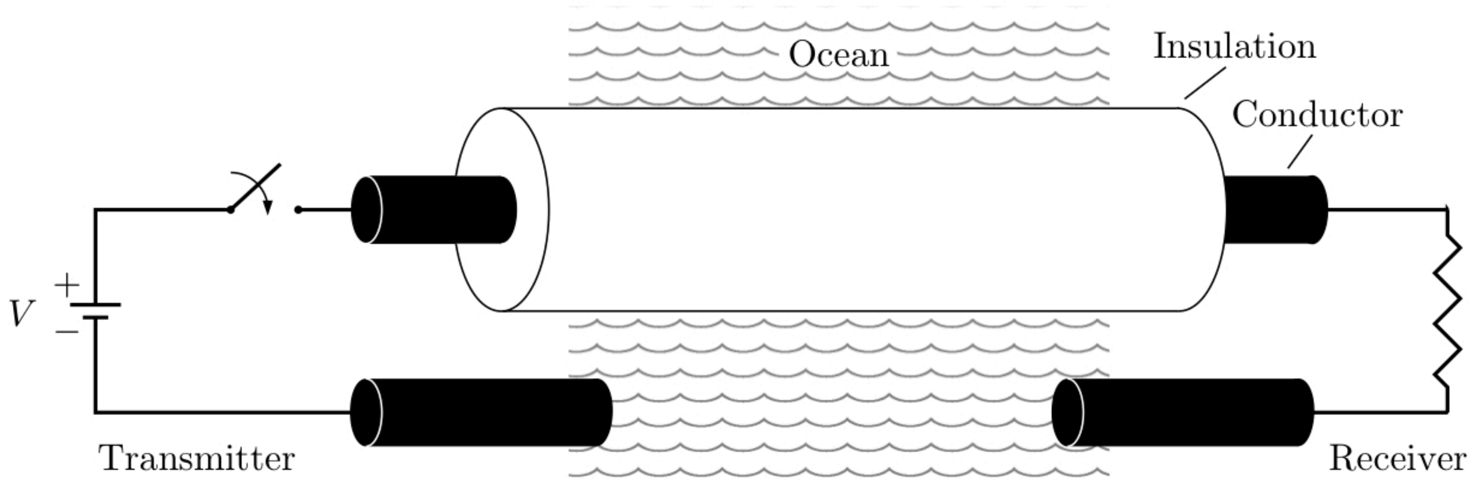


Figure 3.8

→ Axon behaves in fashion similar to a leaky submarine cable

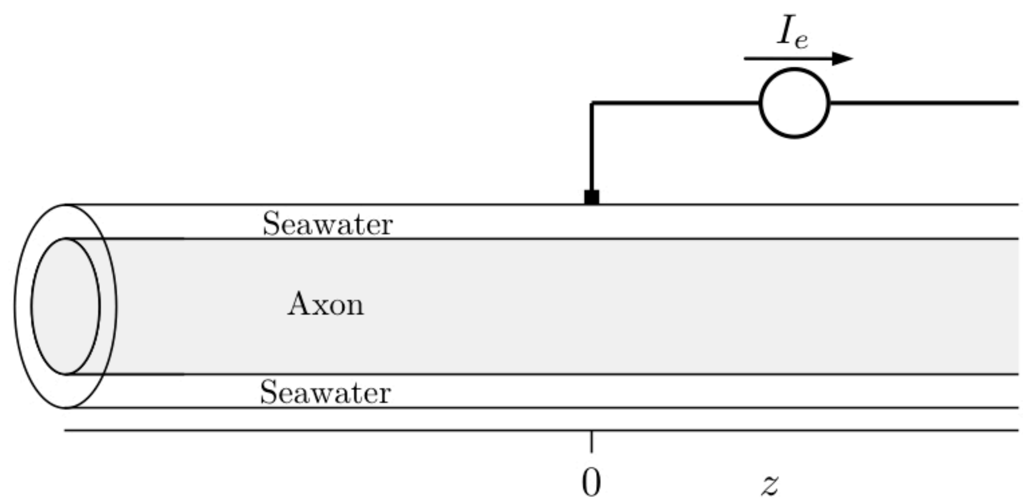
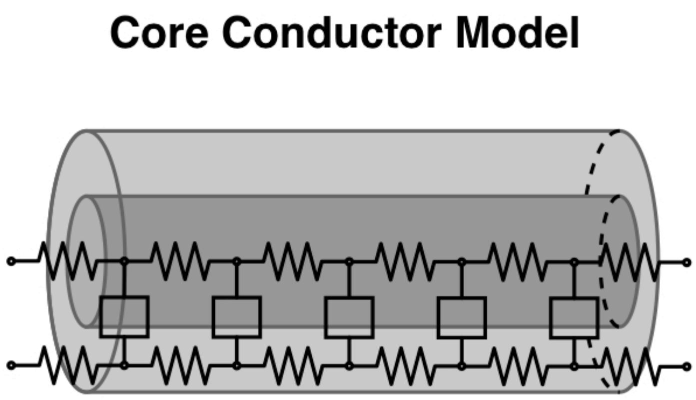
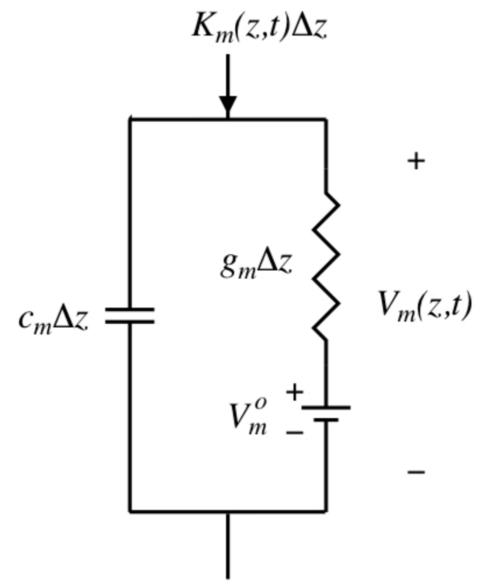


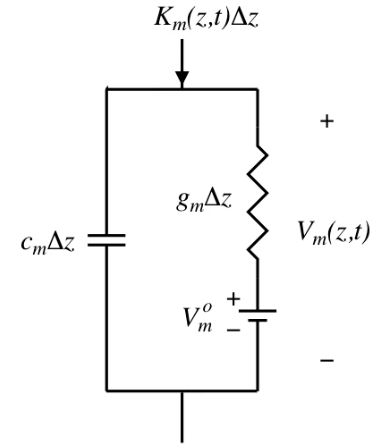
Figure 3.9

Cable Model - Overview





→ Combine together both "models"



For ΔV_m small:

$$K_m = 2\pi a J_m = 2\pi a C_m \frac{dV_m}{dt} + 2\pi a G_m (V_m - V_m^o) = c_m \frac{dV_m}{dt} + g_m (V_m - V_m^o)$$

Combine with core-conductor model:

$$\frac{\partial^2 V_m}{\partial z^2} = (r_o + r_i) K_m - r_o K_e = (r_o + r_i) \left[c_m \frac{\partial V_m}{\partial t} + g_m (V_m - V_m^o) \right] - r_o K_e$$

$$V_m + \frac{c_m}{g_m} \frac{\partial V_m}{\partial t} - \frac{1}{g_m (r_o + r_i)} \frac{\partial^2 V_m}{\partial z^2} = V_m^o + \frac{r_o}{g_m (r_o + r_i)} K_e$$

$$V_m + \frac{c_m}{g_m}\frac{\partial V_m}{\partial t} - \frac{1}{g_m(r_o + r_i)}\frac{\partial^2 V_m}{\partial z^2} = V_m^o + \frac{r_o}{g_m(r_o + r_i)}K_e$$

Introduce two new constants (τ_M and λ_C)

$$V_m + \tau_M \frac{\partial V_m}{\partial t} - \lambda_C^2 \frac{\partial^2 V_m}{\partial z^2} = V_m^o + r_o \lambda_C^2 K_e$$

Let $V_m = v_m + V_m^o$: (*incremental change in memb. potential*)

$$v_m + \tau_M \frac{\partial v_m}{\partial t} - \lambda_C^2 \frac{\partial^2 v_m}{\partial z^2} = r_o \lambda_C^2 K_e \quad \text{(Cable Equation)}$$

Summary

Cable Equation

Let $v_m(z,t) = V_m(z,t) - V_m^o$ and $|v_m(z,t)| << |V_m^o|$:

$$\boxed{v_m(z,t) + \tau_M \frac{\partial v_m(z,t)}{\partial t} - \lambda_C^2 \frac{\partial^2 v_m(z,t)}{\partial z^2} = r_o \lambda_C^2 K_e(z,t)}$$

Note:
Somewhat similar to the diffusion equation
(but not exactly due to extra v_m term)

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2}$$

Constants: τ_M and λ_C

$$V_m + \tau_M \frac{\partial V_m}{\partial t} - \lambda_C^2 \frac{\partial^2 V_m}{\partial z^2} = V_m^o + r_o \lambda_C^2 K_e$$

Space constant (λ_C) - property of cell, not just membrane

$$\lambda_C = \frac{1}{\sqrt{(r_i + r_o) g_m}} \approx \sqrt{\frac{a}{2\rho_i G_m}}$$

(assuming $r_o << r_i$)

Wider axons → Further propagation/less degradation

Time constant (τ_M) – independent of cellular dimensions

$$\tau_M = \frac{c_m}{g_m}$$

Cable Equation

Let $v_m(z,t) = V_m(z,t) - V_m^o$ and $|v_m(z,t)| << |V_m^o|$:

$$\boxed{v_m(z,t) + \tau_M \frac{\partial v_m(z,t)}{\partial t} - \lambda_C^2 \frac{\partial^2 v_m(z,t)}{\partial z^2} = r_o \lambda_C^2 K_e(z,t)}$$

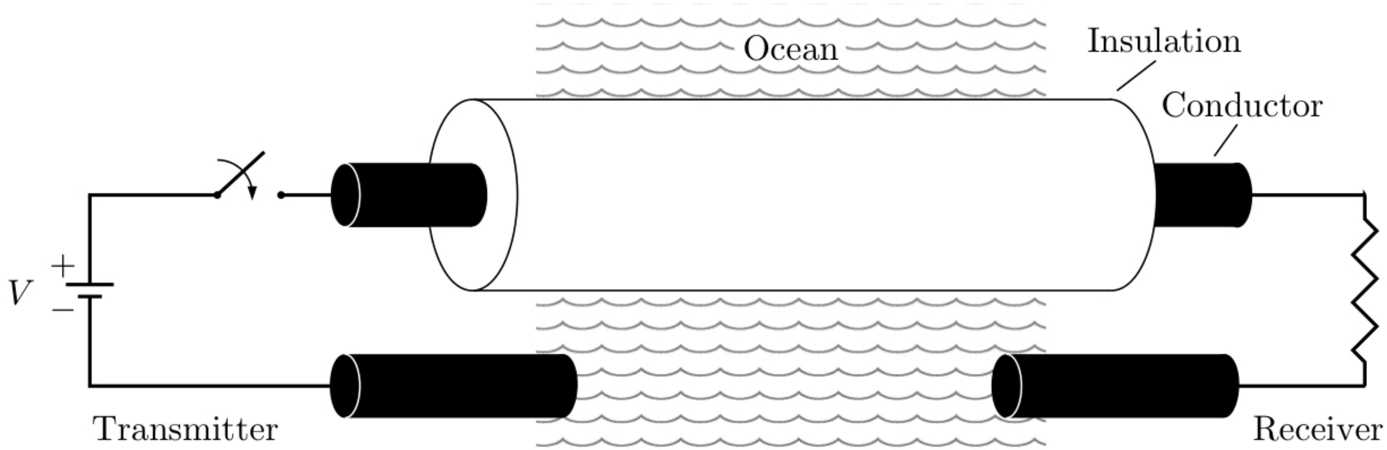


Figure 3.8

Axon ←→ Leaky submarine 'cable'

Cable Model – Solution for spatial impulse

Time-independent solution
→ constantly applied current

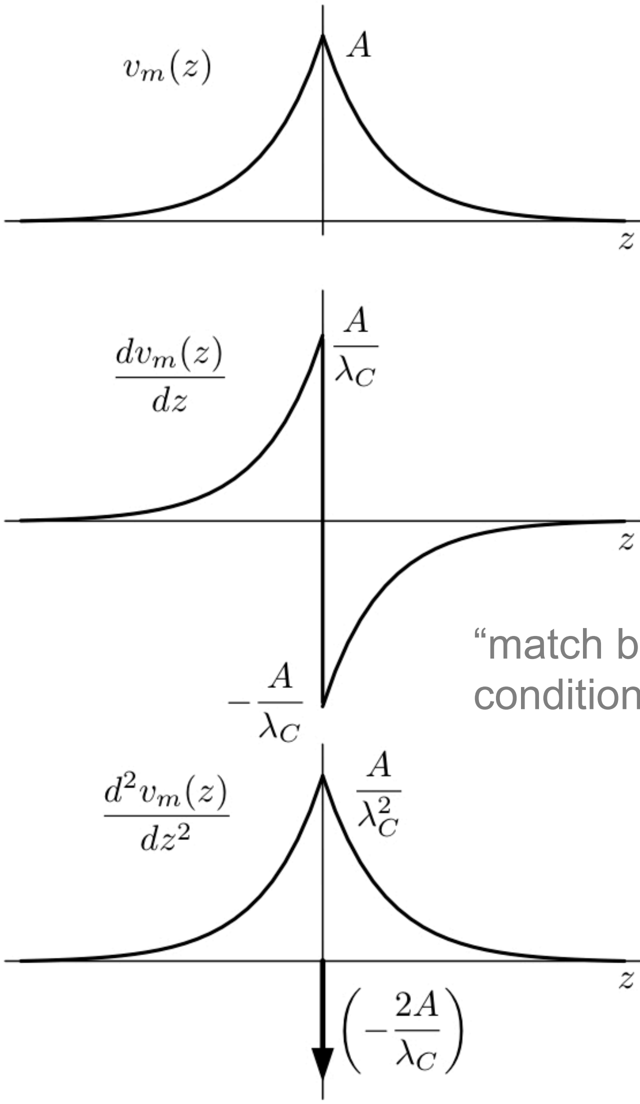


Figure 3.10

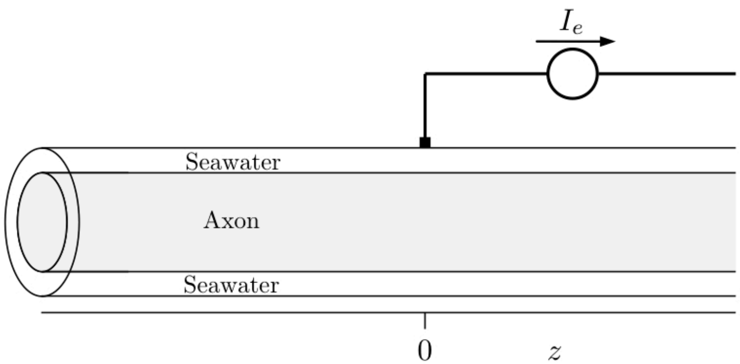


Figure 3.9

"match boundary conditions" at z=0

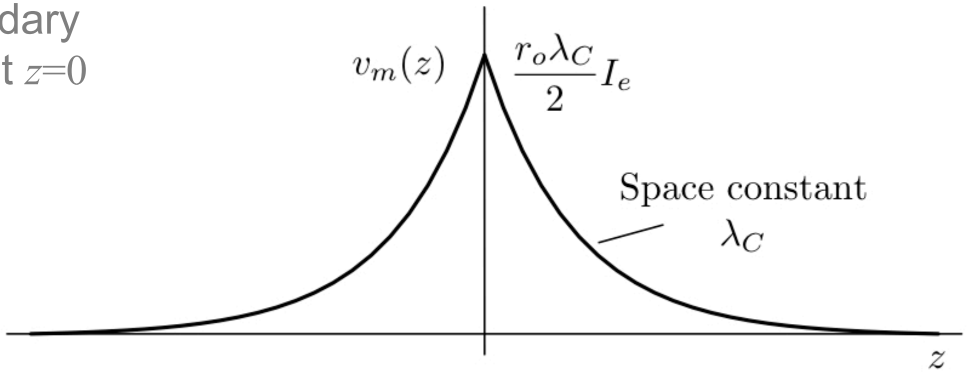


Figure 3.11

→ Amplitude falls off (re space const.)

<u>Cable Model</u> – Space constant

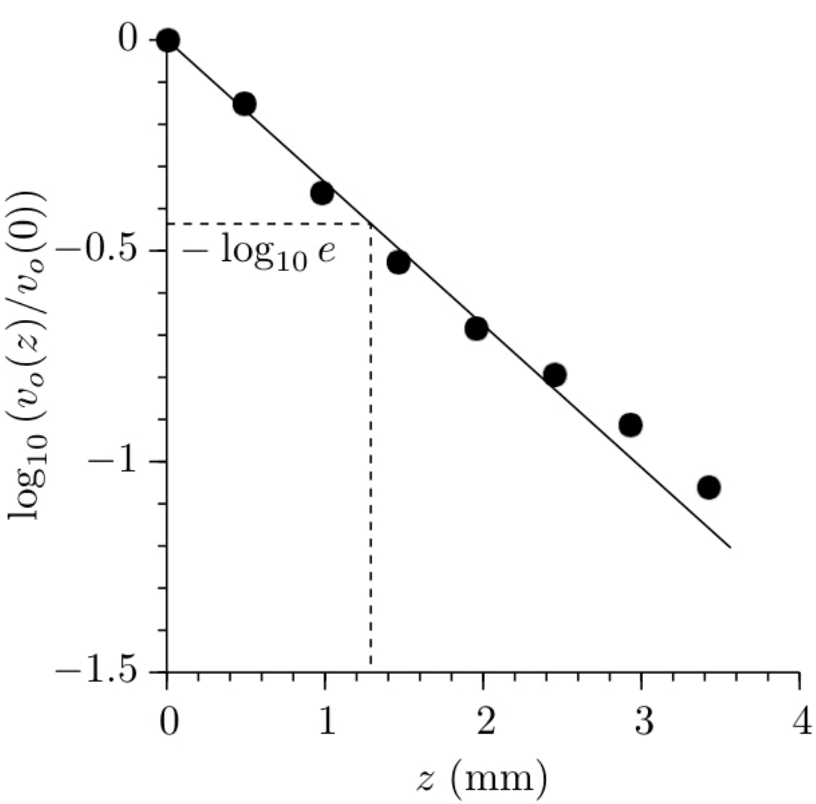


Figure 3.20

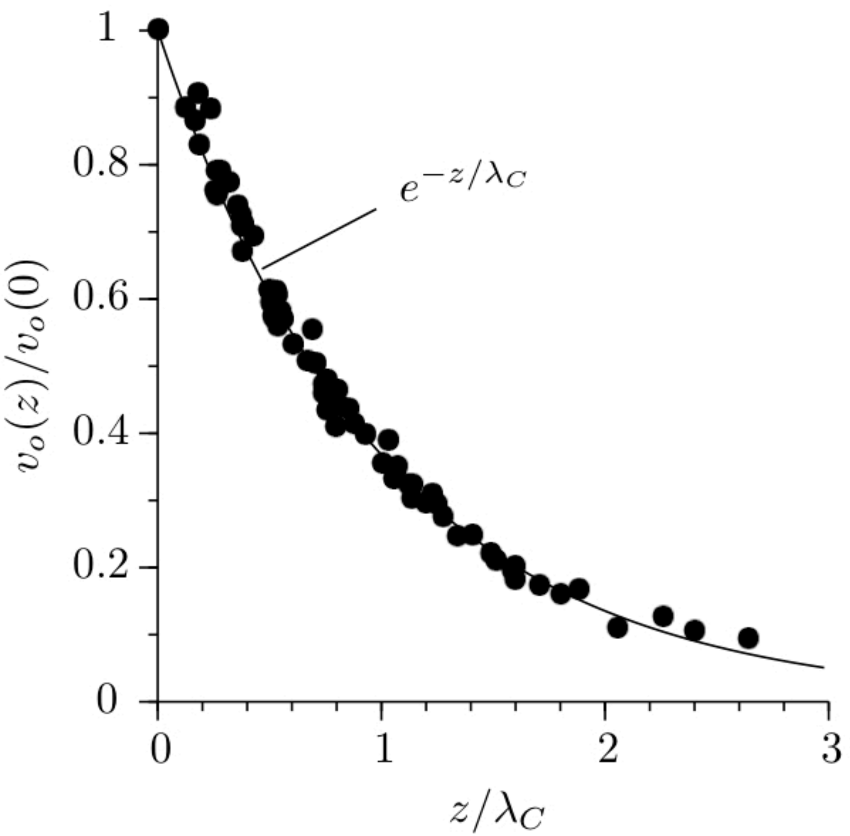


Figure 3.21

→ Space constant (λ_c) typically on order of mm (even less for small unmyelinated fibers)

→ Solutions allow for propagation, but in a decremental fashion

→ Axons alone are not good 'cables' for sending signals long-ish distances!

Cable Model – Solution for temporal & spatial impulse

Assume infinitesimal electrode and $i_e(t)$ brief so that

$$k_e(z,t) = 0 \ ; \quad \text{if } z \neq 0 \text{ or } t \neq 0$$

For $t \neq 0$ or $z \neq 0$

$$v_m(z,t) + \tau_M \frac{\partial v_m}{\partial t} - \lambda_C^2 \frac{\partial^2 v_m}{\partial z^2} = 0$$

Let

$$v_m(z,t) = w(z,t) e^{-t/\tau_M}$$

Then

$$\frac{\partial v_m}{\partial t} = -\frac{1}{\tau_M} w(z,t) e^{-t/\tau_M} + \frac{\partial w}{\partial t} e^{-t/\tau_M}$$

$$\frac{\partial^2 v_m}{\partial z^2} = \frac{\partial^2 w}{\partial z^2} e^{-t/\tau_M}$$

Cable Model – (A) Solution

Substituting,

$$w(z,t)e^{-t/\tau_M} - w(z,t)e^{-t/\tau_M} + \tau_M \frac{\partial w}{\partial t} e^{-t/\tau_M} - \lambda_C^2 \frac{\partial^2 w}{\partial z^2} e^{-t/\tau_M} = 0$$

$$\tau_M \frac{\partial w}{\partial t} = \lambda_C^2 \frac{\partial^2 w}{\partial z^2}$$

Solving cable equation (here w/ change of variable) is like diffusion equation!

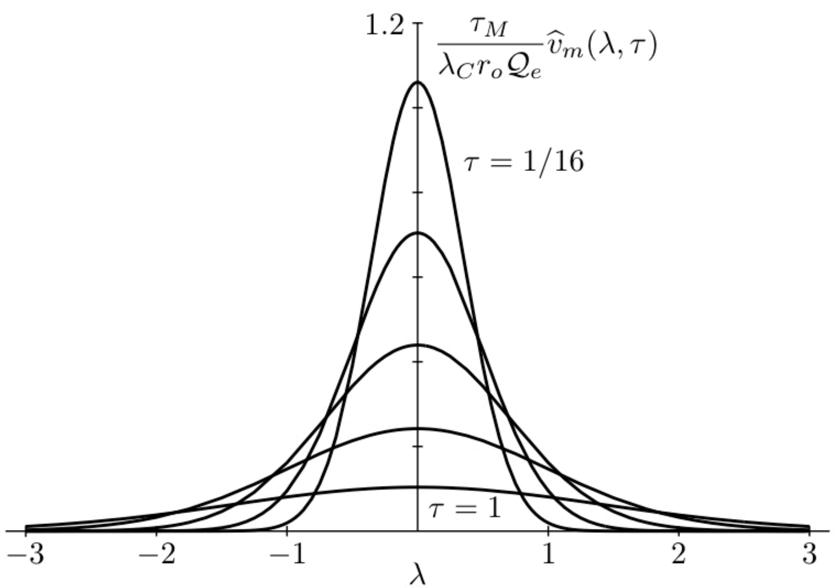


Figure 3.23

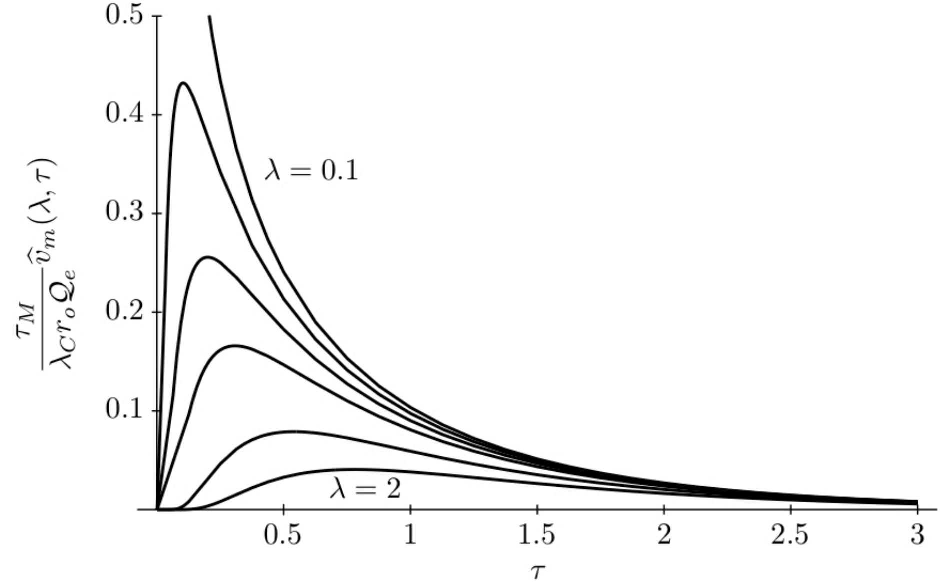


Figure 3.24

Cable Model – (A) Solution

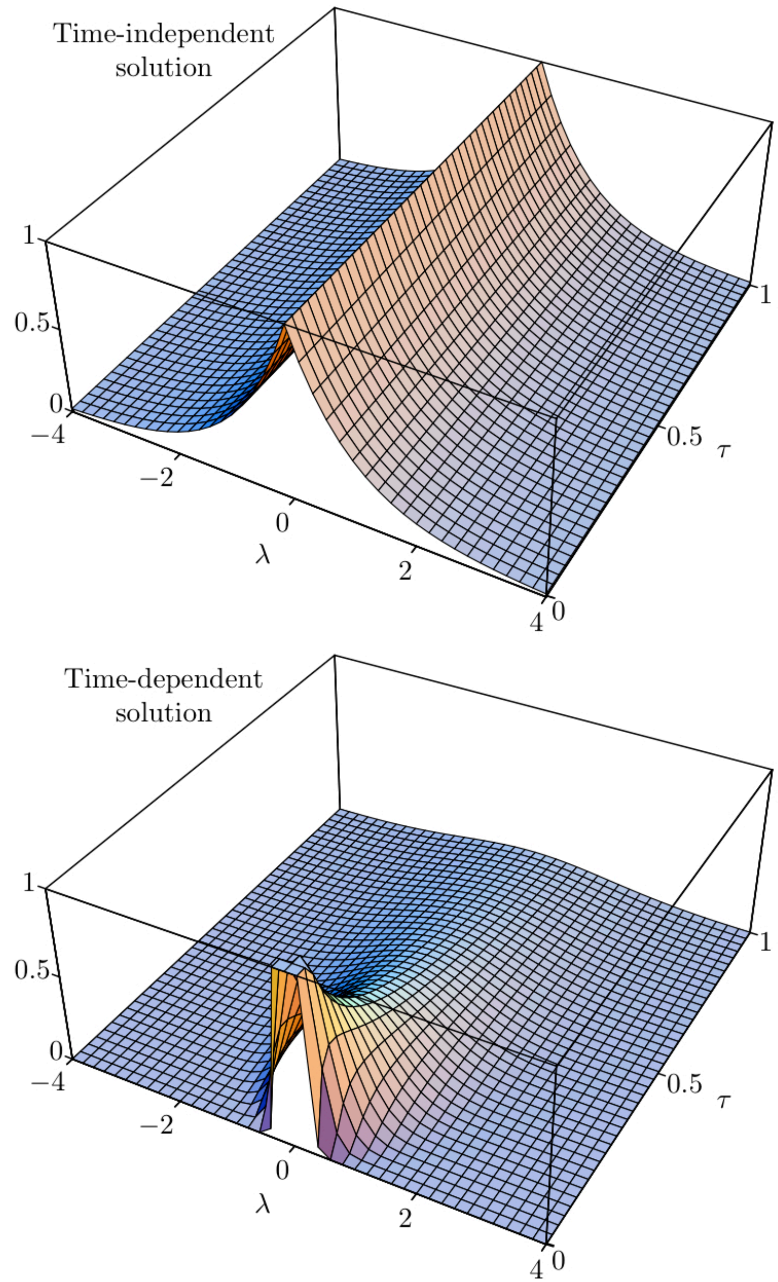


Figure 3.25

Cable Model – General Properties

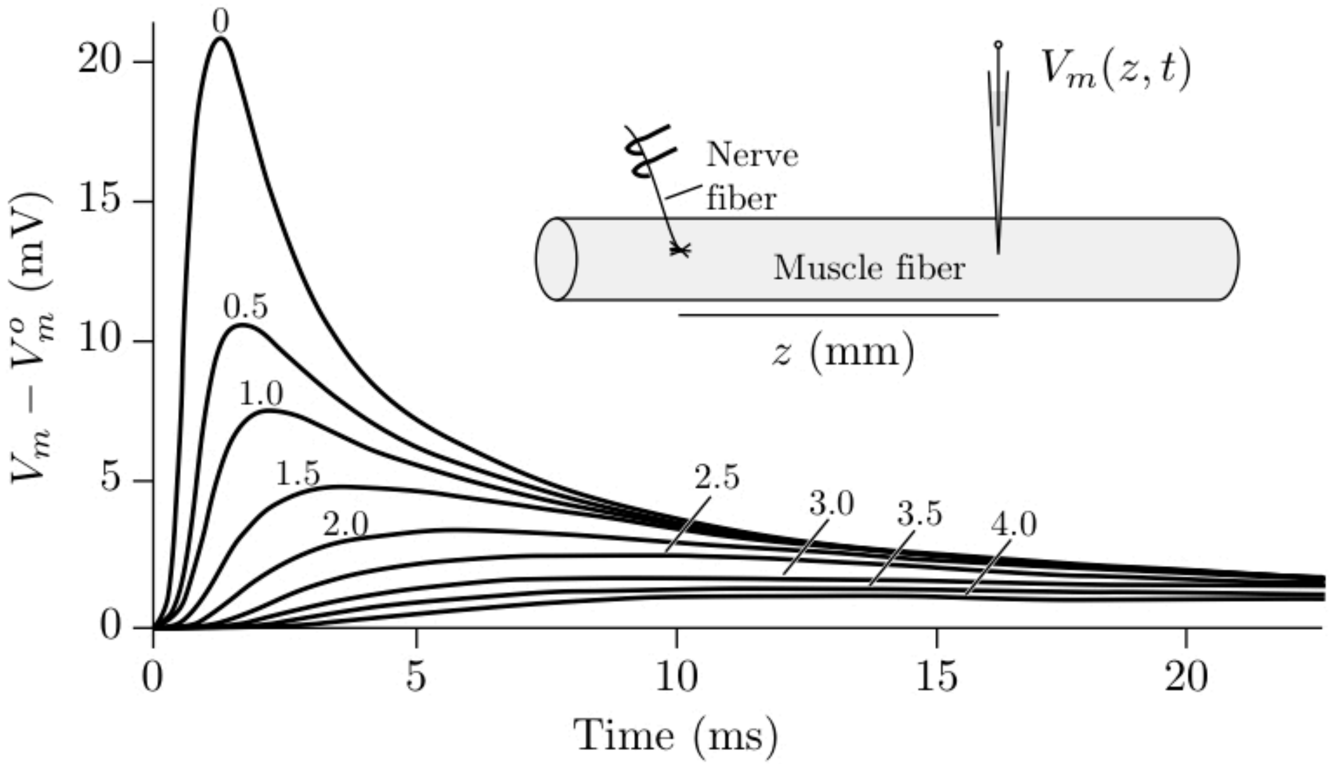


Figure 2.5

→ Solutions allow for propagation, but in a decremental fashion

Cable Model – General Properties

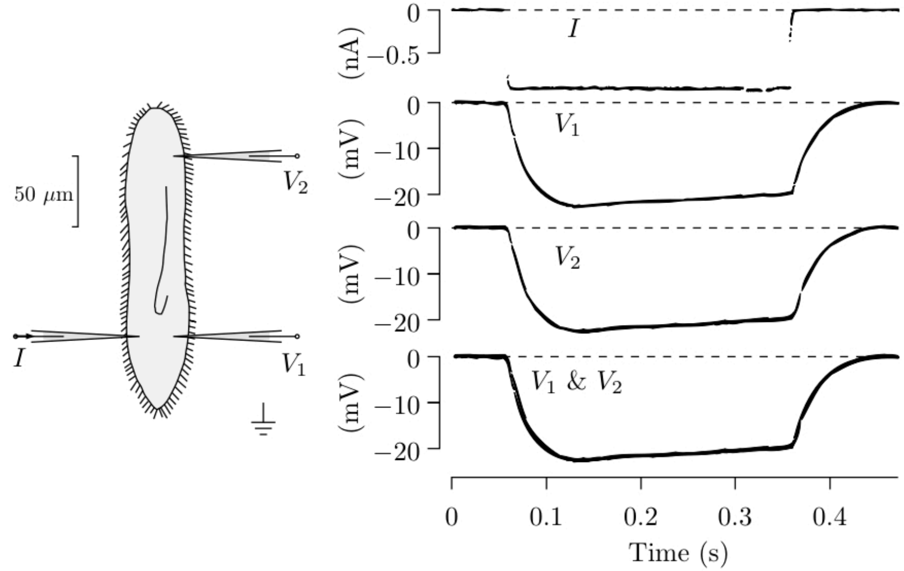


Figure 2.3

→ Cellular dimensions re space constant (λ_C) determine whether a cell is electrically *small* or *large*

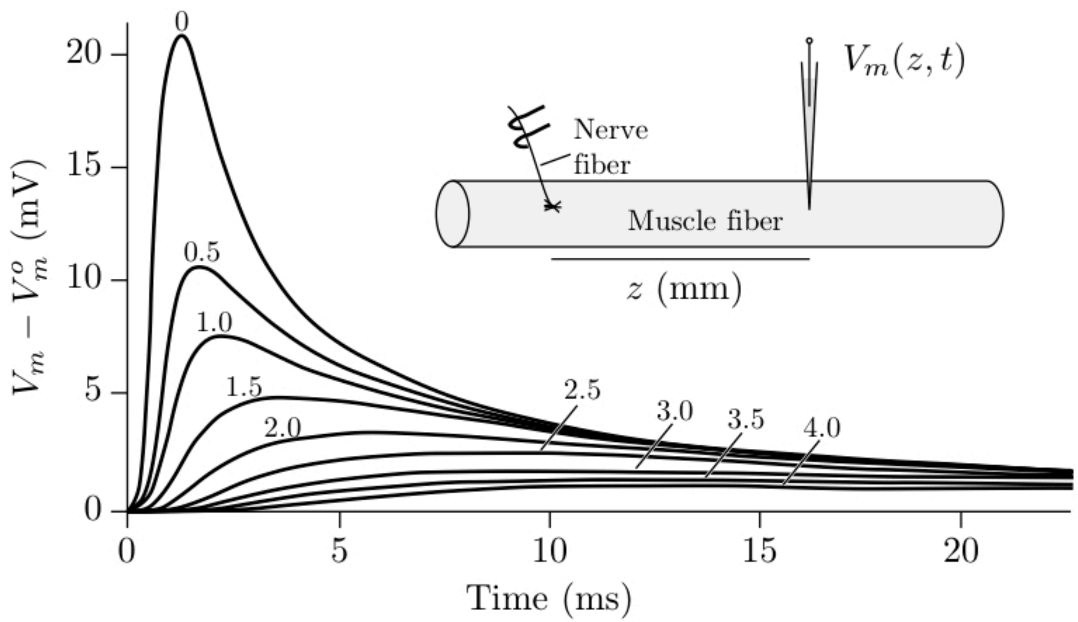


Figure 2.5

Ex.

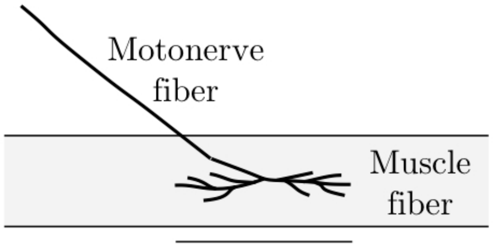
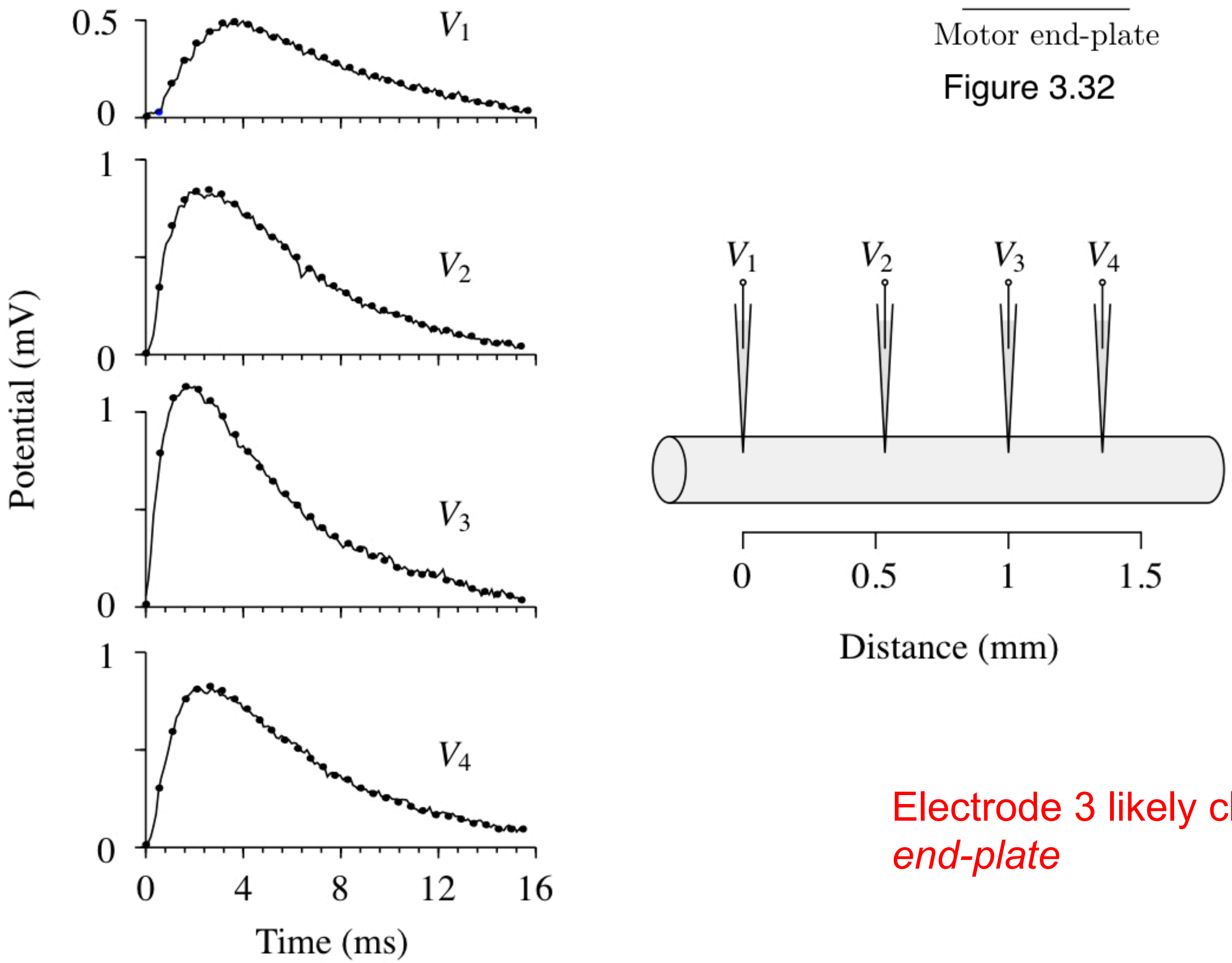


Figure 3.32



Electrode 3 likely closest to *end-plate*

Figure 3.33

Linearity

Linearity → Superpostion

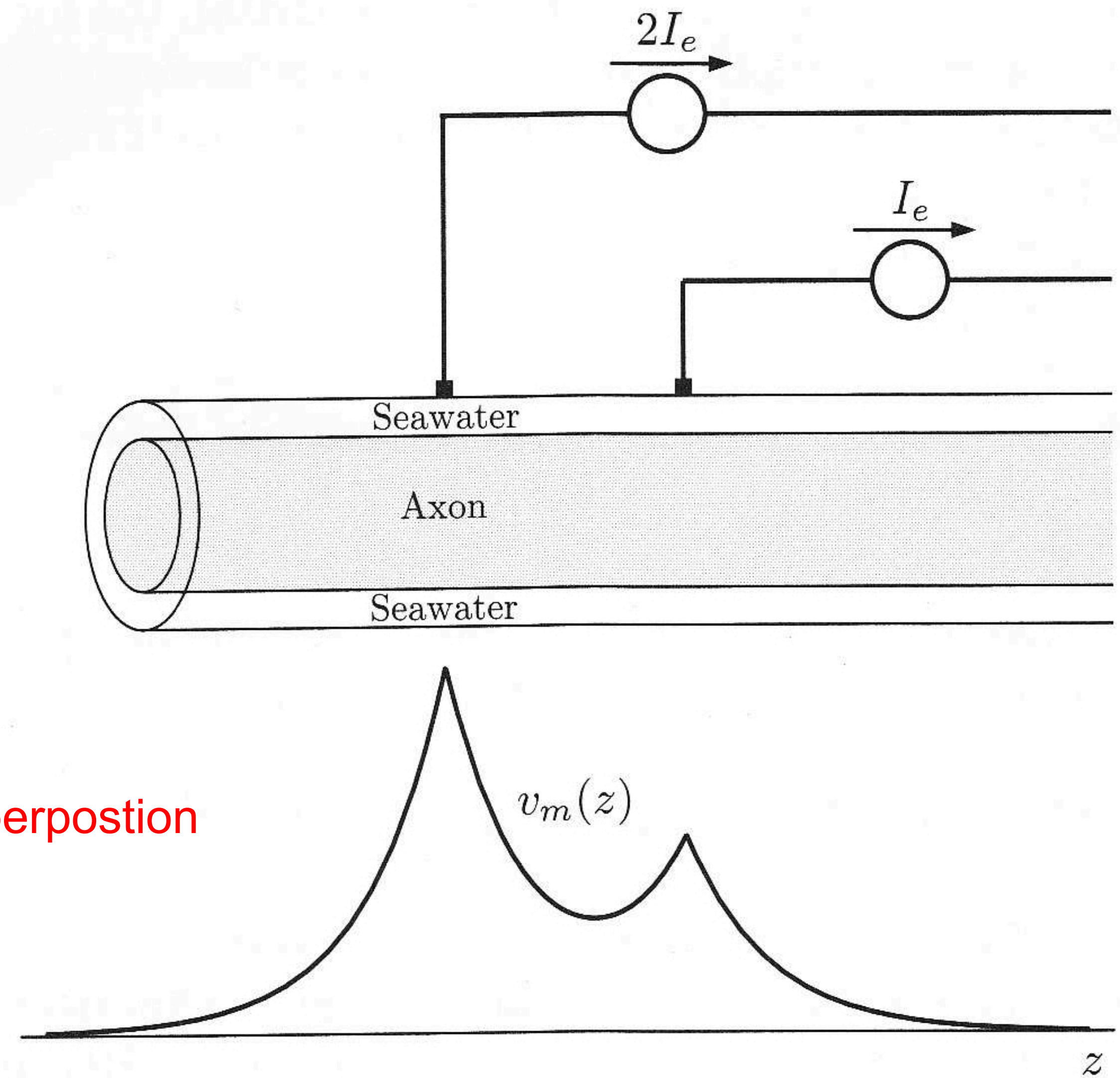


Figure 3.19

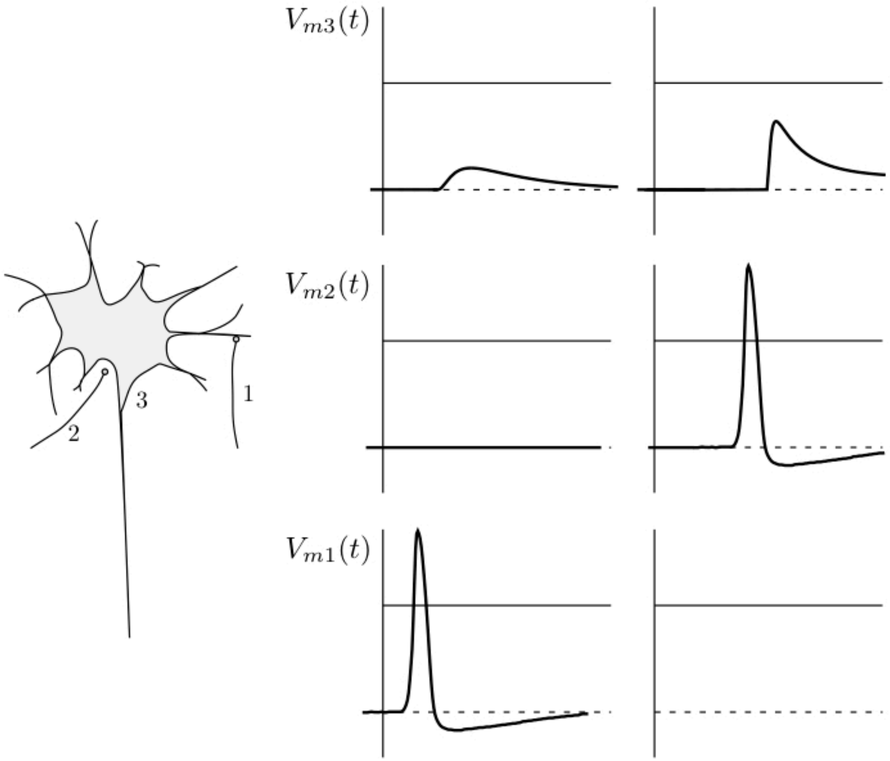


Figure 3.34

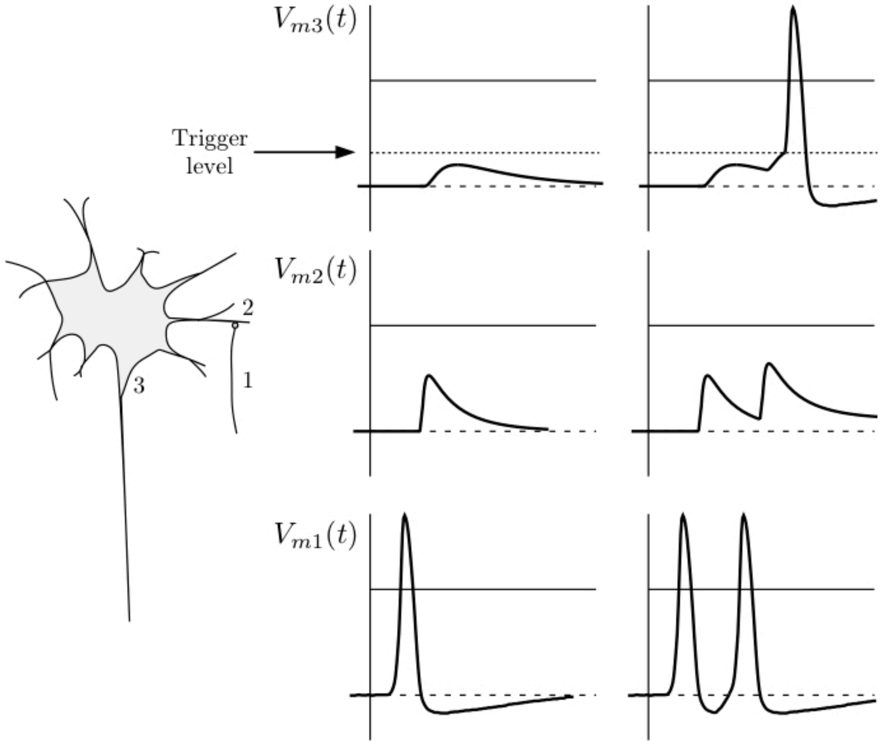


Figure 3.35

“Electronic distance”

“Temporal integration”

→ Key considerations with regard to synapses (i.e., inter-neuron communication)

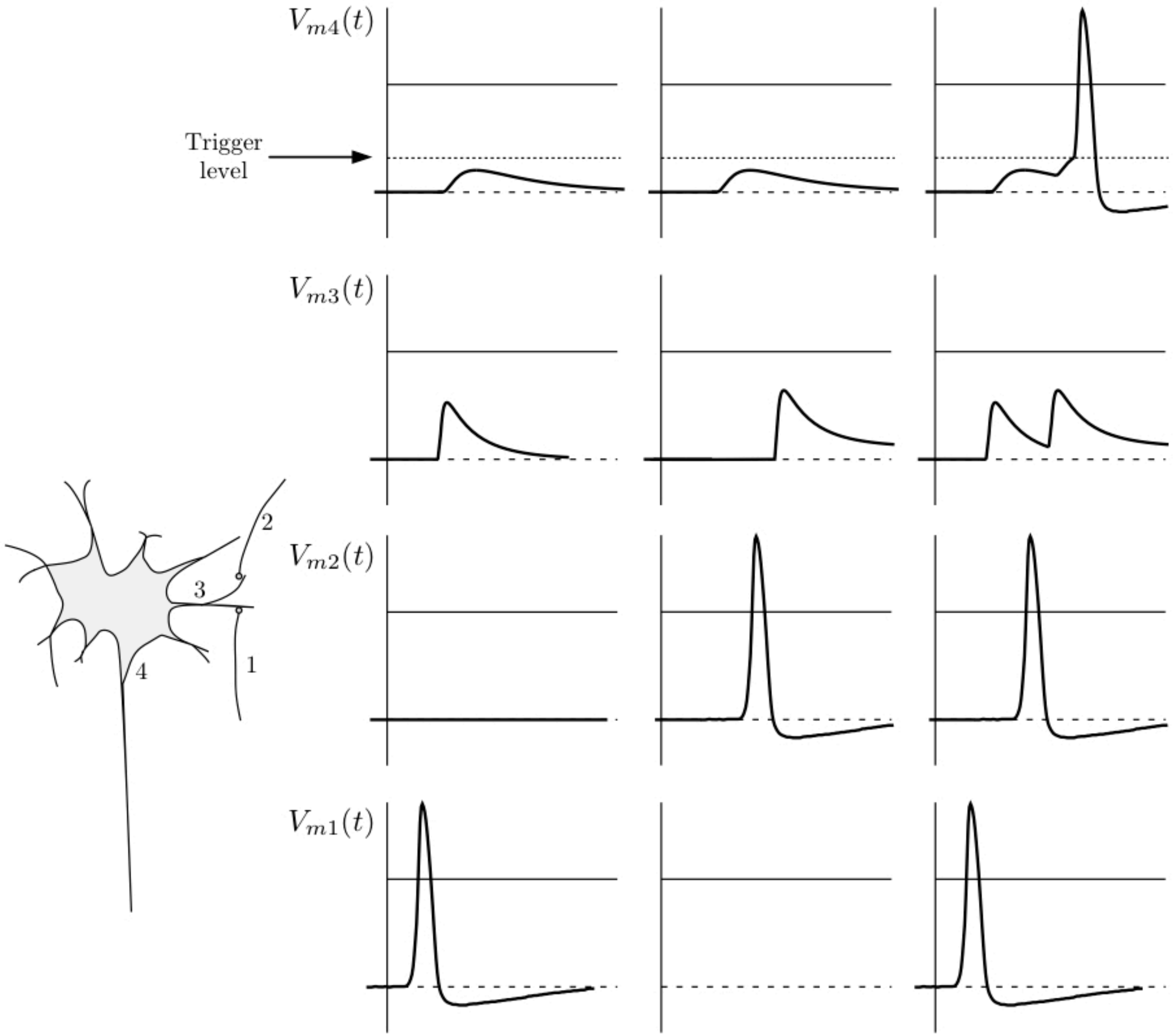
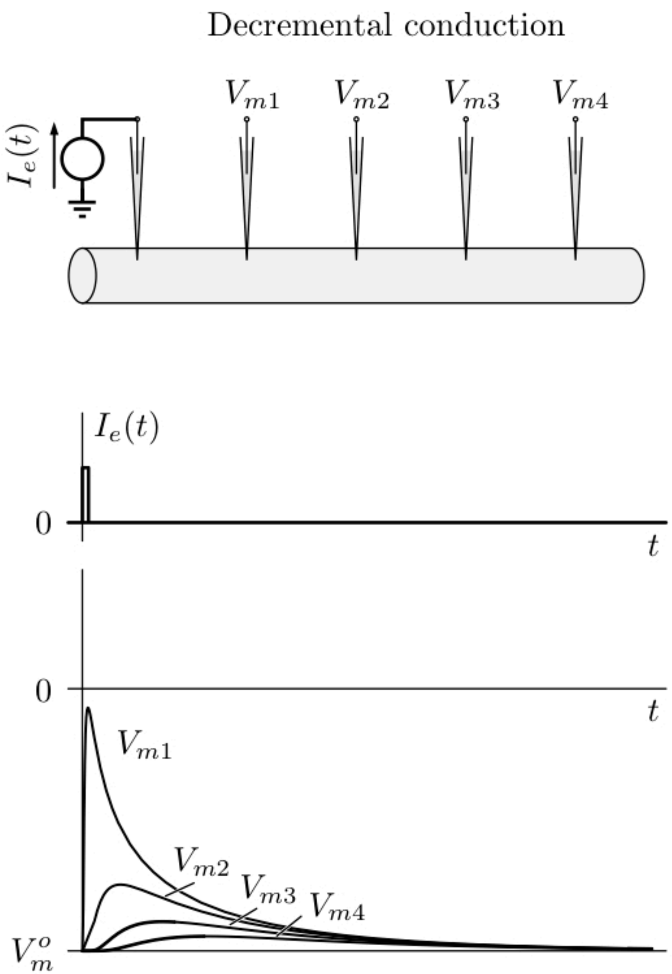


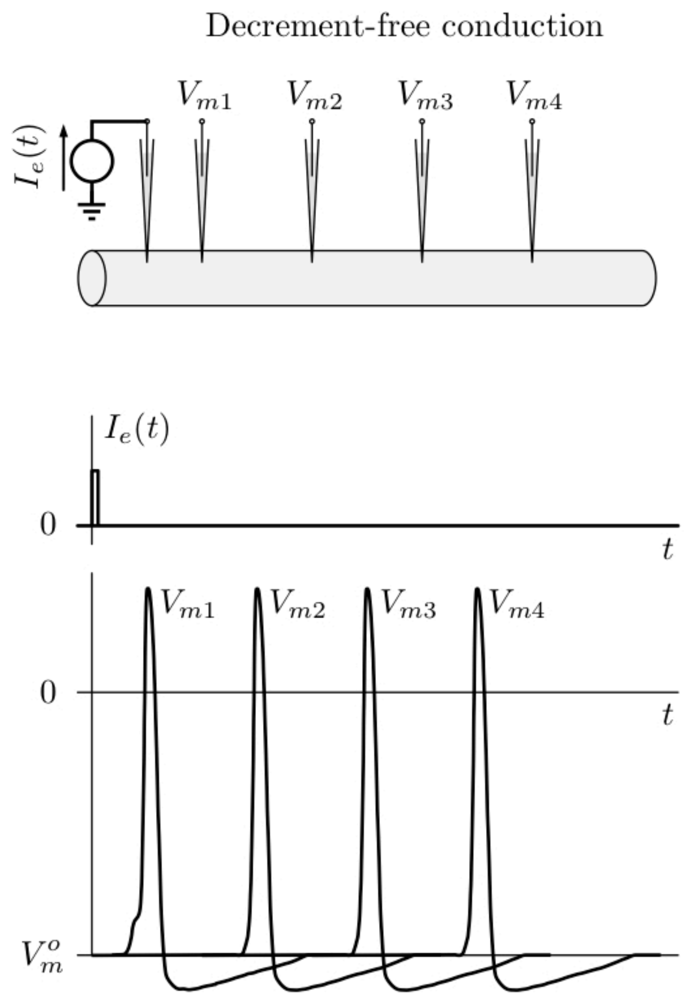
Figure 3.36

Spatial integration

Looking Ahead: Hodgkin-Huxley

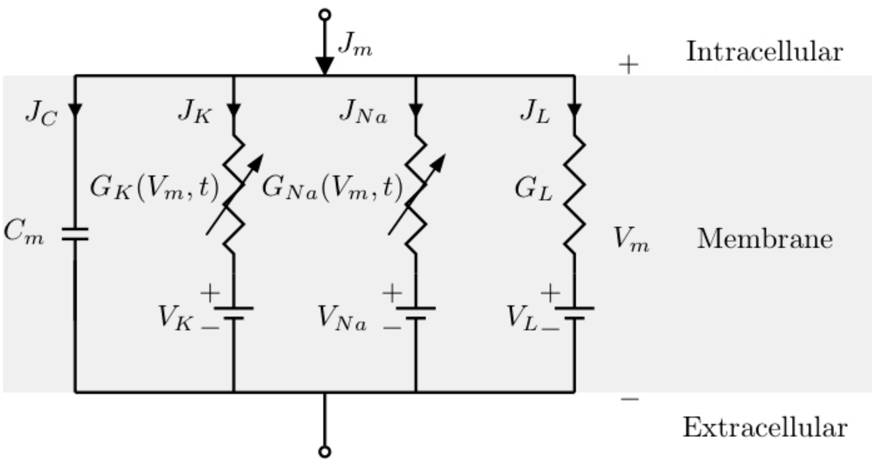


Electrically inexcitable cell



Electrically excitable cell

Hodgkin Huxley model



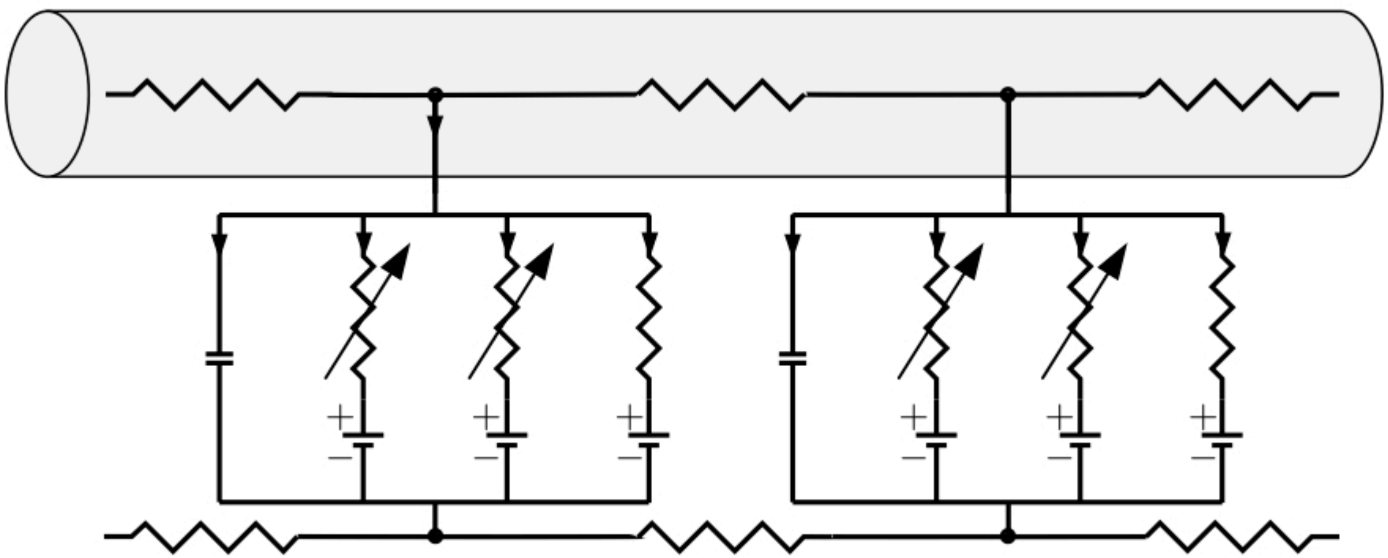


Figure 4.7

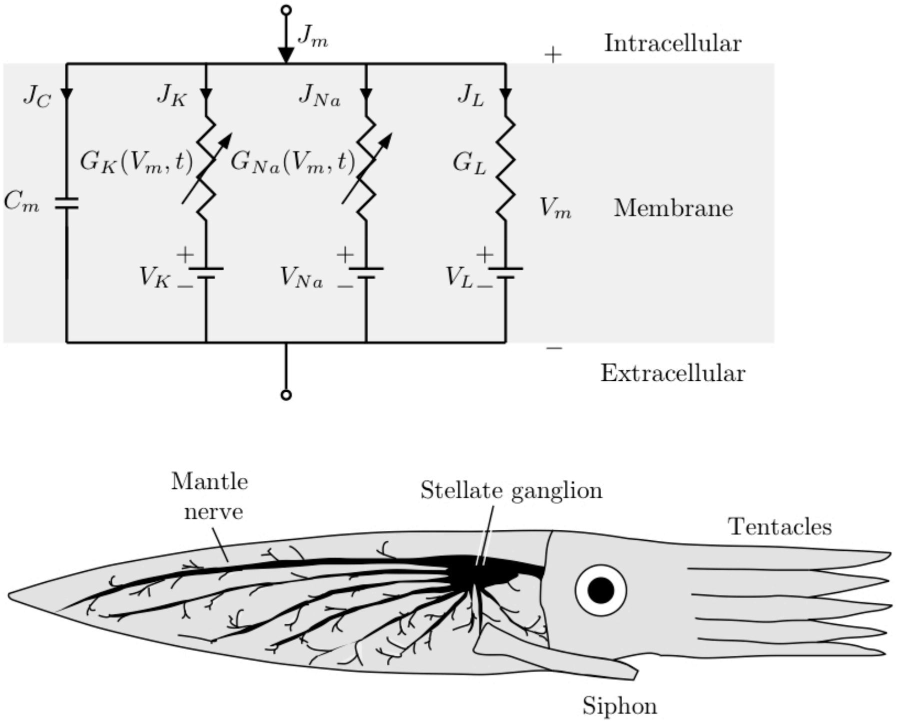


Figure 1.28

$$G_K(V_m,t) = \overline{G}_K n^4(V_m,t)$$

$$G_{Na}(V_m,t) = \overline{G}_{Na} m^3(V_m,t) h(V_m,t)$$

$$n(V_m,t) + \tau_n(V_m)\frac{dn(V_m,t)}{dt} = n_\infty(V_m)$$

$$m(V_m,t) + \tau_m(V_m)\frac{dm(V_m,t)}{dt} = m_\infty(V_m)$$

$$h(V_m,t) + \tau_h(V_m)\frac{dh(V_m,t)}{dt} = h_\infty(V_m)$$

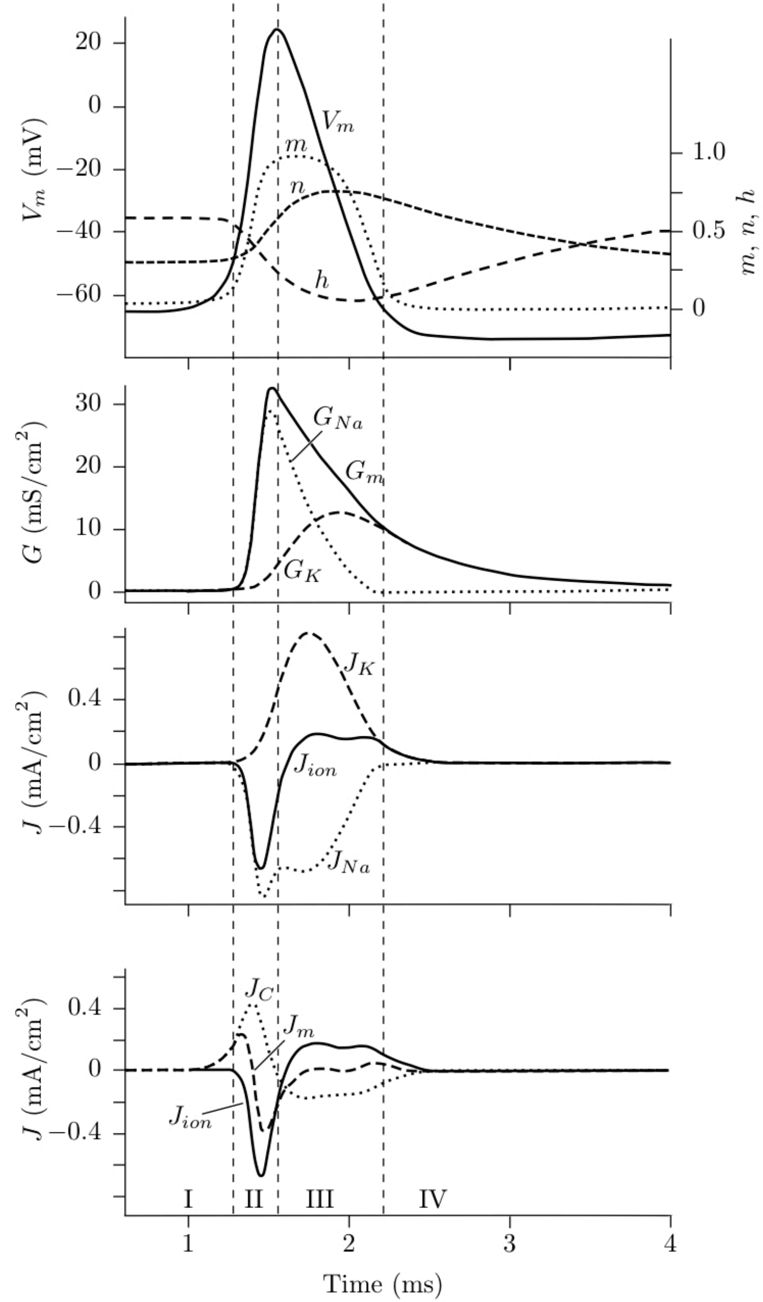


Figure 4.32