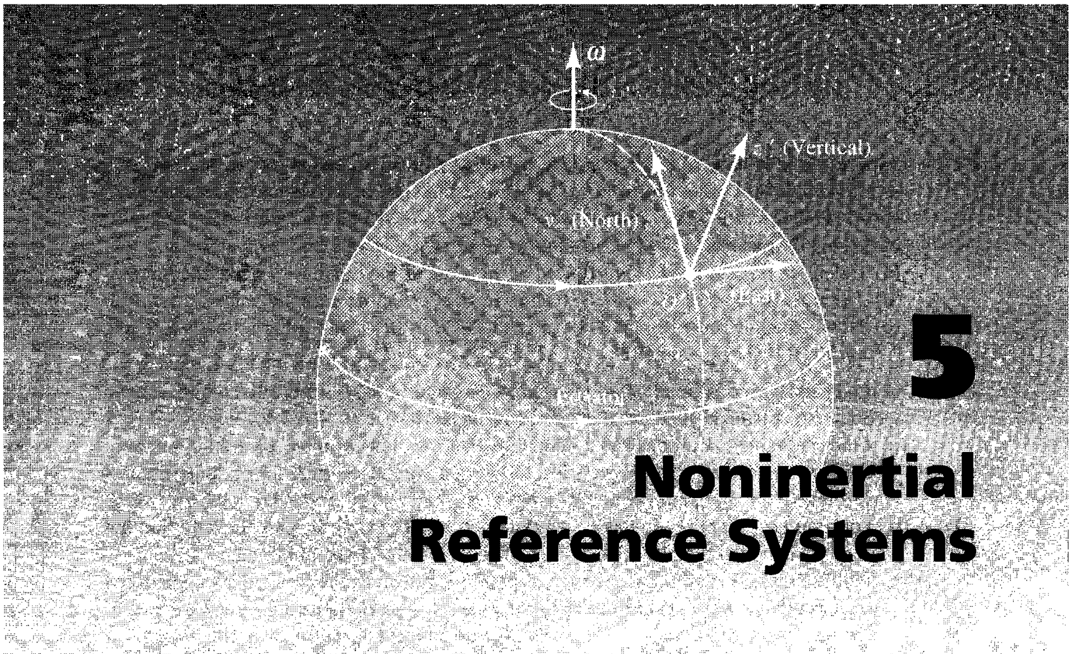

5 Noninertial Reference Systems

"I was sitting in a chair at the patent office in Bern, when all of a sudden a thought occurred to me: If a person falls freely, he will not feel his own weight. I was startled. This simple thought made a deep impression on me. It impelled me toward a theory of gravitation."

— Albert Einstein, *"The Happiest Thought of My Life"*; see A. Pais, *Inward Bound,* New York, Oxford Univ. Press, 1986

5.1 Accelerated Coordinate Systems and Inertial Forces

In describing the motion of a particle, it is frequently convenient, and sometimes necessary, to employ a coordinate system that is not inertial. For example, a coordinate system fixed to the Earth is the most convenient one to describe the motion of a projectile, even though the Earth is accelerating and rotating.

We shall first consider the case of a coordinate system that undergoes pure translation. In Figure 5.1.1 $Oxyz$ are the primary coordinate axes (assumed fixed), and $O'x'y'z'$ are the moving axes. In the case of pure translation, the respective axes Ox and $O'x'$, and so on, remain parallel. The position vector of a particle P is denoted by $\mathbf{r}$ in the fixed system and by $\mathbf{r}'$ in the moving system. The displacement OO' of the moving origin is denoted by $\mathbf{R}_0$. Thus, from the triangle $OO'P$, we have

$$\mathbf{r} = \mathbf{R}_0 + \mathbf{r}' \tag{5.1.1}$$

Taking the first and second time derivatives gives

$$\mathbf{v} = \mathbf{V}_0 + \mathbf{v}' \tag{5.1.2}$$

$$\mathbf{a} = \mathbf{A}_0 + \mathbf{a}' \tag{5.1.3}$$

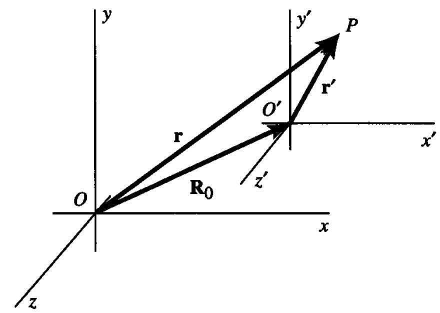

Figure 5.1.1 Relationship between the position vectors for two coordinate systems undergoing pure translation relative to each other.

in which $\mathbf{V}_0$ and $\mathbf{A}_0$ are, respectively, the velocity and acceleration of the moving system, and $\mathbf{v}'$ and $\mathbf{a}'$ are the velocity and acceleration of the particle *in* the moving system.

In particular, if the moving system is not accelerating, so that $\mathbf{A}_0 = 0$, then

$$\mathbf{a} = \mathbf{a}'$$

so the acceleration is the same in either system. Consequently, if the primary system is inertial, Newton's second law $\mathbf{F} = m\mathbf{a}$ becomes $\mathbf{F} = m\mathbf{a}'$ in the moving system; that is, the moving system is also an inertial system (provided it is not rotating). Thus, as far as Newtonian mechanics is concerned, we cannot specify a unique coordinate system; if Newton's laws hold in one system, they are also valid in any other system moving with uniform velocity relative to the first.

On the other hand if the moving system is accelerating, then Newton's second law becomes

$$\mathbf{F} = m\mathbf{A}_0 + m\mathbf{a}' \tag{5.1.4a}$$

or

$$\mathbf{F} - m\mathbf{A}_0 = m\mathbf{a}' \tag{5.1.4b}$$

for the equation of motion in the accelerating system. If we wish, we can write Equation 5.1.4b in the form

$$\mathbf{F}' = m\mathbf{a}' \tag{5.1.5}$$

in which $\mathbf{F}' = \mathbf{F} + (-m\mathbf{A}_0)$. That is, an acceleration $\mathbf{A}_0$ of the reference system can be taken into account by adding an *inertial term* $-m\mathbf{A}_0$ to the force $\mathbf{F}$ and equating the result to the product of mass and acceleration in the moving system. Inertial terms in the equations of motion are sometimes called *inertial forces*, or *fictitious forces*. Such "forces" are not due to interactions with other bodies, rather, they stem from the acceleration of the reference system. Whether or not one wishes to call them forces is purely a matter of taste. In any case, inertial terms are present if a noninertial coordinate system is used to describe the motion of a particle.

EXAMPLE 5.1.1

A block of wood rests on a rough horizontal table. If the table is accelerated in a horizontal direction, under what conditions will the block slip?

Solution:

Let μ_s be the coefficient of static friction between the block and the table top. Then the force of friction $\mathbf{F}$ has a maximum value of $\mu_s mg$, where m is the mass of the block. The condition for slipping is that the inertial force $-m\mathbf{A}_0$ exceeds the frictional force, where $\mathbf{A}_0$ is the acceleration of the table. Hence, the condition for slipping is

$$|-m\mathbf{A}_0| > \mu_s m\mathbf{g}$$

or

$$A_0 > \mu_s g$$

EXAMPLE 5.1.2

A pendulum is suspended from the ceiling of a railroad car, as shown in Figure 5.1.2a. Assume that the car is accelerating uniformly toward the right ($+x$ direction). A noninertial observer, the boy inside the car, sees the pendulum hanging at an angle θ, left of vertical. He believes it hangs this way because of the existence of an inertial force $\mathbf{F}'_x$, which acts on all objects in his accelerated frame of reference (Figure 5.1.2b). An inertial observer, the girl outside the car, sees the same thing. She knows, however, that there is no real force $\mathbf{F}'_x$ acting on the pendulum. She knows that it hangs this way because a net force in the horizontal direction is required to accelerate it at the rate $\mathbf{A}_0$ that she observes (Figure 5.1.2c). Calculate the acceleration $\mathbf{A}_0$ of the car from the inertial observer's point of view. Show that, according to the noninertial observer, $\mathbf{F}'_x = -m\mathbf{A}_0$ is the force that causes the pendulum to hang at the angle θ.

Solution:

The inertial observer writes down Newton's second law for the hanging pendulum as

$$\sum \mathbf{F}_i = m\mathbf{a}$$

$$T\sin\theta = mA_0 \qquad T\cos\theta - mg = 0$$

$$\therefore A_0 = g\tan\theta$$

She concludes that the suspended pendulum hangs at the angle θ because the railroad car is accelerating in the horizontal direction and a horizontal force is needed to make it accelerate. This force is the x-component of the tension in the string. The acceleration of the car is proportional to the tangent of the angle of deflection. The pendulum, thus, serves as a linear accelerometer.

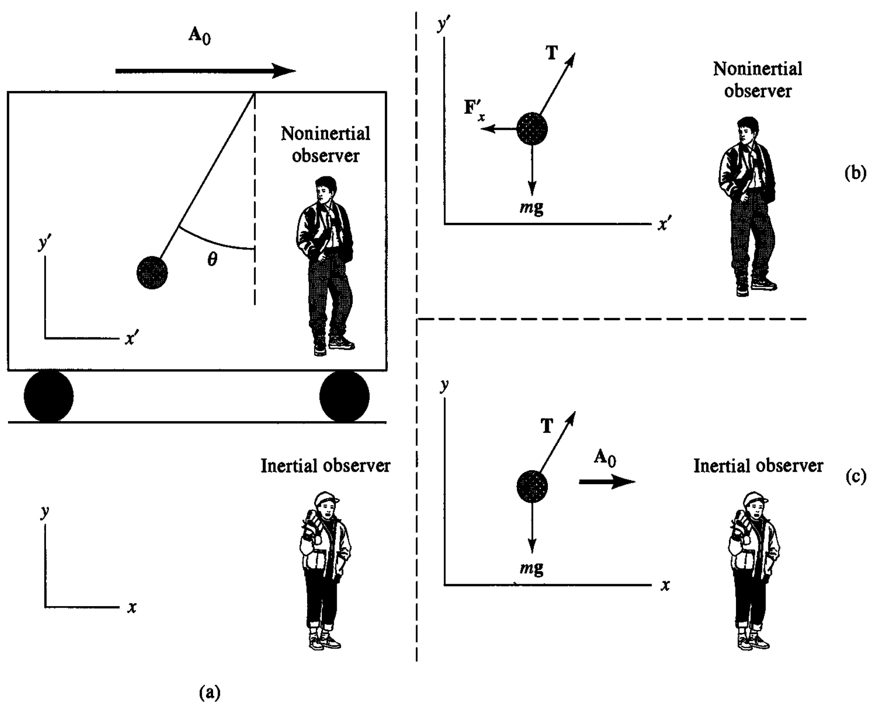

Figure 5.1.2 (a) Pendulum suspended in an accelerating railroad car as seen by (b) the noninertial observer and (c) the inertial observer.

On the other hand the noninertial observer, unaware of the outside world (assume the railroad track is perfectly smooth—no vibration—and that the railroad car has no windows or other sensory clues for another reference point), observes that the pendulum just hangs there, tilted to the left of vertical. He concludes that

$$\sum \mathbf{F}_i' = m\mathbf{a}' = 0$$

$$T\sin\theta - F_x' = 0 \qquad T\cos\theta - mg = 0$$

$$\therefore F_x' = mg\tan\theta$$

All the forces acting on the pendulum are in balance, and the pendulum hangs left of vertical due to the force $\mathbf{F}_x'$ $(= -m\mathbf{A}_0)$. In fact if this observer were to do some more experiments in the railroad car, such as drop balls or stones or whatever, he would see that they would also be deflected to the left of vertical. He would soon discover that the amount of the deflection would be independent of their mass. In other words he would conclude that there was a force, quite like a gravitational one (to be discussed in Chapter 6), pushing things to the left of the car with an acceleration $\mathbf{A}_0$ as well as the force pulling them down with an acceleration $\mathbf{g}$.

EXAMPLE 5.1.3

Two astronauts are standing in a spaceship accelerating upward with an acceleration $\mathbf{A}_0$ as shown in Figure 5.1.3. Let the magnitude of $\mathbf{A}_0$ equal g. Astronaut #1 throws a ball directly toward astronaut #2, who is 10 m away on the other side of the ship. What must be the initial speed of the ball if it is to reach astronaut #2 before striking the floor? Assume astronaut #1 releases the ball at a height $h = 2$ m above the floor of the ship. Solve the problem from the perspective of both (a) a noninertial observer (inside the ship) and (b) an inertial observer (outside the ship).

Solution:

(a) The noninertial observer believes that a force $-m\mathbf{A}_0$ acts upon all objects in the ship. Thus, in the noninertial (x', y') frame of reference, we conclude that the trajectory of the ball is a parabola, that is,

$$x'(t) = \dot{x}'_0 t \qquad y'(t) = y'_0 - \tfrac{1}{2}A_0 t^2$$

$$\therefore y'(x') = y'_0 - \tfrac{1}{2}A_0\left(\frac{x'}{\dot{x}'_0}\right)^2$$

Setting $y'(x')$ equal to zero when $x' = 10$ m and solving for $\dot{x}'_0$ yields

$$\dot{x}'_0 = \left(\frac{A_0}{2y'_0}\right)^{1/2} x'$$

$$= \left(\frac{9.8 \text{ ms}^{-2}}{4 \text{ m}}\right)^{1/2} (10 \text{ m}) = 15.6 \text{ ms}^{-1}$$

(b) The inertial observer sees the picture a little differently. It appears to him that the ball travels at constant velocity in a straight line after it is released and that the floor of the spaceship accelerates upward to intercept the ball. A plot of the vertical position of the ball and the floor of the spaceship is shown schematically in Figure 5.1.4. Both the ball and the rocket have the same initial upward speed $\dot{y}_0$ at the moment the ball is released by astronaut #1.

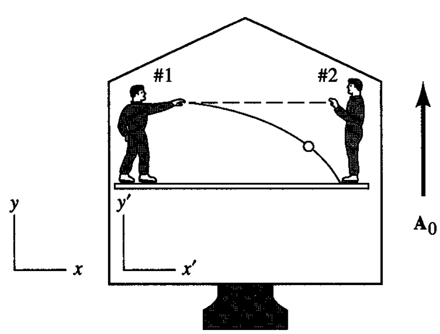

Figure 5.1.3 Two astronauts throwing a ball in a spaceship accelerating at $|\mathbf{A}_0| = |\mathbf{g}|$.

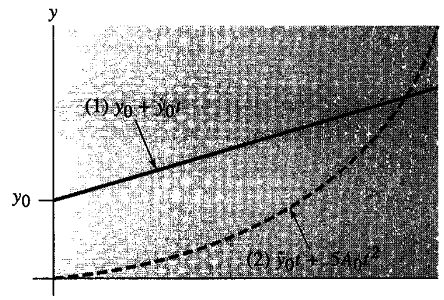

Figure 5.1.4 Vertical position of (1) a ball thrown in an accelerating rocket and (2) the floor of the rocket as seen by an inertial observer.

The vertical positions of the ball and the floor coincide at a time t that depends on the initial height of the ball

$$y_0 + \dot{y}_0 t = \dot{y}_0 t + \tfrac{1}{2} A_0 t^2$$
$$y_0 = \tfrac{1}{2} A_0 t^2$$

During this time t, the ball has traveled a horizontal distance x, where

$$x = \dot{x}_0 t \qquad \text{or} \qquad t = \frac{x}{\dot{x}_0}$$

Inserting this time into the relation for y_0 above yields the required initial horizontal speed of the ball

$$y_0 = \tfrac{1}{2} A_0 \left(\frac{x}{\dot{x}_0} \right)^2$$
$$\dot{x}_0 = \left(\frac{A_0}{2y_0} \right)^{1/2} x$$

Thus, each observer calculates the same value for the initial horizontal velocity, as well they should.

The analysis seems less complex from the perspective of the noninertial observer. In fact the noninertial observer would physically experience the inertial force $-m\mathbf{A}_0$. It would seem every bit as real as the gravitational force we experience here on Earth. Our astronaut might even invent the concept of gravity to "explain" the dynamics of moving objects observed in the spaceship.

5.2 | Rotating Coordinate Systems

In the previous section, we showed how velocities, accelerations, and forces transform between an inertial frame of reference and a noninertial one that is accelerating at a constant rate. In this section and the following one, we show how these quantities transform between an inertial frame and a noninertial one that is rotating as well.

We start our discussion with the case of a primed coordinate system rotating with respect to an unprimed, fixed, inertial one. The axes of the coordinate systems have a common origin (see Figure 5.2.1). At any given instant the rotation of the primed system takes place about some specific axis of rotation, whose direction is designated by a unit vector, $\mathbf{n}$. The instantaneous angular speed of the rotation is designated by ω. The product, $\omega\mathbf{n}$, is the *angular velocity* of the rotating system

$$\boldsymbol{\omega} = \omega\mathbf{n} \tag{5.2.1}$$

The sense direction of the angular velocity vector is given by the right-hand rule (see Figure 5.2.1), as in the definition of the cross product.

The position of any point P in space can be designated by the vector $\mathbf{r}$ in the fixed, unprimed system and by the vector $\mathbf{r}'$ in the rotating, primed system (see Figure 5.2.2).

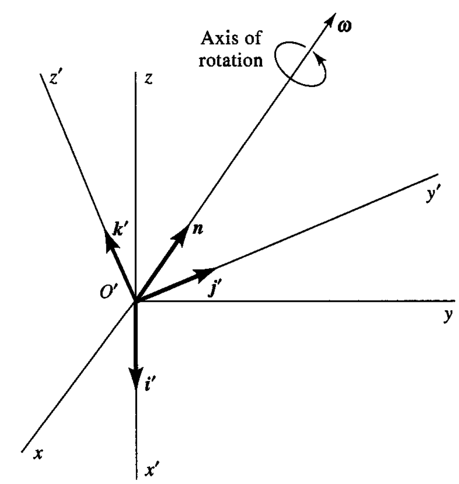

Figure 5.2.1 The angular velocity vector of a rotating coordinate system.

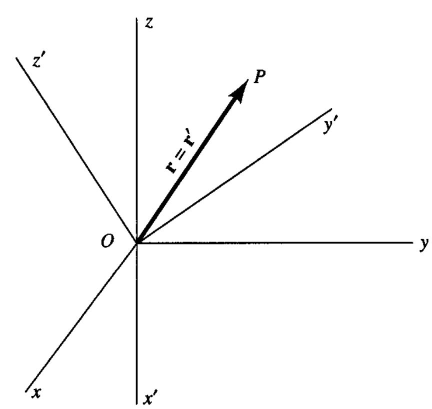

Figure 5.2.2 Rotating coordinate system (primed system).

Because the coordinate axes of the two systems have the same origin, these vectors are equal, that is,

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z = \mathbf{r}' = \mathbf{i}'x' + \mathbf{j}'y' + \mathbf{k}'z' \tag{5.2.2}$$

When we differentiate with respect to time to find the velocity, we must keep in mind the fact that the unit vectors $\mathbf{i}'$, $\mathbf{j}'$, and $\mathbf{k}'$ in the rotating system are *not* constant, whereas the primary unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are. Thus, we can write

$$\mathbf{i}\frac{dx}{dt} + \mathbf{j}\frac{dy}{dt} + \mathbf{k}\frac{dz}{dt} = \mathbf{i}'\frac{dx'}{dt} + \mathbf{j}'\frac{dy'}{dt} + \mathbf{k}'\frac{dz'}{dt} + x'\frac{d\mathbf{i}'}{dt} + y'\frac{d\mathbf{j}'}{dt} + z'\frac{d\mathbf{k}'}{dt} \tag{5.2.3}$$

The three terms on the left-hand side of the preceding equation clearly give the velocity vector $\mathbf{v}$ in the fixed system, and the first three terms on the right are the components of the velocity *in* the rotating system, which we shall call $\mathbf{v}'$, so the equation may be written

$$\mathbf{v} = \mathbf{v}' + x'\frac{d\mathbf{i}'}{dt} + y'\frac{d\mathbf{j}'}{dt} + z'\frac{d\mathbf{k}'}{dt} \tag{5.2.4}$$

The last three terms on the right represent the velocity due to rotation of the primed coordinate system. We must now determine how the time derivatives of the basis vectors are related to the rotation.

To find the time derivatives $d\mathbf{i}'/dt$, $d\mathbf{j}'/dt$, and $d\mathbf{k}'/dt$, consider Figure 5.2.3. Here is shown the change $\Delta\mathbf{i}'$ in the unit vector $\mathbf{i}'$ due to a small rotation $\Delta\theta$ about the axis of rotation. (The vectors $\mathbf{j}'$ and $\mathbf{k}'$ are omitted for clarity.) From the figure we see that the magnitude of $\Delta\mathbf{i}'$ is given by the approximate relation

$$|\Delta\mathbf{i}'| \approx (|\mathbf{i}'|\sin\phi)\Delta\theta = (\sin\phi)\Delta\theta$$

where ϕ is the angle between $\mathbf{i}'$ and $\boldsymbol{\omega}$. Let Δt be the time interval for this change. Then we can write

$$\left|\frac{d\mathbf{i}'}{dt}\right| = \lim_{\Delta t\to 0}\left|\frac{\Delta\mathbf{i}'}{\Delta t}\right| = \sin\phi\frac{d\theta}{dt} = (\sin\phi)\omega \tag{5.2.5}$$

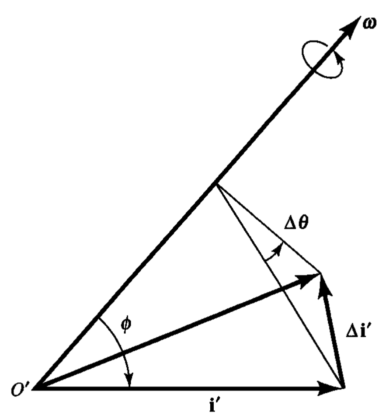

Figure 5.2.3 Change in the unit vector $\mathbf{i}'$ produced by a small rotation $\Delta\theta$.

Now the direction of $\Delta\mathbf{i}'$ is perpendicular to *both* $\boldsymbol{\omega}$ and $\mathbf{i}'$; consequently, from the definition of the cross product, we can write Equation 5.2.5 in vector form

$$\frac{d\mathbf{i}'}{dt} = \boldsymbol{\omega} \times \mathbf{i}' \tag{5.2.6}$$

Similarly, we find $d\mathbf{j}'/dt = \boldsymbol{\omega} \times \mathbf{j}'$, and $d\mathbf{k}'/dt = \boldsymbol{\omega} \times \mathbf{k}'$.

We now apply the preceding result to the last three terms in Equation 5.2.4 as follows:

$$\begin{aligned} x'\frac{d\mathbf{i}'}{dt} + y'\frac{d\mathbf{j}'}{dt} + z'\frac{d\mathbf{k}'}{dt} &= x'(\boldsymbol{\omega} \times \mathbf{i}') + y'(\boldsymbol{\omega} \times \mathbf{j}') + z'(\boldsymbol{\omega} \times \mathbf{k}') \\ &= \boldsymbol{\omega} \times (\mathbf{i}'x' + \mathbf{j}'y' + \mathbf{k}'z') \\ &= \boldsymbol{\omega} \times \mathbf{r}' \end{aligned} \tag{5.2.7}$$

This is the velocity of P due to rotation of the primed coordinate system. Accordingly, Equation 5.2.4 can be shortened to read

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' \tag{5.2.8}$$

or, more explicitly

$$\left(\frac{d\mathbf{r}}{dt}\right)_{fixed} = \left(\frac{d\mathbf{r}'}{dt}\right)_{rot} + \boldsymbol{\omega} \times \mathbf{r}' = \left[\left(\frac{d}{dt}\right)_{rot} + \boldsymbol{\omega} \times \right]\mathbf{r}' \tag{5.2.9}$$

that is, the operation of differentiating the position vector with respect to time in the fixed system is equivalent to the operation of taking the time derivative in the rotating system plus the operation $\boldsymbol{\omega} \times$. A little reflection shows that the same applies to *any* vector $\mathbf{Q}$, that is,

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{fixed} = \left(\frac{d\mathbf{Q}}{dt}\right)_{rot} + \boldsymbol{\omega} \times \mathbf{Q} \tag{5.2.10a}$$

In particular, if that vector is the velocity, then we have

$$\left(\frac{d\mathbf{v}}{dt}\right)_{fixed} = \left(\frac{d\mathbf{v}}{dt}\right)_{rot} + \boldsymbol{\omega} \times \mathbf{v} \tag{5.2.10b}$$

But $\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}'$, so

$$\begin{aligned} \left(\frac{d\mathbf{v}}{dt}\right)_{fixed} &= \left(\frac{d}{dt}\right)_{rot}(\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}') + \boldsymbol{\omega} \times (\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}') \\ &= \left(\frac{d\mathbf{v}'}{dt}\right)_{rot} + \left[\frac{d(\boldsymbol{\omega} \times \mathbf{r}')}{dt}\right]_{rot} + \boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') \\ &= \left(\frac{d\mathbf{v}'}{dt}\right)_{rot} + \left(\frac{d\boldsymbol{\omega}}{dt}\right)_{rot} \times \mathbf{r}' + \boldsymbol{\omega} \times \left(\frac{d\mathbf{r}'}{dt}\right)_{rot} \\ &\quad + \boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') \end{aligned} \tag{5.2.11}$$

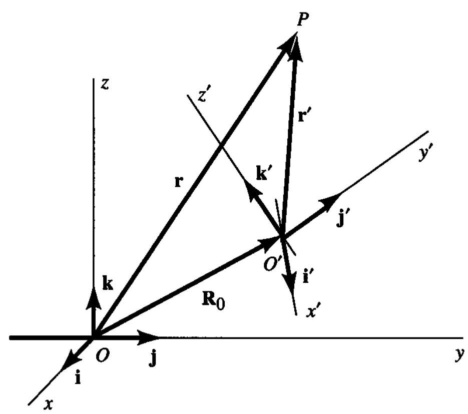

Figure 5.2.4 Geometry for the general case of translation and rotation of the moving coordinate system (primed system).

Now concerning the term involving the time derivative of $\boldsymbol{\omega}$, we have $(d\boldsymbol{\omega}/dt)_{fixed} = (d\boldsymbol{\omega}/dt)_{rot} + \boldsymbol{\omega} \times \boldsymbol{\omega}$. But the cross product of any vector with itself vanishes, so $(d\boldsymbol{\omega}/dt)_{fixed} = (d\boldsymbol{\omega}/dt)_{rot} = \dot{\boldsymbol{\omega}}$. Because $\mathbf{v}' = (d\mathbf{r}'/dt)_{rot}$ and $\mathbf{a}' = (d\mathbf{v}'/dt)_{rot}$, we can express the final result as follows:

$$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') \tag{5.2.12}$$

giving the acceleration in the fixed system in terms of the position, velocity, and acceleration in the rotating system.

In the general case in which the primed system is undergoing *both* translation and rotation (Figure 5.2.4), we must add the velocity of translation $\mathbf{V}_0$ to the right-hand side of Equation 5.2.8 and the acceleration $\mathbf{A}_0$ of the moving system to the right-hand side of Equation 5.2.12. This gives the general equations for transforming from a fixed system to a moving and rotating system:

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \mathbf{V}_0 \tag{5.2.13}$$

$$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \mathbf{A}_0 \tag{5.2.14}$$

The term $2\boldsymbol{\omega} \times \mathbf{v}'$ is known as the *Coriolis acceleration*, and the term $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')$ is called the *centripetal acceleration*. The Coriolis acceleration appears whenever a particle moves in a rotating coordinate system (except when the velocity $\mathbf{v}'$ is parallel to the axis of rotation), and the centripetal acceleration is the result of the particle being carried around a circular path in the rotating system. The centripetal acceleration is always directed toward the axis of rotation and is perpendicular to the axis as shown in Figure 5.2.5. The term $\dot{\boldsymbol{\omega}} \times \mathbf{r}'$ is called the *transverse acceleration*, because it is perpendicular to the position vector $\mathbf{r}'$. It appears as a result of any angular acceleration of the rotating system, that is, if the angular velocity vector is changing in either magnitude or direction, or both.

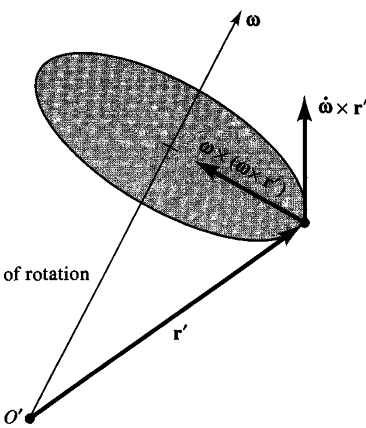

Figure 5.2.5 Illustrating the centripetal acceleration.

EXAMPLE 5.2.1

A wheel of radius b rolls along the ground with constant forward speed V_0. Find the acceleration, relative to the ground, of any point on the rim.

Solution:

Let us choose a coordinate system fixed to the rotating wheel, and let the moving origin be at the center with the x'-axis passing through the point in question, as shown in Figure 5.2.6. Then we have

$$\mathbf{r}' = \mathbf{i}'b \qquad \mathbf{a}' = \ddot{\mathbf{r}}' = 0 \qquad \mathbf{v}' = \dot{\mathbf{r}}' = 0$$

The angular velocity vector is given by

$$\boldsymbol{\omega} = \mathbf{k}'\omega = \mathbf{k}'\frac{V_0}{b}$$

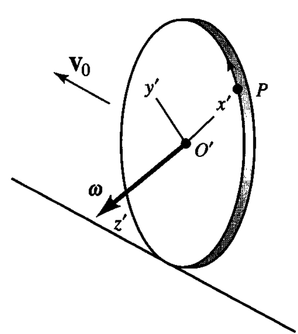

Figure 5.2.6 Rotating coordinates fixed to a rolling wheel.

for the choice of coordinates shown; therefore, all terms in the expression for acceleration vanish except the centripetal term:

$$\begin{aligned}\mathbf{a} &= \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}') = \mathbf{k}'\omega\times(\mathbf{k}'\omega\times\mathbf{i}'b)\\ &= \frac{V_0^2}{b}\mathbf{k}'\times(\mathbf{k}'\times\mathbf{i}')\\ &= \frac{V_0^2}{b}\mathbf{k}'\times\mathbf{j}'\\ &= \frac{V_0^2}{b}(-\mathbf{i}')\end{aligned}$$

Thus, $\mathbf{a}$ is of magnitude V_0^2/b and is always directed toward the center of the rolling wheel.

EXAMPLE 5.2.2

A bicycle travels with constant speed around a track of radius ρ. What is the acceleration of the highest point on one of its wheels? Let V_0 denote the speed of the bicycle and b the radius of the wheel.

Solution:

We choose a coordinate system with origin at the center of the wheel and with the x'-axis horizontal pointing toward the center of curvature C of the track. Rather than have the moving coordinate system rotate with the wheel, we choose a system in which the z'-axis remains vertical as shown in Figure 5.2.7. Thus, the $O'x'y'z'$ system rotates

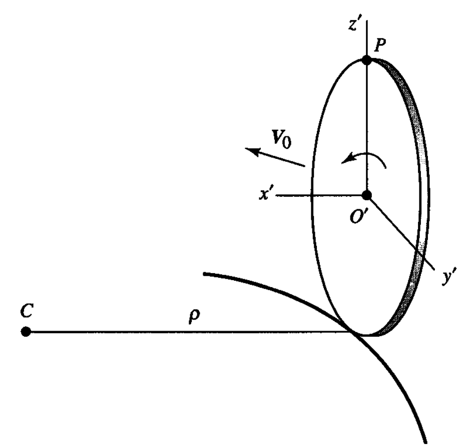

Figure 5.2.7 Wheel rolling on a curved track. The z'-axis remains vertical as the wheel turns.

with angular velocity $\boldsymbol{\omega}$, which can be expressed as

$$\boldsymbol{\omega} = \mathbf{k}'\frac{V_0}{\rho}$$

and the acceleration of the moving origin $\mathbf{A}_0$ is given by

$$\mathbf{A}_0 = \mathbf{i}'\frac{V_0^2}{\rho}$$

Because each point on the wheel is moving in a circle of radius b with respect to the moving origin, the acceleration in the $O'x'y'z'$ system of any point on the wheel is directed toward O' and has magnitude V_0^2/b. Thus, in the moving system we have

$$\ddot{\mathbf{r}}' = -\mathbf{k}'\frac{V_0^2}{b}$$

for the point at the top of the wheel. Also, the velocity of this point in the moving system is given by

$$\mathbf{v}' = -\mathbf{j}'V_0$$

so the Coriolis acceleration is

$$2\boldsymbol{\omega} \times \mathbf{v}' = 2\left(\frac{V_0}{\rho}\mathbf{k}'\right) \times (-\mathbf{j}'V_0) = 2\frac{V_0^2}{\rho}\mathbf{i}'$$

Because the angular velocity $\boldsymbol{\omega}$ is constant, the transverse acceleration is zero. The centripetal acceleration is also zero because

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') = \frac{V_0^2}{\rho^2}\mathbf{k}' \times (\mathbf{k}' \times b\mathbf{k}') = 0$$

Thus, the net acceleration, relative to the ground, of the highest point on the wheel is

$$\mathbf{a} = 3\frac{V_0^2}{\rho}\mathbf{i}' - \frac{V_0^2}{b}\mathbf{k}'$$

5.3 Dynamics of a Particle in a Rotating Coordinate System

The fundamental equation of motion of a particle in an inertial frame of reference is

$$\mathbf{F} = m\mathbf{a} \qquad (5.3.1)$$

where $\mathbf{F}$ is the vector sum of all real, physical forces acting on the particle. In view of Equation 5.2.14, we can write the equation of motion in a noninertial frame of reference as

$$\mathbf{F} - m\mathbf{A}_0 - 2m\boldsymbol{\omega} \times \mathbf{v}' - m\dot{\boldsymbol{\omega}} \times \mathbf{r}' - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') = m\mathbf{a}' \qquad (5.3.2)$$

All the terms from Equation 5.2.14, except $\mathbf{a}'$, have been multiplied by m and transposed to show them as inertial forces added to the real, physical forces $\mathbf{F}$. The $\mathbf{a}'$ term has been multiplied by m also, but kept on the right-hand side. Thus, Equation 5.3.2 represents the dynamical equation of motion of a particle in a noninertial frame of reference subjected to both real, physical forces as well as those inertial forces that appear as a result of the acceleration of the noninertial frame of reference. The inertial forces have names corresponding to their respective accelerations, discussed in Section 5.2. The *Coriolis force* is

$$\mathbf{F}'_{Cor} = -2m\boldsymbol{\omega} \times \mathbf{v}' \tag{5.3.3}$$

The *transverse force* is

$$\mathbf{F}'_{trans} = -m\dot{\boldsymbol{\omega}} \times \mathbf{r}' \tag{5.3.4}$$

The *centrifugal force* is

$$\mathbf{F}'_{centrif} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') \tag{5.3.5}$$

The remaining inertial force $-m\mathbf{A}_0$ appears whenever the (x', y', z') coordinate system is undergoing a translational acceleration, as discussed in Section 5.1.

A noninertial observer in an accelerated frame of reference who denotes the acceleration of a particle by the vector $\mathbf{a}'$ is forced to include any or all of these inertial forces along with the real forces to calculate the correct motion of the particle. In other words, such an observer writes the fundamental equation of motion as

$$\mathbf{F}' = m\mathbf{a}'$$

in which the sum of the vector forces $\mathbf{F}'$ acting on the particle is given by

$$\mathbf{F}' = \mathbf{F}_{physical} + \mathbf{F}'_{Cor} + \mathbf{F}'_{trans} + \mathbf{F}'_{centrif} - m\mathbf{A}_0$$

We have emphasized the real, physical nature of the force term $\mathbf{F}$ in Equation 5.3.2 by appending the subscript *physical* to it here. $\mathbf{F}$ (or $\mathbf{F}_{physical}$) forces are the only forces that a noninertial observer claims are actually acting upon the particle. The inclusion of the remaining four inertial terms depends critically on the exact status of the noninertial frame of reference being used to describe the motion of the particle. They arise because of the inertial property of the matter whose motion is under investigation, rather than from the presence or action of any surrounding matter.

The Coriolis force is particularly interesting. It is present only if a particle is *moving* in a rotating coordinate system. Its direction is always perpendicular to the velocity vector of the particle in the moving system. The Coriolis force thus seems to deflect a moving particle at right angles to its direction of motion. (The Coriolis force has been rather fancifully called "the merry-go-round force." Try walking radially inward or outward on a moving merry-go-round to experience its effect.) This force is important in computing the trajectory of a projectile. Coriolis effects are responsible for the circulation of air around high- or low-pressure systems on Earth's surface. In the case of a high-pressure area,[1] as air spills down from the high, it flows outward and away, deflecting toward the right as it moves into the surrounding low, setting up a clockwise circulation pattern. In the Southern Hemisphere the reverse is true.

[1] A high-pressure system is essentially a bump in Earth's atmosphere where more air is stacked up above some region on Earth's surface than it is for surrounding regions.

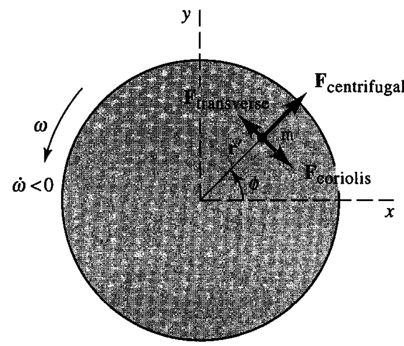

Figure 5.3.1 Inertial forces acting on a mass m moving radially outward on a platform rotating with angular velocity $\boldsymbol{\omega}$ and angular acceleration $\dot{\boldsymbol{\omega}} < 0$. The xy-axes are fixed. The direction of $\boldsymbol{\omega}$ is out of the paper.

The transverse force is present only if there is an angular acceleration (or deceleration) of the rotating coordinate system. This force is always perpendicular to the radius vector $\mathbf{r}'$ in the rotating coordinate system.

The centrifugal force is the familiar one that arises from rotation about an axis. It is directed outward away from the axis of rotation and is perpendicular to that axis. These three inertial forces are illustrated in Figure 5.3.1 for the case of a mass m moving radially outward on a rotating platform, whose rate of rotation is decreasing ($\dot{\omega} < 0$). The z-axis is the axis of rotation, directed out of the paper. That is also the direction of the angular velocity vector $\boldsymbol{\omega}$. Because $\mathbf{r}'$, the radius vector denoting the position of m in the rotating system, is perpendicular to $\boldsymbol{\omega}$, the magnitude of the centrifugal force is $mr'\omega^2$. In general if the angle between $\boldsymbol{\omega}$ and $\mathbf{r}'$ is θ, then the magnitude of the centripetal force is $mr'\omega^2 \sin\theta$ where $r' \sin\theta$ is the shortest distance from the mass to the axis of rotation.

EXAMPLE 5.3.1

A bug crawls outward with a constant speed v' along the spoke of a wheel that is rotating with constant angular velocity $\boldsymbol{\omega}$ about a vertical axis. Find all the apparent forces acting on the bug (see Figure 5.3.2).

Solution:

First, let us choose a coordinate system fixed on the wheel, and let the x'-axis point along the spoke in question. Then we have

$$\dot{\mathbf{r}}' = \mathbf{i}'\dot{x}' = \mathbf{i}'v'$$

$$\ddot{\mathbf{r}}' = 0$$

for the velocity and acceleration of the bug as described in the rotating system. If we choose the z'-axis to be vertical, then

$$\boldsymbol{\omega} = \mathbf{k}'\omega$$

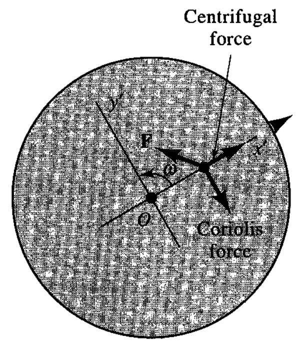

Figure 5.3.2 Forces on an insect crawling outward along a radial line on a rotating wheel.

The various forces are then given by the following:

$$-2m\boldsymbol{\omega}\times\dot{\mathbf{r}}' = -2m\omega v'(\mathbf{k}'\times\mathbf{i}') = -2m\omega v'\mathbf{j}' \qquad \textit{Coriolis force}$$
$$-m\dot{\boldsymbol{\omega}}\times\mathbf{r}' = 0 \qquad (\boldsymbol{\omega} = \text{constant}) \qquad \textit{transverse force}$$
$$\begin{aligned}-m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}') &= -m\omega^2[\mathbf{k}'\times(\mathbf{k}'\times\mathbf{i}'x')] \qquad \textit{centrifugal force}\\ &= -m\omega^2(\mathbf{k}'\times\mathbf{j}'x')\\ &= m\omega^2x'\mathbf{i}'\end{aligned}$$

Thus, Equation 5.3.2 reads

$$\mathbf{F} - 2m\omega v'\mathbf{j}' + m\omega^2x'\mathbf{i}' = 0$$

Here $\mathbf{F}$ is the real force exerted on the bug by the spoke. The forces are shown in Figure 5.3.2.

EXAMPLE 5.3.2

In Example 5.3.1, find how far the bug can crawl before it starts to slip, given the coefficient of static friction μ_s between the bug and the spoke.

Solution:

Because the force of friction $\mathbf{F}$ has a maximum value of $\mu_s mg$, slipping starts when

$$|\mathbf{F}| = \mu_s mg$$

or

$$[(2m\omega v')^2 + (m\omega^2x')^2]^{1/2} = \mu_s mg$$

On solving for x', we find

$$x' = \frac{\left[\mu_s^2g^2 - 4\omega^2(v')^2\right]^{1/2}}{\omega^2}$$

for the distance the bug can crawl before slipping.

EXAMPLE 5.3.3

A smooth rod of length l rotates in a plane with a constant angular velocity $\boldsymbol{\omega}$ about an axis fixed at the end of the rod and perpendicular to the plane of rotation. A bead of mass m is initially positioned at the stationary end of the rod and given a slight push such that its initial speed directed down the rod is $\epsilon = \omega l$ (see Figure 5.3.3). Calculate how long it takes for the bead to reach the other end of the rod.

Solution:

The best way to solve this problem is to examine it from the perspective of an (x', y') frame of reference rotating with the rod. If we let the x'-axis lie along the rod, then the problem is one-dimensional along that direction. The only real force acting on the bead is $\mathbf{F}$, the reaction force that the rod exerts on the bead. It points perpendicular to the rod, along the y'-direction as shown in Figure 5.3.3. $\mathbf{F}$ has no x'-component because there is no friction. Thus, applying Equation 5.3.2 to the bead in this rotating frame, we obtain

$$\mathrm{F}\mathbf{j}' - 2m\omega\mathbf{k}' \times \dot{x}'\mathbf{i}' - m\omega\mathbf{k}' \times (\omega\mathbf{k}' \times x'\mathbf{i}') = m\ddot{x}'\mathbf{i}'$$

$$\mathrm{F}\mathbf{j} - 2m\omega\dot{x}'\mathbf{j}' + m\omega^2 x'\mathbf{i}' = m\ddot{x}'\mathbf{i}'$$

The first inertial force in the preceding equation is the Coriolis force. It appears in the expression because of the bead's velocity $\dot{x}'\mathbf{i}'$ along the x'-axis in the rotating frame. Note that it balances out the reaction force $\mathbf{F}$ that the rod exerts on the bead. The second inertial force is the centrifugal force, $m\omega^2 x'$. From the bead's perspective, this force shoves it down the rod. These ideas are embodied in the two scalar equivalents of the above vector equation

$$\mathrm{F} = 2m\omega\dot{x}' \qquad m\omega^2 x' = m\ddot{x}'$$

Solving the second equation above yields $x'(t)$, the position of the bead along the rod as a function of time

$$x'(t) = Ae^{\omega t} + Be^{-\omega t}$$
$$\dot{x}'(t) = \omega Ae^{\omega t} - \omega Be^{-\omega t}$$

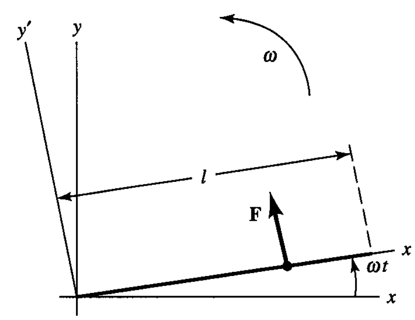

Figure 5.3.3 Bead sliding along a smooth rod rotating at constant angular velocity $\boldsymbol{\omega}$ about an axis fixed at one end.

The boundary conditions, $x'(t=0)=0$ and $\dot{x}'(t=0)=\epsilon$, allow us to determine the constants A and B

$$x'(0) = 0 = A + B \qquad \dot{x}'(0) = \epsilon = \omega(A - B)$$

$$A = -B = \frac{\epsilon}{2\omega}$$

which lead to the explicit solution

$$x'(t) = \frac{\epsilon}{2\omega}(e^{\omega t} - e^{-\omega t})$$

$$= \frac{\epsilon}{\omega}\sinh \omega t$$

The bead flies off the end of the rod at time T, where

$$x'(T) = \frac{\epsilon}{\omega}\sinh \omega T = l$$

$$T = \frac{1}{\omega}\sinh^{-1}\left(\frac{\omega l}{\epsilon}\right)$$

Because the initial speed of the bead is $\epsilon = \omega l$, the preceding equation becomes

$$T = \frac{1}{\omega}\sinh^{-1}(1) = \frac{0.88}{\omega}$$

5.4 | Effects of Earth's Rotation

Let us apply the theory developed in the foregoing sections to a coordinate system that is moving with the Earth. Because the angular speed of Earth's rotation is 2π radians per day, or about 7.27×10^{-5} radians per second, we might expect the effects of such rotation to be relatively small. Nevertheless, it is the spin of the Earth that produces the equatorial bulge; the equatorial radius is some 13 miles greater than the polar radius.

Static Effects: The Plumb Line

Let us consider the case of a plumb bob that is normally used to define the direction of the local "vertical" on the surface of the Earth. We discover that the plumb bob hangs perpendicular to the local surface (discounting bumps and surface irregularities). Because of the Earth's rotation, however, it does not point toward the center of the Earth unless it is suspended somewhere along the equator or just above one of the poles. Let us describe the motion of the plumb bob in a local frame of reference whose origin is at the position of the bob. Our frame of reference is attached to the surface of the Earth. It is undergoing translation as well as rotation. The translation of the frame takes place along a circle whose radius is $\rho = r_e \cos \lambda$, where r_e is the radius of the Earth and λ is the geocentric latitude of the plumb bob (see Figure 5.4.1).

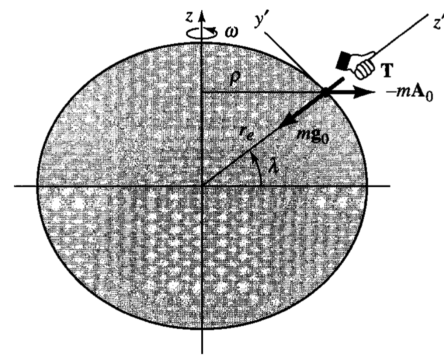

Figure 5.4.1 Gravitational force $m\mathbf{g}_0$, inertial force $-m\mathbf{A}_0$, and tension $\mathbf{T}$ acting on a plumb bob hanging just above the surface of the Earth at latitude λ.

Its rate of rotation is $\boldsymbol{\omega}$, the same as that of the Earth about its axis. Let us now examine the terms that make up Equation 5.3.2. The acceleration of the bob $\mathbf{a}'$ is zero; the bob is at rest in the local frame of reference. The centrifugal force on the bob relative to our local frame is zero because $\mathbf{r}'$ is zero; the origin of the local coordinate system is centered on the bob. The transverse force is zero because $\dot{\boldsymbol{\omega}} = 0$; the rotation of the Earth is constant. The Coriolis force is zero because $\mathbf{v}'$, the velocity of the plumb bob, is zero; the plumb bob is at rest in the local frame. The only surviving terms in Equation 5.3.2 are the real forces $\mathbf{F}$ and the inertial term $-m\mathbf{A}_0$, which arises because the local frame of reference is accelerating. Thus,

$$\mathbf{F} - m\mathbf{A}_0 = 0 \tag{5.4.1}$$

The rotation of the Earth causes the acceleration of the local frame. In fact, the situation under investigation here is entirely analogous to that of Example 5.1.2—the linear accelerometer. There, the pendulum bob did not hang vertically because it experienced an inertial force directed opposite to the acceleration of the railroad car. The case here is almost completely identical. The bob does not hang on a line pointing toward the center of the Earth because the inertial force $-m\mathbf{A}_0$ throws it outward, away from Earth's axis of rotation. This force, like the one of Example 5.1.2, is also directed opposite to the acceleration of the local frame of reference. It arises from the centripetal acceleration of the local frame toward Earth's axis. The magnitude of this force is $m\omega^2 r_e \cos\lambda$. It is a maximum when $\lambda = 0$ at the Earth's equator and a minimum at either pole when $\lambda = \pm 90°$. It is instructive to compare the value of the acceleration portion of this term, $A_0 = \omega^2 r_e \cos\lambda$, to g, the acceleration due to gravity. At the equator, it is $3.4 \times 10^{-3} g$ or less than 1% of g.

$\mathbf{F}$ is the vector sum of all real, physical forces acting on the plumb bob. All forces, including the inertial force $-m\mathbf{A}_0$, are shown in the vector diagram of Figure 5.4.2a. The tension $\mathbf{T}$ in the string balances out the real gravitational force $m\mathbf{g}_0$ and the inertial force $-m\mathbf{A}_0$. In other words

$$(\mathbf{T} + m\mathbf{g}_0) - m\mathbf{A}_0 = 0 \tag{5.4.2}$$

Now, when we hang a plumb bob, we normally think that the tension $\mathbf{T}$ balances out the local force of gravity, which we call $m\mathbf{g}$. We can see from Equation 5.4.2 and Figure 5.4.2b

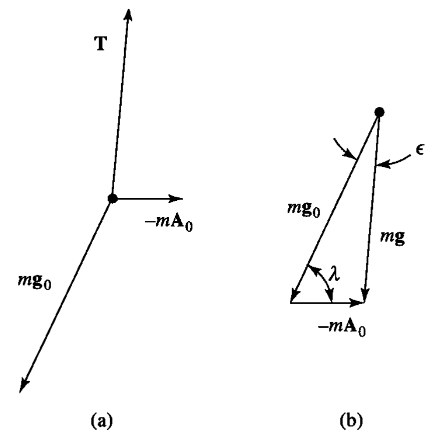

Figure 5.4.2 (a) Forces acting on a plumb bob at latitude λ. (b) Forces defining the weight of the plumb bob, $m\mathbf{g}$.

that $m\mathbf{g}$ is actually the vector sum of the real gravitational force $m\mathbf{g}_0$ and the inertial force $-m\mathbf{A}_0$. Thus,

$$m\mathbf{g}_0 - m\mathbf{g} - m\mathbf{A}_0 = 0 \qquad \therefore\ \mathbf{g} = \mathbf{g}_0 - \mathbf{A}_0 \tag{5.4.3}$$

As can be seen from Figure 5.4.2b, the local acceleration $\mathbf{g}$ due to gravity contains a term $\mathbf{A}_0$ due to the rotation of the Earth. The force $m\mathbf{g}_0$ is the true force of gravity and is directed toward the center of the Earth. The inertial reaction $-m\mathbf{A}_0$, directed away from Earth's axis, causes the direction of the plumb line to deviate by a small angle ϵ away from the direction toward Earth's center. The plumb line direction defines the local direction of the vector $\mathbf{g}$. The shape of the Earth is also defined by the direction of $\mathbf{g}$. Hence, the plumb line is always perpendicular to Earth's surface, which is not shaped like a true sphere but is flattened at the poles and bulged outward at the equator as depicted in Figure 5.4.1.

We can easily calculate the value of the angle ϵ. It is a function of the geocentric latitude of the plumb bob. Applying the law of sines to Figure 5.4.2b, we have

$$\frac{\sin\epsilon}{m\omega^2 r_e \cos\lambda} = \frac{\sin\lambda}{mg} \tag{5.4.4a}$$

or, because ϵ is small

$$\sin\epsilon \approx \epsilon = \frac{\omega^2 r_e}{g}\cos\lambda\,\sin\lambda = \frac{\omega^2 r_e}{2g}\sin 2\lambda \tag{5.4.4b}$$

Thus, ϵ vanishes at the equator ($\lambda = 0$) and the poles ($\lambda = \pm 90°$) as we have already surmised. The maximum deviation of the direction of the plumb line from the center of the Earth occurs at $\lambda = 45°$ where

$$\epsilon_{max} = \frac{\omega^2 r_e}{2g} \approx 1.7\times 10^{-3}\ \text{radian} \approx 0.1° \tag{5.4.4c}$$

In this analysis, we have assumed that the real gravitational force $m\mathbf{g}_0$ is constant and directed toward the center of the Earth. This is not valid, because the Earth is not a true sphere. Its cross section is approximately elliptical as we indicated in Figure 5.4.1; therefore, $\mathbf{g}_0$ varies with latitude. Moreover, local mineral deposits, mountains, and so on, affect the value of $\mathbf{g}_0$. Clearly, calculating the shape of the Earth (essentially, the angle ϵ as a function of λ) is difficult. A more accurate solution can only be obtained numerically. The corrections to the preceding analysis are small.

Dynamic Effects: Motion of a Projectile

The equation of motion for a projectile near the Earth's surface (Equation 5.3.2) can be written

$$m\ddot{\mathbf{r}}' = \mathbf{F} + m\mathbf{g}_0 - m\mathbf{A}_0 - 2m\boldsymbol{\omega} \times \dot{\mathbf{r}}' - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') \tag{5.4.5}$$

where $\mathbf{F}$ represents any applied forces other than gravity. From the static case considered above, however, the combination $m\mathbf{g}_0 - m\mathbf{A}_0$ is called $m\mathbf{g}$; hence, we can write the equation of motion as

$$m\ddot{\mathbf{r}}' = \mathbf{F} + m\mathbf{g} - 2m\boldsymbol{\omega} \times \dot{\mathbf{r}}' - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') \tag{5.4.6}$$

Let us consider the motion of a projectile. If we ignore air resistance, then $\mathbf{F} = 0$. Furthermore, the term $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')$ is very small compared with the other terms, so we can ignore it. The equation of motion then reduces to

$$m\ddot{\mathbf{r}}' = m\mathbf{g} - 2m\boldsymbol{\omega} \times \dot{\mathbf{r}}' \tag{5.4.7}$$

in which the last term is the Coriolis force.

To solve the preceding equation we choose the directions of the coordinate axes $O'x'y'z'$ such that the z'-axis is vertical (in the direction of the plumb line), the x'-axis is to the east, and the y'-axis points north (Figure 5.4.3). With this choice of axes, we have

$$\mathbf{g} = -\mathbf{k}'g \tag{5.4.8}$$

The components of $\boldsymbol{\omega}$ in the primed system are

$$\omega_{x'} = 0 \qquad \omega_{y'} = \omega\cos\lambda \qquad \omega_{z'} = \omega\sin\lambda \tag{5.4.9}$$

The cross product is, therefore, given by

$$\begin{aligned}\boldsymbol{\omega} \times \dot{\mathbf{r}}' &= \begin{vmatrix} \mathbf{i}' & \mathbf{j}' & \mathbf{k}' \\ \omega_{x'} & \omega_{y'} & \omega_{z'} \\ \dot{x}' & \dot{y}' & \dot{z}' \end{vmatrix} \\ &= \mathbf{i}'(\omega\dot{z}'\cos\lambda - \omega\dot{y}'\sin\lambda) + \mathbf{j}'(\omega\dot{x}'\sin\lambda) + \mathbf{k}'(-\omega\dot{x}'\cos\lambda)\end{aligned} \tag{5.4.10}$$

Using the results for $\boldsymbol{\omega} \times \dot{\mathbf{r}}'$ in Equation 5.4.10 and canceling the m's and equating components, we find

$$\ddot{x}' = -2\omega(\dot{z}'\cos\lambda - \dot{y}'\sin\lambda) \tag{5.4.11a}$$

$$\ddot{y}' = -2\omega(\dot{x}'\sin\lambda) \tag{5.4.11b}$$

$$\ddot{z}' = -g + 2\omega\dot{x}'\cos\lambda \tag{5.4.11c}$$

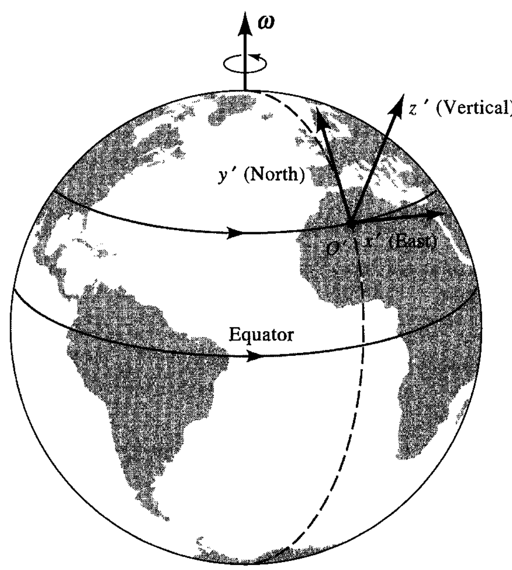

Figure 5.4.3 Coordinate axes for analyzing projectile motion.

for the component differential equations of motion. These equations are not of the separated type, but we can integrate once with respect to t to obtain

$$\dot{x}' = -2\omega(z' \cos\lambda - y' \sin\lambda) + \dot{x}'_0 \qquad (5.4.12a)$$

$$\dot{y}' = -2\omega x' \sin\lambda + \dot{y}'_0 \qquad (5.4.12b)$$

$$\dot{z}' = -gt + 2\omega x' \cos\lambda + \dot{z}'_0 \qquad (5.4.12c)$$

The constants of integration $\dot{x}'_0, \dot{y}'_0$, and $\dot{z}'_0$ are the initial components of the velocity. The values of $\dot{y}'$ and $\dot{z}'$ from Equations 5.4.12b and c may be substituted into Equation 5.4.11a. The result is

$$\ddot{x}' = 2\omega gt \cos\lambda - 2\omega(\dot{z}'_0 \cos\lambda - \dot{y}'_0 \sin\lambda) \qquad (5.4.13)$$

where terms involving ω^2 have been ignored. We now integrate again to get

$$\dot{x}' = \omega gt^2 \cos\lambda - 2\omega t(\dot{z}'_0 \cos\lambda - \dot{y}'_0 \sin\lambda) + \dot{x}'_0 \qquad (5.4.14)$$

and finally, by a third integration, we find x' as a function of t:

$$x'(t) = \tfrac{1}{3}\omega gt^3 \cos\lambda - \omega t^2(\dot{z}'_0 \cos\lambda - \dot{y}'_0 \sin\lambda) + \dot{x}'_0 t + x'_0 \qquad (5.4.15a)$$

The preceding expression for x' may be inserted into Equations 5.4.12b and c. The resulting equations, when integrated, yield

$$y'(t) = \dot{y}'_0 t - \omega \dot{x}'_0 t^2 \sin\lambda + y'_0 \qquad (5.4.15b)$$

$$z'(t) = -\tfrac{1}{2}gt^2 + \dot{z}'_0 t + \omega \dot{x}'_0 t^2 \cos\lambda + z'_0 \qquad (5.4.15c)$$

where, again, terms of order ω^2 have been ignored.

In Equations 5.4.15a–c, the terms involving ω express the effect of Earth's rotation on the motion of a projectile in a coordinate system fixed to the Earth.

EXAMPLE 5.4.1

Falling Body

Suppose a body is dropped from rest at a height h above the ground. Then at time $t = 0$ we have $\dot{x}_0' = \dot{y}_0' = \dot{z}_0' = 0$, and we set $x_0' = y_0' = 0, z_0' = h$ for the initial position. Equations 5.4.15a–c then reduce to

$$x'(t) = \tfrac{1}{3}\omega g t^3 \cos\lambda$$
$$y'(t) = 0$$
$$z'(t) = -\tfrac{1}{2}gt^2 + h$$

Thus, as it falls, the body drifts to the east. When it hits the ground ($z' = 0$), we see that $t^2 = 2h/g$, and the eastward drift is given by the corresponding value of $x'(t)$, namely,

$$x_h' = \tfrac{1}{3}\omega\left(\frac{8h^3}{g}\right)^{1/2} \cos\lambda$$

For a height of, say, 100 m at a latitude of 45°, the drift is

$$\tfrac{1}{3}(7.27\times10^{-5}\ \mathrm{s^{-1}})(8\times100^3\ \mathrm{m^3}/9.8\ \mathrm{m\cdot s^{-2}})^{1/2}\ \cos 45° = 1.55\times10^{-2}\ \mathrm{m} = 1.55\ \mathrm{cm}$$

Because Earth turns to the east, common sense would seem to say that the body should drift westward. Can the reader think of an explanation?

EXAMPLE 5.4.2

Deflection of a Rifle Bullet

Consider a projectile that is fired with high initial speed v_0 in a nearly horizontal direction, and suppose this direction is east. Then $\dot{x}_0' = v_0$ and $\dot{y}_0' = \dot{z}_0' = 0$. If we take the origin to be the point from which the projectile is fired, then $x_0' = y_0' = z_0' = 0$ at time $t = 0$. Equation 5.4.15b then gives

$$y'(t) = -\omega v_0 t^2 \sin\lambda$$

which says that the projectile veers to the south or to the right in the Northern Hemisphere ($\lambda > 0$) and to the left in the Southern Hemisphere ($\lambda < 0$). If H is the horizontal range of the projectile, then we know that $H \approx v_0 t_1$, where t_1 is the time of flight. The transverse deflection is then found by setting $t = t_1 = H/v_0$ in the above expression for $y'(t)$. The result is

$$\Delta \approx \frac{\omega H^2}{v_0}|\sin\lambda|$$

for the magnitude of the deflection. This is the same for *any* direction in which the projectile is initially aimed, provided the trajectory is flat. This follows from the fact that the magnitude of the horizontal component of the Coriolis force on a body traveling parallel to the ground is independent of the direction of motion. (See Problem 5.12.) Because the deflection is proportional to the square of the horizontal range, it becomes of considerable importance in long-range gunnery.

*5.5 Motion of a Projectile in a Rotating Cylinder

Here is one final example concerning the dynamics of projectiles in rotating frames of reference. The example is rather involved and makes use of applied numerical techniques. We hope its inclusion gives you a better appreciation for the connection between the geometry of straight-line, force-free trajectories seen in an inertial frame of reference and the resulting curved geometry seen in a noninertial rotating frame of reference. The inertial forces that appear in a noninertial frame lead to a curved trajectory that may be calculated from the perspective of an inertial frame solely on the basis of geometrical considerations. This must be the case if the validity of Newton's laws of motion is to be preserved in noninertial frames of reference. Such a realization, although completely obvious with hindsight, should not be trivialized. It was ultimately just this sort of realization that led Einstein to formulate his general theory of relativity.

EXAMPLE 5.5.1

In several popular science fiction novels[2] spacecraft capable of supporting entire populations have been envisioned as large, rotating toroids or cylinders. Consider a cylinder of radius $R = 1000$ km and, for our purposes here, infinite length. Let it rotate about its axis with an angular velocity of $\omega = 0.18°/\text{s}$. It completes one revolution every 2000 s. This rotation rate leads to an apparent centrifugal acceleration for objects on the interior surface of $\omega^2 R$ equal to 1 g. Imagine several warring factions living on the interior of the cylinder. Let them fire projectiles at each other.

(a) Show that when projectiles are fired at low speeds ($v \ll \omega R$) and low "altitudes" at nearby points (say, $\Delta r' \leq R/10$), the equations of motion governing the resulting trajectories are identical to those of a similarly limited projectile on the surface of the Earth.

(b) Find the general equations of motion for a projectile of unlimited speed and range using cylindrical coordinates rotating with the cylinder.

(c) Find the trajectory h versus ϕ' of a projectile fired vertically upward with a velocity $v' = \omega R$ in this noninertial frame of reference. $h = R - r'$ is the altitude of the projectile and ϕ' is its angular position in azimuth relative to the launch point. Calculate the angle Φ where it lands relative to the launch point. Also, calculate the maximum height H reached by the projectile.

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[2] For example, *Rendezvous with Rama* and *Rama II* by Arthur C. Clarke (Bantam Books) or *Titan* by John Varley (Berkeley Books).

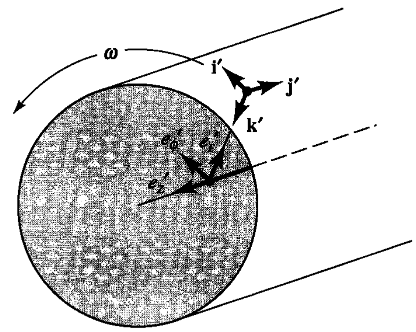

Figure 5.5.1 Coordinates denoted by unit vectors $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ on the interior surface of a rotating cylinder. Unit vectors $\mathbf{e}_{r'}$, $\mathbf{e}_{\phi'}$, $\mathbf{e}_{z'}$ denote cylindrical coordinates. Each set is embedded in and rotates with the cylinder.

(d) Finally, calculate h versus ϕ' solely from the geometrical basis that an inertial observer would employ to predict what the noninertial observer would see. Show that this result agrees with that of part (c), calculated from the perspective of the noninertial observer. In particular, show that Φ and H agree.

Solution:

(a) Because we first consider short, low-lying trajectories, we choose Cartesian coordinates (x', y', z') denoted by the unit vectors $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ attached to and rotating with the cylinder shown in Figure 5.5.1.

The coordinate system is centered on the launch point. Because no real force is acting on the projectile after it is launched, the mass m common to all remaining terms in Equation 5.3.2 can be stripped and the equation then written in terms of accelerations only

$$-\mathbf{A}_0 - 2\boldsymbol{\omega} \times \mathbf{v}' - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') = \mathbf{a}' \tag{5.5.1}$$

The transverse acceleration is zero because the cylinder rotates at a constant rate. The first term on the left is the acceleration of the coordinate system origin. It is given by

$$\mathbf{A}_0 = \omega^2 R\mathbf{k}' \tag{5.5.2}$$

The second term is the Coriolis acceleration and is given by

$$\begin{aligned}\mathbf{a}_{Cor} &= 2\boldsymbol{\omega} \times \mathbf{v}' = 2(-\mathbf{j}'\omega) \times (\mathbf{i}'\dot{x}' + \mathbf{j}'\dot{y}' + \mathbf{k}'\dot{z}') \\ &= 2\omega\dot{x}'\mathbf{k}' - 2\omega\dot{z}'\mathbf{i}'\end{aligned} \tag{5.5.3}$$

The third term is the centrifugal acceleration given by

$$\begin{aligned}\mathbf{a}_{centrif} &= -\mathbf{j}'\omega \times [(-\mathbf{j}'\omega) \times \mathbf{r}'] \\ &= \mathbf{j}'\omega \times [(\mathbf{j}'\omega) \times (\mathbf{i}'x' + \mathbf{j}'y' + \mathbf{k}'z')] \\ &= \mathbf{j}'\omega \times (-\mathbf{k}'\omega x' + \mathbf{i}'\omega z') \\ &= -\mathbf{i}'\omega^2 x' - \mathbf{k}'\omega^2 z'\end{aligned} \tag{5.5.4}$$

After gathering all appropriate terms, the x'-, y'-, and z'-components of the resultant acceleration become

$$\begin{aligned} \ddot{x}' &= 2\omega \dot{z}' + \omega^2 x' \\ \ddot{y}' &= 0 \\ \ddot{z}' &= -2\omega \dot{x}' + \omega^2 z' - \omega^2 R \end{aligned} \tag{5.5.5}$$

If projectiles are limited in both speed and range such that

$$|\dot{x}'| \sim |\dot{z}'| \ll \omega R \qquad |x'| \sim |z'| \ll R \tag{5.5.6}$$

and recalling that the rotation rate of the cylinder has been adjusted to $\omega^2 R = g$, the above acceleration components reduce to

$$\ddot{x}' \approx 0 \qquad \ddot{y}' = 0 \qquad \ddot{z}' \approx -g \tag{5.5.7}$$

which are equivalent to the equations of motion for a projectile of limited speed and range on the surface of the Earth.

(b) In this case no limit is placed on projectile velocity or range. We describe the motion using cylindrical coordinates (r', ϕ', z') attached to and rotating with the cylinder as indicated in Figure 5.5.1. r' denotes the radial position of the projectile measured from the central axis of the cylinder; ϕ' denotes its azimuthal position and is measured from the radius vector directed outward to the launch point; z' represents its position along the cylinder ($z' = 0$ corresponds to the z'-position of the launch point). The overall position, velocity, and acceleration of the projectile in cylindrical coordinates are given by Equations 1.12.1–1.12.3. We can use these relations to evaluate all the acceleration terms in Equation 5.5.1. The term $\mathbf{A}_0$ is zero, because the rotating coordinate system is centered on the axis of rotation. The Coriolis acceleration is

$$\begin{aligned} 2\boldsymbol{\omega} \times \boldsymbol{v}' &= 2\omega \boldsymbol{e}_{z'} \times (\dot{r}' \boldsymbol{e}_{r'} + r' \dot{\phi}' \boldsymbol{e}_{\phi'} + \dot{z}' \boldsymbol{e}_{z'}) \\ &= 2\omega \dot{r}' (\boldsymbol{e}_{z'} \times \boldsymbol{e}_{r'}) + 2\omega r' \dot{\phi}' (\boldsymbol{e}_{z'} \times \boldsymbol{e}_{\phi'}) \\ &= 2\omega \dot{r}' \boldsymbol{e}_{\phi'} - 2\omega r' \dot{\phi}' \boldsymbol{e}_{r'} \end{aligned} \tag{5.5.8}$$

The centrifugal acceleration is

$$\begin{aligned} \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') &= \omega^2 \boldsymbol{e}_{z'} \times [\boldsymbol{e}_{z'} \times (r' \boldsymbol{e}_{r'} + z' \boldsymbol{e}_{z'})] \\ &= \omega^2 \boldsymbol{e}_{z'} \times r' \boldsymbol{e}_{\phi'} \\ &= -\omega^2 r' \boldsymbol{e}_{r'} \end{aligned} \tag{5.5.9}$$

We can now rewrite Equation 5.5.1 in terms of components by gathering together all the previous corresponding elements and equating them to those in Equation 1.12.3

$$\begin{aligned} \ddot{r}' - r' \dot{\phi}'^2 &= 2\omega r' \dot{\phi}' + \omega^2 r' \\ 2\dot{r}' \dot{\phi}' + r' \ddot{\phi}' &= -2\omega \dot{r}' \\ \ddot{z}' &= 0 \end{aligned} \tag{5.5.10}$$

In what follows we ignore the z'-equation of motion because it contains no nonzero acceleration terms and simply gives rise to a "drift" along the axis of the cylinder of any trajectory seen in the $r'\phi'$ plane. Finally, we rewrite the radial and azimuthal equations in such a way that we can more readily see the dependency of the acceleration upon velocities and positions

$$\ddot{r}' = 2\omega r'\dot{\phi}' + (\omega^2 + \dot{\phi}'^2)r' \qquad (5.5.11a)$$

$$\ddot{\phi}' = -\frac{2\dot{r}'}{r'}(\omega + \dot{\phi}') \qquad (5.5.11b)$$

(c) Before solving these equations of motion for a projectile fired vertically upward (from the viewpoint of a cylinder dweller), we investigate the situation from the point of view of an inertial observer located outside the rotating cylinder. The rotational speed of the cylinder is ωR. If the projectile is fired vertically upward with a speed ωR from the point of view of the noninertial observer, the inertial observer sees the projectile launched with a speed $v = \sqrt{2}\,\omega R$ at 45° with respect to the vertical. Furthermore, according to this observer, no real forces act on the projectile. Travel appears to be in a straight line. Its flight path is a chord of a quadrant. This situation is depicted in Figure 5.5.2.

As can be seen in Figure 5.5.2, by the time the projectile reaches a point in its trajectory denoted by the vector $\mathbf{r}'$, the cylinder has rotated such that the launch point a has moved to the position labeled b. Therefore, the inertial observer concludes that the noninertial observer thinks that the projectile has moved through the angle ϕ' and attained an altitude of $R - r'$. When the projectile lands, the noninertial observer finds that the projectile has moved through a total angle of $\Phi = \pi/2 - \omega T$, where T is the total time of flight. But $T = L/v = \sqrt{2}R/(\sqrt{2}\,\omega R) = 1/\omega$, or $\omega T = 1$ radian. Hence, the apparent deflection angle should be $\Phi = \pi/2 - 1$ radians, or about 32.7°. The maximum height reached by the projectile occurs midway through its trajectory when $\omega t + \phi' = \pi/4$ radians. At this point $r' = R/\sqrt{2}$ or $H = R - R/\sqrt{2} = 290$ km. At least, this is what the inertial observer believes the noninertial observer would see. Let us see what the noninertial observer does see according to Newton's laws of motion.

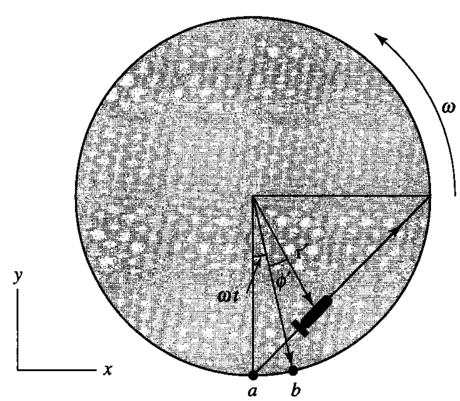

Figure 5.5.2 Trajectory of a projectile launched inside a rotating cylinder at 45° with respect to the "vertical" from the point of view of an inertial observer.

We have used *Mathematica* to solve the differential equations of motion (Equation 5.5.11a and b) numerically as in Example 4.3.2 and the result is shown in Figure 5.5.3.

It can be seen that the projectile is indeed launched vertically upward according to the rotating observer. But the existence of the centrifugal and Coriolis inertial forces causes the projectile to accelerate back toward the surface and toward the east, in the direction of the angular rotation of the cylinder. Note that the rotating observer concludes that the vertically launched projectile has been pushed sideways by the Coriolis force such that it lands 32.7° to the east of the launch point. The centrifugal force has limited its altitude to a maximum value of 290 km. Each value is in complete agreement with the conclusion of the noninertial observer. Clearly, an intelligent military, aware of the dynamical equations of motion governing projectile trajectories on this cylindrical world, could launch all their missiles vertically upward and hit any point around the cylinder by merely adjusting launch velocities. (Positions located up or down the cylindrical axis could be hit by tilting the launcher in that direction and firing the projectile at the required initial and thereafter constant axial velocity $\dot{z}_0$.)

(d) The inertial observer calculates the trajectory seen by the rotating observer in the following way: first, look at Figure 5.5.4. It is a blow-up of the geometry illustrated in Figure 5.5.2.

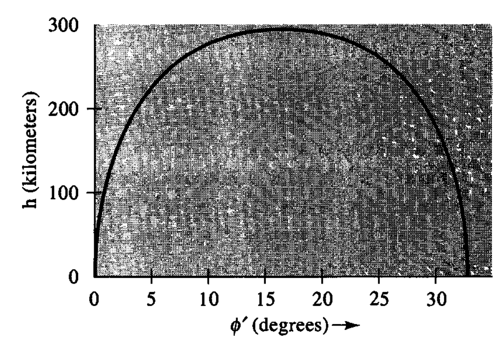

Figure 5.5.3 Trajectory of a projectile fired vertically upward (toward the central axis) from the interior surface of a large cylinder rotating with an angular velocity $\boldsymbol{\omega}$, such that $\omega^2 R = g$.

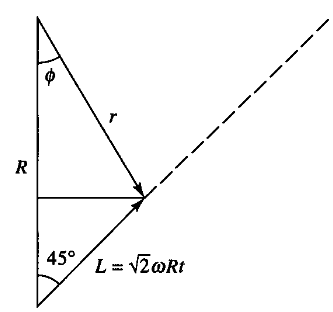

Figure 5.5.4 Geometry used to calculate the trajectory seen by a rotating observer according to an inertial observer.

ϕ is the azimuthal angle of the projectile as measured in the fixed, inertial frame. The azimuthal angle in the noninertial frame is $\phi' = \phi - \omega t$ (see Figure 5.5.2). As can be seen from the geometry of Figure 5.5.4, we can calculate the functional dependency of ϕ upon time

$$\tan\phi(t) = \frac{L(t)\sin 45°}{R - L(t)\cos 45°} = \frac{L(t)}{\sqrt{2}R - L(t)}$$
$$= \frac{\sqrt{2}\,\omega Rt}{\sqrt{2}R - \sqrt{2}\,\omega Rt} = \frac{\omega t}{1-\omega t} \tag{5.5.12}$$

The projectile appears to be deflected toward the east by the angle ϕ' as a function of time given by

$$\phi'(t) = \phi(t) - \omega t = \tan^{-1}\left(\frac{\omega t}{1-\omega t}\right) - \omega t \tag{5.5.13}$$

The dependency of r' on time is given by

$$\begin{aligned} r'^2(t) &= [L(t)\sin 45°]^2 + [R - L(t)\cos 45°]^2 \\ &= L(t)^2 + R^2 - \sqrt{2}L(t)R \\ &= 2(\omega Rt)^2 + R^2 - \sqrt{2}(\sqrt{2}\omega Rt)R \\ &= R^2[1 - 2\omega t(1-\omega t)] \\ \therefore r'(t) &= R[1 - 2\omega t(1-\omega t)]^{1/2} \end{aligned} \tag{5.5.14}$$

These final two parametric equations $r'(t)$ and $\phi'(t)$ describe a trajectory that the inertial observer predicts the noninertial observer should see. If we let time evolve and then plot $h = R - r'$ versus ϕ', we obtain exactly the same trajectory shown in Figure 5.5.3. That trajectory was calculated by the noninertial observer who used Newton's dynamical equations of motion in the rotating frame of reference. Thus, we see the equivalence between the curved geometry of straight lines seen from the perspective of an accelerated frame of reference and the existence of inertial forces that produce that geometry in the accelerated frame.

5.6 | The Foucault Pendulum

In this section we study the effect of Earth's rotation on the motion of a pendulum that is free to swing in any direction, the so-called *spherical pendulum*. As shown in Figure 5.6.1, the applied force acting on the pendulum bob is the vector sum of the weight $m\mathbf{g}$ and the tension $\mathbf{S}$ in the cord. The differential equation of motion is then

$$m\ddot{\mathbf{r}}' = m\mathbf{g} + \mathbf{S} - 2m\boldsymbol{\omega} \times \dot{\mathbf{r}}' \tag{5.6.1}$$

Here we ignored the term $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')$. It is vanishingly small in this context. Previously, we worked out the components of the cross product $\boldsymbol{\omega} \times \mathbf{r}'$ (see Equation 5.4.10). Now the x'- and y'-components of the tension can be found simply by noting that the

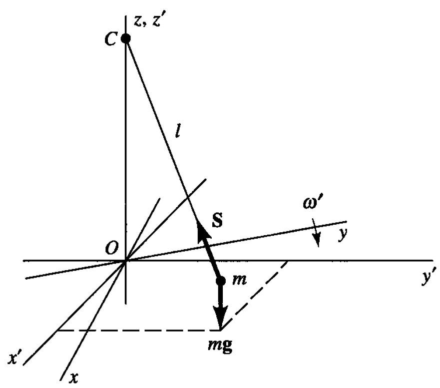

Figure 5.6.1 The Foucault pendulum.

direction cosines of the vector **S** are $-x'/l$, $-y'/l$, and $-(l-z')/l$, respectively. Consequently $S_x = -x'S/l$, $S_y = -y'S/l$, and the corresponding components of the differential equation of motion (5.6.1) are

$$m\ddot{x}' = \frac{-x'}{l}S - 2m\omega(\dot{z}'\cos\lambda - \dot{y}'\sin\lambda) \tag{5.6.2a}$$

$$m\ddot{y}' = \frac{-y'}{l}S - 2m\omega\dot{x}'\sin\lambda \tag{5.6.2b}$$

We are interested in the case in which the amplitude of oscillation of the pendulum is small so that the magnitude of the tension S is very nearly constant and equal to mg. Also, we shall ignore $\dot{z}'$ compared with $\dot{y}'$ in Equation 5.6.2a. The $x'y'$ motion is then governed by the following differential equations:

$$\ddot{x}' = -\frac{g}{l}x' + 2\omega'\dot{y}' \tag{5.6.3a}$$

$$\ddot{y}' = -\frac{g}{l}y' - 2\omega'\dot{x}' \tag{5.6.3b}$$

in which we have introduced the quantity $\omega' = \omega\sin\lambda = \omega_{z'}$, which is the local vertical component of Earth's angular velocity.

Again we are confronted with a set of differential equations of motion that are not in separated form. A heuristic method of solving the equations is to transform to a new coordinate system $Oxyz$ that rotates relative to the primed system in such a way as to cancel the vertical component of Earth's rotation, namely, with angular rate $-\omega'$ about the vertical axis as shown in Figure 5.4.3. Thus, the unprimed system has no rotation about the vertical axis. The equations of transformation are

$$x' = x\cos\omega't + y\sin\omega't \tag{5.6.4a}$$

$$y' = -x\sin\omega't + y\cos\omega't \tag{5.6.4b}$$

On substituting the expressions for the primed quantities and their derivatives from the preceding equations into Equations 5.6.3a and b, the following result is obtained, after collecting terms and dropping terms involving ω'^2,

$$\left(\ddot{x}+\frac{g}{l}x\right)\cos\omega't+\left(\ddot{y}+\frac{g}{l}y\right)\sin\omega't=0 \tag{5.6.5}$$

and an identical equation, except that the sine and cosine are reversed. Clearly, the preceding equation is satisfied if the coefficients of the sine and cosine terms both vanish, namely,

$$\ddot{x}+\frac{g}{l}x=0 \tag{5.6.6a}$$

$$\ddot{y}+\frac{g}{l}y=0 \tag{5.6.6b}$$

These are the differential equations of the two-dimensional harmonic oscillator discussed previously in Section 4.4. Thus, the path, projected on the xy plane, is an ellipse with *fixed* orientation in the unprimed system. In the primed system the path is an ellipse that undergoes a steady precession with angular speed $\omega' = \omega \sin \lambda$.

In addition to this type of precession, there is another *natural* precession of the spherical pendulum, which is ordinarily much larger than the rotational precession under discussion. However, if the pendulum is carefully started by drawing it aside with a thread and letting it start from rest by burning the thread, the natural precession is rendered negligibly small.[3]

The rotational precession is clockwise in the Northern Hemisphere and counterclockwise in the Southern. The period is $2\pi/\omega' = 2\pi/(\omega \sin \lambda) = 24/\sin \lambda$ h. Thus, at a latitude of 45°, the period is (24/0.707) h = 33.94 h. The result was first demonstrated by the French physicist Jean Foucault in Paris in the year 1851. The Foucault pendulum has come to be a traditional display in major planetariums throughout the world.

Problems

5.1 A 120-lb person stands on a bathroom spring scale while riding in an elevator. If the elevator has (a) upward and (b) downward acceleration of $g/4$, what is the weight indicated on the scale in each case?

5.2 An ultracentrifuge has a rotational speed of 500 rps. (a) Find the centrifugal force on a 1-μg particle in the sample chamber if the particle is 5 cm from the rotational axis. (b) Express the result as the ratio of the centrifugal force to the weight of the particle.

5.3 A plumb line is held steady while being carried along in a moving train. If the mass of the plumb bob is m, find the tension in the cord and the deflection from the local vertical if the train is accelerating forward with constant acceleration $g/10$. (Ignore any effects of Earth's rotation.)

5.4 If, in Problem 5.3, the plumb line is not held steady but oscillates as a simple pendulum, find the period of oscillation for small amplitude.

5.5 A hauling truck is traveling on a level road. The driver suddenly applies the brakes, causing the truck to decelerate by an amount $g/2$. This causes a box in the rear of the truck to slide forward. If the coefficient of sliding friction between the box and the truckbed is $\frac{1}{3}$, find the acceleration of the box relative to **(a)** the truck and **(b)** the road.

[3]The natural precession will be discussed briefly in Chapter 10.