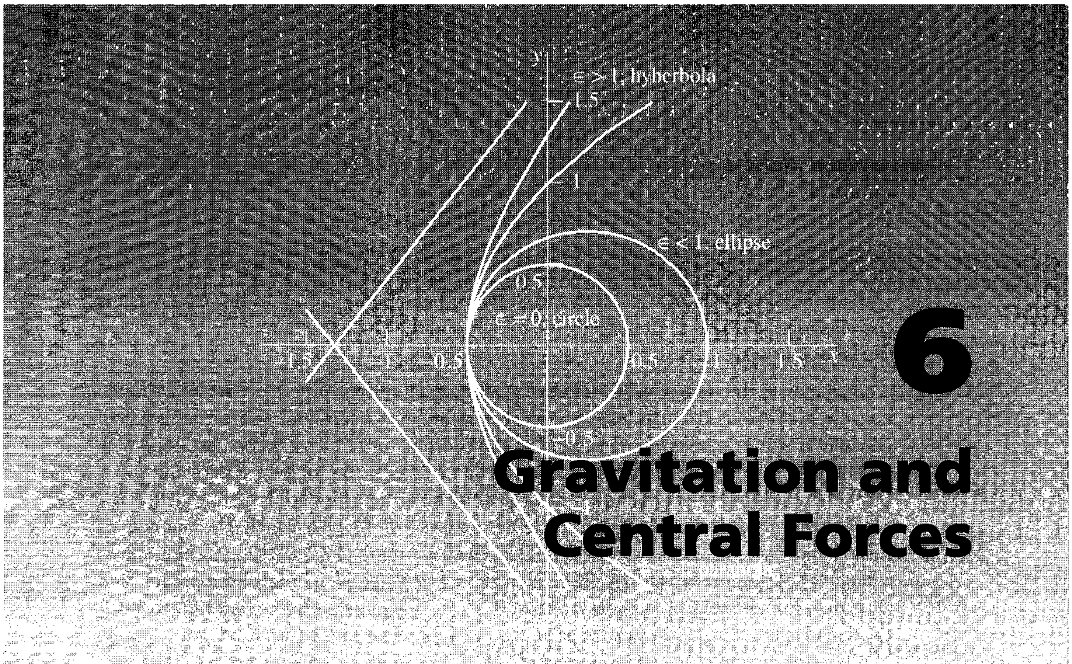

6 Gravitation and Central Forces

"We have explained the phenomena of the heavens and of our sea by the power of gravity, but have not yet assigned the cause of this power . . . I have not been able to discover the cause of those properties of gravity from phenomena, and I frame no hypotheses;—"

—Sir Isaac Newton, *The Principia*, 1687; Florian Cajori's translation, Berkeley, Univ. of Calif. Press, 1966

"Gravity must be a scholastic occult quality or the effect of a miracle."

—Gottfried Wilhelm Leibniz; See *Let Newton Be!*, by J. Fauvel, R. Flood, M. Shorthand, and R. Wilson, Oxford Univ. Press, 1988

6.1 | Introduction

Throughout the year ancient peoples observed the five visible planets slowly move through the fixed constellations of the zodiac in a fairly regular fashion. But occasionally, at times that occurred with astonishing predictability, they mysteriously halted their slow forward progression, suddenly reversing direction for as long as a few weeks before again resuming their steady march through the sky. This apparent quirk of planetary behavior is called *retrograde motion*. Unmasking its origin would consume the intellectual energy of ancient astronomers for centuries to come. Indeed, horribly complicated concoctions from minds shackled by philosophical dogma and fuzzy notions of physics, such as the cycles and epicycles of Ptolemy (125 C.E.) and others of like-minded mentality, would serve as models of physical reality for more than 2000 years. Ultimately, Nicolaus Copernicus (1473–1543) demonstrated that retrograde motion was nothing other than a simple consequence of the relative motion between Earth and the other planets each moving in a heliocentric orbit. Nonetheless, even Copernicus could not purge himself of the Ptolemaic epicycles, constrained by the dogma of uniform circular motion and the requirement of obtaining agreement between the observed and predicted irregularities of planetary motion.

It was not until Johannes Kepler (1571–1630) turned loose his potent intellect on the problem of solving the orbit of Mars, an endeavor that was to occupy him intensely for 20 years, that for the first time in history, scientists glimpsed the precise mathematical nature of the heavenly motions. Kepler painstakingly constructed a concise set of three mathematical laws that accurately described the orbits of the planets around the Sun. These three laws of planetary motion were soon seen by Newton as nothing other than simple consequences of the interplay of a law of universal gravitation with three fundamental laws of mechanics that Newton had developed mostly from Galileo's investigations of the motions of terrestrial objects. Thus, Newton was to incorporate the physical workings of all the heavenly bodies within a framework of natural law that resided on Earth. The world would never be seen in quite the same way again.

Newton's Law of Universal Gravitation

Newton formally announced the law of universal gravitation in the *Principia*, published in 1687. He actually worked out much of the theory at his family home in Woolsthorpe, England, as early as 1665–1666, during a six-month hiatus from Cambridge University, which was closed while a plague ravaged most of London.

The law can be stated as follows:

Every particle in the universe attracts every other particle with a force whose magnitude is proportional to the product of the masses of the two particles and inversely proportional to the square of the distance between them. The direction of the force lies along the straight line connecting the two particles.

We can express the law vectorially by the equation

$$\mathbf{F}_{ij} = G\frac{m_i m_j}{r_{ij}^2}\left(\frac{\mathbf{r}_{ij}}{r_{ij}}\right) \tag{6.1.1}$$

where $\mathbf{F}_{ij}$ is the force on particle i of mass m_i exerted by particle j of mass m_j. The vector $\mathbf{r}_{ij}$ is the directed line segment running from particle i to particle j, as shown in Figure 6.1.1. The law of action and reaction requires that $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$. The constant of proportionality G

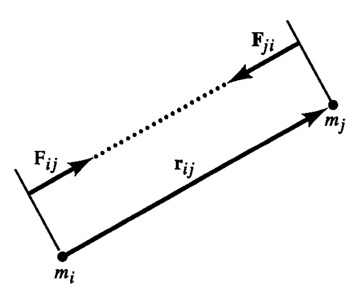

Figure 6.1.1 Action and reaction in Newton's law of gravity.

is known as the *universal constant of gravitation*. Its value is determined in the laboratory by carefully measuring the force between two bodies of known mass. The internationally accepted value at present is, in SI units,

$$G = (6.67259 \pm 0.00085) \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$$

All our present knowledge of the masses of astronomical bodies, including Earth, is based on the value of this fundamental constant.[1]

This law is an example of a general class of forces termed *central;* that is, forces whose lines of action either emanate from or terminate on a single point or center. Furthermore, if the magnitude of the force, as is the case with gravitation, is independent of any direction, the force is isotropic. The behavior of such a force may be visualized in the following way: Imagine being confined to a hypothetical, spherical surface centered about a massive particle that serves as a source of gravity. When walking around that surface, one would discover that the force of attraction would always be directed toward the center, and the magnitude of this force would be independent of position on the spherical surface. Nothing about this force could be used to determine position on the sphere.

The main purpose of this chapter is to study the motion of a particle subject to a central, isotropic force with particular emphasis on the force of gravity. In carrying out this study, we follow Newton's original line of inquiry, which led to the formulation of his universal law of gravitation. In so doing, we hope to engender an appreciation for the tremendous depth of Newton's intellectual achievement.

Gravitation: An Inverse-Square Law?

While home at Woolsthorpe in 1665, Newton took up the studies that were to occupy him for the rest of his life: mathematics, mechanics, optics, and gravitation. Perhaps the most classic image we have of Newton depicts him sitting under an apple tree and being struck by a falling apple. This visual image is meant to convey the notion of Newton pondering the nature of gravity, most probably wondering whether or not the force that caused the apple to fall could be the same one that held the Moon in its orbit about the Earth.

Galileo, who very nearly postulated the law of inertia in its Newtonian form, inexplicably failed to apply it correctly to the motion of heavenly objects. He missed the most fundamental point of circular motion, namely, that objects moving in circles are accelerated inward and, therefore, require a resultant force in that direction. By Newton's time, a number of natural philosophers had come to the conclusion that some sort of force was required, not to accelerate a planet or satellite inward toward its parent body, but to "maintain it in its orbit." In 1665, the Italian astronomer Giovanni Borelli had presented a theory of the motion of the Galilean Moons of Jupiter in which he stated that the centrifugal

[1] G is the least accurately known of all the basic physical constants. This stems from the fact that the gravitational force between two bodies of laboratory size is extremely small. For a review of the current situation regarding the determination of G, see an article by J. Maddox, *Nature*, **30,** 723 (1984). Also, H. de Boer, "Experiments Relating to the Newtonian Gravitational Constant," in B. N. Taylor and W. D. Phillips, eds., *Precision Measurements and Fundamental Constants* (Natl. Bur. Stand. U.S., Spec. Publ., 617, 1984).

force of a Moon's orbital motion was exactly in equilibrium with the attractive force of Jupiter.[2]

Newton was the first to realize that Earth's Moon was not "balanced in its orbit" but was undergoing a centripetal acceleration toward Earth that had to be caused by a centripetal force. Newton surmised that this force was the same one that attracted all Earth-bound objects toward its surface. This had to be the case, because the kinematical behavior of the Moon was no different from that of any object falling toward Earth. The falling Moon never hits Earth because the Moon has such a large tangential velocity that, as it falls a given distance, it moves far enough sideways that Earth's surface has curved away by that same distance. No one at the time even remotely suspected that the centripetal acceleration of the Moon and the gravitational acceleration of an apple falling on the surface of Earth had a common origin.

Newton demonstrated that if a falling apple could also be given a large enough horizontal velocity, its motion would be identical to that of the orbiting Moon (the apple's orbit would just be closer to Earth), thus making the argument for a common origin of an attractive gravitational force even more convincing. Newton further reasoned that the centripetal acceleration of an apple put in orbit about Earth just above its surface would be identical to its gravitational free-fall acceleration. (Imagine an apple shot horizontally out of a powerful cannon. Let there be no air resistance. If the initial horizontal speed of the projected apple were adjusted just right, the apple would never hit Earth because Earth's surface would fall away at the same rate that the apple would fall toward it, just as is the case for the orbiting Moon. In other words, the apple would be in orbit and its centripetal acceleration would exactly equal the g of an apple falling from rest.) Thus, with this single brilliant mental leap, Newton was about to uncover the first and one of the most beautiful of all unifying principles in physics, the law of universal gravitation.

The critical question for Newton was figuring out just how this attractive force depended on distance away from Earth's center. Newton knew that the strength of Earth's attractive force was proportional to the acceleration of falling objects at whatever distance from Earth they happened to be. The Moon's acceleration toward Earth is $a = v^2/r$, where v is the speed of the Moon and r is the radius of its circular orbit. (This is equal to the local value of g.) Newton deduced, with the aid of Kepler's third law (the square of the orbital period τ^2 is proportional to the cube of the distance from the center of the orbit r^3), that this acceleration should vary as $1/r^2$. For example, if the Moon were 4 times farther away from Earth than it actually is, then by Kepler's third law its period of revolution would be 8 times longer, and its orbital speed 2 times slower; consequently, its centripetal acceleration would be 16 times less than it is—or weaker as the inverse square of the distance.

[2]Recall from Chapter 5 that centrifugal force is an inertial force exerted on an object in a rotating frame of reference. In the context here, it arises from the centripetal acceleration of a Galilean moon traveling in essentially a circular orbit around Jupiter. To most pre-Newtonian thinkers, the centrifugal force acting on planets or satellites was a real one. Many of their arguments centered on the nature of the force required to "balance out" the centrifugal force. They completely missed the point that from the perspective of an inertial observer, the satellite was undergoing centripetal acceleration inward. They were thus arguing from the perspective of a noninertial observer, although none of them had such a precise understanding regarding the distinction.

Newton thus hypothesized that the local value of g for all falling objects and, hence, the attractive force of gravity, should vary accordingly. To confirm this hypothesis Newton had to calculate the centripetal acceleration of the Moon, compare it to the acceleration g of a falling apple, and see if the ratio were equal to that of the inverse square of their respective distances from the center of the Earth. The Moon's distance is 60 Earth radii. The force of Earth's gravity must, therefore, weaken by a factor of 3600. The rate of fall of an apple must be 3600 times larger than that of the Moon or, put another way, the distance an apple falls in 1 s should equal the distance the Moon falls toward Earth in 1 min, the distance of fall being proportional to time squared. Unfortunately, Newton made a mistake in carrying out this calculation. He assumed that an angle of 1° subtended an arc length of 60 miles on the surface of Earth. He got this from a sailor's manual, the only book at hand. (This distance is, in fact, 60 nautical miles, or 69 English miles.) Setting this equal to 60 English miles of 5280 ft each, however, he computed the Moon's distance of fall in 1 s to be 0.0036 ft, or 13 ft in 1 min. Through Galileo's experiments with falling bodies, repeated later with more accuracy, an apple (or any other body) had been measured to fall about 15 ft in 1 s on Earth. The values are very close, differing by about 1 part in 8, but such a difference was great enough that Newton abandoned his brilliant idea! Later, he was to use the correct values, get it exactly right, and, thus, demonstrate an inverse-square law for the law of gravity.

Proportional to Mass?

Newton also concluded that the force of gravity acting on any object must be proportional to its mass (as opposed to, say, mass squared or something else). This conclusion is derivable from his second law of motion and Galileo's finding that the rate of fall of all objects is independent of their weight and composition. For example, let the force of gravity of Earth acting on some object of inertial mass m be proportional to that mass. Then, according to Newton's second law of motion, $F_{grav} = k \cdot m/r^2 = m \cdot a = m \cdot g$. Thus, $g = k/r^2$. The masses cancel out in this dynamical equation, and the acceleration g depends only on some constant k (which, in some way, must depend on the mass of the Earth but, obviously, is the same for all bodies attracted to the Earth) and the distance r to the center of the Earth. So all bodies fall with the same acceleration regardless of their mass or composition. The gravitational force must be directly proportional to the inertial mass, or this precise cancellation would not occur. Then all falling bodies would exhibit mass-dependent accelerations, contrary to all experiments designed to test such a hypothesis. In fact the equivalence of inertial and gravitational mass of all objects is one of the cornerstones of Einstein's general theory of relativity. For Newton this equivalence remained a mystery to his death.

Product of Masses, Universality?

Newton also realized that if the force of gravity were to obey his third law of motion and if the force of gravity were proportional to the mass of the object being attracted, then it must also be proportional to the mass of the attracting object. Such a requirement leads us inevitably to the conclusion that the law of gravity must, therefore, be "universal";

that is, every object in the universe must attract (albeit very weakly, in most cases) every other object in the universe. Let us see how this comes about. Imagine two masses m_1 and m_2 separated by a distance r. The forces of attraction on 1 by 2 and on 2 by 1 are $F_{12} = k_2 m_1/r^2$ and $F_{21} = k_1 m_2/r^2$, where k_1 and k_2 are "constants" that, as we are forced to conclude, must depend on the mass of the attracting object. According to Newton's third law, these forces have to be equal in magnitude (and opposite in direction); therefore, $k_2 m_1 = k_1 m_2$ or $k_2/k_1 = m_2/m_1$. To ensure the equality of this ratio, the strength of the attraction of gravity must be proportional to the mass of the attractive body, that is, $k_i = Gm_i$. Thus, the force of gravity between two particles is a central, isotropic law of force possessing a wonderful symmetry: particle 1 attracts particle 2 and particle 2 attracts particle 1 with a magnitude and direction, obeying Newton's third law, proportional to the product of each of their masses and varying inversely as the square of their distance of separation. This conclusion was the work of true genius!

6.2 | Gravitational Force between a Uniform Sphere and a Particle

Newton did not publish the *Principia* until 1687. There was one particular problem that bothered him and made him reluctant to publish. We quickly glossed over that problem in our preceding discussion. Newton derived the inverse-square law by assuming that the relevant distance of separation between two objects, such as Earth and Moon, is the distance between their respective geometrical centers. This does not seem to be unreasonable for spherical objects like the Sun and the planets, or Earth and Moon, whose distances of separation are large compared with their radii. But what about Earth and the apple? If you or I were the apple, looking around at all the stuff in Earth attracting us, we would see lines of gravitational force tugging on us from directions all over the place. There is stuff to the east and stuff to the west whose directions of pull differ by 180°. Who is to say that when we properly add up all the force vectors, due to all this attractive stuff, we get a resultant vector that points to the center of Earth and whose strength depends on the mass of Earth and inversely on the distance to its center squared, as though all Earth's mass were completely concentrated at its center?

Yet, this is the way it works out. It is a tricky problem in calculus that requires a vector sum of infinitessimal contributions over an infinite number of mass elements that lead to a finite result. At that time no one knew calculus because Newton had just invented it, probably to solve this very problem! He was understandably reluctant to publish such a proof, couched in a framework of nonexistent mathematics. Because everyone knows calculus in our present age of enlightenment, we will go ahead and use it to solve the problem, proving that, for any uniform spherical body or any spherically symmetric distribution of matter, the gravitational force exerted by it on any external particle can be calculated by simply assuming that the entire mass of the distribution acts as though concentrated at its geometric center. Only an inverse-square force law works this way.

Consider first a thin uniform shell of mass M and radius R. Let r be the distance from the center O to a test particle P of mass m (Fig. 6.2.1). We assume that $r > R$. We shall divide the shell into circular rings of width $R\,\Delta\theta$ where, as shown in the figure, the angle

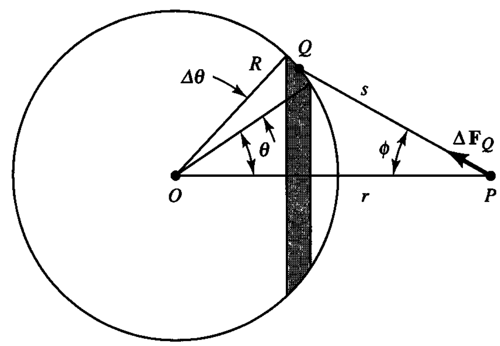

Figure 6.2.1 Coordinates for calculating the gravitational field of a spherical shell.

POQ is denoted by θ, Q being a point on the ring. The circumference of our representative ring element is, therefore, $2\pi R \sin\theta$, and its mass ΔM is given by

$$\Delta M \approx \rho 2\pi R^2 \sin\theta\, \Delta\theta \tag{6.2.1}$$

where ρ is the mass per unit area of the shell.

Now the gravitational force exerted on P by a small subelement Q of the ring (which we regard as a particle) is in the direction PQ. Let us resolve this force $\Delta \mathbf{F}_Q$ into two components, one component along PO, of magnitude $\Delta F_Q \cos\phi$, the other perpendicular to PO, of magnitude $\Delta F_Q \sin\phi$. Here ϕ is the angle OPQ, as shown in Figure 6.2.1. From symmetry we can easily see that the vector sum of all of the perpendicular components exerted on P by the whole ring vanishes. The force $\Delta \mathbf{F}$ exerted by the entire ring is, therefore, in the direction PO, and its magnitude ΔF is obtained by summing the components $\Delta F_Q \cos\phi$. The result is

$$\Delta F = G\frac{m\,\Delta M}{s^2}\cos\phi = G\frac{m 2\pi\rho R^2 \sin\theta \cos\phi}{s^2}\Delta\theta \tag{6.2.2}$$

where s is the distance PQ (the distance from the particle P to the ring) as shown. The magnitude of the force exerted on P by the whole shell is then obtained by taking the limit of $\Delta\theta$ and integrating

$$F = Gm2\pi\rho R^2 \int_0^\pi \frac{\sin\theta \cos\phi\, d\theta}{s^2} \tag{6.2.3}$$

The integral is most easily evaluated by expressing the integrand in terms of s. From the triangle OPQ we have, from the law of cosines,

$$r^2 + R^2 - 2rR\cos\theta = s^2 \tag{6.2.4}$$

Differentiating, because both R and r are constant, we have,

$$rR \sin\theta\, d\theta = s\, ds \tag{6.2.5}$$

Also, in the same triangle OPQ, we can write

$$\cos\phi = \frac{s^2 + r^2 - R^2}{2rs} \tag{6.2.6}$$

On performing the substitutions given by the preceding two equations, and changing the limits of integration from $[0, \pi]$ to $[r - R, r + R]$ we obtain

$$\begin{aligned} F &= Gm2\pi\rho R^2 \int_{r-R}^{r+R} \frac{s^2 + r^2 - R^2}{2Rr^2 s^2} ds \\ &= \frac{GmM}{4Rr^2} \int_{r-R}^{r+R} \left(1 + \frac{r^2 - R^2}{s^2}\right) ds \\ &= \frac{GmM}{r^2} \end{aligned} \tag{6.2.7}$$

where $M = 4\pi\rho R^2$ is the mass of the shell. We can then write vectorially

$$\mathbf{F} = -G\frac{Mm}{r^2}\mathbf{e}_r \tag{6.2.8}$$

where $\mathbf{e}_r$ is the unit radial vector from the origin O. The preceding result means that a uniform spherical shell of matter attracts an external particle as if the whole mass of the shell were concentrated at its center. This is true for every concentric spherical portion of a solid uniform sphere. *A uniform spherical body, therefore, attracts an external particle as if the entire mass of the sphere were located at the center.* The same is true also for a nonuniform sphere provided the density depends only on the radial distance r.

The gravitational force on a particle located *inside* a uniform spherical shell is zero. The proof is left as an exercise (see Problem 6.2).

6.3 | Kepler's Laws of Planetary Motion

Kepler's laws of planetary motion were a landmark in the history of physics. They played a crucial role in Newton's development of the law of gravitation. Kepler deduced these laws from a detailed analysis of planetary motions, primarily the motion of Mars, the closest outer planet and one whose orbit is, unlike that of Venus, highly elliptical. Mars had been most accurately observed, and its positions on the celestial sphere dutifully recorded by Kepler's irascible but brilliant patron, Tycho de Brahe (1546–1601). Kepler even used some sightings that had been made by the early Greek astronomer Hipparchus (190–125 B.C.E.). Kepler's three laws are:

I. Law of Ellipses (1609)
The orbit of each planet is an ellipse, with the Sun located at one of its foci.

II. Law of Equal Areas (1609)
A line drawn between the Sun and the planet sweeps out equal areas in equal times as the planet orbits the Sun.

III. Harmonic Law (1618)
The square of the sidereal period of a planet (the time it takes a planet to complete one revolution about the Sun relative to the stars) is directly proportional to the cube of the semimajor axis of the planet's orbit.

The derivation of these laws from Newton's theories of gravitation and mechanics was one of the most stupendous achievements in the annals of science. A number of Newton's colleagues who were prominent members of the British Royal Society were convinced that the Sun exerted a force of gravitation on the planets, that the strength of that force must diminish by the square of the distance between the Sun and the planet, and that this fact could be used to explain Kepler's laws. (Kepler's second law is, however, a statement that the angular momentum of a planet in orbit is conserved, a consequence only of the central nature of the gravitational force, not its inverse-square feature.) The trouble was, as noted by Edmond Halley (1656–1742) over lunch with Robert Hooke (1635–1703) and Christopher Wren (1632–1723) in January of 1684, that no one could make the connection mathematically. Part of the problem was that no one, except the silent Newton, could show that the gravitational forces of spherical bodies could be treated as though they emanated from and terminated on their geometric centers. Hooke brashly stated that he could prove the fact that the planets traveled in elliptical orbits but had not told anyone how to do it so that they might, in attempting a solution themselves, appreciate the magnitude of the problem. Wren offered a prize of 40 shillings—in those days the price of an expensive book—to the one who could produce such a proof within two months. Neither Hooke, nor anyone else, won the prize!

In August of 1684, while visiting Cambridge, Halley stopped in to see Newton and asked him what would be the shape of the planets' orbits if they were subject to an inverse-square attractive force by the Sun? Newton replied, without hesitation, "An ellipse!" Halley wanted to know how Newton knew this, and Newton said that he had calculated it years ago. Halley was stunned. They looked through thousands of Newton's papers but could not find the calculation. Newton told Halley that he would redo it and send it to him.

Newton had actually done the calculation five years earlier, in 1679, stimulated in part by Robert Hooke, the aforementioned claimant to the inverse-square law, who had written Newton with questions about the trajectory of objects falling toward a gravitationally attractive body. Unfortunately, there was a mistake in the calculation of Newton's written reply to Hooke. Hooke, with glee, pointed out the mistake, and the angry Newton, concentrating on the problem with renewed vigor, apparently straightened things out. These subsequent calculations, however, also contained a mistake, which is perhaps why Newton failed to find them when queried by Halley. At any rate, Newton furiously attacked the problem again and within three months sent Halley a paper in which he correctly derived all of Kepler's laws from an inverse-square law of gravitation and the laws of mechanics. Thus was the *Principia* born. In the sections that follow we, too, derive Kepler's laws from Newton's fundamental principles.

6.4 | Kepler's Second Law: Equal Areas

Conservation of Angular Momentum

Kepler's second law is nothing other than the statement that the angular momentum of a planet about the Sun is a conserved quantity. To show this, we first define angular momentum and then show that its conservation is a general consequence of the central nature of the gravitational force.

The *angular momentum* of a particle located a vector distance $\mathbf{r}$ from a given origin and moving with momentum $\mathbf{p}$ is defined to be the quantity $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. The time derivative of this quantity is

$$\frac{d\mathbf{L}}{dt} = \frac{d(\mathbf{r} \times \mathbf{p})}{dt} = \mathbf{v} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} \tag{6.4.1}$$

but

$$\mathbf{v} \times \mathbf{p} = \mathbf{v} \times m\mathbf{v} = m\mathbf{v} \times \mathbf{v} = 0 \tag{6.4.2}$$

Thus,

$$\mathbf{r} \times \mathbf{F} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} = \frac{d\mathbf{L}}{dt} \tag{6.4.3}$$

where we have used Newton's second law, $\mathbf{F} = d\mathbf{p}/dt$.

The cross product $\mathbf{N} = \mathbf{r} \times \mathbf{F}$ is the moment of force, or torque, on the particle about the origin of the coordinate system. If $\mathbf{r}$ and $\mathbf{F}$ are collinear, this cross product vanishes and so does $\mathbf{L}$. The angular momentum $\mathbf{L}$, in such cases, is a constant of the motion. This is quite obviously the case for a particle (or a planet) subject to a *central force* $\mathbf{F}$, that is, one that either emanates or terminates from a single point and whose line of action lies along the radius vector $\mathbf{r}$.

Furthermore, because the vectors $\mathbf{r}$ and $\mathbf{v}$ define an "instantaneous" plane within which the particle moves, and because the angular momentum vector $\mathbf{L}$ is normal to this plane and is constant in both magnitude and direction, the orientation of this plane is fixed in space. Thus, the problem of motion of a particle in a central field is really a two-dimensional problem and can be treated that way without any loss of generality.

Angular Momentum and Areal Velocity of a Particle Moving in a Central Field

As previously mentioned, Kepler's second law, the constancy of the *areal velocity,* $\dot{A}$, of a planet about the Sun, depends only upon the central nature of the gravitational force and not upon how the strength of the force varies with radial distance from the Sun. Here we show that this law is equivalent to the more general result that the angular momentum of any particle moving in a central field of force is conserved, as shown in the preceding section.

To do so, we first calculate the magnitude of the angular momentum of a particle moving in a central field. We use polar coordinates to describe the motion. The velocity of the particle is

$$\mathbf{v} = \mathbf{e}_r\dot{r} + \mathbf{e}_\theta r\dot{\theta} \tag{6.4.4}$$

where $\mathbf{e}_r$ is the unit radial vector and $\mathbf{e}_\theta$ is the unit transverse vector. (see Figure 6.4.1(a)).

The magnitude of the angular momentum is

$$L = |\mathbf{r} \times m\mathbf{v}| = |r\mathbf{e}_r \times m(\dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta)| \tag{6.4.5}$$

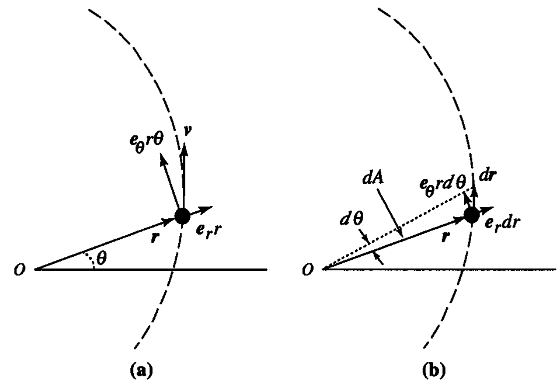

Figure 6.4.1 (a) Angular momentum $L = |\mathbf{r} \times m\mathbf{v}|$ of a particle moving in a central field. (b) Area $dA = \frac{1}{2}\,|\mathbf{r} \times d\mathbf{r}|$ swept out by the radius vector $\mathbf{r}$ of the particle as it moves in a central field.

Because $|\mathbf{e}_r \times \mathbf{e}_r| = 0$ and $|\mathbf{e}_r \times \mathbf{e}_\theta| = 1$, we find

$$L = mr^2\dot{\theta} = \text{constant} \tag{6.4.6}$$

for *any particle moving in a central field of force* including a planet moving in the gravitational field of the Sun.

Now, we calculate the "areal velocity," $\dot{A}$, of the particle. Figure 6.4.1(b) shows the triangular area, dA, swept out by the radius vector $\mathbf{r}$ as a planet moves a vector distance $d\mathbf{r}$ in a time dt along its trajectory relative to the origin of the central field. The area of this small triangle is

$$dA = \tfrac{1}{2}\,|\mathbf{r} \times d\mathbf{r}| = \tfrac{1}{2}|r\mathbf{e}_r \times (\mathbf{e}_r dr + \mathbf{e}_\theta r d\theta)| = \tfrac{1}{2}r(rd\theta) \tag{6.4.7}$$

(**Note:** any increment of motion along the radial direction, $\mathbf{e}_r$, does not add to, or subtract from, the area dA—nor does it contribute anything to the angular momentum of the particle about the center of force.)

Thus, the areal velocity, or the rate at which "area is swept out" by the radius vector pointing to the moving particle is

$$\frac{dA}{dt} = \dot{A} = \tfrac{1}{2}r^2\dot{\theta} = \frac{L}{2m} = \text{constant} \tag{6.4.8}$$

An equivalent way to see this relation is to note that, because $d\mathbf{r} = \mathbf{v}dt$, Equation 6.4.7 can be written as

$$dA = \tfrac{1}{2}\,|\mathbf{r} \times d\mathbf{r}| = \tfrac{1}{2}|\mathbf{r} \times \mathbf{v}dt| = \frac{L}{2m}dt \tag{6.4.9}$$

which also reduces to Equation 6.4.8.

Thus, the areal velocity, $\dot{A}$, of a particle moving in a central field is directly proportional to its angular momentum and, therefore, is also a constant of the motion, exactly as Kepler discovered for planets moving in the central gravitational field of the Sun.

EXAMPLE 6.4.1

Let a particle be subject to an attractive central force of the form $f(r)$ where r is the distance between the particle and the center of the force. Find $f(r)$ if all circular orbits are to have identical areal velocities, $\dot{A}$.

Solution:

Because the orbits are circular, the acceleration, $\ddot{\mathbf{r}}$, has no transverse component and is entirely in the radial direction. In polar coordinates, it is given by Equation 1.11.10

$$a_r = \ddot{r} - r\dot{\theta}^2 = -r\dot{\theta}^2$$

because $\ddot{r} = 0$. Thus,

$$-mr\dot{\theta}^2 = f(r)$$

Because the areal velocity is the same for all circular orbits, then the angular momentum of the particle, $L = mr^2\dot{\theta}$, must be also. Multiplying and dividing the above expression by the factor, r^3, yields the relation

$$-\frac{mr^4\dot{\theta}^2}{r^3} = -\frac{L^2}{mr^3} = f(r)$$

or in terms of the areal velocity, $\dot{A} = L/2m$

$$f(r) = -\frac{4m\dot{A}^2}{r^3}$$

Therefore, the attractive force for which all circular orbits have identical areal velocities (and angular momenta) is the *inverse r-cube*.

6.5 | Kepler's First Law: The Law of Ellipses

To prove Kepler's first law, we develop a general differential equation for the orbit of a particle in any central, isotropic field of force. Then we solve the orbital equation for the specific case of an inverse-square law of force.

First we express Newton's differential equations of motion using two-dimensional polar coordinates instead of three, remembering from our previous discussion that no loss of generality is incurred because the motion is confined to a plane. The equation of motion in polar coordinates is

$$m\ddot{\mathbf{r}} = f(r)\mathbf{e}_r \tag{6.5.1}$$

where $f(r)$ is the central, isotropic force that acts on the particle of mass m. It is a function only of the scalar distance r to the force center (hence, it is isotropic), and its direction is along the radius vector (hence, it is central). As shown in Equations 1.11.9 and 1.11.10, the radial component of $\ddot{\mathbf{r}}$ is $\ddot{r} - r\dot{\theta}^2$ and the transverse component is $2\dot{r}\dot{\theta} + r\ddot{\theta}$. Thus, the component differential equations of motion are

$$m(\ddot{r} - r\dot{\theta}^2) = f(r) \tag{6.5.2a}$$

$$m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0 \tag{6.5.2b}$$

From the latter equation it follows that (see Equation 1.11.11)

$$\frac{d}{dt}(r^2\dot{\theta}) = 0 \tag{6.5.3}$$

or

$$r^2\dot{\theta} = \text{constant} = l \tag{6.5.4}$$

From Equation 6.4.6 we see that

$$|l| = \frac{L}{m} = |\mathbf{r} \times \mathbf{v}| \tag{6.5.5}$$

Thus, l is the angular momentum per unit mass. Its constancy is simply a restatement of a fact that we already know, namely, that the angular momentum of a particle is constant when it is moving under the action of a central force.

Given a certain radial force function $f(r)$, we could, in theory, solve the pair of differential equations (Equations 6.5.2a and b) to obtain r and θ as functions of t. Often one is interested only in the path in space (the *orbit*) without regard to the time t. To find the equation of the orbit, we use the variable u defined by

$$r = \frac{1}{u} \tag{6.5.6}$$

Then

$$\dot{r} = -\frac{1}{u^2}\dot{u} = -\frac{1}{u^2}\dot{\theta}\frac{du}{d\theta} = -l\frac{du}{d\theta} \tag{6.5.7}$$

The last step follows from the fact that

$$\dot{\theta} = lu^2 \tag{6.5.8}$$

according to Equations 6.5.4 and 6.5.6.

Differentiating a second time, we obtain

$$\ddot{r} = -l\frac{d}{dt}\frac{du}{d\theta} = -l\frac{d\theta}{dt}\frac{d}{d\theta}\frac{du}{d\theta} = -l\dot{\theta}\frac{d^2u}{d\theta^2} = -l^2u^2\frac{d^2u}{d\theta^2} \tag{6.5.9}$$

Substituting the values found for r, $\dot{\theta}$, and $\ddot{r}$ into Equation 6.5.2a, we obtain

$$m\left[-l^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}(l^2u^4)\right] = f(u^{-1}) \tag{6.5.10a}$$

which reduces to

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{ml^2u^2} f(u^{-1}) \tag{6.5.10b}$$

Equation 6.5.10b is the *differential equation of the orbit* of a particle moving under a central force. The solution gives u (hence, r) as a function of θ. Conversely, if one is given the polar equation of the orbit, namely, $r = r(\theta) = u^{-1}$, then the force function can be found by differentiating to get $d^2u/d\theta^2$ and inserting this into the differential equation.

EXAMPLE 6.5.1

A particle in a central field moves in the spiral orbit

$$r = c\theta^2$$

Determine the force function.

Solution:

We have

$$u = \frac{1}{c\theta^2}$$

and

$$\frac{du}{d\theta} = \frac{-2}{c}\theta^{-3} \qquad \frac{d^2u}{d\theta^2} = \frac{6}{c}\theta^{-4} = 6cu^2$$

Then from Equation 6.5.10b

$$6cu^2 + u = -\frac{1}{ml^2u^2} f(u^{-1})$$

Hence,

$$f(u^{-1}) = -ml^2(6cu^4 + u^3)$$

and

$$f(r) = -ml^2\left(\frac{6c}{r^4} + \frac{1}{r^3}\right)$$

Thus, the force is a combination of an inverse cube and inverse–fourth power law.

EXAMPLE 6.5.2

In Example 6.5.1 determine how the angle θ varies with time.

Solution:

Here we use the fact that $l = r^2\dot{\theta}$ is constant. Thus,

$$\dot{\theta} = lu^2 = l\frac{1}{c^2\theta^4}$$

or

$$\theta^4 d\theta = \frac{l}{c^2} dt$$

and so, by integrating, we find

$$\frac{\theta^5}{5} = lc^{-2}t$$

where the constant of integration is taken to be zero, so that $\theta = 0$ at $t = 0$. Then we can write

$$\theta = \alpha t^{1/5}$$

where α = constant = $(5lc^{-2})^{1/5}$.

Inverse-Square Law

We can now solve Equation 6.5.10b for the orbit of a particle subject to the force of gravity. In this case

$$f(r) = -\frac{k}{r^2} \tag{6.5.11}$$

where the constant $k = GMm$. In this chapter we always assume that $M \gg m$ and remains fixed in space. The small mass m is the one whose orbit we calculate. (A modification in our treatment is required when $M \approx m$, or at least not much greater than m. We present such a treatment in Chapter 7.) The equation of the orbit (Equation 6.5.10b) then becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{k}{ml^2} \tag{6.5.12}$$

Equation 6.5.12 has the same form as the one that describes the simple harmonic oscillator, but with an additive constant. The general solution is

$$u = A\cos(\theta - \theta_0) + \frac{k}{ml^2} \tag{6.5.13}$$

or

$$r = \frac{1}{k/ml^2 + A\cos(\theta - \theta_0)} \tag{6.5.14}$$

The constants of integration, A and θ_0, are determined from initial conditions or from the values of the position and velocity of the particle at some particular instant of time. The value of θ_0, however, can always be adjusted by a simple rotation of the coordinate system used to measure the polar angle of the particle. Consistent with convention, we set $\theta_0 = 0$, which corresponds to a direction toward the point of the particle's closest approach

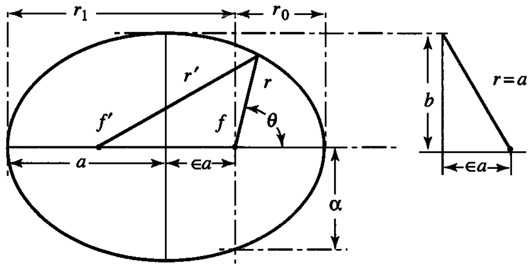

Figure 6.5.1 The ellipse

f, f'	The two foci of the ellipse
a	Semimajor axis
b	Semiminor axis: $b = (1-\epsilon^2)^{1/2}a$
ϵ	Eccentricity: each focus displaced from center by ϵa
α	Latus rectum: Distance of focus from point on the ellipse perpendicular to major axis: $\alpha = (1-\epsilon^2)\,a$
r_0	Distance from the focus to the pericenter: $r_0 = (1-\epsilon)a$
r_1	Distance from the focus to the apocenter: $r_1 = (1+\epsilon)a$

to the origin. We can then rewrite Equation 6.5.14 as

$$r = \frac{ml^2/k}{1+(Aml^2/k)\cos\theta} \tag{6.5.15}$$

This equation describes an ellipse (see Figure 6.5.1) with the origin at one of its foci, in the case where the motion of the particle is bound.

An ellipse is defined to be the locus of points whose total distance from two *foci*, f and f', is a constant, that is,

$$r + r' = constant = 2a \tag{6.5.16}$$

where a is the semimajor axis of the ellipse, and the two foci are offset from its center, each by an amount ϵa. ϵ is called the *eccentricity* of the ellipse. We can show that Equation 6.5.15 is equivalent to this fundamental definition of the ellipse by first finding a relation between r and r' using the Pythagorean theorem (see Figure 6.5.1).

$$\begin{aligned} r'^2 &= r^2\sin^2\theta + (2\epsilon a + r\cos\theta)^2 \\ &= r^2 + 4\epsilon a(\epsilon a + r\cos\theta) \end{aligned} \tag{6.5.17}$$

and then substituting the fundamental definition, $r' = 2a - r$, into it to obtain

$$r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos\theta} \tag{6.5.18a}$$

As can be seen from Figure 6.5.1, at $\theta = \pi/2$, $r = a(1-\epsilon^2) = \alpha$ the *latus rectum* of the ellipse. Hence, Equation 6.5.18a can be put into the form

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \tag{6.5.18b}$$

Thus, it is equivalent to Equation 6.5.15 with

$$\alpha = \frac{ml^2}{k} \tag{6.5.19}$$

and

$$\epsilon = \frac{Aml^2}{k} \tag{6.5.20}$$

Even though Equations 6.5.18a and b were derived for an ellipse, they are more general than that: they actually describe any *conic section* and all possible orbits other than elliptical around the gravitational source.

A conic section is formed by the intersection of a plane and a cone (see Figure 6.5.2a–d). The angle of tilt between the plane and the axis of the cone determines the resulting section. This angle is related to the eccentricity ϵ in Equations 6.5.18a and b. When $0 < \epsilon < 1$, Equations 6.5.18a and b describe an ellipse. It is formed when the angle between the plane and the axis of the cone is less than $\pi/2$ but greater than β, where β is the generating angle of the cone (see Figure 6.5.2b). The expression for a circle, $r = a$, is retrieved when $\epsilon = 0$, and it is formed when the plane is perpendicular to the cone's axis (see Figure 6.5.2a). As $\epsilon \to 1, a \to \infty$, but the product $\alpha = a(1 - \epsilon^2)$ remains finite and when $\epsilon = 1$ Equation 6.5.18a and b then describe a parabola. The angle between the plane and the axis of the cone is then equal to β (see Figure 6.5.2c). Finally, when $\epsilon > 1$, Equations 6.5.18a and b describe a hyperbola, and the angle of tilt lies between 0 and β (see Figure 6.5.2d). The different conic sections as seen by an observer positioned perpendicular to the plane in each of Figures 6.5.2a–d are shown in Figure 6.5.2e. They correspond to different possible orbits of the particle.

In reference to the elliptical orbits of the planets around the Sun (see Figure 6.5.1), r_0, the *pericenter* of the orbit, is called the *perihelion*, or distance of closest approach to the Sun; r_1, the *apocenter* of the orbit, is called the *aphelion*, or the distance at which the planet is farthest from the Sun. The corresponding distances for the orbit of the Moon around the Earth—and for the orbits of the Earth's artificial satellites—are called the *perigee* and *apogee*, respectively. From Equation 6.5.18b, it can be seen that these are the values of r at $\theta = 0$ and $\theta = \pi$, respectively:

$$r_0 = \frac{\alpha}{1+\epsilon} \tag{6.5.21a}$$

$$r_1 = \frac{\alpha}{1-\epsilon} \tag{6.5.21b}$$

The orbital eccentricities of the planets are quite small. (See Table 6.5.1.) For example, in the case of Earth's orbit $\epsilon = 0.017$, $r_0 = 91{,}000{,}000$ mi, and $r_1 = 95{,}000{,}000$ mi. On the other hand, the comets generally have large orbital eccentricities (highly elongated orbits). Halley's Comet, for instance, has an orbital eccentricity of 0.967 with a perihelion distance of only 55,000,000 mi, while at aphelion it is beyond the orbit of Neptune. Many comets (the nonrecurring type) have parabolic or hyperbolic orbits.

The energy of the object is the primary factor that determines whether or not its orbit is an open (parabola, hyperbola) or closed (circle, ellipse) conic section. "High"-energy

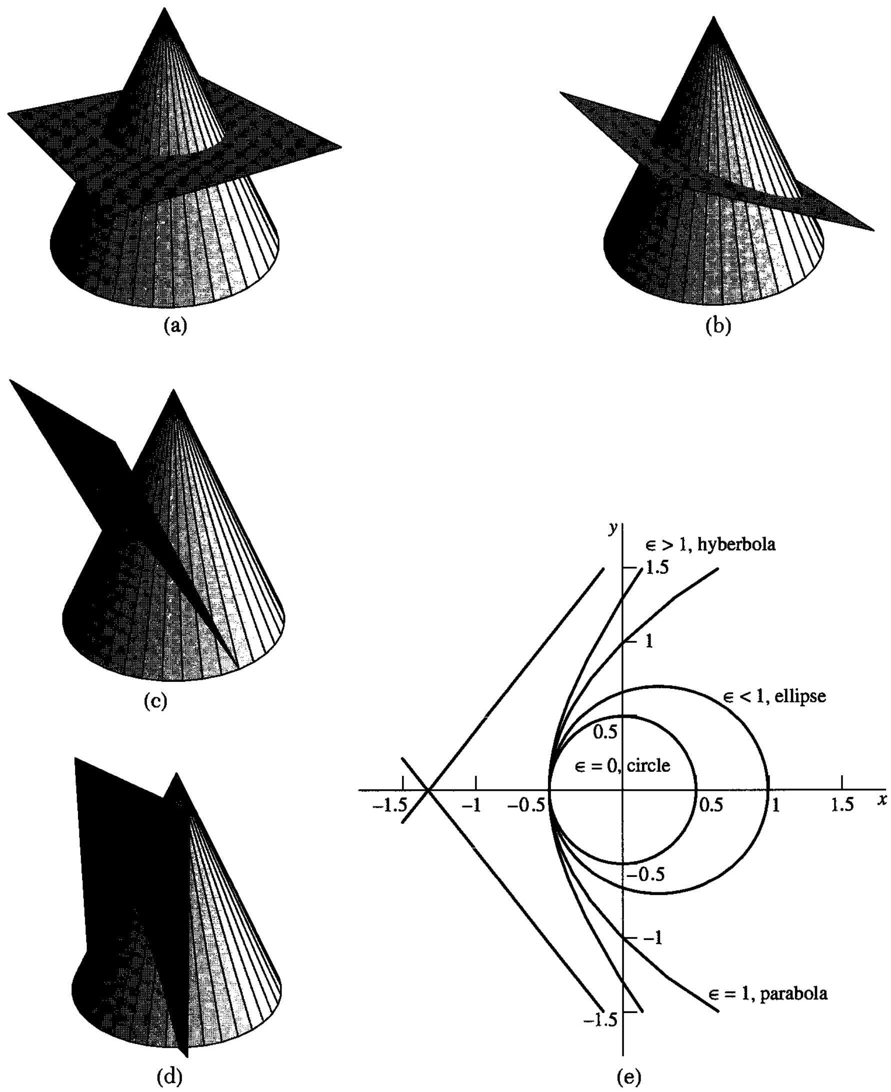

Figure 6.5.2 (a) Circle: $\epsilon = 0$. (b) Ellipse: $\epsilon < 1$. (c) Parabola: $\epsilon = 1$. (d) Hyperbola: $\epsilon > 1$. (e) The family of conic sections.

objects follow open, unbound orbits, and "low"-energy objects follow closed, bound ones. We treat this subject in greater detail in Section 6.10. Using the language of a noninertial observer, perfectly circular orbits correspond to a situation when the gravitational and centrifugal forces of a planet are exactly balanced. It should surprise you to see that the orbits of the planets are nearly circular.

It is very difficult to envision just how such a situation might arise from initial conditions. If a planet is hurtling around the Sun a little too fast, the centrifugal force slightly outweighs the gravitational force, and the planet moves away from the Sun a little bit. In doing so, it slows down until the force of gravity begins to overwhelm the centrifugal force. The planet then falls back in a little bit closer to the Sun, picking up speed along the way. The centrifugal force builds up to a point where it again outweighs the force of gravity and the process repeats itself. Thus, elliptical orbits can be seen as the result of a continuing tug of war between the slightly unbalanced gravitational and centrifugal forces that inevitably occurs whenever the tangential velocity of the planet is not adjusted just so. These forces must grow and shrink in such a way that the stability of the orbit is ensured. The criterion for stability is discussed in Section 6.13.

One way in which these two forces could be perfectly balanced all the way around the orbit would be if the planet started off just right; that is, very special initial conditions would have to have been set up in the beginning, so to speak. It is difficult to imagine how any natural process could have established such nearly perfect prerequisites. Hence, if planets are bound to the Sun at all, one would think that they would be most likely to travel in elliptical orbits just like Kepler said, unless something happened during the course of solar system evolution that brought the planets into circular orbits. We leave it to the student to think about just what sort of thing might do this.

EXAMPLE 6.5.3

Calculate the speed of a satellite in circular orbit about Earth.

Solution:

In the case of circular motion, the orbital radius is given by $r_c = a = \alpha = ml^2/k$ because the eccentricity is zero (Equations 6.5.18a and 6.5.19). In Earth's gravitational field, the force constant is $k = GM_e m$ in which M_e is the mass of Earth and m is the mass of the satellite. The angular momentum of the satellite per unit mass $l = v_c r_c$, where v_c is the speed of the satellite. Thus,

$$r_c = \frac{m(v_c r_c)^2}{GM_e m}$$

$$\therefore v_c^2 = \frac{GM_e}{r_c}$$

As shown in Example 2.3.2, the product GM_e can be found by noting that the force of gravity at Earth's surface is $mg = GM_e/R_e^2$ or $GM_e = gR_e^2$, where R_e is the radius of Earth. Thus, the speed of a satellite in circular orbit is

$$v_c = \left(\frac{gR_e^2}{r_c}\right)^{1/2}$$

For satellites in low-lying orbits close to Earth's surface, $r_c \approx R_e$, so the speed is $v_c \approx (gR_e)^{1/2} = (9.8\ \mathrm{ms}^{-1} \times 6.4 \times 10^6\ \mathrm{m})^{1/2} = 7920$ m/s, or about 8 km/s.

EXAMPLE 6.5.4

The most energy-efficient way to send a spacecraft to the Moon is to boost its speed while it is in circular orbit about the Earth such that its new orbit is an ellipse. The boost point is the perigee of the ellipse, and the point of arrival at the Moon is the apogee (see Figure 6.5.3). Calculate the percentage increase in speed required to achieve such an orbit. Assume that the spacecraft is initially in a low-lying circular orbit about Earth. The distance between Earth and the Moon is approximately $60R_e$, where R_e is the radius of Earth.

Solution:

The radius and speed of the craft in its initial circular orbit was calculated in Example 6.5.3. That radius is the perigee distance of the new orbit, $r_0 = R_e$. Let v_0 be the velocity required at perigee to send the craft to an apogee at $r_1 = 60R_e$. Because the eccentricity of the initial circular orbit is zero, we have (Equations 6.5.19, 6.5.21a)

$$r_0 = \frac{\alpha_c}{\epsilon+1} = \alpha_c = \frac{ml_c^2}{k}$$

But the angular momentum per unit mass for the circular orbit, l_c (Equation 6.5.4), is a constant and can be set equal to

$$l_c = r^2\dot{\theta} = r_0^2\dot{\theta}_0 = r_0 v_c$$

On substituting this into the preceding expression, we obtain

$$r_0 = \frac{k}{mv_c^2}$$

After the speed boost from v_c to v_0 at perigee, from Equation 6.5.21a we obtain an elliptical orbit of eccentricity ϵ given by

$$\epsilon = \frac{\alpha}{r_0} - 1 = \frac{ml^2}{kr_0} - 1 = \frac{mv_0^2 r_0}{k} - 1$$

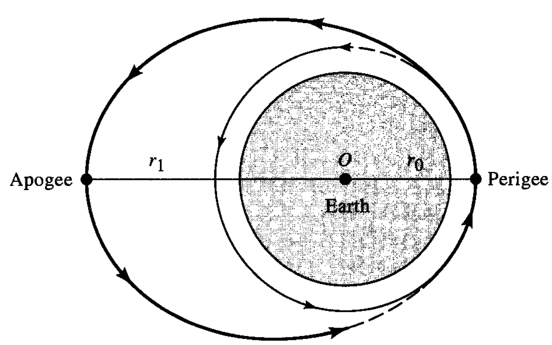

Figure 6.5.3 Spacecraft changing from a circular to an elliptical orbit.

where we made use of the fact that the new angular momentum per unit mass is $l = v_0 r_0$. Inserting the previous expression for r_0 into the preceding one gives us the ratio of the speeds required to achieve the new eccentric orbit

$$\left(\frac{v_0}{v_c}\right)^2 = \epsilon + 1$$

We can find the eccentricity in terms of the distances of perigee and apogee from the geometry of an ellipse

$$r_1 = (1+\epsilon)a = (1+\epsilon)\frac{(r_1 + r_0)}{2}$$

$$\therefore (1+\epsilon) = \left(\frac{v_0}{v_c}\right)^2 = \frac{2r_1}{r_1 + r_0}$$

Putting in numbers for the required orbit, we obtain

$$\frac{v_0}{v_c} = \sqrt{\frac{2r_1}{r_1 + r_0}} = \sqrt{\frac{120R_e}{61R_e}} = 1.40$$

Thus, a 40% boost up to a speed of about 11.2 km/s is required.

6.6 | Kepler's Third Law: The Harmonic Law

Why is Kepler's third law, relating orbital period to distance from the Sun, called the harmonic law? Kepler's work, more than that of any of the other great scientists who were trying to unlock the mysteries of planetary motion, is a wonderful illustration of the profound effect that intense hunger for knowledge and personal belief have on the growth of science. Kepler held the conviction that the world, which had treated him so harshly at times, nonetheless, was fundamentally a beautiful place. Kepler believed in the Pythagorean doctrine of celestial harmony. The world was a tumultuous place, and the planets were discordant only because humankind had not yet learned how to hear the true harmony of the worlds. In his work *Harmonice Mundi* (*The Harmony of the World*), Kepler, like the Pythagoreans almost 2000 years earlier, tried to connect the planetary motions with all fields of abstraction and harmony: geometrical figures, numbers, and musical harmonies. In this attempt he failed. But in the midst of all this work, indicative of his yearnings and strivings, we find his final precious jewel, always cited as Kepler's third law, the harmonic law. It was this law that gave us the sheets to the music of the spheres.

We show how the third law can be derived from Newton's laws of motion and the inverse-square law of gravity. Starting with Equation 6.4.7b, Kepler's second law

$$\dot{A} = \frac{L}{2m} \tag{6.6.1}$$

we can relate the area of the orbit to its period and angular momentum per unit mass, $l = L/m$, by integrating the areal velocity over the entire orbital period

$$\int_0^\tau \dot{A}dt = A = \frac{l}{2}\tau \qquad \tau = \frac{2A}{l} \tag{6.6.2}$$

The industrious student can easily prove that the area of an ellipse is πab. Thus, we have (see Figure 6.5.1)

$$\tau = \frac{2\pi ab}{l} = \frac{2\pi a^2\sqrt{1-\epsilon^2}}{l} \tag{6.6.3}$$

or

$$\begin{aligned}\tau^2 &= \frac{4\pi^2 a^4}{l^2}(1-\epsilon^2)\\ &= \frac{4\pi^2 a^4}{l^2}\frac{\alpha}{a} = 4\pi^2 a^3 \frac{\alpha}{l^2}\end{aligned} \tag{6.6.4}$$

On inserting the relation $\alpha = ml^2/k$ (Equation 6.5.19) and the force constant $k = GM_\odot m$, appropriate for planetary motion about the Sun (whose mass is $M_\odot$), into Equation 6.6.4, we arrive at Kepler's third law:

$$\tau^2 = \frac{4\pi^2}{GM_\odot}a^3 \tag{6.6.5}$$

—the square of a planet's orbital period is proportional to the cube of its "distance" from the Sun. The relevant "distance" is the semimajor axis a of the elliptical orbit. In the case of a circular orbit, this distance reduces to the radius.

The constant $4\pi^2/GM_\odot$ is the same for all objects in orbit about the Sun, regardless of their mass.[3] If distances are measured in astronomical units ($1\ \text{AU} = 1.50 \cdot 10^8$ km) and periods are expressed in Earth years, then $4\pi^2/GM_\odot = 1$. Kepler's third law then takes the very simple form $\tau^2 = a^3$. Listed in Table 6.6.1 are the periods and their squares, the semimajor axes and their cubes, along with the eccentricities of all the planets.

(**Note:** Most of the planets have nearly circular orbits, with the exception of Pluto, Mercury, and Mars.)

[3] Actually the Sun and a planet orbit their common center of mass (if only a two-body problem is being considered). A more accurate treatment of the orbital motion is carried out in Section 7.3 where it is shown that a more correct value for this "constant" is given by $4\pi^2/G(M_\odot + m)$, where m is the mass of the orbiting planet. The correction is very small.

TABLE 6.6.1 Planetary Data

Planet	Period $\tau(yr)$	Square $\tau^2(yr^2)$	Semimajor Axis a(AU)	Cube a^3(AU^3)	Eccentricity ϵ
Mercury	0.241	**0.0581**	0.387	**0.0580**	0.206
Venus	0.615	**0.378**	0.723	**0.378**	0.007
Earth	1.000	**1.000**	1.000	**1.000**	0.017
Mars	1.881	**3.538**	1.524	**3.540**	0.093
Jupiter	11.86	**140.7**	5.203	**140.8**	0.048
Saturn	29.46	**867.9**	9.539	**868.0**	0.056
Uranus	84.01	**7058.**	19.18	**7056.**	0.047
Neptune	164.8	**27160.**	30.06	**27160.**	0.009
Pluto	247.7	**61360.**	39.440	**61350.**	0.249

EXAMPLE 6.6.1

Find the period of a comet whose semimajor axis is 4 AU.

Solution:

With τ measured in years and a in astronomical units, we have

$$\tau = 4^{3/2} \text{ years} = 8 \text{ years}$$

About 20 comets in the solar system have periods like this, whose aphelia lie close to Jupiter's orbit. They are known as Jupiter's family of comets. They do not include Halley's Comet.

EXAMPLE 6.6.2

The altitude of a near circular, low earth orbit (LEO) satellite is about 200 miles.
(a) Calculate the period of this satellite.

Solution:

For circular orbits, we have

$$\frac{GM_E m}{R^2} = m\frac{v^2}{R} = m\frac{4\pi^2 R^2/\tau^2}{R}$$

Solving for τ

$$\tau^2 = \frac{4\pi^2}{GM_E}R^3$$

which—no surprise—is Kepler's third law for objects in orbit about the Earth. Let $R = R_E + h$ where h is the altitude of the satellite above the Earth's surface. Then

$$\tau^2 = \frac{4\pi^2}{GM_E} R_E^3 \left(1 + \frac{h}{R_E}\right)^3$$

But $GM_E/R_E^2 = g$, so we have

$$\tau = 2\pi\sqrt{\frac{R_E}{g}}\left(1 + \frac{h}{R_E}\right)^{3/2} \approx 2\pi\sqrt{\frac{R_E}{g}}\left(1 + \frac{3h}{2R_E}\right)$$

Putting in numbers $R_E = 6371$ km, $h = 322$ km, we get $\tau \approx 90.8$ min ≈ 1.51 hr.

There is another way to do this if you realize that Kepler's third law, being a derivative of Newton's laws of motion and his law of gravitation, applies to any set of bodies in orbit about another. The Moon orbits the Earth once every 27.3 days[4] at a radius of 60.3 R_E. Thus, scaling Kepler's third law to these values (1 month = 27.3 days and 1 lunar unit (LU) = 60.3 R_E), we have

$$\tau^2 \text{ (months)} = R^3 \ (LU)$$

Thus, for our LEO satellite $R = \frac{6693}{6371} R_E = 1.051 R_E = \frac{1.051 R_E}{60.3\ R_E/LU} = 0.01743\ LU$

$$\tau \text{ (months)} = R^{3/2}[LU] = (0.01743)^{3/2} \text{ months} = 0.002301 \text{ months} \equiv 1.51 \text{ hr}$$

(b) A geosynchronous satellite orbits the Earth in its equatorial plane with a period of 24 hr. Thus, it seems to hover above a fixed point on the ground (which is why you can point your TV satellite receiver dish towards a fixed direction in the sky). What is the radius of its orbit?

Solution:

Using Kepler's third law again, we get

$$R_{geo} = \tau^{2/3} = \left(\frac{1}{27.3}\right)^{2/3} = 0.110\ LU \equiv 6.65 R_E \approx 42{,}400 \text{ km}$$

Universality of Gravitation

A tremendous triumph of Newtonian physics ushered in the 19th century—Urbain Jean Leverrier's (1811–1877) discovery of Neptune. It signified a turning point in the history of science, when the newly emerging methodology, long embroiled in a struggle with biblical ideas, began to dominate world concepts. The episode started when Alexis Bouvard, a farmer's boy from the Alps, came to Paris to study science and there perceived irregularities in the motion of Uranus that could not be accounted for by the attraction

[4] This is the *sidereal month*, or the time it takes the Moon to complete one orbit of 360 degrees relative to the stars. The month that we are all familiar with is the *synodic month* of 29.5 days, which is the time it takes for the Moon to go through all of its phases.

of the other known planets. In the years to follow, as the irregularities in the motion of Uranus mounted up, the opinion became fairly widespread among astronomers that there had to be an unknown planet disturbing the motion of Uranus.

In 1842–1843 John Couch Adams (1819–1892), a gifted student at Cambridge, began work on this problem, and by September of 1845 he presented Sir George Airy (1801–1892), the Astronomer Royal, and James Challis, the director of the Cambridge Observatory, the likely coordinates of the offending, but then unknown, planet. It seemed impossible to them that a mere student, armed only with paper and pencil, could take observations of Uranus, invoke the known laws of physics, and predict the existence and precise location of an undiscovered planet. Besides, Airy had grave doubts concerning the validity of the inverse square law of gravity. In fact, he believed that the law of gravity fell off faster than inverse square at great distances. Thus, somewhat understandably, Airy was reluctant to place much credence in Adams' work, and so the two great men chose to ignore him, sealing forever their fate as the astronomers who failed to discover Neptune.

It was about this time that Leverrier began work on the problem. By 1846, he had calculated the orbit of the unknown planet and made a precise prediction of its position on the celestial sphere. Airy and Challis saw that Leverrier's result miraculously agreed with the prediction of Adams. Challis immediately initiated a search for the unknown planet in the suspicious sector of the sky, but owing to Cambridge's lack of detailed star maps in that area, the search was laboriously painstaking and the data reduction problem prodigious. Had Challis proceeded with vigor and tenacity, he most assuredly would have found Neptune, for it was there on his photographic plates. Unfortunately, he dragged his feet. By this time, however, an impatient Leverrier had written Johann Galle (1812–1910), astronomer at the Berlin Observatory, asking him to use their large refractor to examine the stars in the suspect area to see if one showed a disc, a sure signature of a planet. A short time before the arrival of Leverrier's letter containing this request, the Berlin Observatory had received a detailed star map of this sector of the sky from the Berlin Academy. On receipt of Leverrier's letter, September 23, 1846, the map was compared with an image of the sky taken that night, and the planet was identified as a foreign star of eighth magnitude, not seen on the star map. It was named Neptune. Newtonian physics had triumphed in a way never seen before—laws of physics had been used to make a verifiable prediction to the world at large, an unexpected demonstration of the power of science.

Since that time, celestial objects observed at increasingly remote distances continue to exhibit behavior consistent with the laws of Newtonian physics (ignoring those special cases involving large gravitational fields or very extreme distances, each requiring treatment with general relativity). The behavior of binary star systems within our galaxy serves as a classic example. Such stars are bound together gravitationally, and their orbital dynamics are well described by Newtonian mechanics. We discuss them in the next chapter. So strongly do we believe in the universality of gravitation and the laws of physics, that apparent violations by celestial objects, as in the case of Uranus and the subsequent discovery of Neptune, are usually greeted by searches for unseen disturbances. Rarely do we instead demand the overthrow of the laws of physics. (Although, astonishingly enough, two famous examples discussed subsequently in this chapter had precisely this effect and helped revolutionize physics.)

The more likely scenario, ferreting out the unseen disturbance, is currently in progress in many areas of contemporary astronomical research, as illustrated by the

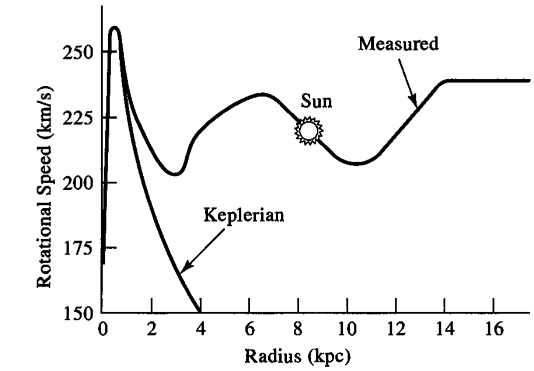

Figure 6.6.1 Galactic rotation curve. The Sun's speed is about 220 km/s and its distance from the galactic center is about 8.5 kpc (≈28,000 light years).

search for dark matter in the universe. One of the reasons we think that an enormous amount of unseen matter fills the universe (perhaps 10 times as much as is visible) can be gleaned from the dynamics of spiral galaxies—thin disc-shaped, rotating aggregates of as many as 100 billion stars. At first sight the rotation curve for spiral galaxies (a plot of the rotational velocity of the stars as a function of their radial distance from the galactic center) seems to violate Kepler's laws. An example of such a curve is shown in Figure 6.6.1. Most of the luminous matter of a spiral galaxy is contained in its central nucleus whose extent is on the order of several thousand light years in radius. The rest of the luminous matter is in the spiral arms that extend in radius out to a distance of about 50,000 light years. The whole thing slowly rotates about its center of gravity, exactly as one would expect for a self-gravitating conglomerate of stars, gas, dust, and so on. The surprising thing about the rotation curve in Figure 6.6.1 is that it is apparently non-Keplerian.

We can illustrate what we mean by this with a simple example. Assume that the entire galactic mass is concentrated within a nucleus of radius R and that stars fill the nucleus at uniform density. This is an oversimplification, but a calculation based on such a model should serve as a guide for what we might expect a rotation curve to look like. The rotational velocity of stars at some radius $r < R$ within the nucleus is determined only by the amount of mass M within the radius r. Stars external to r have no effect. Because the density of stars within the total nuclear radius R is constant, we calculate M as

$$M = \tfrac{4}{3}\pi\rho r^3 \tag{6.6.6}$$

where

$$\rho = \frac{M_{gal}}{\left(\frac{4}{3}\right)\pi R^3} \tag{6.6.7}$$

and from Newton's second law

$$\frac{GMm}{r^2} = \frac{mv^2}{r} \tag{6.6.8}$$

for the gravitational force exerted on a star of mass m at a distance r from the center of the nucleus by the mass M interior to that distance r. Solving for v, we get

$$v = \sqrt{GM_{gal}/R^3}\; r \tag{6.6.9}$$

or, the rotational velocity of stars at $r < R$ is proportional to r. For stars in the spiral arms at distances $r > R$, we obtain

$$\frac{GM_{gal}m}{r^2} = \frac{mv^2}{r} \tag{6.6.10}$$

thus,

$$v = \sqrt{\frac{GM_{gal}}{r}} \tag{6.6.11}$$

or, the rotational velocity of stars at $r > R$ is proportional to $1/\sqrt{r}$. This is what we mean by Keplerian rotation. It is the way the velocities of planets depend on their distance from the Sun. We show such a curve in Figure 6.6.1, where we have assumed that the entire mass of the galaxy is uniformly distributed in a sphere whose radius is 1 kpc [1 parsec (pc) = 3.26 light years (ly)].

Let us examine the measured rotation curve. Initially, it climbs rapidly from zero at the galactic center to about 250 km/s at 1 kpc, more or less as expected, but the astonishing thing is that the curve does not fall off in the expected Keplerian manner. It stays more or less flat all the way out to the edges of the spiral arms (the zero on the vertical axis has been suppressed so the curve is flatter than it appears to be). The conclusion is inescapable. As we move away from the galactic center, we must "pick up" more and more matter within any given radius, which causes even the most remote objects in the galaxy to orbit at velocities that exceed those expected for a highly centralized matter distribution. Because most of the luminosity of a galaxy comes from its nucleus, we conclude that dark, unseen matter must permeate spiral galaxies all the way out to their very edges and beyond. (In fact it should be a simple matter to deduce the radial distribution of dark matter required to generate this flat rotation curve.) Of course, Newton's laws could be wrong, but we think not. It looks like a case of Neptune revisited.

6.7 Potential Energy in a Gravitational Field: Gravitational Potential

In Example 2.3.2 we showed that the inverse-square law of force leads to an inverse first power law for the potential energy function. In this section we derive this same relationship in a more physical way.

Let us consider the work W required to move a test particle of mass m along some prescribed path in the gravitational field of another particle of mass M.

We place the particle of mass M at the origin of our coordinate system, as shown in Figure 6.7.1a. Because the force $\mathbf{F}$ on the test particle is given by $\mathbf{F} = -(GMm/r^2)\mathbf{e}_r$, then

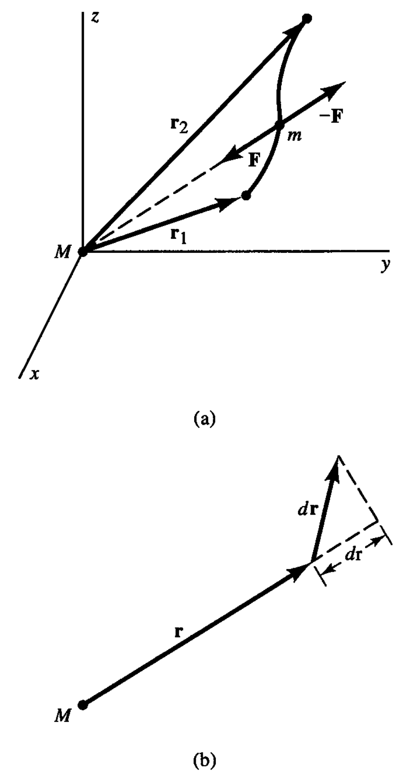

Figure 6.7.1 Diagram for finding the work required to move a test particle in a gravitational field.

to overcome this force an external force $-\mathbf{F}$ must be applied. The work dW done in moving the test particle through a displacement $d\mathbf{r}$ is, thus, given by

$$dW = -\mathbf{F} \cdot d\mathbf{r} = \frac{GMm}{r^2}\mathbf{e}_r \cdot d\mathbf{r} \tag{6.7.1}$$

Now we can resolve $d\mathbf{r}$ into two components: $\mathbf{e}_r\, dr$ parallel to $\mathbf{e}_r$ (the radial component) and the other at right angles to $\mathbf{e}_r$ (Figure 6.7.1b). Clearly,

$$\mathbf{e}_r \cdot d\mathbf{r} = dr \tag{6.7.2}$$

and so W is given by

$$W = GMm\int_{r_1}^{r_2} \frac{dr}{r^2} = -GMm\left(\frac{1}{r_2} - \frac{1}{r_1}\right) \tag{6.7.3}$$

where r_1 and r_2 are the radial distances of the particle at the beginning and end of the path. Thus, the work is independent of the particular path taken; it depends only on the endpoints. This verifies a fact we already knew, namely that a central force described by an inverse-square law is conservative.

We can, thus, define the potential energy of a test particle of mass m at a given point in the gravitational field of another particle of mass M as the work done in moving the test particle from some arbitrary reference position r_1 to the position r_2. We take the reference position to be $r_1 = \infty$. This assignment is usually a convenient one, because the gravitational force between two particles vanishes when they are separated by ∞. Thus, putting $r_1 = \infty$ and $r_2 = r$ in Equation 6.7.3, we have

$$V(r) = GMm\int_{\infty}^{r} \frac{dr}{r^2} = -\frac{GMm}{r} \tag{6.7.4}$$

Like the gravitational force, the gravitational potential energy of two particles separated by ∞ also vanishes. Note that for finite separations, the gravitational potential energy is negative.

Both the gravitational force and potential energy between two particles involve the concept of action at a distance. Newton himself was never able to explain or describe the mechanism by which such a force worked. We do not attempt to either, but we would like to introduce the concept of field in such a way that forces and potential energies can be thought of as being generated not by actions at a distance but by local actions of matter with an existing field. To do this, we introduce the quantity Φ, called the *gravitational potential*

$$\Phi = \lim_{m\to 0}\left(\frac{V}{m}\right) \tag{6.7.5}$$

In essence Φ is the gravitational potential energy per unit mass that a very small test particle would have in the presence of other surrounding masses. We take the limit as $m \to 0$ to ensure that the presence of the test particle does not affect the distribution of the other matter and change the thing we are trying to define. Clearly, the potential should depend only on the magnitude of the other masses and their positions in space, not those of the particle we are using to test for the presence of gravitation. We can think of the potential as a scalar function of spatial coordinates, $\Phi(x, y, z)$, or a field, set up by all the other surrounding masses. We test for its presence by placing the test mass m at any point (x, y, z). The potential energy of that test particle is then given by

$$V(x,y,z) = m\Phi(x,y,z) \tag{6.7.6}$$

We can think of this potential energy as being generated by the local interaction of the mass m and the field Φ that is present at the point (x, y, z).

The gravitational potential at a distance r from a particle of mass M is

$$\Phi = -\frac{GM}{r} \tag{6.7.7}$$

If we have a number of particles $M_1, M_2, \ldots, M_i, \ldots$ located at positions $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_i, \ldots$, then the gravitational potential at the point $\mathbf{r}(x, y, z)$ is the sum of the gravitational potentials of all the particles, that is,

$$\Phi(x,y,z) = \sum \Phi_i = -G\sum \frac{M_i}{s_i} \tag{6.7.8}$$

in which s_i is the distance of the field point $\mathbf{r}(x, y, z)$ from the position $\mathbf{r}_i(x_i, y_i, z_i)$ of the ith particle

$$s_i = |\mathbf{r} - \mathbf{r}_i| \tag{6.7.9}$$

We define a vector field $\mathbf{g}$, called the ***gravitational field intensity***, in a way that is completely analogous to the preceding definition of the gravitational potential scalar field

$$\mathbf{g} = \lim_{m \to 0} \left(\frac{\mathbf{F}}{m} \right) \tag{6.7.10}$$

Thus, the gravitational field intensity is the gravitational force per unit mass acting on a test particle of mass m positioned at the point (x, y, z). Clearly, if the test particle experiences a gravitational force given by

$$\mathbf{F} = m\mathbf{g} \tag{6.7.11}$$

then we know that other nearby masses are responsible for the presence of the local field intensity $\mathbf{g}$.[5]

The relationship between field intensity and the potential is the same as that between the force $\mathbf{F}$ and the potential energy V, namely

$$\mathbf{g} = -\nabla\Phi \tag{6.7.12a}$$

$$\mathbf{F} = -\nabla V \tag{6.7.12b}$$

The gravitational field intensity can be calculated by first finding the potential function from Equation 6.7.8 and then calculating the gradient. This method is usually simpler than the method of calculating the field directly from the inverse-square law. The reason is that the potential energy is a scalar sum, whereas the field intensity is given by a vector sum. The situation is analogous to the theory of electrostatic fields. In fact one can apply any of the corresponding results from electrostatics to find gravitational fields and potentials with the proviso, of course, that there are no negative masses analogous to negative charge.

EXAMPLE 6.7.1

Potential of a Uniform Spherical Shell

As an example, let us find the potential function for a uniform spherical shell.

Solution:

By using the same notation as that of Figure 6.2.1, we have

$$\Phi = -G\int \frac{dM}{s} = -G\int \frac{2\pi\rho R^2 \sin\theta \, d\theta}{s}$$

[5] $\mathbf{g}$ is the local acceleration of a mass m due to gravity. On the surface of Earth, its value is 9.8 m/s^2 and is primarily due to the mass of the Earth.

From the relation between s and θ that we used in Equation 6.2.5, we find that the preceding equation may be simplified to read

$$\Phi = -G\frac{2\pi\rho R^2}{rR}\int_{r-R}^{r+R} ds = -\frac{GM}{r} \qquad (6.7.13)$$

where M is the mass of the shell. This is the same potential function as that of a single particle of mass M located at O. Hence, the gravitational field outside the shell is the same as if the entire mass were concentrated at the center. It is left as a problem to show that, with an appropriate change of the integral and its limits, the potential inside the shell is constant and, hence, that the field there is zero.

EXAMPLE 6.7.2

Potential and Field of a Thin Ring

We now wish to find the potential function and the gravitational field intensity in the plane of a thin circular ring.

Solution:

Let the ring be of radius R and mass M. Then, for an exterior point lying in the plane of the ring, Figure 6.7.2, we have

$$\Phi = -G\int\frac{dM}{s} = -G\int_0^{2\pi}\frac{\mu R\,d\theta}{s}$$

where μ is the linear mass density of the ring. To evaluate the integral, we first express s as a function of θ using the law of cosines

$$s^2 = R^2 + r^2 - 2Rr\cos\theta$$

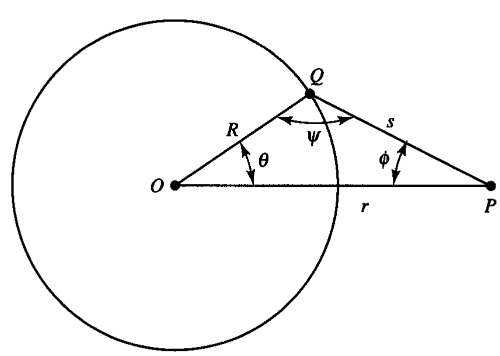

Figure 6.7.2 Coordinates for calculating the gravitational field of a ring.

The integral becomes

$$\Phi = -2R\mu G\int_0^{\pi}\frac{d\theta}{(r^2+R^2-2Rr\cos\theta)^{1/2}}$$
$$= -\frac{2R\mu G}{r}\int_0^{\pi}\frac{d\theta}{[1+(R^2/r^2)-2(R/r)\cos\theta)]^{1/2}}$$

First, let us use the so-called far field approximation $r > R$ and expand the integrand in a power series of $x(= R/r)$, making certain to keep all terms of order x^2.

$$\Phi = -2x\mu G\int_0^{\pi}\left[\left(1-\tfrac{1}{2}x^2+x\cos\theta\right)+\tfrac{3}{8}(x^2-2x\cos\theta)^2+\cdots\right]d\theta$$
$$= -2x\mu G\int_0^{\pi}\left(1-\tfrac{1}{2}x^2+x\cos\theta+\tfrac{3}{2}x^2\cos^2\theta-\tfrac{3}{2}x^3\cos\theta+\tfrac{3}{8}x^4+\cdots\right)d\theta$$

Now, dropping all terms of order x^3 or higher and noting that the term containing $\cos\theta$ has zero integral over a half cycle, we obtain

$$\Phi = -2x\mu G\left(\pi+\pi\frac{x^2}{4}+\cdots\right.$$
$$= \frac{-2\pi R\mu G}{r}\left(1+\frac{R^2}{4r^2}+\cdots\right.$$
$$= -\frac{GM}{r}\left(1+\frac{R^2}{4r^2}+\cdots\right.$$

The field intensity at a distance r from the center of the ring is in the radial direction (because Φ is not a function of θ) and is given by

$$\mathbf{g} = -\frac{\partial\Phi}{\partial r}\mathbf{e}_r = -\frac{GM}{r^2}\left(1+\tfrac{3}{4}\left(\frac{R}{r}\right)^2+\cdots\right)\mathbf{e}_r$$

The field is not given by an inverse-square law. If $r \gg R$, the term in parentheses approaches unity, and the field intensity approaches the inverse-square field of a single particle of mass M. This is true for a finite-sized body of any shape; that is, for distances large compared with the linear dimensions of the body, the field intensity approaches that of a single particle of mass M.

The potential for a point near the center of the ring can be found by invoking the near field, or $r < R$, approximation. The solution proceeds more or less as before, but in this case we expand the preceding integrand in powers of r/R to obtain

$$\Phi = -\frac{GM}{R}\left(1+\frac{r^2}{4R^2}+\cdots\right.$$

$\mathbf{g}$ can again be found by differentiation

$$\mathbf{g} = \left(\frac{GM}{2R^3}r\right)\mathbf{e}_r+\cdots$$

Thus, a ring of matter exerts an approximately linear *repulsive* force, directed away from the center, on a particle located somewhere near the center of the ring. It is easy to see that this must be so. Imagine that you are a small mass at the center of such a ring of radius R with a field of view both in front of you and behind you that subtends some definite angle. If you move slowly a distance r away from the center, the matter you see attracting you in the forward direction diminishes by a factor of r, whereas the matter you see attracting you from behind grows by r. But because the force of gravity from any material element falls as $1/r^2$, the force exerted on you by the forward mass and the backward mass varies as $1/r$, and, thus, the force difference between the two of them is $[1/(R-r) - 1/(R+r)]$ or proportional to r for $r < R$. The gravitational force of the ring repels objects from its center.

6.8 | Potential Energy in a General Central Field

We showed previously that a central field of the inverse-square type is conservative. Let us now consider the question of whether or not any (isotropic) central field of force is conservative. A general isotropic central field can be expressed in the following way:

$$\mathbf{F} = f(r)\mathbf{e}_r \tag{6.8.1}$$

in which $\mathbf{e}_r$ is the unit radial vector. To apply the test for conservativeness, we calculate the curl of $\mathbf{F}$. It is convenient here to employ spherical coordinates for which the curl is given in Appendix F. We find

$$\nabla \times \mathbf{F} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \mathbf{e}_r & \mathbf{e}_\theta r & \mathbf{e}_\phi r \sin\theta \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ F_r & rF_\theta & rF_\phi \sin\theta \end{vmatrix} \tag{6.8.2}$$

For our central force $F_r = f(r)$, $F_\theta = 0$, and $F_\phi = 0$. The curl then reduces to

$$\nabla \times \mathbf{F} = \frac{\mathbf{e}_\theta}{r \sin\theta}\frac{\partial f}{\partial \phi} - \frac{\mathbf{e}_\phi}{r}\frac{\partial f}{\partial \theta} = 0 \tag{6.8.3}$$

The two partial derivatives both vanish because $f(r)$ does not depend on the angular coordinates ϕ and θ. Thus, the curl vanishes, and so the general central field defined by Equation 6.8.1 is conservative. We recall that the same test was applied to the inverse-square field in Example 4.2.5.

We can now define a potential energy function

$$V(r) = -\int_{r_{ref}}^{r} \mathbf{F} \cdot d\mathbf{r} = -\int_{r_{ref}}^{r} f(r)dr \tag{6.8.4}$$

where the lower limit r_{ref} is the reference value of r at which the potential energy is defined to be zero. For inverse-power type forces, r_{ref} is often taken to be at infinity. This allows us to calculate the potential energy function, given the force function. Conversely, if we know the potential energy function, we have

$$f(r) = -\frac{dV(r)}{dr} \tag{6.8.5}$$

giving the force function for a central field.

6.9 | Energy Equation of an Orbit in a Central Field

The square of the speed is given in polar coordinates from Equation 1.11.7

$$\mathbf{v}\cdot\mathbf{v} = v^2 = \dot{r}^2 + r^2\dot{\theta}^2 \tag{6.9.1}$$

Because a central force is conservative, the total energy $T + V$ is constant and is given by

$$\tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E = \text{constant} \tag{6.9.2}$$

We can also write Equation 6.9.2 in terms of the variable $u = 1/r$. From Equations 6.5.7 and 6.5.8 we obtain

$$\tfrac{1}{2}ml^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] + V(u^{-1}) = E \tag{6.9.3}$$

The preceding equation is called *the energy equation of the orbit.*

EXAMPLE 6.9.1

In Example 6.5.1 we had for the spiral orbit $r = c\theta^2$

$$\frac{du}{d\theta} = \frac{-2}{c}\theta^{-3} = -2c^{1/2}u^{3/2}$$

so the energy equation of the orbit is

$$\tfrac{1}{2}ml^2(4cu^3 + u^2) + V = E$$

Thus,

$$V(r) = E - \tfrac{1}{2}ml^2\left(\frac{4c}{r^3} + \frac{1}{r^2}\right)$$

This readily gives the force function of Example 6.5.1, because $f(r) = -dV/dr$.

6.10 | Orbital Energies in an Inverse-Square Field

The potential energy function for an inverse-square force field is

$$V(r) = -\frac{k}{r} = -ku \tag{6.10.1}$$

so the energy equation of the orbit (Equation 6.9.3) becomes

$$\tfrac{1}{2}ml^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] - ku = E \tag{6.10.2}$$

Solving for $du/d\theta$, we first get

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2E}{ml^2} + \frac{2ku}{ml^2} \tag{6.10.3a}$$

and then

$$\frac{du}{d\theta} = \sqrt{\frac{2E}{ml^2} + \frac{2ku}{ml^2} - u^2} \tag{6.10.3b}$$

Separating variables yields

$$d\theta = \frac{du}{\sqrt{\frac{2E}{ml^2} + \frac{2ku}{ml^2} - u^2}} \tag{6.10.3c}$$

We introduce three constants, a, b, and c

$$a = -1 \qquad b = \frac{2k}{ml^2} \qquad c = \frac{2E}{ml^2} \tag{6.10.4}$$

to write Equation 6.10.3c in a standard form to carry out its integration

$$\theta - \theta_0 = \int \frac{du}{\sqrt{au^2 + bu + c}} = \frac{1}{\sqrt{-a}} \cos^{-1}\left(-\frac{b + 2au}{\sqrt{b^2 - 4ac}}\right) \tag{6.10.5}$$

where θ_0 is a constant of integration. Rewriting Equation 6.10.5 first gives us

$$-\frac{b + 2au}{\sqrt{b^2 - 4ac}} = \cos\left[\sqrt{-a}(\theta - \theta_0)\right] \tag{6.10.6a}$$

and then solving for u

$$u = \frac{\sqrt{b^2 - 4ac}}{-2a} \cos\left[\sqrt{-a}(\theta - \theta_0)\right] + \frac{b}{-2a} \tag{6.10.6b}$$

Now we replace u with $1/r$ and insert the values for the constants a, b, and c from Equation 6.10.4 into Equation 6.10.6b.

$$\frac{1}{r} = \frac{1}{2}\sqrt{\frac{4k^2}{m^2l^4} + \frac{8E}{ml^2}} \cos(\theta - \theta_0) + \frac{k}{ml^2} \tag{6.10.7a}$$

and factoring out the quantity k/ml^2 yields

$$\frac{1}{r} = \frac{k}{ml^2}\left[\sqrt{1 + \frac{2Eml^2}{k^2}} \cos(\theta - \theta_0) + 1\right] \tag{6.10.7b}$$

which, upon simplifying, yields the polar equation of the orbit analogous to Equation 6.5.18a,

$$r = \frac{ml^2/k}{1 + \sqrt{1 + 2Eml^2/k^2} \cos(\theta - \theta_0)} \tag{6.10.7c}$$

If, as before, we set $\theta_0 = 0$, which defines the direction toward the orbital pericenter to be the reference direction for measuring polar angles, and we compare Equation 6.10.7c with 6.5.18b, we see that again it represents a conic section whose eccentricity is

$$\epsilon = \sqrt{1 + \frac{2E}{k}\frac{ml^2}{k}} \tag{6.10.8}$$

From Equations 6.5.18a, b and 6.5.19, $\alpha = ml^2/k = (1 - \epsilon^2)a$ and on inserting these relations into Equation 6.10.8, we see that

$$-\frac{2E}{k} = \frac{1-\epsilon^2}{\alpha} = \frac{1}{a} \tag{6.10.9}$$

or

$$E = -\frac{k}{2a} \tag{6.10.10}$$

Thus, the total energy of the particle determines the semimajor axis of its orbit. We now see from Equation 6.10.8, as stated in Section 6.5, that the total energy E of the particle completely determines the particular conic section that describes the orbit:

$E < 0$	$\epsilon < 1$	*closed orbits (ellipse or circle)*
$E = 0$	$\epsilon = 1$	*parabolic orbit*
$E > 0$	$\epsilon > 1$	*hyperbolic orbit*

Because $E = T + V$ and is constant, the closed orbits are those for which $T < |V|$, and the open orbits are those for which $T \geq |V|$.

In the Sun's gravitational field the force constant $k = GM_\odot m$, where $M_\odot$ is the mass of the Sun and m is the mass of the body. The total energy is then

$$\frac{mv^2}{2} - \frac{GM_\odot m}{r} = E = \text{constant} \tag{6.10.11}$$

so the orbit is an ellipse, a parabola, or a hyperbola depending on whether v^2 is less than, equal to, or greater than the quantity $2GM_\odot/r$, respectively.

EXAMPLE 6.10.1

A comet is observed to have a speed v when it is a distance r from the Sun, and its direction of motion makes an angle ϕ with the radius vector from the Sun, Figure 6.10.1. Find the eccentricity of the comet's orbit.

Solution:

To use the formula for the eccentricity (Equation 6.10.8), we need the square of the angular momentum constant l. It is given by

$$l^2 = |\mathbf{r} \times \mathbf{v}|^2 = (rv \sin\phi)^2$$

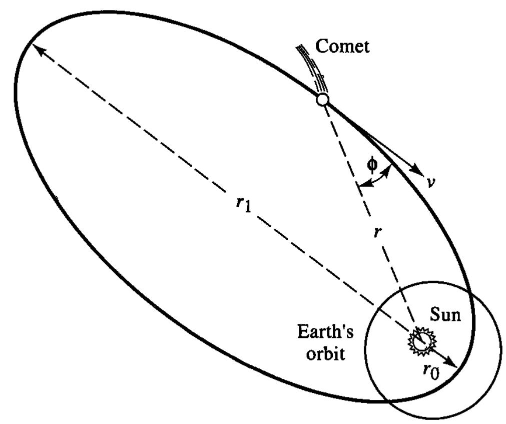

Figure 6.10.1 Orbit of a comet.

The eccentricity, therefore, has the value

$$\epsilon = \left[1 + \left(v^2 - \frac{2GM_\odot}{r}\right)\left(\frac{rv\sin\phi}{GM_\odot}\right)^2\right]^{1/2}$$

Note that the mass m of the comet cancels out. Now the product $GM_\odot$ can be expressed in terms of Earth's speed v_e and orbital radius a_e (assuming a circular orbit), namely

$$GM_\odot = a_e v_e^2$$

The preceding expression for the eccentricity then becomes

$$\epsilon = \left[1 + \left(\mathsf{V}^2 - \frac{2}{\mathsf{R}}\right)(\mathsf{RV}\ \sin\phi)^2\right]^{1/2}$$

where we have introduced the *dimensionless ratios*

$$\mathsf{V} = \frac{v}{v_e} \qquad \mathsf{R} = \frac{r}{a_e}$$

which simplify the computation of ϵ.

As a numerical example, let v be one-half the Earth's speed, let r be four times Earth–Sun distance, and $\phi = 30°$. Then $\mathsf{V} = 0.5$ and $\mathsf{R} = 4$, so the eccentricity is

$$\epsilon = [1 + (0.25 - 0.5)(4 \times 0.5 \times 0.5)^2]^{1/2} = (0.75)^{1/2} = 0.866$$

For an ellipse the quantity $(1 - \epsilon^2)^{-1/2}$ is equal to the ratio of the major (long) axis to the minor (short) axis. For the orbit of the comet in this example this ratio is $(1 - 0.75)^{-1/2} = 2$, or 2:1, as shown in Figure 6.10.1.

EXAMPLE 6.10.2

When a spacecraft is placed into geosynchronous orbit (Example 6.6.2), it is first launched, along with a propulsion stage, into a near circular, low earth orbit (LEO) using an appropriate booster rocket. Then the propulsion stage is fired and the spacecraft is

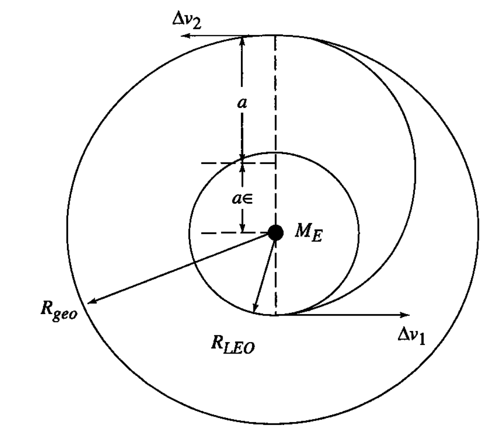

Figure 6.10.2 Boosting a satellite from low earth orbit (LEO) to a geosynchronous (geo) orbit.

transferred to an elliptic orbit designed to take it to geosynchronous altitude at orbital apogee (see Figure 6.10.2). At apogee, the propulsion stage is fired again to take it out of its elliptical orbit and put it into a circular, geocentric (geo) orbit. Thus, two velocity boosts are required of the propulsion stage: (a) Δv_1, to move the satellite from its circular LEO into the elliptical transfer orbit and (b) Δv_2, to circularize the orbit of the satellite at the geosynchronous altitude. Calculate the required velocity boosts, Δv_1 and Δv_2.

(a) Solution:

We essentially solved this problem in Example 6.5.4. We do it here in a slightly different way. First, we note that the radii of the two circular orbits and the semimajor axis of the transfer elliptical orbit (see Figure 6.10.2) are related

$$R_{LEO} + R_{geo} = 2a$$

From the figure we see that $R_{LEO} = a(1 - \epsilon)$ and $R_{geo} = a(1 + \epsilon)$ are the perigee and apogee distances of the transfer orbit.

Now, we use the energy equation (6.10.11) to calculate the velocity at perigee, v_p, of the spacecraft, after boost to the transfer elliptical orbit. The energy of the elliptical orbit is

$$E = -\frac{GM_E m}{2a} = \frac{1}{2}mv_p^2 - \frac{GM_E m}{a(1-\epsilon)}$$

Solving for v_p gives

$$v_p^2 = \frac{GM_E}{a}\left(\frac{1+\epsilon}{1-\epsilon}\right)$$

Substituting for a, $1 + \epsilon$ and $1 - \epsilon$ into the above gives

$$v_a^2 = \frac{2GM_E}{R_{LEO} + R_{geo}}\left(\frac{R_{geo}}{R_{LEO}}\right)$$

Now, we can calculate the velocity of the satellite in *circular* LEO from the condition

$$\frac{mv_{LEO}^2}{R_{LEO}} = \frac{GM_E m}{R_{LEO}^2}$$

or

$$v_{LEO}^2 = \frac{GM_E}{R_{LEO}}$$

After a little algebra, we find that the required velocity boost is

$$\Delta v_1 = v_p - v_{LEO} = \sqrt{\frac{GM_E}{R_{LEO}}}\left[\sqrt{\frac{2R_{geo}}{R_{LEO}+R_{geo}}} - 1\right]$$

Remembering that $g = GM_E/R_E^2$ we have

$$\Delta v_1 = R_E\sqrt{\frac{g}{R_{LEO}}}\left[\sqrt{\frac{2R_{geo}}{R_{LEO}+R_{geo}}} - 1\right]$$

Putting in numbers: $R_E = 6371$ km, $R_{LEO} = 6693$ km, $R_{geo} = 42{,}400$ km we get

$$\Delta v_1 = 8{,}600 \text{ km/hr}$$

(b) Solution:

The energy of the spacecraft at apogee is

$$E = -\frac{GM_E m}{2a} = \tfrac{1}{2}mv_a^2 - \frac{GM_E m}{a(1+\epsilon)}$$

Solving for the velocity at apogee, v_a

$$v_a^2 = \frac{GM_E}{a}\left(\frac{1-\epsilon}{1+\epsilon}\right)$$

Substituting for a, $1+\epsilon$ and $1-\epsilon$ into the above gives

$$v_a^2 = \frac{2GM_E}{R_{LEO}+R_{geo}}\left(\frac{R_{LEO}}{R_{geo}}\right)$$

As before, the condition for a *circular* orbit at this radius is

$$\frac{mv_{geo}^2}{R_{geo}} = \frac{GM_E m}{R_{geo}^2}$$

Thus,

$$v_{geo}^2 = \frac{GM_E}{R_{geo}}$$

$$\Delta v_2 = v_{geo} - v_a = \sqrt{\frac{GM_E}{R_{geo}}}\left[1 - \sqrt{\frac{2R_{LEO}}{R_{LEO}+R_{geo}}}\right] = R_E\sqrt{\frac{g}{R_{geo}}}\left[1 - \sqrt{\frac{2R_{LEO}}{R_{LEO}+R_{geo}}}\right]$$

Putting in numbers, we get

$$\Delta v_2 = 5269 \text{ km/hr}$$

Note, the total boost, $\Delta v_1 + \Delta v_2 = 8{,}600$ km/hr + 5269 km/hr = 13,869 km/hr, required of the spacecraft propulsion system to place it into geo orbit is almost 50% of the boost required by the launcher to place it into LEO!

6.11 Limits of the Radial Motion: Effective Potential

We have seen that the angular momentum of a particle moving in any isotropic central field is a constant of the motion, as expressed by Equations 6.5.4 and 6.5.5 defining l. This fact allows us to write the general energy equation (Equation 6.9.2) in the following form:

$$\frac{m}{2}\left(\dot{r}^2 + \frac{l^2}{r^2}\right) + V(r) = E \qquad (6.11.1a)$$

or

$$\frac{m}{2}\dot{r}^2 + U(r) = E \qquad (6.11.1b)$$

in which

$$U(r) = \frac{ml^2}{2r^2} + V(r) \qquad (6.11.1c)$$

The function $U(r)$ defined here is called the *effective potential*. The term $ml^2/2r^2$ is called the *centrifugal potential*. Looking at Equation 6.11.1b we see that, as far as the radial motion is concerned, the particle behaves in exactly the same way as a particle of mass m moving in one-dimensional motion under a potential energy function $U(r)$. As in Section 3.3 in which we discussed harmonic motion, the limits of the radial motion (turning points) are given by setting $\dot{r} = 0$ in Equation 6.11.1b. These limits are, therefore, the roots of the equation

$$U(r) - E = 0 \qquad (6.11.2a)$$

or

$$\frac{ml^2}{2r^2} + V(r) - E = 0 \qquad (6.11.2b)$$

Furthermore, the *allowed* values of r are those for which $U(r) \leq E$, because $\dot{r}^2$ is necessarily positive or zero.

Thus, it is possible to determine the range of the radial motion without knowing anything about the orbit. A plot of $U(r)$ is shown in Figure 6.11.1. Also shown are the radial limits r_0 and r_1 for a particular value of the total energy E. The graph is drawn for the inverse-square law, namely,

$$U(r) = \frac{ml^2}{2r^2} - \frac{k}{r} \qquad (6.11.3)$$

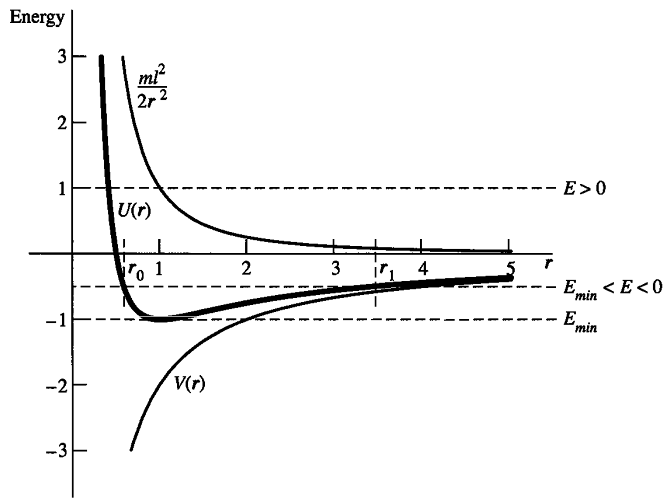

Figure 6.11.1 The effective potential for the inverse-square law of force and limits of the radial motion.

In this case Equation 6.11.2a, on rearranging terms, becomes

$$-2Er^2 - 2kr + ml^2 = 0 \tag{6.11.4}$$

which is a quadratic equation in r. The two roots are

$$r_{1,0} = \frac{k \pm (k^2 + 2Eml^2)^{1/2}}{-2E} \tag{6.11.5}$$

giving the maximum (upper sign) and minimum (lower sign) values of the radial distance r under the inverse-square law of force.

When $E < 0$, the orbits are bound, the two roots are both positive, and the resulting orbit is an ellipse in which r_0 and r_1 are the pericenter and apocenter, respectively. When the energy is equal to its minimum possible value

$$E_{min} = -\frac{k^2}{2ml^2} \tag{6.11.6}$$

Equation 6.11.5 then has a single root given by

$$r_0 = -\frac{k}{2E_{min}} \tag{6.11.7}$$

and the orbit is a circle. Note, that this result can also be obtained from Equation 6.10.10 ($a = -k/2E$) because $r_0 = a$ in the case of a circular orbit. When $E \geq 0$, Equation 6.11.5 has only a single, positive real root corresponding to a parabola ($E = 0$) or a hyperbola ($E > 0$).

Because the effective potential of the particle is axially symmetric, its shape in two dimensions can be formed by rotating the curve in Figure 6.11.1 about the vertical

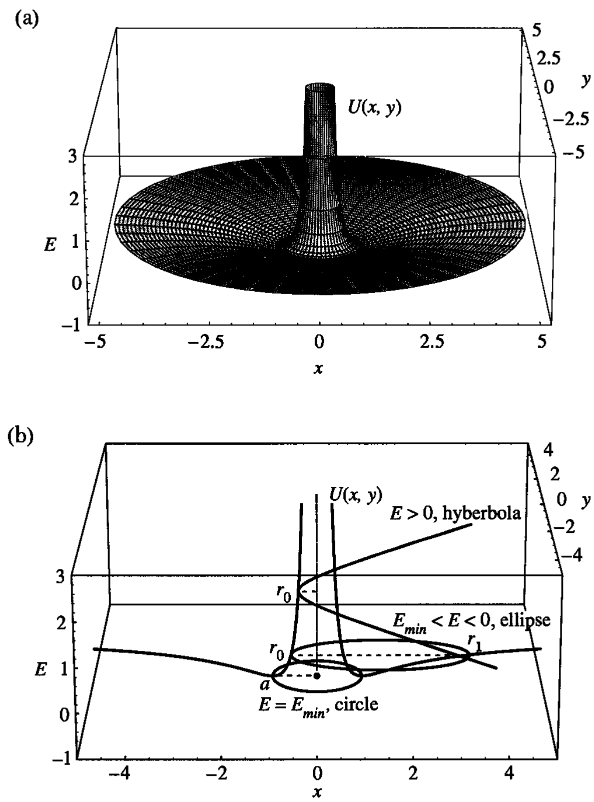

Figure 6.11.2 (a) The effective potential for the inverse-square law of force in two dimensions. (b) The relationship between total energy E, the effective potential, and the resulting orbits.

(Energy) axis. The resulting two-dimensional shape is shown in Figure 6.11.2a. The orbit of the particle can be visualized as taking place in that figure, constrained to a horizontal plane of constant energy E. If the energy of the particle is $E < 0$, the intersection of the plane and the effective potential surface forms an inner circle of radius r_0 that marks a central, impenetrable centrifugal barrier and an outer circle of radius r_1, that marks the farthest point of escape from the center of attraction. The motion of the particle is bound to a region between these two limits. If the energy of the particle is equal to its minimum possible value $E = E_{min}$, these two circles converge to a single one that traces out the minimum of the effective potential surface. The particle is constrained to that circle. If the energy of the particle is $E \geq 0$, the intersection forms only a single circle of radius r_0 around the centrifugal barrier and the particle is only inhibited from passing into that region; otherwise, it is free to escape to infinity along either a parabolic or hyperbolic trajectory.

Figure 6.11.2b suppresses the display of the effective potential *mesh* in Figure 6.11.2a (except for the two radial curves in the $\pm x$ direction) and shows the circular, elliptical, and hyperbolic orbits that result when the energy (in appropriately scaled units) is equal to -1, $-\frac{1}{2}$, and $+1$ units, respectively. Note that, in all cases, the particle is constrained to move in a region of its plane of constant energy, in which the value of its energy exceeds the value of the effective potential at that point.

EXAMPLE 6.11.1

Find the semimajor axis of the orbit of the comet of Example 6.10.1.

Solution:

Equation 6.10.10 gives directly

$$a = \frac{k}{-2E} = \frac{GM_{\odot}m}{-2\left(\frac{mv^2}{2} - \frac{GM_{\odot}m}{r}\right)}$$

where m is the mass of the comet. Clearly, m again cancels out. Also, as stated previously, $GM_{\odot} = a_e v_e^2$. So the final result is the simple expression

$$a = \frac{a_e}{(2/\mathsf{R}) - \mathsf{V}^2}$$

where R and V are as defined in Example 6.10.1.

For the previous numerical values, $\mathsf{R} = 4$ and $\mathsf{V} = 0.5$, we find $a = a_e/[0.5 - (0.5)^2] = 4a_e$.

Examples 6.10.1 and 6.10.2 bring out an important fact, namely, that the orbital parameters are independent of the mass of a body. Given the same initial position, speed, and direction of motion, a grain of sand, a coasting spaceship, or a comet would all have identical orbits, provided that no other bodies came near enough to have an effect on the motion of the body. (We also assume, of course, that the mass of the body in question is small compared with the Sun's mass.)

6.12 | Nearly Circular Orbits in Central Fields: Stability

A circular orbit is possible under any attractive central force, but not all central forces result in *stable* circular orbits. We wish to investigate the following question: If a particle traveling in a circular orbit suffers a slight disturbance, does the ensuing orbit remain close to the original circular path? To answer the query, we refer to the radial differential equation of motion (Equation 6.5.2a). Because $\dot{\theta} = l/r^2$, we can write the radial equation as follows:

$$m\ddot{r} = \frac{ml^2}{r^3} + f(r) \tag{6.12.1}$$

[This is the same as the differential equation for one-dimensional motion under the effective potential $U(r) = (ml^2/2r^2) + V(r)$, so that $m\ddot{r} = -dU(r)/dr = (ml^2/r^3) - dV(r)/dr$.]

Now for a circular orbit, r is constant, and $\ddot{r} = 0$. Thus, calling a the radius of the circular orbit, we have

$$-\frac{ml^2}{a^3} = f(a) \tag{6.12.2}$$

for the force at $r = a$. It is convenient to express the radial motion in terms of the variable x defined by

$$x = r - a \tag{6.12.3}$$

The differential equation for radial motion then becomes

$$m\ddot{x} = ml^2(x+a)^{-3} + f(x+a) \tag{6.12.4}$$

Expanding the two terms involving $x + a$ as power series in x, we obtain

$$m\ddot{x} = ml^2a^{-3}\left(1 - 3\frac{x}{a} + \cdots\right) + [f(a) + f'(a)x + \cdots] \tag{6.12.5}$$

Equation 6.12.5, by virtue of the relation shown in Equation 6.12.2, reduces to

$$m\ddot{x} + \left[\frac{-3}{a}f(a) - f'(a)\right]x = 0 \tag{6.12.6}$$

if we ignore terms involving x^2 and higher powers of x. Now, if the coefficient of x (the quantity in brackets) in Equation 6.12.6 is positive, then the equation is the same as that of the simple harmonic oscillator. In this case the particle, if perturbed, oscillates harmonically about the circle $r = a$, so the circular orbit is a stable one. On the other hand, if the coefficient of x is negative, the motion is nonoscillatory, and the result is that x eventually increases exponentially with time; the orbit is unstable. (If the coefficient of x is zero, then higher terms in the expansion must be included to determine the stability.) Hence, we can state that a circular orbit of radius a is stable if the force function $f(r)$ satisfies the inequality

$$f(a) + \frac{a}{3}f'(a) < 0 \tag{6.12.7}$$

For example, if the radial force function is a power law, namely,

$$f(r) = -cr^n \tag{6.12.8}$$

then the condition for stability reads

$$-ca^n - \frac{a}{3}cna^{n-1} < 0 \tag{6.12.9}$$

which reduces to

$$n > -3 \tag{6.12.10}$$

Thus, the inverse-square law ($n = -2$) gives stable circular orbits, as does the law of direct distance ($n = 1$). The latter case is that of the two-dimensional isotropic harmonic oscillator. For the inverse–fourth power ($n = -4$) circular orbits are unstable. It can be shown that circular orbits are also unstable for the inverse-cube law of force ($n = -3$). To show this it is necessary to include terms of higher power than 1 in the radial equation. (See Problem 6.26.)

6.13 Apsides and Apsidal Angles for Nearly Circular Orbits

An *apsis*, or *apse*, is a point in an orbit at which the radius vector assumes an extreme value (maximum or minimum). The perihelion and aphelion points are the apsides of planetary orbits. The angle swept out by the radius vector between two consecutive apsides is called the *apsidal angle*. Thus, the apsidal angle is π for elliptic orbits under the inverse-square law of force.

In the case of motion in a nearly circular oribt, we have seen that r oscillates about the circle $r = a$ (if the orbit is stable). From Equation 6.12.6 it follows that the period τ_r of this oscillation is given by

$$\tau_r = 2\pi \left[\frac{m}{-(3/a)f(a) - f'(a)} \right]^{1/2} \tag{6.13.1}$$

The apsidal angle in this case is just the amount by which the polar angle θ increases during the time that r oscillates from a minimum value to the succeeding maximum value. This time is clearly $\tau_r/2$. Now $\dot{\theta} = l/r^2$; therefore, $\dot{\theta}$ remains approximately constant, and we can write

$$\dot{\theta} \approx \frac{l}{a^2} = \left[-\frac{f(a)}{ma} \right]^{1/2} \tag{6.13.2}$$

The last step in Equation 6.13.2 follows from Equation 6.12.2; hence, the apsidal angle is given by

$$\psi = \tfrac{1}{2}\tau_r \dot{\theta} = \pi \left[3 + a\frac{f'(a)}{f(a)} \right]^{-1/2} \tag{6.13.3}$$

Thus, for the power law of force $f(r) = -cr^n$, we obtain

$$\psi = \pi(3+n)^{-1/2} \tag{6.13.4}$$

The apsidal angle is independent of the size of the orbit in this case. The orbit is *reentrant*, or repetitive, in the case of the inverse-square law ($n = -2$) for which $\psi = \pi$ and in the case of the linear law ($n = 1$) for which $\psi = \pi/2$. If, however, say $n = 2$, then $\psi = \pi/\sqrt{5}$, which is an irrational multiple of π, and so the motion does not repeat itself.

If the law of force departs slightly from the inverse-square law, then the apsides either advances or regresses steadily, depending on whether the apsidal angle is slightly greater or slightly less than π. (See Figure 6.13.1.)

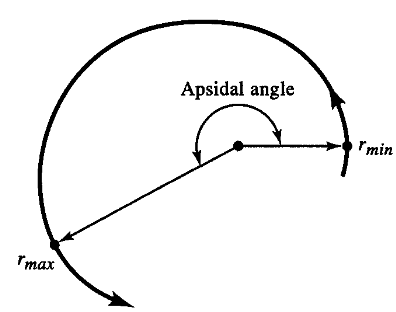

Figure 6.13.1 Illustrating the apsidal angle.

EXAMPLE 6.13.1

Let us assume that the gravitational force field acting on the planet Mercury takes the form

$$f(r) = -\frac{k}{r^2} + \epsilon r$$

where ϵ is very small. The first term is the gravitational field due to the Sun, and the second term is a repulsive perturbation due to a surrounding ring of matter. We assume this matter distribution as a simple model to represent the gravitational effects of all the other planets, primarily Jupiter. The perturbation is linear for points near the Sun and in the plane of the surrounding ring, as previously explained in Example 6.7.2. The apsidal angle, from Equation 6.13.3, is

$$\begin{aligned}\Psi &= \pi\left(3 + a\frac{2ka^{-3} + \epsilon}{-ka^{-2} + \epsilon a}\right)^{-1/2} \\ &= \pi\left(\frac{1 - 4k^{-1}\epsilon a^3}{1 - k^{-1}\epsilon a^3}\right)^{-1/2} = \pi\left(1 - \frac{\epsilon}{k}a^3\right)^{1/2}\left(1 - 4\frac{\epsilon}{k}a^3\right)^{-1/2} \\ &\approx \pi\left(1 - \tfrac{1}{2}\frac{\epsilon}{k}a^3\right)\left(1 + 2\frac{\epsilon}{k}a^3\right) \\ &\approx \pi\left(1 + \tfrac{3}{2}\frac{\epsilon}{k}a^3\right)\end{aligned}$$

In the last step, we used the binomial expansion theorem to expand the terms in brackets in powers of ϵ/k and kept only the first-order term. The apsidal angle advances if ϵ is positive and regresses if it is negative.

By 1877, Urbain Leverrier, using perturbation methods, had succeeded in calculating the gravitational effects of all the known planets on one another's orbit. Depending on the planet, the apsidal angles were found to advance or regress in good agreement with theory with the sole exception of the planet Mercury. Observations of Mercury's solar transits since 1631 indicated an advance of the perihelion of its orbit by 565″ of arc per century. According to Leverrier, it should advance only 527″ per century, a discrepancy of 38″. Simon Newcomb (1835–1909), chief of the office for the American Nautical Almanac, improved Leverrier's calculations, and by the beginning of the 20th century, the accepted values for the advance of Mercury's perihelion per century were 575″ and 534″, respectively, or a discrepancy of 41″ ± 2″ of arc. Leverrier himself had decided that the discrepancy was real and that it could be accounted for by an as yet unseen planet with a diameter of about 1000 miles circling the Sun within Mercury's orbit at a distance of about 0.2 AU. (You can easily extend the preceding example to show that an interior planet would lead to an advance in the perihelion of Mercury's orbit by the factor δ/ka^2.) Leverrier called the unseen planet Vulcan. No such planet was found.

Another possible explanation was put forward by Asaph Hall (1829–1907), the discoverer of the satellites of Mars in 1877. He proposed that the exponent in Newton's law of gravitation might not be exactly 2, that instead, it might be 2.0000001612 and that this would do the trick. Einstein was to comment that the discrepancy in Mercury's orbit "could be explained by means of classic mechanics only on the assumption of hypotheses which have little probability and which were devised solely for this purpose." The discrepancy, of course, was nicely explained by Einstein himself in a paper presented to the Berlin Academy in 1915. The paper was based on Einstein's calculations of general relativity even before he had fully completed the theory. Thus, here we have the highly remarkable event of a discrepancy between observation and existing theory leading to the confirmation of an entirely new superceding theory.

If the Sun were oblate (football-shaped) enough, its gravitational field would depart slightly from an inverse-square law, and the perihelion of Mercury's orbit would advance. Measurements to date have failed to validate this hypothesis as a possible explanation. Similar effects, however, have been observed in the case of artificial satellites in orbit about Earth. Not only does the perihelion of a satellite's orbit advance, but the plane of the orbit precesses if the satellite is not in Earth's equatorial plane. Detailed analysis of these orbits shows that Earth is basically "pear-shaped and somewhat lumpy."

6.14 Motion in an Inverse-Square Repulsive Field: Scattering of Alpha Particles

Ironically one of the crowning achievements of Newtonian mechanics contained its own seeds of destruction. In 1911, Ernest Rutherford (1871–1937), attempting to solve the problem of the scattering of alpha particles by thin metal foils, went for help back to the very source of classical mechanics, the *Principia* of Sir Isaac Newton. Paradoxically, in the process of finding a solution to the problem based on classical mechanics, the idea of the nuclear atom was born, an idea that would forever remain incomprehensible within the confines of the classic paradigm. A complete, self-consistent theory of the nuclear atom would emerge only when many of the notions of Newtonian mechanics were given up and replaced by the novel and astounding concepts of quantum mechanics. It is not that Newtonian mechanics was "wrong"; its concepts, which worked so well time and again when applied to the macroscopic world of falling balls and orbiting planets, simply broke down when applied to the microscopic world of atoms and nuclei. Indeed, the architects of the laws of quantum physics constructed them in such a way that the results of calculations based on the new laws agreed with those of Newtonian mechanics when applied to problems in the macroscopic world. The domain of Newtonian physics would be seen to be merely limited, rather than "wrong," and its practitioners from that time on would now have to be aware of these limits.

In the early 1900s the atom was thought to be a sort of distributed blob of positive charge within which were embedded the negatively charged electrons discovered in 1897 by J. J. Thomson (1856–1940). The model was first suggested by Lord Kelvin (1824–1907) in 1902 but mathematically refined a year later by Thomson. Thomson developed the model with emphasis on the mechanical and electrical stability of the system. In his honor it became known as the *Thomson atom.*

In 1907, Rutherford accepted a position at the University of Manchester where he encountered Hans Geiger (1882–1945), a bright young German experimental physicist, who was about to embark on an experimental program designed to test the validity of the Thomson atom. His idea was to direct a beam of the recently discovered alpha particles emitted from radioactive atoms toward thin metal foils. A detailed analysis of the way they scattered should provide information on the structure of the atom. With the help of Ernest Marsden, a young undergraduate, Geiger would carry out these investigations over several years. Things behaved more or less as expected, except there were many more large-angle scatterings than could be accounted for by the Thomson model. In fact, some of the alpha particles scattered completely backward at angles of 180°. When Rutherford heard of this, he was dumbfounded. It was as though an onrushing freight train had been hurled backward on striking a chicken sitting in the middle of the track.

In searching for a model that would lead to such a large force being exerted on a fast-moving projectile, Rutherford envisioned a comet swinging around the Sun and coming back out again, just like the alpha particles scattered at large angles. This suggested the idea of a hyperbolic orbit for a positively charged alpha particle attracted by a negatively charged nucleus. Of course, Rutherford realized that the only important thing in the dynamics of the problem was the inverse-square nature of the law, which, as we have seen, leads to conic sections as solutions for the orbit. Whether the force is attractive or repulsive is completely irrelevant. Rutherford then remembered a theorem about conics from geometry that related the eccentricity of the hyperbola to the angle between its asymptotes. Using this relation, along with conservation of angular momentum and energy, he obtained a complete solution to the alpha particle-scattering problem, which agreed well with the data of Geiger and Marsden. Thus, the current model of the nuclear atom was born.

We solve this problem next but be aware that an identical solution could be obtained for an attractive force. The solution says nothing about the sign of the nuclear charge. The sign becomes obvious from other arguments.

Consider a particle of charge q and mass m (the incident high-speed particle) passing near a heavy particle of charge Q (the nucleus, assumed fixed). The incident particle is repelled with a force given by Coulomb's law:

$$f(r) = \frac{Qq}{r^2} \tag{6.14.1}$$

where the position of Q is taken to be the origin. (We shall use cgs electrostatic units for Q and q. Then r is in centimeters, and the force is in dynes.) The differential equation of the orbit (Equation 6.5.12) then takes the form

$$\frac{d^2u}{d\theta^2} + u = -\frac{Qq}{ml^2} \tag{6.14.2}$$

and so the equation of the orbit is

$$u^{-1} = r = \frac{1}{A\cos(\theta - \theta_0) - Qq/ml^2} \tag{6.14.3}$$

We can also write the equation of the orbit in the form given by Equation 6.10.7c, namely,

$$r = \frac{ml^2Q^{-1}q^{-1}}{-1+(1+2Eml^2Q^{-2}q^{-2})^{1/2}\cos(\theta-\theta_0)} \tag{6.14.4}$$

because $k = -Qq$. The orbit is a hyperbola. This may be seen from the physical fact that the energy E is always greater than zero in a repulsive field of force. (In our case $E = \frac{1}{2}mv^2 + Qq/r$.) Hence, the eccentricity ϵ, the coefficient of $\cos(\theta - \theta_0)$, is greater than unity, which means that the orbit must be hyperbolic.

The incident particle approaches along one asymptote and recedes along the other, as shown in Figure 6.14.1. We have chosen the direction of the polar axis such that the initial position of the particle is $\theta = 0$, $r = \infty$. It is clear from either of the two equations of the orbit that r assumes its minimum value when $\cos(\theta - \theta_0) = 1$, that is, when $\theta = \theta_0$. Because $r = \infty$ when $\theta = 0$, then r is also infinite when $\theta = 2\theta_0$. Hence, the angle between the two asymptotes of the hyperbolic path is $2\theta_0$, and the angle θ_s through which the incident particle is deflected is given by

$$\theta_s = \pi - 2\theta_0 \tag{6.14.5}$$

Furthermore, in Equation 6.14.4 the denominator vanishes at $\theta = 0$ and $\theta = 2\theta_0$. Thus,

$$-1 + (1 + 2Eml^2Q^{-2}q^{-2})^{1/2}\cos\theta_0 = 0 \tag{6.14.6}$$

from which we readily find

$$\tan\theta_0 = (2Em)^{1/2}lQ^{-1}q^{-1} = \cot\frac{\theta_s}{2} \tag{6.14.7}$$

The last step follows from the angle relationship given above.

In applying Equation 6.14.7 to scattering problems, the constant l is usually expressed in terms of another quantity b called the *impact parameter*. The impact parameter is the

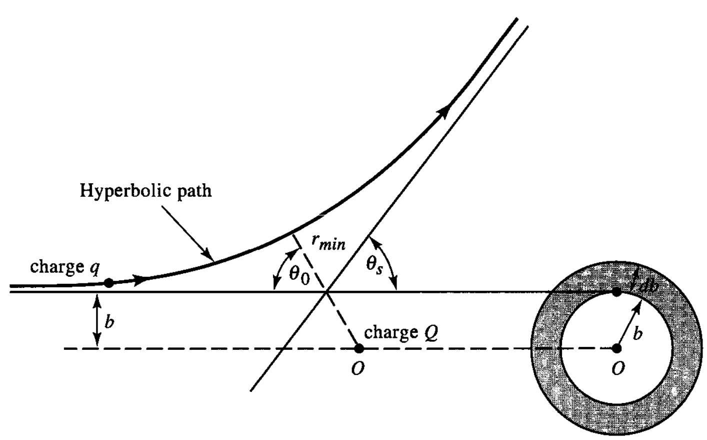

Figure 6.14.1 Hyperbolic path (orbit) of a charged particle moving in the inverse-square repulsive force field of another charged particle.

perpendicular distance from the origin (scattering center) to the initial line of motion of the particle, as shown in Figure 6.14.1. Thus

$$|l| = |\mathbf{r} \times \mathbf{v}| = bv_0 \tag{6.14.8}$$

where v_0 is the initial speed of the particle. We know also that the energy E is constant and is equal to the initial kinetic energy $\frac{1}{2}mv_0^2$, because the initial potential energy is zero ($r = \infty$). Accordingly, we can write the scattering formula (Equation 6.14.7) in the form

$$\cot\frac{\theta_s}{2} = \frac{bmv_0^2}{Qq} = \frac{2bE}{Qq} \tag{6.14.9}$$

giving the relationship between the scattering angle and the impact parameter.

In a typical scattering experiment a beam of particles is projected at a target, such as a thin foil. The nuclei of the target atoms are the scattering centers. The fraction of incident particles that are deflected through a given angle θ_s can be expressed in terms of a *differential scattering cross section* $\sigma(\theta_s)$ defined by the equation

$$\frac{dN}{N} = n\sigma(\theta_s)\,d\Omega \tag{6.14.10}$$

Here dN is the number of incident particles scattered through an angle between θ_s and $\theta_s + d\theta_s$, N is the total number of incident particles, n is the number of scattering centers per unit area of the target foil, and $d\Omega$ is the element of solid angle corresponding to the increment $d\theta_s$. Thus, $d\Omega = 2\pi \sin\theta_s\, d\theta_s$.

Now an incident particle approaching a scattering center has an impact parameter lying between b and $b + db$ if the projection of its path lies in a ring of inner radius b and outer radius $b + db$ (see Figure 6.14.1). The area of this ring is $2\pi b\, db$. The total number of such particles must correspond to the number scattered through a given angle, that is

$$dN = Nn\sigma(\theta_s)2\pi \sin\theta_s\, d\theta_s = Nn2\pi b\, db \tag{6.14.11}$$

Thus,

$$\sigma(\theta_s) = \frac{b}{\sin\theta_s}\left|\frac{db}{d\theta_s}\right| \tag{6.14.12}$$

To find the scattering cross section for charged particles, we differentiate with respect to θ_s in Equation 6.14.9:

$$\frac{1}{2\sin^2\left(\frac{\theta_s}{2}\right)} = \frac{2E}{Qq}\left|\frac{db}{d\theta_s}\right| \tag{6.14.13}$$

(The absolute value sign is inserted because the derivative is negative.) By eliminating b and $|db/d\theta_s|$ among Equations 6.14.9, .12, and .13 and using the identity

$$\sin\theta_s = 2\sin(\theta_s/2)\cos(\theta_s/2),$$

we find the following result:

$$\sigma(\theta_s) = \frac{Q^2q^2}{16E^2}\frac{1}{\sin^4(\theta_s/2)} \tag{6.14.14}$$

This is the famous Rutherford scattering formula. It shows that the differential cross section varies as the inverse fourth power of $\sin(\theta_s/2)$. Its experimental verification in the first part of this century marked one of the early milestones of nuclear physics.

EXAMPLE 6.14.1

An alpha particle emitted by radium ($E = 5$ million eV $= 5 \times 10^6 \times 1.6 \times 10^{-12}$ erg) suffers a deflection of 90° on passing near a gold nucleus. What is the value of the impact parameter?

Solution:

For alpha particles $q = 2e$, and for gold $Q = 79e$, where e is the elementary charge. (The charge carried by a single electron is $-e$.) In cgs units $e = 4.8 \times 10^{-10}$ esu. Thus, from Equation 6.14.9

$$b = \frac{Qq}{2E}\cot 45^\circ = \frac{2\times 79\times(4.8)^2\times 10^{-20}\ \text{cm}}{2\times 5\times 1.6\times 10^{-6}}$$
$$= 2.1\times 10^{-12}\ \text{cm}$$

EXAMPLE 6.14.2

Calculate the distance of closest approach of the alpha particle in Example 6.14.1.

Solution:

The distance of closest approach is given by the equation of the orbit (Equation 6.14.4) for $\theta = \theta_0$; thus,

$$r_{min} = \frac{ml^2Q^{-1}q^{-1}}{-1+(1+2Eml^2Q^{-2}q^{-2})^{1/2}}$$

On using Equation 6.14.9 and a little algebra, the preceding equation can be written

$$r_{min} = \frac{b\cot(\theta_s/2)}{-1+[1+\cot^2(\theta_s/2)]^{1/2}} = \frac{b\cos(\theta_s/2)}{1-\sin(\theta_s/2)}$$

Thus, for $\theta_s = 90°$, we find $r_{min} = 2.41\, b = 5.1 \times 10^{-12}$ cm.

Notice that the expressions for r_{min} become indeterminate when $l = b = 0$. In this case the particle is aimed directly at the nucleus. It approaches the nucleus along a straight line, and, being continually repelled by the Coulomb force, its speed is reduced to zero when it reaches a certain point r_{min}, from which point it returns along the same straight line. The angle of deflection is 180°. The value of r_{min} in this case is found by using the fact that the energy E is constant. At the turning point the potential energy is Qq/r_{min}, and the kinetic energy is zero. Hence, $E = \frac{1}{2}mv_0^2 = Qq/r_{min}$, and

$$r_{min} = \frac{Qq}{E}$$

For radium alpha particles and gold nuclei we find $r_{min} \approx 10^{-12}$ cm when the angle of deflection is 180°. The fact that such deflections are actually observed shows that the order of magnitude of the radius of the nucleus is at least as small as 10^{-12} cm.

Problems

6.1 Find the gravitational attraction between two solid lead spheres of 1 kg mass each if the spheres are almost in contact. Express the answer as a fraction of the weight of either sphere. (The density of lead is 11.35 g/cm^3.)

6.2 Show that the gravitational force on a test particle inside a thin uniform spherical shell is zero
(a) By finding the force directly
(b) By showing that the gravitational potential is constant

6.3 Assuming Earth to be a uniform solid sphere, show that if a straight hole were drilled from pole to pole, a particle dropped into the hole would execute simple harmonic motion. Show also that the period of this oscillation depends only on the density of Earth and is independent of the size. What is the period in hours? ($R_{earth} = 6.4 \times 10^6$ m.)

6.4 Show that the motion is simple harmonic with the same period as the previous problem for a particle sliding in a straight, smooth tube passing obliquely through Earth. (Ignore any effects of rotation.)

6.5 Assuming a circular orbit, show that Kepler's third law follows directly from Newton's second law and his law of gravity: $GMm/r^2 = mv^2/r$.

6.6 (a) Show that the radius for a circular orbit of a synchronous (24-h) Earth satellite is about 6.6 Earth radii.
(b) The distance to the Moon is about 60.3 Earth radii. From this calculate the length of the sidereal month (period of the Moon's orbital revolution).

6.7 Show that the orbital period for an Earth satellite in a circular orbit just above Earth's surface is the same as the period of oscillation of the particle dropped into a hole drilled through Earth (see Problem 6.3).

6.8 Calculate Earth's velocity of approach toward the Sun, when it is at an extremum of the latus rectum through the Sun. Take the eccentricity of Earth's orbit to be $\frac{1}{60}$ and its semimajor axis to be 93,000,000 miles (see Figure 6.5.1).

6.9 If the solar system were embedded in a uniform dust cloud of density ρ, show that the law of force on a planet a distance r from the center of the Sun would be given by

$$F(r) = -\frac{GMm}{r^2} - \left(\frac{4}{3}\right)\pi \rho m G r$$

6.10 A particle moving in a central field describes the spiral orbit $r = r_0 e^{k\theta}$. Show that the force law is inverse cube and that θ varies logarithmically with t.

6.11 A particle moves in an inverse-cube field of force. Show that, in addition to the exponential spiral orbit of Problem 6.10, two other types of orbit are possible and give their equations.

6.12 The orbit of a particle moving in a central field is a circle passing through the origin, namely $r = r_0 \cos\theta$. Show that the force law is inverse–fifth power.